

## Richard G. Brown

# Precalculus with Discrete Mathematics and Data Analysis



2

McDougal Lippell  
Houghton Mifflin

STATE \_\_\_\_\_  
 PROVINCE \_\_\_\_\_  
 COUNTY \_\_\_\_\_  
 PARISH \_\_\_\_\_  
 SCHOOL DISTRICT \_\_\_\_\_  
 OTHER \_\_\_\_\_

Enter information  
in spaces  
to the left as  
instructed

[illegible]

1. Teachers should see that the pupil's name is clearly written in ink in the spaces above in every book issued.
2. The following terms should be used in recording the condition of the book: New; Good; Fair; Poor; Bad.



## *FEATURES AND BENEFITS*

### **Advanced Mathematics**

#### **Precalculus with Discrete Mathematics and Data Analysis**

To lay the groundwork for further study of mathematics at the college level, **all standard precalculus topics** are presented, as well as substantial new material on **discrete mathematics** and **data analysis**, pp. iii – xiii (contents).

Numerous **applications** lessons, examples, and exercises establish the importance of mathematics to everyday life and a variety of scientific and technical fields, pp. 43 – 48, 698 – 700.

**Integration of technology** throughout the lesson presentations, examples, and exercises fosters effective learning and prepares students for participation in a technological society, pp. 75, 131, 690.

The wide variety of types and multiple levels of **exercises** meet many different teaching and learning needs. New communication exercises, including **Reading, Writing, Discussion, and Visual Thinking** exercises, are labeled for easy reference, as are **application exercises**, pp. 107, 129, 154.

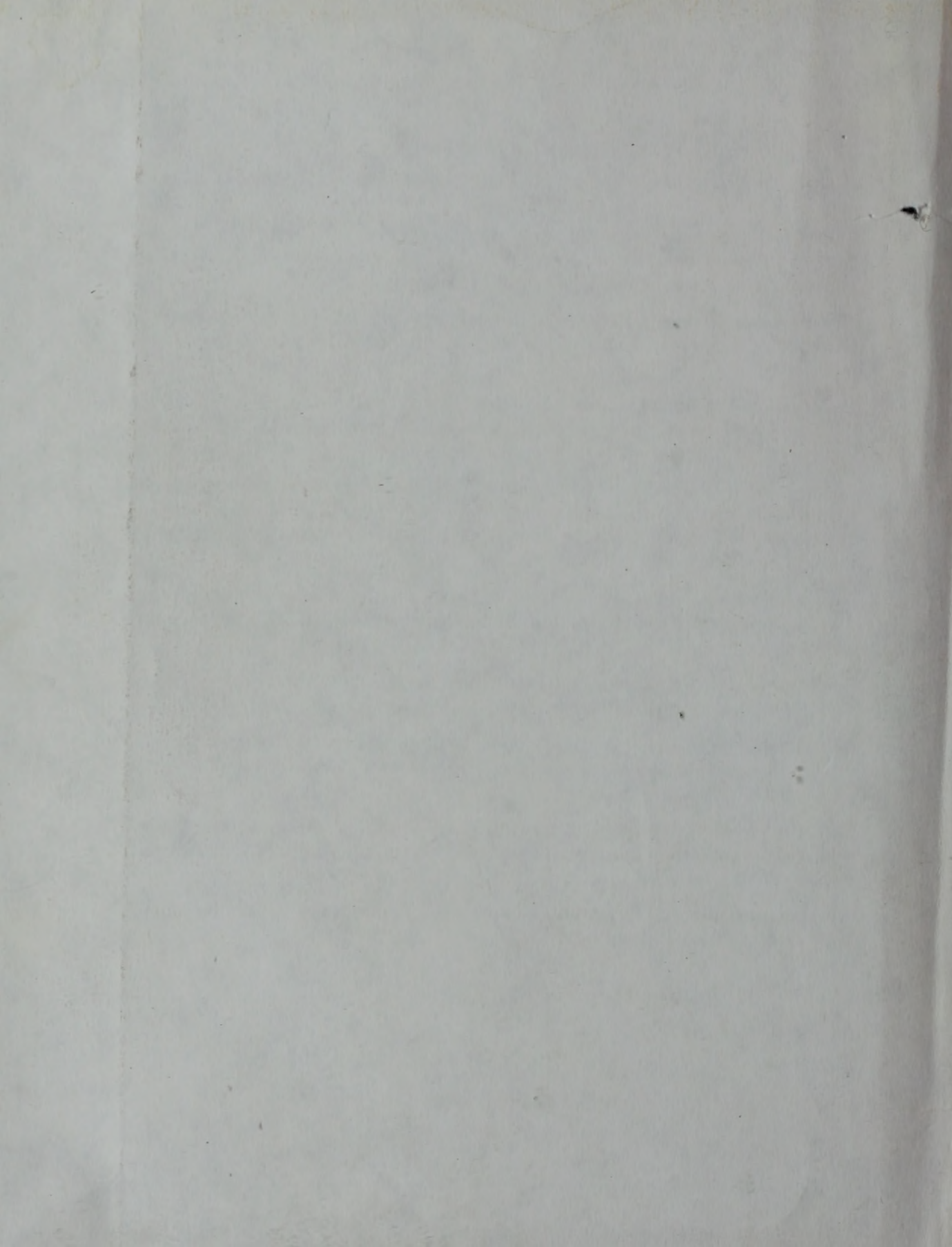
Numerous **worked-out examples**, **important results in tinted boxes**, **Class Exercises**, **Chapter Summaries**, **Chapter Tests**, and **Cumulative Reviews** make the text accessible and easy to use, pp. 126, 172, 176.

**Activities** in lesson presentations, **Investigation and Research exercises**, and end-of-chapter **Projects** are especially useful in promoting active learning, pp. 19, 156, 210.

The flexible outline, incorporation of technology, applications orientation, and provisions for active student learning meet **contemporary standards** (see Key Topics and Approaches, pp. xiv – xvii).

**Careers in Mathematics** pages, **Biographies**, and other text references, establish the **multicultural nature of mathematics** and its importance to people of varied interests and background today, pp. xxvi, 366, 537.

A section on **College Entrance Examinations** gives students an excellent opportunity to prepare for these important examinations, pp. 790 – 799.



# **A**dvanced **M**athematics

*Precalculus  
with Discrete Mathematics  
and Data Analysis*

**Richard G. Brown**



**Editorial  
Adviser**

Andrew M. Gleason

**Teacher  
Consultants**

Martha A. Brown  
Dane R. Camp  
Maria F. G. Fierro  
Wallis Green  
Linda Hunter  
Carolyn Kennedy

**McDougal Littell Inc.**

A Houghton Mifflin Company  
Evanston, Illinois Boston Dallas



## AUTHOR

**Richard G. Brown**, Mathematics Teacher, Phillips Exeter Academy, Exeter, New Hampshire. A teacher and author, Mr. Brown has taught a wide range of mathematics courses for both students and teachers at several schools and universities. His affiliations have included the Newton (Massachusetts) High School, the University of New Hampshire, Arizona State University, and the North Carolina School for Science and Mathematics during the school year beginning in 1983. Currently a member of the COMAP Consortium Council, he is an active participant in professional mathematics organizations and the author of mathematics texts and journal articles.

## EDITORIAL ADVISER

**Andrew M. Gleason**, Hallis Professor of Mathematics and Natural Philosophy, Harvard University. Professor Gleason is a well-known research mathematician and a member of the National Academy of Sciences. He has served as President of the American Mathematical Society.

## TEACHER CONSULTANTS

**Martha A. Brown**, Supervisor of Mathematics, Prince George's County Public Schools, Prince George's County, Maryland

**Dane R. Camp**, Mathematics Teacher, Dawner's Grove High School, Dawner's Grove, Illinois

**Maria F. G. Fierro**, Mathematics Department Chairperson, Cerritos High School, Cerritos, California

**Wallis Green**, Mathematics Teacher, C. E. Jordan High School, Durham, North Carolina

**Linda Hunter**, Mathematics Department Chairperson, Douglas MacArthur High School, San Antonio, Texas

**Carolyn Kennedy**, Mathematics Teacher, West High School, Columbus, Ohio

## TECHNOLOGY CONSULTANT

**Wade Ellis, Jr.**, Mathematics Instructor, West Valley College, Saratoga, California

## ACKNOWLEDGEMENTS

The author wishes to thank **Jonathan Choate**, Mathematics Teacher, Gratan School, Gratan, Massachusetts, for contributing the material on dynamical systems in Chapter 19. The author also wishes to thank **Donna DiFranco**, Professor of Mathematics, Bentley College, Waltham, Massachusetts, for contributing the projects that follow Chapters 5, 9, 12, 18, and 19.

Copyright © 1997, 1994 by Houghton Mifflin Company. All rights reserved.

No part of this work may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying and recording, or by any information storage or retrieval system without prior written permission of McDougal Littell Inc. unless such copying is expressly permitted by federal copyright law. Address inquiries to Manager, Rights and Permissions, McDougal Littell Inc., P.O. Box 1667, Evanston, IL 60204.



# Introduction

## Introduction

Dear Student,

Welcome to this course in *Advanced Mathematics*! This year you will explore many new topics as well as extend those you have studied in previous courses.

I have written this book with the goal of making mathematics clear, interesting, and relevant. As a result, you will see many real-world applications of the topics you study. For example:

- Exponential functions model population growth, decline of natural resources, and cost of a college education.
- Logarithmic functions measure the intensity of earthquakes, the loudness of music, and the brightness of stars.
- Trigonometric functions describe AM/FM radio waves, the pattern of the tides, and the daily change in the time of sunset.
- Discrete mathematics provides techniques for calculating the return on a financial investment, deciding which mix of products to manufacture, and predicting the course of a flu epidemic.
- Probability theory predicts potential gains and losses with car insurance, business decisions, and even multiple choice tests.
- Statistics helps organize and analyze many types of numerical information, such as percentile ranks, sports data, and advertising claims, that bombard us daily.

As you see the wide range of fields that use mathematics, think about how mathematics will play a role in your own future. Take full advantage of this course to prepare for the many opportunities ahead.

I'd like to offer you the advice that I offer my own students:

Math is not a spectator sport!

Don't just watch other people do mathematics! Stay actively involved by doing the activities, participating in classroom discussions and group work, reading the text and examples, and working on the exercises and projects.

I hope you find the course useful, stimulating, and enjoyable.

Sincerely,

*Richard G. Brown*  
Richard G. Brown



## CHAPTERS 1–6

## CHAPTERS 7-11

## CHAPTERS 12–18

## CHAPTERS 19–20



# Functions, Graphs, and Applications

## CHAPTER 1

### LINEAR AND QUADRATIC FUNCTIONS

#### Linear Functions


1-1	Points and Lines	1
1-2	Slopes of Lines	7
1-3	Finding Equations of Lines	14
1-4	Linear Functions and Models	19


#### Quadratic Functions

1-5	The Complex Numbers	25
1-6	Solving Quadratic Equations	30
1-7	Quadratic Functions and Their Graphs	37
1-8	Quadratic Models	43

<b>Activities</b>	10, 19
-------------------	--------

#### Technology

 Graphing Calculator/Software	3, 5, 36, 40, 41, 42
--	-------------------------

 Calculator Exercises	13, 29
--	--------

 Computer Exercises	29, 36
--	--------

#### TESTS AND REVIEW

Chapter Summary	48
Chapter Test	50

## CHAPTER 2

**POLYNOMIAL FUNCTIONS****Zeros and Factors of Polynomial Functions**

2-1	Polynomials	53
2-2	Synthetic Division; The Remainder and Factor Theorems	58

**Graphs; Maximums and Minimums**

2-3	Graphing Polynomial Functions	62
2-4	Finding Maximums and Minimums of Polynomial Functions	68


**Polynomial Equations**

2-5	Using Technology to Approximate Roots of Polynomial Equations	75
2-6	Solving Polynomial Equations by Factoring	80
2-7	General Results for Polynomial Equations	85


**Activities**

62, 63, 64

**Technology**

	Graphing Calculator/Software	62, 63, 64, 65, 67, 69, 72, 74, 75, 78, 93
---	------------------------------	--

	Calculator Exercises	74
---	----------------------	----

	Computer Exercises	58, 79
---	--------------------	--------

HISTORICAL DEVELOPMENT	88
------------------------	----

## TESTS AND REVIEW

Chapter Summary	91
-----------------	----

Chapter Test	93
--------------	----

## CHAPTER 3

**INEQUALITIES****Inequalities in One Variable**

3-1	Linear Inequalities; Absolute Value	95
3-2	Polynomial Inequalities in One Variable	100

**Inequalities in Two Variables**

3-3	Polynomial Inequalities in Two Variables	104
3-4	Linear Programming	108
<b>Activities</b>		101

**Activities****Technology**

	Graphing Calculator/Software	102, 104
---	------------------------------	----------

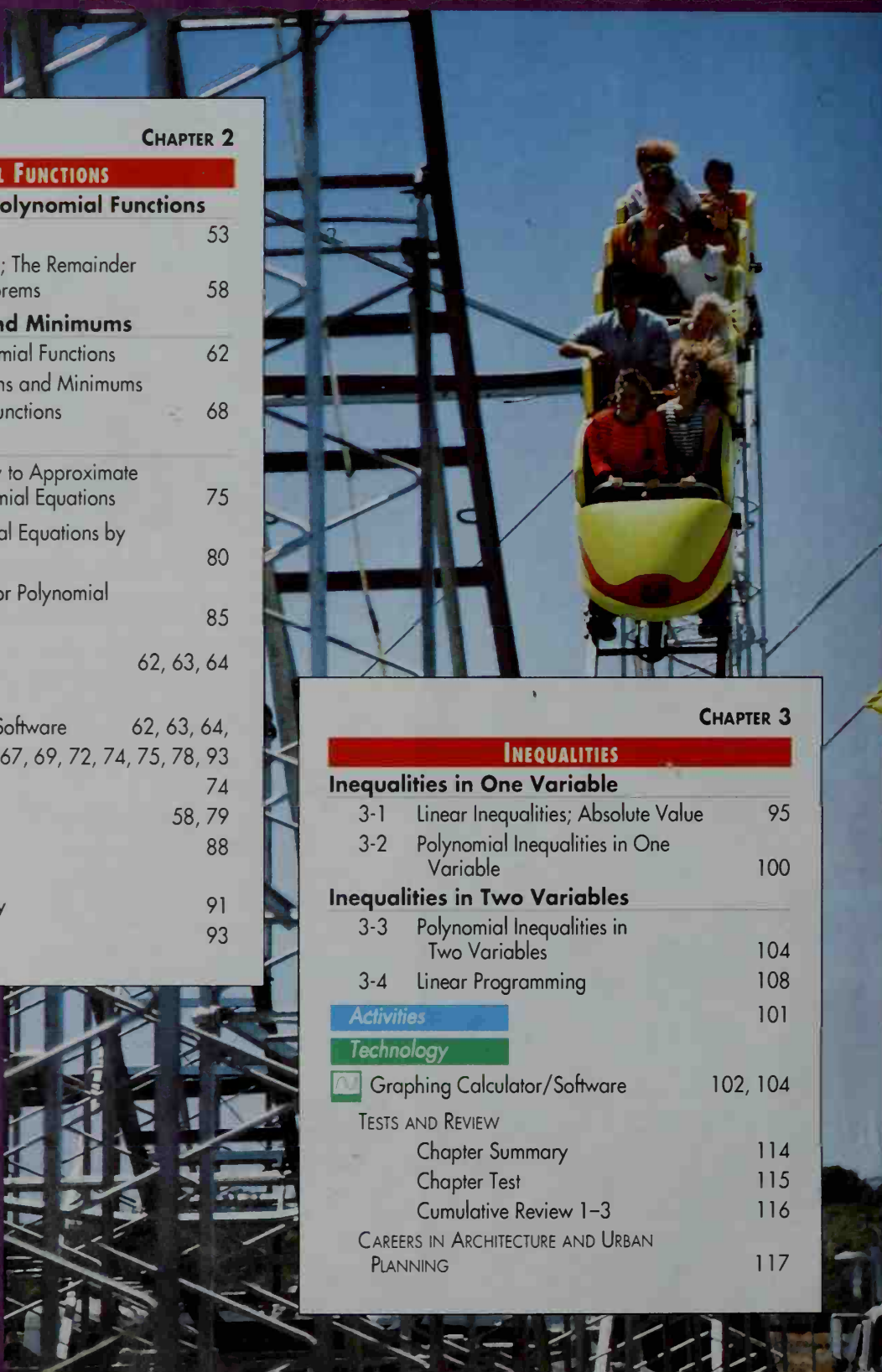
## TESTS AND REVIEW

Chapter Summary	114
-----------------	-----

Chapter Test	115
--------------	-----

Cumulative Review 1-3	116
-----------------------	-----

CAREERS IN ARCHITECTURE AND URBAN PLANNING	117
--	-----





**FUNCTIONS****Properties of Functions**

4-1	Functions	119
4-2	Operations on Functions	124

**Graphs and Inverses of Functions**


4-3	Reflecting Graphs; Symmetry	131
4-4	Periodic Functions; Stretching and Translating Graphs	138
4-5	Inverse Functions	146

**Applications of Functions**

4-6	Functions of Two Variables	151
4-7	Forming Functions from Verbal Descriptions	157

<b>Activities</b>	131
-------------------	-----

**Technology**

 Graphing Calculator/Software	127, 129, 130, 131, 138, 145, 150, 158, 162, 164, 167
--	---

 Calculator Exercises	150
--	-----

**TESTS AND REVIEW**

Chapter Summary	165
Chapter Test	166

## CHAPTER 5

**EXPONENTS AND LOGARITHMS****Exponents**


5-1	Growth and Decay: Integral Exponents	169
5-2	Growth and Decay: Rational Exponents	175
5-3	Exponential Functions	180
5-4	The Number $e$ and the Function $e^x$	186

**Logarithms**

5-5	Logarithmic Functions	191
5-6	Laws of Logarithms	197
5-7	Exponential Equations; Changing Bases	203

<b>Activities</b>	180, 187
-------------------	----------

**Technology**

 Graphing Calculator/Software	185, 187, 190, 195, 201, 205, 206, 207, 210
--	---

 Computer Exercises	190
--	-----

**TESTS AND REVIEW**

Chapter Summary	208
Chapter Test	209

**Project**

Newton's Law of Cooling	210
-------------------------	-----


## CHAPTER 6

**ANALYTIC GEOMETRY**

Introduction	213
6-1 Coordinate Proofs	214
6-2 Equations of Circles	219
6-3 Ellipses	225
6-4 Hyperbolas	231
6-5 Parabolas	238
6-6 Systems of Second-Degree Equations	242
6-7 A New Look at Conic Sections	247

<b>Activities</b>	226
-------------------	-----

**Technology**

 Graphing Calculator/Software	220, 223, 224, 228, 229, 230, 235, 241, 242, 245
---	--

 Computer Exercises	231
--	-----

**COMMUNICATION**

Reading	246
---------	-----

<b>BIOGRAPHICAL NOTE</b>	237
--------------------------	-----

**TESTS AND REVIEW**

Chapter Summary	252
Chapter Test	253
Cumulative Review 4-6	254





# Trigonometry

## CHAPTER 7

### TRIGONOMETRIC FUNCTIONS

#### Angles, Arcs, and Sectors



7-1	Measurement of Angles	257
7-2	Sectors of Circles	263

#### The Trigonometric Functions

7-3	The Sine and Cosine Functions	268
7-4	Evaluating and Graphing Sine and Cosine	275
7-5	The Other Trigonometric Functions	282
7-6	The Inverse Trigonometric Functions	286
		269

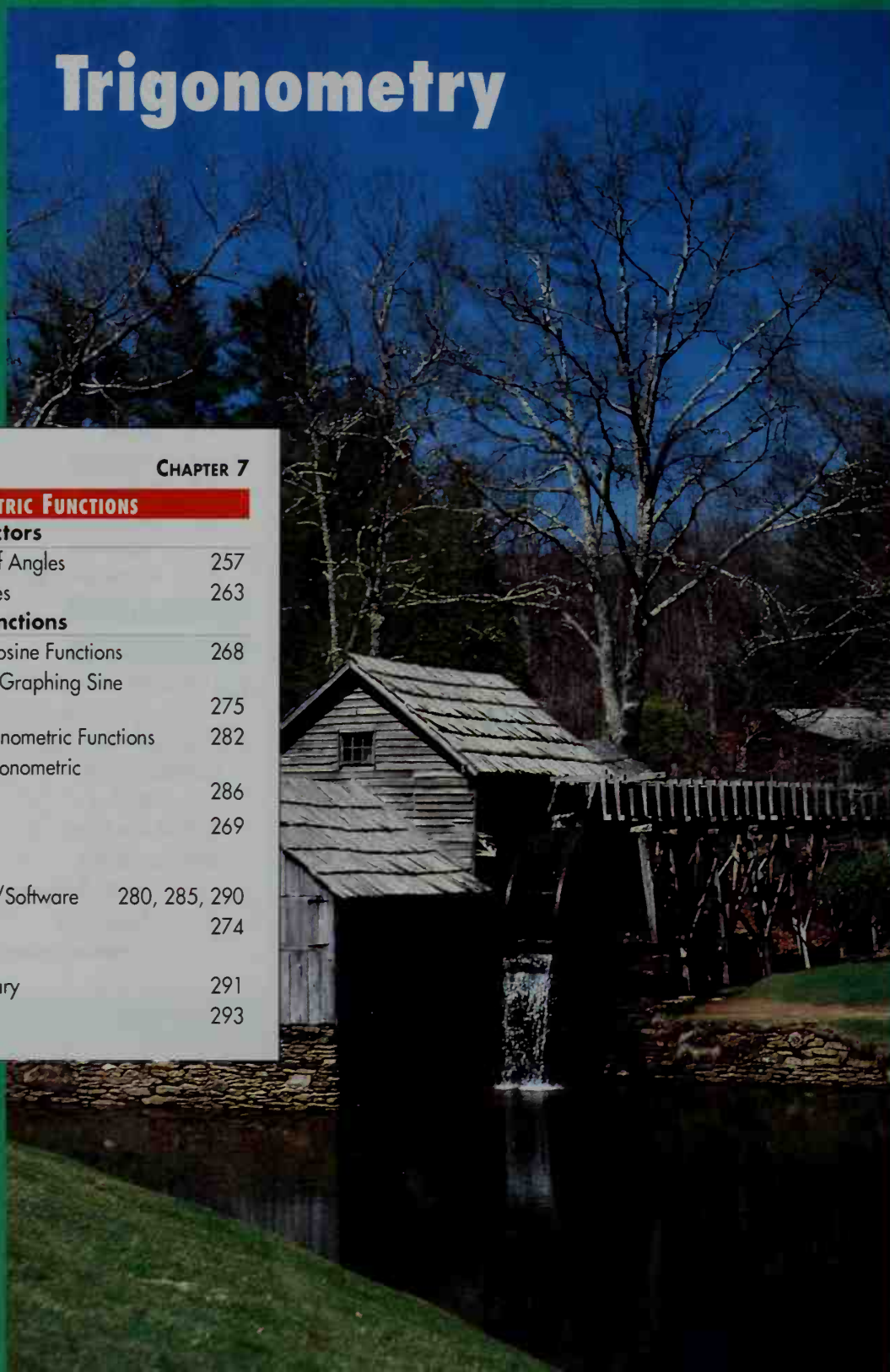
#### Activities

#### Technology

	Graphing Calculator/Software	280, 285, 290
	Computer Exercises	274

#### TESTS AND REVIEW

Chapter Summary	291
Chapter Test	293



## CHAPTER 8

**TRIGONOMETRIC EQUATIONS AND APPLICATIONS****Equations and Applications of Sine Waves**



8-1	Simple Trigonometric Equations	295
8-2	Sine and Cosine Curves	301
8-3	Modeling Periodic Behavior	308

**Identities and Equations**

8-4	Relationships Among the Functions	317
8-5	Solving More Difficult Trigonometric Equations	323

<b>Activities</b>	318, 319
-------------------	----------

**Technology**

	Graphing Calculator/Software	298, 300, 303, 306, 310, 314, 315, 319, 321, 325, 326, 327
	Computer Exercises	322
	BIOGRAPHICAL NOTE	323
	TESTS AND REVIEW	
	Chapter Summary	328
	Chapter Test	328



## CHAPTER 9

**TRIANGLE TRIGONOMETRY**

9-1	Solving Right Triangles	331
9-2	The Area of a Triangle	339
9-3	The Law of Sines	345
9-4	The Law of Cosines	350
	Mixed Trigonometry Exercises	355
9-5	Applications of Trigonometry to Navigation and Surveying	359

<b>Activities</b>	346
-------------------	-----

**Technology**

	Graphing Calculator/Software	334, 338, 340, 343, 344
	Computer Exercises	338, 355
	TESTS AND REVIEW	
	Chapter Summary	364
	Chapter Test	365
<b>Project</b>		
	The People of Mathematics	366



## CHAPTER 10

**TRIGONOMETRIC ADDITION FORMULAS**

10-1	Formulas for $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$	369
10-2	Formulas for $\tan(\alpha \pm \beta)$	375
10-3	Double-Angle and Half-Angle Formulas	380
10-4	Solving Trigonometric Equations	386

<b>Activities</b>	380
-------------------	-----

**Technology**

	Graphing Calculator/Software	374, 378, 384, 386, 388, 389, 390, 391
	Computer Exercises	391
	COMMUNICATION Discussion	379
	BIOGRAPHICAL NOTE	375
	TESTS AND REVIEW	
	Chapter Summary	391
	Chapter Test	393



## CHAPTER 11

**POLAR COORDINATES AND COMPLEX NUMBERS**

11-1	Polar Coordinates and Graphs	395
11-2	Geometric Representation of Complex Numbers	403
11-3	Powers of Complex Numbers	407
11-4	Roots of Complex Numbers	412

<b>Activities</b>	409
-------------------	-----

**Technology**

	Graphing Calculator/Software	397, 400, 401
	Computer Exercises	402
	BIOGRAPHICAL NOTE	411
	TESTS AND REVIEW	
	Chapter Summary	414
	Chapter Test	415
	Cumulative Review 7-11	416

# Discrete Mathematics and Data Analysis

## CHAPTER 12

### VECTORS AND DETERMINANTS

#### Properties and Basic Operations

12-1	Geometric Representation of Vectors	419
12-2	Algebraic Representation of Vectors	426
12-3	Vector and Parametric Equations: Motion in a Plane	432
12-4	Parallel and Perpendicular Vectors; Dot Product	441

#### Three Dimensions



12-5	Vectors in Three Dimensions	446
12-6	Vectors and Planes	452

#### Determinants and Their Applications

12-7	Determinants	458
12-8	Applications of Determinants	461
12-9	Determinants and Vectors in Three Dimensions	465

<b>Activities</b>	434, 459
-------------------	----------

#### Technology

	Graphing Calculator/Software	434, 437, 438, 439
	Computer Exercises	440, 467

#### TESTS AND REVIEW

Chapter Summary	468
Chapter Test	469

#### Project

Uniform Circular Motion	470
-------------------------	-----





**CHAPTER 13****SEQUENCES AND SERIES****Finite Sequences and Series**


13-1 Arithmetic and Geometric Sequences	473
13-2 Recursive Definitions	479
13-3 Arithmetic and Geometric Series and Their Sums	486

**Infinite Sequences and Series**

13-4 Limits of Infinite Sequences	493
13-5 Sums of Infinite Series	500
13-6 Sigma Notation	506
13-7 Mathematical Induction	510

**Activities** 474, 480

**Technology**

 Graphing Calculator/Software	480, 483, 485 486, 489
--	---------------------------

 Calculator Exercises	498, 506
--	----------

 Computer Exercises	499, 510
--	----------

BIOGRAPHICAL NOTE 499

## TESTS AND REVIEW

Chapter Summary	514
Chapter Test	515

**CHAPTER 14****MATRICES****Matrix Operations**


14-1 Matrix Addition and Scalar Multiplication	517
14-2 Matrix Multiplication	523
14-3 Applying Matrices to Linear Systems	530

**Applications of Matrices**

14-4 Communication Matrices	537
14-5 Transition Matrices	543
14-6 Transformation Matrices	551

**Activities** 553

**Technology**

 Graphing Calculator/Software	536, 541, 542, 547, 550
--	----------------------------

BIOGRAPHICAL NOTE 537

## TESTS AND REVIEW

Chapter Summary	560
Chapter Test	560
Cumulative Review 12-14	562

CHAPTER 15

**COMBINATORICS**

15-1	Venn Diagrams	565
15-2	The Multiplication, Addition, and Complement Principles	571
15-3	Permutations and Combinations	578
15-4	Permutations with Repetition; Circular Permutations	583
Mixed	Combinatorics Exercises	587
15-5	The Binomial Theorem; Pascal's Triangle	590

**Activities**

**Technology**

////	Computer Exercises	583
------	--------------------	-----

TESTS AND REVIEW

Chapter Summary	594
Chapter Test	595

CHAPTER 16

**PROBABILITY**

**Finding Probabilities**



16-1	Introduction to Probability	597
16-2	Probability of Events Occurring Together	605
16-3	The Binomial Probability Theorem	613
16-4	Probability Problems Solved with Combinations	619

**Applications of Probability**

16-5	Working with Conditional Probability	624
16-6	Expected Value	630

**Activities**

**Technology**

	Graphing Calculator/Software	618
	Computer Exercises	618
	COMMUNICATION	
	Reading	623
	TESTS AND REVIEW	
	Chapter Summary	636
	Chapter Test	636

**CHAPTER 17****STATISTICS****Descriptive Statistics**

17-1	Tables, Graphs, and Averages	639
17-2	Box-and-Whisker Plots	648
17-3	Variability	653
17-4	The Normal Distribution	660

**Inferential Statistics**

17-5	Sampling	669
17-6	Confidence Intervals for Surveys and Polls	674

**Activities**

662

**Technology**

////	Computer Exercises	674
------	--------------------	-----

	BIOGRAPHICAL NOTE	660
--	-------------------	-----

**TESTS AND REVIEW**

	Chapter Summary	680
--	-----------------	-----

	Chapter Test	681
--	--------------	-----


**CHAPTER 18****CURVE FITTING AND MODELS**

18-1	Introduction to Curve Fitting; The Least-Squares Line	683
18-2	Fitting Exponential Curves	691
18-3	Fitting Power Curves	698
18-4	Choosing the Best Model	705

**Activities**

700

**Technology**

	Graphing Calculator/Software	684, 689, 690, 693, 700, 705
---	------------------------------	---------------------------------

**TESTS AND REVIEW**

	Chapter Summary	709
--	-----------------	-----

	Chapter Test	710
--	--------------	-----

	Cumulative Review 15–18	711
--	-------------------------	-----

	CAREERS IN GENETICS AND STATISTICS	713
--	------------------------------------	-----

**Project**

	Preference Testing	714
--	--------------------	-----



# Limits and Introduction to Calculus

## CHAPTER 19

### LIMITS, SERIES, AND ITERATED FUNCTIONS

#### Limits

19-1	Limits of Functions	717
19-2	Graphs of Rational Functions	726

#### Series

19-3	Using Technology to Approximate the Area under a Curve	729
19-4	Power Series	733


#### Iterated Functions

19-5	Analyzing Orbits	737
19-6	Applications of Iterated Functions	744

#### Activities

726, 730, 737, 739

#### Technology

	Graphing Calculator/Software	722, 724, 726, 728, 729, 731, 745, 746, 748, 749, 750
--	------------------------------	---

BIOGRAPHICAL NOTE	736
-------------------	-----

#### TESTS AND REVIEW

Chapter Summary	748
Chapter Test	749

#### Project

Chaos in the Complex Plane	750
----------------------------	-----

CHAPTER 20

**AN INTRODUCTION TO CALCULUS**

20-1	The Slope of a Curve	757
20-2	Using Derivatives in Curve Sketching	764
20-3	Extreme Value Problems	769
20-4	Velocity and Acceleration	774
	<b>Activities</b>	769

COMMUNICATION		
	Visual Thinking	763
TESTS AND REVIEW		
	Chapter Summary	781
	Chapter Test	782
	Cumulative Review 19–20	783

GEOMETRY REVIEW	784
PROPERTIES OF THE REAL NUMBER SYSTEM	788
TRIGONOMETRY REVIEW	789
COLLEGE ENTRANCE EXAMINATIONS TABLES	790
800	
APPENDIX 1	
Graph Theory	822
APPENDIX 2	
The Median-Median Line	830
APPENDIX 3	
Descartes' Rule of Signs and Bounds for Real Roots	835
APPENDIX 4	
Partial Fractions	838
APPENDIX 5	
Tangents and Normals to Conic Sections	842
APPENDIX 6	
Graphing Calculator Applications	846
<b>Refresher Exercises</b>	854
LIST OF SYMBOLS	864
VISUAL GLOSSARY OF FUNCTIONS	866
GLOSSARY	868
INDEX	874
ANSWERS TO SELECTED EXERCISES	

# Applications

"When am I ever going to use this?"

Business  
Business Communications  
Consumer Economics  
Economics  
Finance  
Industrial Design  
Linear Programming  
Manufacturing  
Operations Research  
Statistics

Social Science

Archaeology  
Education  
Polling  
Psychology  
Transportation

- 37. Finance** A house bought for \$50,000 in 1980 was sold to the nearest percent, what was the annual rate of appreciation of the house?
- 38. Consumer Economics** The price of firewood four years ago was \$140 per cord. Today, a cord of wood costs \$182. To the nearest percent, what has been the annual rate of increase in the cost?
- 39. Economics** The consumer price index (CPI) is a measure of the average cost of goods and services. The United States government set the index at 100 for the period 1982-1984. In 1988, the index was 118.3. What was the average annual rate of increase (to the nearest tenth of a percent) from 1984 to 1988?
- 40. Research** Look in an almanac to find the current consumer price index. Then determine the average annual rate of increase (to the nearest tenth of a percent) for this index since 1984. (See Exercise 39.)
- 41. Education** Yearly expenses at a state university have increased from \$14,000 to \$18,500 in the last 4 years. What has been the average annual growth rate in expenses? If this growth rate continues, what will the expenses be 4 years from now?
- 42. Research** Find the present cost of something that interests you and its cost several years ago. Find the average annual growth rate.



Solve:

43. a.  $(4x)^3 = 9^6$       b.  $3^{4x} = 9^6$       44. a.  $(4 - x)^{1/2} = 2$   
45.  $\frac{2^{x^2}}{2^x} = 64$       46.  $\frac{8^{x^2}}{(5^x)^2} = 125$       47.  $\sqrt{\frac{9^{x^2+1}}{27^x}} = 81$       48.  $\sqrt{\frac{8^{x^2+1}}{16^x}} = 3$   
Factor. (In Exercise 49(a), factor out  $a^{1/2}b^{1/2}$ .)  
49. a.  $a^{1/2}b^{1/2} - a^{1/2}b^{1/2}$       b.  $a^{1/2}b^{-1/2} - a^{1/2}b^{-1/2}$   
50. a.  $(x-1)^{1/2} - (x-1)^{-1/2}$       b.  $(x+1)^{1/2} - (x+1)^{-1/2}$   
51. a.  $(x^2+1)^{1/2} - x^2(x^2+1)^{1/2}$       b.  $(x^2+2)^{1/2} - (x^2+2)^{-1/2}$   
52. a.  $(2x+1)^{2/3} - 4(2x+1)^{-1/3}$       b.  $(1+x^2)^{1/2} - (1+x^2)^{-1/2}$

Exponents

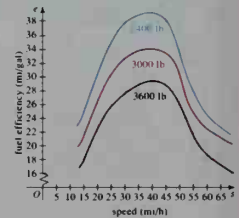
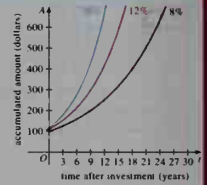
Daily Life

Carpentry  
Forestry  
Horticulture  
Landscaping  
Music  
Plumbing  
Recreation  
Sports

9. **Discussion** The amount that a person pays for auto insurance is a function of many variables. Discuss what some of these variables are.
10. a. If  $f(x, y) = \sqrt{x^2 + y^2}$ , find  $f(3, 4)$ ,  $f(-4, 3)$ , and  $f(0, 5)$ .  
b. Sketch some constant curves of the function  $f$ .
11. Name several pairs  $(a, b)$  not in the domain of  $f(a, b) = \frac{a+b}{a-b}$ .

## WRITTEN EXERCISES

- A 1. Consumer Economics** If \$100 is invested at interest rate  $r$  compounded annually, then the accumulated amount  $t$  years later is  $A = 100(1+r)^t$ .
- a. The formula shows that  $A$  is a function of  $t$  and  $r$ .
- b. The graphs of  $A$  versus  $t$  for several constant values of  $r$  are shown at the right. The equation of the rightmost curve is  $A = 100(1.08)^t$ . What are the equations of the other two curves?
- c. About how many years does it take to double your money at 8% interest compounded annually? at 12% interest? at 16% interest?
- 2. Consumer Economics** The graph at the right below shows the fuel efficiency at various speeds for 2400 lb, 3000 lb, and 3600 lb cars. The fuel efficiency is given in miles per gallon (mi/gal).
- a. Fuel efficiency is a function of  $t$  and  $w$ .
- b. At what other speed will a 3000 lb car have approximately the same fuel efficiency as it has when traveling at 25 mi/h?
- c. At what speeds does a 3600 lb car have approximately the same fuel efficiency as a 2400 lb car traveling at 55 mi/h?
- d. Regardless of what a car weighs, at approximately what speed is maximum fuel efficiency reached?



154 Chapter Four

15.  $|x - 4| < 3$       16.  $|x - 7| > 3$   
17.  $|x + 7| \geq 3$       18.  $|x + 2| < -1$   
19.  $|x - 8| = 4$       20.  $|x + 3| = 7$   
21.  $|2x - 4| \leq 5$       22.  $|3x - 9| \geq 9$   
23.  $|4x + 8| \leq 9$       24.  $|6 - 3x| < 12$

- B 25. Horticulture** Plant experts advise that gardens kept indoors must have high humidity, plenty of sunlight during the day, and cool temperatures at night. The recommended nighttime temperature range in degrees Fahrenheit is  $60^\circ \leq F \leq 65^\circ$ . Given that  $C = \frac{5}{9}(F - 32)$ , express the corresponding temperature range in degrees Celsius.



26. a. **Writing** Use the geometric definition of  $|x|$  and the definition of a combined inequality to write a sentence that gives the meaning of the expression  $2 < |x| < 4$  and graph its solution.  
b. Solve the inequality  $2 < |x| < 4$  and graph its solution.  
c. Solve the inequality  $1 \leq |x| < 5$  and graph its solution.

In Exercises 27-32, solve the given inequality and graph its solution.

27.  $1 \leq |x - 4| \leq 3$       28.  $2 < |x - 6| \leq 5$   
29.  $0 \leq |x - 7| < 2$       30.  $0 < |x + 3| < 1$   
31.  $\frac{1}{x} < 2$  (Hint: Consider two cases.)      32.  $\frac{1}{x-1} > 4$

Use the following definition of  $|x|$  to complete Exercises 33-37:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Solve.

- C 33.**  $|x| + |x - 2| = 2$       34.  $|x| + |x - 2| > 5$
35. Show that  $|ab| = |a| \cdot |b|$  when (1)  $a > 0$  and  $b > 0$ ; (2)  $a < 0$  and  $b > 0$ ; (3)  $a > 0$  and  $b < 0$ ; and (4)  $a < 0$  and  $b < 0$ .
36. a. Give three examples illustrating the triangle inequality:  
 $|a + b| \leq |a| + |b|$   
b. **Investigation** In Chapter 11 we will define the absolute value of a complex number as follows:  $|a + yi| = \sqrt{a^2 + y^2}$ . Decide whether the triangle inequality holds if  $a$  and  $b$  are complex numbers.
37. Use the triangle inequality stated in Exercise 36(a) to prove:  
a.  $|a - b| \leq |a| + |b|$       b.  $||a| - |b|| \leq |a - b|$



## Earth Science

Conservation  
Geography  
Geology  
Meteorology  
Oceanography

- 52. Geology** The Richter scale was proposed in 1935 by Charles Richter. It was refined in 1979. The Richter magnitude,  $R$ , of an earthquake is given by,

$$R = 0.67 \log (0.37E) + 1.46,$$

where  $E$  is the energy in  $\text{kW} \cdot \text{h}$  released by the earthquake.

- Show that  $E = 2.7 \cdot 10^{(R - 1.46)/0.67}$ .
  - Show that if  $R$  increases by 1 unit,  $E$  increases by a factor of about 31.
- 53.** If  $9^{99}$  is multiplied out and is typed on a strip of paper, 3 digits per centimeter, about how many kilometers long would the paper be? (Hint: The common logarithm of a number can tell you how many digits the number has.)
- 54.** Suppose that all you know about a function  $f$  is that  $f(ab) = f(a) + f(b)$  for all positive numbers  $a$  and  $b$ .
- Find  $f(1)$ .
  - Prove that  $f(a^2) = 2f(a)$  and  $f(a^3) = 3f(a)$ . What generalization does this suggest?
  - Prove that  $f(\sqrt{a}) = \frac{1}{2}f(a)$  and  $f(\sqrt[3]{a}) = \frac{1}{3}f(a)$ . What generalization does this suggest?
  - Prove that  $f\left(\frac{1}{b}\right) = -f(b)$ .
  - Prove that  $f\left(\frac{a}{b}\right) = f(a) - f(b)$ .
  - Try to find a function  $f$  that satisfies the original equation.
  - If  $f(10) = 1$ , find the values of  $x$  for which  $f(x) = 2$  and  $f(x) = 3$ .



The San Andreas fault

## 5-7 Exponential Equations; Changing Bases

**Objective** To solve exponential equations and to change logarithms from one base to another.

An **exponential equation** is an equation that contains a variable in the exponent. Here are exponential equations you can solve from Section 5-2:

$$2^{x-3} = 8 \quad 9^{2t} = 3\sqrt{3}$$

These exponential equations are special because both sides of each equation can easily be expressed as powers of the same number. Usually exponential equations cannot be solved this way. In this section, you will see how to use logarithms to solve exponential equations involving a variable such as time,  $t$ .

Exponents and Logarithms 203

- 8. Investigation** Draw two vectors  $a$  and  $b$ . Sketch  $a - b$ ,  $3(a - b)$ ,  $\frac{1}{3}a - 3b$ . What do you notice?
- 9. Navigation** A ship travels 700 km west from port and then 240 km due so before it is disabled. Illustrate this in a vector diagram. Use trigonometry find the course that a rescue ship must take from port in order to reach disabled ship.
- 10. Physics** On graph paper, make a diagram that illustrates a force of 8 N  $\text{ne}$  and a force of 6 N  $\text{west}$  acting on a body. Illustrate the resultant sum of the two forces and estimate its strength. Then, using trigonometry, determine the approximate direction (as a number of degrees west of north) of this force.
- 11. Aviation** On graph paper, make a diagram that illustrates the velocity of an airplane heading east at 400 knots. Illustrate a wind velocity of 50 knots blowing toward the northeast. If the airplane encounters this wind, illustrate its resultant velocity. Estimate the resultant speed and direction of the airplane. (The direction is the angle the resultant makes with due north measured clockwise from north.)
- 12. a. Sports** A swimmer leaves point A swimming south across a river at 2 km/h. The river is 4 km wide and flows east at the rate of 1 km/h. Make a vector diagram showing her resultant velocity.  
b. Calculate her resultant speed.  
c. How long will it take her to swim across the river? How far east does she swim?
- 13. Aviation** On graph paper, make a vector diagram showing an airplane heading southwest at 600 knots and encountering a wind blowing from the west. Show the plane's resultant velocity when the wind blows at (a) 30 knots, (b) 60 knots, and (c) 90 knots.
- 14. Navigation** On graph paper, make a vector diagram showing a motorboat heading east at 10 knots. Add to your diagram a vector representing a current moving southeast. Show the boat's resultant velocity if the current moves at (a) 2 knots, (b) 3 knots, and (c) 4 knots.
- 15. a. Navigation** Make a diagram showing the result of sailing a ship 3 mi on a course of  $040^\circ$  followed by sailing it 8 mi on a course of  $100^\circ$ .  
b. From your diagram, estimate the distance of the ship from its starting point.  
c. Find the exact distance of the ship from its starting point by using the law of cosines.



## WRITTEN EXERCISES

In Exercises 1–4, state any errors that you think might occur in the sampling situation discussed.

- A**
- A senator explains a vote in favor of a new bill by stating that 60% of the mail received favored the bill.
  - A city newspaper asks readers to mail in a form on which they can choose one of three ways to finance improvements to the zoo. Based on the responses, the newspaper reports that financing the improvements by raising the admission charge is the first choice of the majority of the citizens.
  - A radio talk show host invites listeners to telephone the station and talk about their feelings on a proposed highway to be built in their county.
  - A newspaper reporter randomly stops people going in a grocery store and asks, "Does your family use newspaper coupons?"
  - In Farmington High School, there are 360 ninth- or tenth-grade students and 320 eleventh- or twelfth-grade students. A poll shows that 12 out of 40 ninth- or tenth-grade students intend to vote for Lahey as student council president. It also shows that 24 out of 40 eleventh- or twelfth-grade students intend to vote for Lahey. Estimate the percent of the student body favoring Lahey.
  - Twenty-five percent of a city's employees live in the city and 75% live in the surrounding suburbs. The mayor conducts a small survey asking 10 employees who live in the city and 10 who live in the suburbs if they would prefer increases in pay or increases in benefits. Eight of the 10 city dwellers and four of the 10 suburbanites preferred an increase in pay. Estimate the percent of city employees preferring an increase in pay to an increase in benefits.
- B**
- Estimate the total number of years of service of 200 factory workers based on a sample of size  $n = 5$  where the years of service reported are 15, 8, 20, 5, 12.
  - Manufacturing** In one hour, a factory packaged 150 boxes of light bulbs. Estimate the total number of defective light bulbs packaged that hour based on a sample of 8 boxes containing the following numbers of defective bulbs: 5, 3, 5, 1, 2, 9, 4, 3.
  - Ecology** The State Conservation Department wants to estimate the number of deer in a mountainous area. It captures and tags 80 deer and then releases them. Later it captures 156 deer and finds that 12 are tagged. Estimate the number of deer in the area.



Statistics 6

## Life Science

Agriculture  
Animal Science  
Biology  
Ecology  
Medicine  
Nutrition  
Physiology

## Physical Science

Astronomy  
Aviation  
Chemistry  
Civil Engineering  
Electronics  
Engineering  
Mechanics  
Navigation  
Optics  
Physics  
Space Science  
Surveying  
Telecommunications  
Thermodynamics

# Problem Solving

“Did you ever try to solve a problem but didn't know where to begin?”

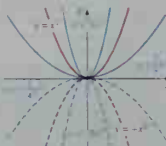
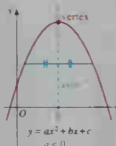
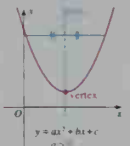
## 1-7 Quadratic Functions and Their Graphs

**Objective** To define and graph quadratic functions

The graph of the quadratic function  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ , is the set of points  $(x, y)$  that satisfy the equation  $y = ax^2 + bx + c$ . This graph is a parabola, a curve that can be seen, for example, in the cables of a suspension bridge and in the path of a thrown ball. Parabolas can also be defined geometrically, as you will see in Chapter 6.



If a graph has an axis of symmetry, then when you fold the graph along this axis, the two halves of the graph coincide. The graph of a quadratic function has a vertical axis of symmetry, or axis. The vertex of the parabola is the point where the axis of symmetry intersects the parabola. If  $a > 0$ , the parabola opens upward, and the function has a minimum value. If  $a < 0$ , the parabola opens downward, and the function has a maximum value. The bigger  $|a|$  is, the narrower the parabola is. In the figure at the far right below, the graph of  $y = 3x^2$  is narrower than the graph of  $y = x^2$ .



The  $y$ -intercept of a parabola with equation  $y = ax^2 + bx + c$  is  $c$ . The  $x$ -intercepts are the solutions of the equation  $ax^2 + bx + c = 0$ . They may have two, one, or no real solutions, depending on the discriminant  $b^2 - 4ac$ . We have shown in the discussion of the quadratic formula that

## Sections

include mathematical modeling to help you describe real-world situations.

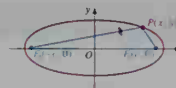
### Activity

Fasten the ends of a string to a piece of cardboard with thumbtacks. Make sure the string has some slack. Keeping the string taut, draw a curve on the cardboard as shown. Describe the curve traced by the pencil point  $P$ . Repeat the experiment by moving the tacks farther apart or closer together.



If  $F_1(c, 0)$  and  $F_2(-c, 0)$  are two fixed points in the plane and  $a$  is a constant,  $0 < c \leq a$ , then the set of all points  $P$  in the plane such that  $PF_1 + PF_2 = 2a$

is an ellipse. This is the geometric definition of an ellipse. The points  $F_1$  and  $F_2$  are called the foci of the ellipse. In the Activities section, you will see how to construct an ellipse.



## Activities

get you involved and help build understanding. A hands-on activity where you construct an ellipse.

## 10-4 Solving Trigonometric Equations

**Objective** To use identities to solve trigonometric equations

The acceleration of a body falling toward the Earth's surface is called acceleration due to gravity, often denoted by  $g$ . In theoretical physics,  $g$  is usually considered constant. However,  $g$  is not actually constant but varies slightly with latitude. A good approximation to the value of  $g$  can be found by using the following formula, which expresses  $g$  in terms of  $\theta$ , the latitude in degrees:

$$g = 9.80665(1 + 0.005288 \sin^2 \theta - 0.000006 \sin^2 2\theta)$$

For example, if you live in Chicago, which has a latitude of  $42^\circ\text{N}$ ,  $\sin \theta \approx 0.6691$  and  $\sin 2\theta \approx 0.9945$ . Therefore,  $g \approx 9.8036 \text{ m/s}^2$ .

As you can see from this example, some problems involve trigonometric equations that have multiples of angles or numbers. The following suggests two methods that may be helpful in solving such equations. The first method gives a graphical method using a graphing calculator or computer, and the second method gives ways for solving the equation algebraically.



Gravity acting on water creates this waterfall.

### Methods for Solving the Trigonometric Equation $f(x) = g(x)$

#### Method 1

Use a graphing calculator or computer to graph  $y = f(x)$  and  $y = g(x)$  on the same set of axes. Use the zoom or trace feature to find the  $x$ -coordinates of any intersection points of the two graphs.

#### Method 2

Use the following guidelines:

- It may be helpful to draw a quick sketch of  $y = f(x)$  and  $y = g(x)$  to see roughly where the solutions are.
- If the equation involves functions of  $2x$  and  $x$ , transform the functions of  $2x$  into functions of  $x$  by using identities.
- If the equation involves functions of  $2x$  only, it is usually better to solve for  $2x$  directly and then solve for  $x$ .
- Be careful not to lose roots when you divide both sides of an equation by a function of the variable. Review the discussion about losing roots on pages 32–33.

## Sections

present alternative methods of solving problems.

23. If 4 is a zero of  $f(x) = 3x^3 + kx - 2$ , find the value of  $k$ .
24. If 2i is a zero of  $f(x) = x^4 + x^2 + a$ , find the value of  $a$ .
25. A quadratic polynomial  $P$  has leading coefficient  $-2$ , a constant term of 6, and no linear term. Find the zeros of  $P$ .
26. The leading coefficient of a cubic polynomial  $P$  is 2, and the coefficient of the linear term is  $-5$ . If  $P(0) = 7$  and  $P(2) = 21$ , find  $P(3)$ .
27. a. What are the zeros of the quadratic function  $f(x) = 2(x-1)(x-4)$ ?  
b. Find a quadratic function with zeros 2 and 3.  
c. Find a cubic function with zeros 0, 2, and 3.
28. a. What are the zeros of the quadratic function  $f(x) = 3(x+1)(x-2)$ ?  
b. Find a quadratic function with zeros 3 and  $-4$ .  
c. Find a quartic function with zeros  $\pm 3$  and  $\pm 4$ .
29. If  $f(x) = 7x + 2$ , find: a.  $f(9.2) - f(8.2)$       b.  $f(x+1) - f(x)$
30. If  $g(x) = 3 - 8x$ , find: a.  $g(6.25) - g(4.25)$       b.  $g(x+2) - g(x)$
31. If  $f(x) = mx + k$ , show that the value of  $\frac{f(x+h) - f(x)}{h}$ , where  $h \neq 0$ , does not depend on  $x$  or  $h$ . Interpret this result graphically.
32. If  $f(x) = x^2$  and  $h \neq 0$ , find the value of  $\frac{f(x+h) - f(x)}{h}$ . Is the value of this expression independent of the values of  $x$  and  $h$ , as it was for the function of Exercise 31?
33. **Investigation** Multiply several pairs of nonzero polynomials. What is the relationship between the degree of the product and the degrees of the factors? How can you use this relationship to justify saying that the polynomial 0 has no degree, even though all other constant polynomials have degree 0?
34. a. Consider the following table of values for the quadratic function  $f(x) = x^2 + 2x + 3$ .

$x$	0	1	2	3	4	5
$f(x)$	3	6	11	18	27	38
differences						

- What pattern do you observe in the differences?
- b. Make a difference table similar to the one above for  $g(x) = 2x^2 - 3x - 1$ . What pattern do you observe in the differences?
- c. Do the differences in the values of  $g(x)$  look like those you observed in parts a and b?

35. **Investigation** Experiment with difference tables. Can you detect a pattern?

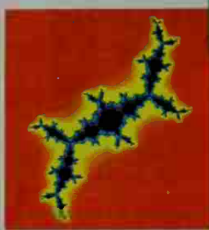
## Investigations

help you discover relationships and connections in mathematics.

### PROJECT

#### Chaos in the Complex Plane

If you've explored some of the dynamical systems presented in Sections 19-5 and 19-6, then you're probably well aware that iterating a function—no matter how simple its rule—can produce unexpectedly complicated orbits for certain seeds. The results are even more unusual and exciting when a function's domain and range, which until now were sets of real numbers, are allowed to be sets of complex numbers instead. In fact, iterating a function with a complex domain and range can produce exceptionally intricate and beautiful graphic images, like the one shown.



Filled-in Julia set of  $f(z) = z^2 - 0.112 + 0.861i$

#### Materials:

- a computer or programmable calculator with graphics mode (preferably color graphics)
- a printer that can print a graphics screen

#### Complex Orbits

Suppose we have a function  $f(z)$  where  $z$  is a complex number. Iterating  $f(z)$  using some complex seed  $z_0$  gives the orbit of  $z_0$ :

$$\begin{aligned} z_0 & \\ z_1 &= f(z_0) \\ z_2 &= f(z_1) = f^2(z_0) \\ z_3 &= f(z_2) = f^3(z_0) \\ &\vdots \end{aligned}$$

For example, the orbit of  $z_0 = 1 + 2i$  is:

$$\begin{aligned} z_0 &= 1 + 2i \\ z_1 &= (1 + 2i)^2 = -3 + 4i \\ z_2 &= (-3 + 4i)^2 = -7 - 24i \\ z_3 &= (-7 - 24i)^2 = -527 + \dots \end{aligned}$$

The calculation of such orbits is tedious. At the top of the next page is a calculator.  $f(z) = z^2$  for any seed  $z_0$ .

750 Project

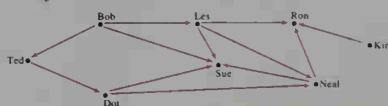
## Projects

provide you with opportunities to explore interesting topics through experiments. Discover how patterns can produce exceptionally intricate and beautiful graphic images.

- For some parts of the following exercises, you may wish to use computer software or a calculator that performs matrix operations.

7. a. Ship A sends messages to Ship B. Ship B sends and receives messages from Ship C and E. Ship D sends and receives messages from Ship C. Ship E receives messages from Ships A and C. Draw this network.  
b. Write the matrix  $M$  that models this communication network, labeling rows and columns alphabetically.  
c. Find  $M^2$ . Explain what the element in the fifth row, second column means.  
d. Find the matrix that represents the number of ways messages can be sent from one ship to another using at most one relay.  
e. Reason from the network diagram what the last element in row 4 of  $M^3$  is. (You do not need to calculate  $M^3$ .)
8. a. Suppose radios of varying quality are used by forest rangers to communicate with each other in a large national park. Make a diagram showing five rangers, some pairs having two-way communication, some pairs having one-way communication, and some having no communication.  
b. Write a matrix  $M$  to model the communication between the forest rangers. Find  $M + M^2$ . What information does this matrix give you?

9. The diagram below models a rumor network.



If a person has only outgoing arrows, then that person is a transmitter of the rumor. A person having only incoming arrows is a receiver of the rumor. A person having both outgoing and incoming arrows is a relay point for the spreading of the rumor.

- a. Identify the transmitters, receivers, and relays for this network.  
b. How can you identify a transmitter by looking at the corresponding rumor matrix? How can you identify a receiver?
10. Let  $M$  be a matrix that illustrates communication among several Ham radio operators. If  $M^2$  contains no zeros, what can you conclude?



## Current technology

is used to solve complex real-world problems.



# R

# Reasoning

"What do you think?"

"Prove it to me!"

"You need to decide."

Decision  
making

17. **Writing** In each of Exercises 15 and 16 you made a prediction. Which of these predictions do you think has a better chance of being correct? Write a few sentences explaining your reasoning.

The table shows statistics for the ten players of a college basketball team in its opening game of the season. Use the data to complete Exercises 18–20.

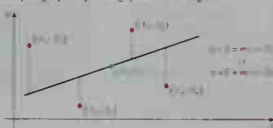
minutes played	32	30	25	21	21	18	16	15	12	10
points scored	22	15	11	8	6	4	8	10	4	6
fouls committed	4	3	5	2	3	2	2	1	0	1

18. Use computer software to make a scatter plot, draw the least-squares line, find its equation, and find the correlation coefficient for each set of data.

x	y
minutes	points
minutes	fouls
fouls	points

19. **Writing** Refer to the results of Exercise 20. Does the positive correlation between fouls committed and points scored suggest a cause-and-effect relationship between fouls and points? As a coach, would you advise playing more defensively and thus committing more fouls as a way of increasing your players' points scored? Write a short paragraph explaining your reasoning.

20. The diagram at the right shows a scatter plot of a set of data and a line with slope  $m$  that contains the point  $(3, y)$ .



- a. For each point  $(x_i, y_i)$ , let  $d_i$  represent the vertical distance between the point and the line, that is,  $d_i = |y_i - y|$ . Show that:

$$d_i = |y_i - y| = m(x_i - 3)$$

42. **Transportation** A hauling company needs to determine whether a large house trailer can be moved along a highway that passes under a bridge with an opening in the shape of a parabolic arch, 12 m wide at the base and 6 m high in the center. If the trailer is 9 m wide and 3.2 m tall (measured from the ground to the top of the trailer), will it fit under the bridge?



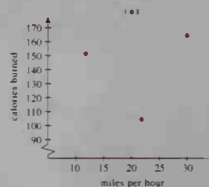
## 8 Quadratic Models

**Objective** To model real-world situations using quadratic functions

Scientists are fascinated by the ability of birds to travel great distances without stopping. Using wind tunnels to monitor parakeets' oxygen consumption and carbon dioxide production in flight, they have investigated the rate at which parakeets expend energy in level flight. The data in the table below show the number of calories burned per gram hour the number of calories needed to move each gram body weight for one hour) for three different flight speeds.

flight speed (miles per hour)	10	15	20	25	30
calories burned per gram hour	100	120	140	160	180

If we want to predict the bird's energy at some other flight speed, we will want to find a good mathematical model based on the data. The information in the table is plotted on the graph in figure (a). Note that the flight speed increases, the energy expended decreases and then increases. Clearly, a linear model would not be suitable to use. We can find many kinds of curves that pass through these data points. This process is called *curve-fitting*, which will be discussed in more detail in Chapter 9. One possible curve is a parabola with equation  $C(x) = ax^2 + bx + c$ . To find the values of  $a$ ,  $b$ , and  $c$ , substitute the data from the graph into this equation, as shown in the next page.



13. In the random-number table shown below, the probability that each digit, 0–9, occurs in any given position is 0.1. We can use the table to perform Monte Carlo simulations in order to find certain probabilities empirically.

49487	52802	28667	62058	87822	14704	18519	17889	45869	14454
29480	91539	46317	84803	86056	62812	33584	70391	77749	64906
25252	97738	23901	11106	86864	55808	22557	23214	15021	54268
02431	42193	94960	19620	29188	05863	92900	06836	13433	21709
69414	89353	70724	67893	23218	72452	03095	68333	13751	37260

For example, to use Monte Carlo simulations to find the probability of getting 0 "heads" in 3 tosses of a coin, first associate the digits 0–4 with the outcome "heads" and the digits 5–9 with the outcome "tails." Then the first row of the random-number table represents 16 simulations of 3 tosses of a coin.

49487	52802	28667	62058	87822	14704	18519	17889	45869	14454
000	000	000	000	000	000	000	000	000	000

- a. Of the 16 simulations, 5 produce 0 "heads." Thus, based on 16 simulations,  $P(0 \text{ "heads"}) = 0.3125$ . Compare this probability to the theoretical probability of getting 0 "heads" in 3 tosses of a coin (see Exercise 11d).  
b. Begin with the last 2 digits of the first row and perform 14 more simulations. Using all 30 simulations, find the probability of getting 0 "heads."  
c. How many simulations can be done using the whole table? Based on these simulations, what is the probability of getting 0 "heads?"  
d. **Discussion** When you use Monte Carlo simulations to approximate the theoretical probability of an event, will increasing the number of simulations always improve the accuracy of the approximation? Explain.

14. a. **Discussion** In order to use Monte Carlo simulations (see Exercise 13) to find the probabilities asked for in Exercises 2–12, how should the digits 0–9 be assigned to the possible outcomes for each exercise? (Hint: It may be helpful to eliminate some digits. Cross off all occurrences of those digits in the table before actually doing a simulation.)  
b. Perform Monte Carlo simulations for Exercise 2 and Exercise 7. Compare the probabilities you find to the theoretical probabilities found using the binomial probability theorem.

15. a. **Sports** A basketball player's free-throw percent is 750. What is the probability that she scores on exactly 4 of her next 5 free throws?  
b. **Discussion** Using the binomial probability theorem to complete part (a) assumes that the probability of a successful free throw is always 750. Is this assumption valid? Explain.



Discuss the  
validity  
of a  
conjecture  
using math

# Follow a logical argument

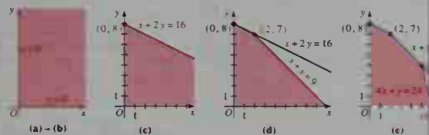
Suppose that a small TV manufacturing company produces console and portable TV's using three different machines, A, B, and C. The table below shows how many hours are required on each machine per day in order to produce a console TV or a portable TV.

Machines	Console	Portable	Hours available
A	1 h	2 h	16
B	1 h	1 h	9
C	4 h	1 h	24

These requirements for the different machines can be described by the following inequalities, where  $x$  represents the number of console TV's and  $y$  represents the number of portable TV's:

- a.  $x \geq 0$
  - b.  $y \geq 0$
  - c.  $x + 2y \leq 16$
  - d.  $x + y \leq 9$
  - e.  $4x + y \leq 24$
- The number of TV's cannot be negative.
- Machine A needs 1 hour for each console TV and 2 hours for each portable. Thus, for  $x$  console TV's and  $y$  portable TV's, Machine A needs  $x + 2y$  hours. Since this machine is available for at most 16 hours a day,  $x + 2y \leq 16$ .
- The last two inequalities are similar to the one above. Machines B and C are available for at most 9 hours and 24 hours respectively.

The possible solutions to any problem subject to these constraints would have to satisfy all five inequalities. The easiest way to find these values is to find the region that is formed by the graphs of the system of inequalities. A step-by-step procedure is shown below.



Any point  $(x, y)$  in graph (e) is called a *feasible solution*. The set of these points is called the *feasible region*.

Inequalities

You may find it helpful to have a graphing calculator to complete Exercises 35 and 36.

35. a. Predict how the graph of  $y = \log_b(\frac{1}{x})$  is related to the graph of  $y = \log_b x$ .
- b. Check your prediction by graphing  $y = \log x$  and  $y = \log(\frac{1}{x})$  on a single set of axes.
36. a. Predict how the graph of  $y = \log_b x^2$  is related to the graph of  $y = \log_b x$ .
- b. Check your prediction by graphing  $y = \log x$  and  $y = \log x^2$  on a single set of axes.
37. a. If  $f(x) = \log_2 x$ , show that  $f(2x) = f(x) + 1$ .
- b. Part (a) shows that horizontally shrinking the graph of  $f$  by a factor of 2 is equivalent to  $\frac{1}{2}$ .
38. a. If  $f(x) = \log_3 x$ , show that  $f(\frac{x}{3}) = f(x) - 1$ .
- b. Make a statement similar to that in part (b) of Exercise 37.
39. Suppose that  $\log y = ax + b$ , where  $a$  and  $b$  are real numbers and  $a$  is nonzero. Express  $y$  in terms of  $x$ . What kind of expression defines  $\log y$  in terms of  $x$ ? What kind of expression defines  $y$  in terms of  $x$ ?
40. **Conservation** The expiration time  $T$  of a natural resource is the time remaining until it is all used. If one assumes that the current growth rate of consumption remains constant, then the expiration time in years is given by

$$T = \frac{1}{r} \ln \left( \frac{R}{C} + 1 \right),$$

where  $C$  = current consumption,  $r$  = current growth rate of consumption, and  $R$  = resource size. Suppose that the world's consumption of oil is growing at the rate of 7% per year ( $r = 0.07$ ) and current consumption is approximately  $17 \times 10^9$  barrels per year. Find the expiration time for the following estimates of  $R$ .

- $10^9$  barrels of remaining oil
- $10^8$  barrels of remaining oil
- $10^7$  barrels of remaining oil



Exponents and Logarithms 201

9. **Biology** A colony of bacteria decays so that the population  $t$  days from now is given by  $A(t) = 1000(\frac{1}{2})^{t/4}$ .
- a. What is the amount present when  $t = 0$ ?
- b. How much will be present in 4 days?
- c. What is the half-life?

10. **Discussion** Can the data below be described as exponential growth?

$x$	0	1	2	3	4
$y$	4	4.4	4.8	5.3	5.8

Justify your answer by using a graph, an equation, or a logical argument. Can you think of a situation that the data might describe?



## WRITTEN EXERCISES

Evaluate each expression with a calculator.

1.  $6^n$  and  $\pi^6$
2.  $3 \cdot 6^{\sqrt{2}}$  and  $\sqrt{2}^{3/6}$

Find an exponential function having the given values.

3.  $f(0) = 3$ ,  $f(1) = 15$
4.  $f(0) = 5$ ,  $f(3) = 40$
5.  $f(0) = 64$ ,  $f(2) = 4$
6.  $f(0) = 80$ ,  $f(4) = 5$

7. **Physics** The half-life of a radioactive isotope is 4 days. If 3.2 kg are present now, how much will be present after
- a. 4 days? b. 8 days? c. 20 days? d.  $t$  days?

8. **Physics** The half-life of radium is about 1600 years. If 1 kg is present now, how much will be present after
- a. 3200 years? b. 16,000 years? c. 800 years? d.  $t$  years?

9. **Physics** The table shows the amount  $A(t)$  in grams of a radioactive element present after  $t$  days. Suppose that  $A(t)$  decays exponentially.

$t(\text{days})$	0	2	4	6	8	10
$A(t)$	320	226	160	115	80	57

- a. What is the half-life of the element?
- b. About how much will be present after 16 days?
- c. Find an equation for  $A(t)$ .

## Make and test conjectures

## Construct and present valid arguments

# Communication

“Have you ever had trouble explaining something because you didn't fully understand it?”

## Discussion

Further your understanding of math by discussing it with others.

### Careers in Genetics and Statistics

The twentieth century has seen unprecedented progress in our ability to understand, control, and cure disease. Many fields of study contribute to medical research, and the methods of data analysis play an important role.

#### GENETICS

Genetics, the study of heredity, encompasses a number of areas of research. Examples include population genetics, which uses statistical analysis to find patterns in gene distribution and the occurrence of genetic diseases; microbial genetics, which explores the process of mutation by studying rapidly multiplying microorganisms; and cytogenetics, which looks at hereditary activity at the cellular level.

Agnes Stroud-Lee, a University of Chicago Ph.D., has done research in cytogenetics at the Los Alamos National Laboratory. By observing the effects of radiation and toxic chemicals on chromosomes, she sought clues to abnormalities in genetic material. Her results have improved our understanding of birth defects and cancer. “I have made roads,” she says. “... other scientists can use my work to further their pursuit of science.” Ms. Stroud-Lee retired in 1979.



Agnes Stroud-Lee

#### STATISTICS

Statisticians are experts in the design of experiments, surveys, and questionnaires used to generate accurate and useful data, and in the meaningful analysis of data. Their skills are indispensable to medical and health research.

The National Center for Health Statistics is an important source of information for researchers and policymakers. NCHS analysis of data reveals connections between such varied health factors as socioeconomic status, type of health care used, nutrition, exposure to risks, and age.

Joe Fred Gonzalez, Jr., a mathematical statistician for NCHS whose work includes developing estimation procedures and national sample survey designs, always found math “challenging and intellectually stimulating.” His desire to understand applications of mathematics led him to an M.S. degree in statistics from the George Washington University. Believing in the value of education, Mr. Gonzalez is also a part-time university and college professor.



Joe Fred Gonzalez, Jr.

Careers in Genetics and Statistics 713

## Research

Investigate interesting facts and applications of mathematics.

### CLASS EXERCISES

- As in Example 1, suppose a die is rolled. Find each probability.
  - $P(\text{perfect square})$
  - $P(\text{factor of } 60)$
  - $P(\text{negative number})$
- If a card is drawn at random from a standard deck of 52 cards, what is the probability of getting:
  - the queen of hearts?
  - a heart?
  - a queen?
  - a red card?
  - a face card?
  - a red face card?
- If two dice are rolled, what is the probability that both show the same number?
- If two dice are rolled, find the probability of getting:
  - a sum of 3
  - a sum of 4
  - a sum of 3 or 4



- Meteorology** If the probability of rain tomorrow is 40%, what is the probability of no rain tomorrow?
- Occupational Safety** If the probability of no accidents in a manufacturing plant during one month is 0.82, what is the probability of at least one accident?
- Discussion** In the solution of Example 3, another possible sample space is the set of the 11 possible sums for the two dice:
 
$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Since two of these 11 possibilities correspond to a sum of 9 or 10, some people might be tempted to say that the probability of a sum of 9 or 10 is  $\frac{2}{11}$ . Discuss why this reasoning is incorrect.

- A penny, a nickel, and a dime are each tossed.
  - Does the set  $S = \{0, 1, 2, 3\}$ , which gives the four different numbers of “heads” that could come up, satisfy the definition of a sample space?
  - What is wrong with reasoning that since one of the four sample points in part (a) corresponds to 2 “heads,” then  $P(2 \text{ “heads”}) = \frac{1}{4}$ ?
- A card is picked at random from a standard deck. Explain why the set  $S = \{\text{club, spade, red card, face card}\}$  is not a sample space.
- Comment on the following reasoning:

There are 3 states whose names begin with the letter C (California, Colorado, and Connecticut); call them “C-states.” Likewise, there are 3 “O-states” (Ohio, Oklahoma, and Oregon). Thus, if a person is chosen at random from the U.S. population, that person has the same probability of being from a “C-state” as from an “O-state.”

602 Chapter Sixteen

Find two angles, one positive and one negative, that are coterminal with each given angle.

- |                        |                     |                     |                      |
|------------------------|---------------------|---------------------|----------------------|
| 17. a. $500^\circ$     | b. $-60^\circ$      | c. $\frac{\pi}{4}$  | d. $-\frac{2\pi}{3}$ |
| 18. a. $1000^\circ$    | b. $-100^\circ$     | c. $\frac{4\pi}{3}$ | d. $-\frac{\pi}{6}$  |
| 19. a. $28.5^\circ$    | b. $116.3^\circ$    | c. $-60.4^\circ$    | d. $-315.3^\circ$    |
| 20. a. $38.4^\circ$    | b. $127.6^\circ$    | c. $-50.8^\circ$    | d. $-320.7^\circ$    |
| 21. a. $360^\circ 30'$ | b. $-90^\circ 40'$  | c. $3^\circ 21'$    | d. $115^\circ 15'$   |
| 22. a. $180^\circ 20'$ | b. $-270^\circ 30'$ | c. $11^\circ 44'$   | d. $172^\circ 11'$   |

- Give an expression in terms of the integer  $n$  for the measure of all angles that are coterminal with an angle of  $29.7^\circ$ .
- Give an expression in terms of the integer  $n$  for the measure of all angles that are coterminal with an angle of  $-116^\circ 10'$ .

Each of Exercises 25–30 gives the speed of a revolving gear. Find (a) the number of degrees per minute through which each gear turns and (b) the number of radians per minute. Give answers to the nearest hundredth.

- |             |              |              |
|-------------|--------------|--------------|
| 25. 35 rpm  | 26. 27 rpm   | 27. 2.5 rpm  |
| 28. 6.5 rpm | 29. 14.6 rpm | 30. 19.8 rpm |

- Reading** On page 257, you were told that when a car with wheels of radius 14 in. is driven at 35 mi/h, the wheels turn at an approximate rate of 420 rpm. Show how to obtain this rate of turn.

- Retraction** Suppose you can ride a bicycle a distance of 5 mi in 15 min. If you ride at a constant speed and if the bicycle's wheels have diameter 27 in., find the wheels' approximate rate of turn (in rpm).

- Research** Consult an encyclopedia or an atlas to see how points on a world map are located by using latitude and longitude coordinates given in degrees, minutes, and seconds.
  - If you travel south from a given point on Earth, about how many miles do you have to go to traverse an angle of  $1^\circ$ ?
  - Explain why your answer to part (a) might be different if you travel west instead of south.
- Research** Consult a book of astronomy or a star atlas to see how stars on a celestial map are located by using angles of right ascension and declination. Describe how each of these angles is measured, and give examples.



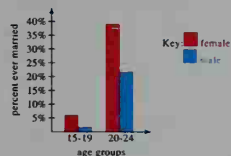
262 Chapter Seven



22. **Geography** The table below gives data on the percent of the U.S. population ever married.

Age group	15-19	20-24	25-29	30-34	35-44	45-54
Percent of females ever married	6	39	71	84	92	95
Percent of males ever married	2	22	57	75	89	94

- a. At the right is part of a comparative histogram showing two side-by-side bars for each of the age groups. Complete the histogram.



- b. **Writing** Write a paragraph explaining what the data show.

23. **Nutrition** The table below gives data on the recommended energy intake, in kilocalories (Cal), for average females and males in various age groups.

Age group	11-14	15-18	19-22	23-50	51-75	≥76
Energy needs (in Cal) for females	2200	2100	2100	2000	1800	1600
Energy needs (in Cal) for males	2700	2800	2900	2700	2400	2050

- a. Draw a comparative histogram showing two side-by-side bars for each of the age groups. (See Exercise 22 for an example of such a histogram.)

- b. **Writing** Write a paragraph explaining what the data show.

24. For a group of 10 teenagers, the mean age is 17.1, the median is 16.5, and the mode is 16. If a 21-year-old joins the group, give the mean, median, and mode for the ages of the 11 people.



Statistics 647

## Writing

Bring together your math skills and your writing skills. Express, in your own words, the meaning and understanding of concepts.

### WRITTEN EXERCISES

For Exercises 1-4, suppose a card is drawn from a well-shuffled standard deck of 52 cards. Find the probability of drawing each of the following.

1. a. a black card      b. a spade      c. not a spade
  2. a. a black face card      b. a black jack      c. not a black jack
  3. a. a red diamond      b. a black diamond      c. not a black diamond
  4. a. a jack      b. a jack or king      c. neither jack nor king
5. Mr. and Mrs. Smith each bought 10 raffle tickets. Each of their three children bought 4 tickets. If 4280 tickets were sold in all, what is the probability that the grand prize winner is:
- a. Mr. or Mrs. Smith?      b. one of the 5 Smiths?      c. none of the Smiths?
6. One of the integers between 11 and 20, inclusive, is picked at random. What is the probability that the integer is:
- a. even?      b. divisible by 3?      c. a prime?

7. New York City is divided into five boroughs: Manhattan, Queens, the Bronx, Brooklyn, and Staten Island. Suppose that a New York City telephone number is randomly chosen.

- a. Explain why the probability that it is a Manhattan telephone number is not  $\frac{1}{5}$ .
- b. What do you need to know in order to find the correct probability in part (a)?

8. Suppose that a member of the U.S. Senate and a member of the U.S. House of Representatives are randomly chosen to be photographed with the President. Explain why  $\frac{1}{50}$  is the probability that the senator is from Iowa, and why  $\frac{1}{100}$  is not the probability that the representative is from Iowa.

In Exercises 9-12, use the table on page 601, which gives the 36 equally likely outcomes when two dice are rolled. Find the probability of each event.

9. a. Sum is 6.      b. Sum is 7.      c. Sum is 8.
10. a. Sum is even.      b. Sum is 12.      c. Sum is less than 12.
11. The two dice show different numbers.
12. The red die shows a greater number than the white die.

Probability 603

## Exercises

Combine your knowledge of math and other subjects to explain interesting problems.



29. a. If  $f(x) = x^{2/3}$ , find  $f'(x)$ .

- b. For what values of  $x$  is  $f'(x)$  undefined?

- c. Sketch the graph of  $f(x)$ . (If you use a computer or graphing calculator, you may need to enter the rule for the function as  $(x^2)^{1/3}$  to obtain the complete graph.) Explain how the graph of  $f(x)$  supports the result of part (b).

30. Repeat Exercise 29 using the function  $f(x) = x^{1/3}$ .

In Exercises 31-36, find a function that has the given derivative.

31.  $f'(x) = 4x^3$       32.  $f'(x) = 5x^4$       33.  $g'(x) = 3x^5$
34.  $g'(x) = x^7$       35.  $h'(x) = 3\sqrt{x}$       36.  $h'(x) = \sqrt[3]{x}$

37. a. Use the binomial theorem (page 591) to write the first three terms and the last term in the expansion of  $(x + h)^n$ .

- b. Use part (a) and the definition of  $f'(x)$  to prove Theorem 2 on page 760 for positive integral values of  $n$ .  
(Note: The binomial theorem can be generalized to apply to any nonzero real number  $n$ . This form of the binomial theorem can then be used to give a general proof of Theorem 2.)

38. Prove Theorem 3 on page 760. (Hint: Use the definition of  $f'(x)$ .)

39. Prove Theorem 4 on page 760. (Hint: Use the definition of  $f'(x)$ .)

### COMMUNICATION: Visual Thinking

The saying "a picture is worth a thousand words" is often true in mathematics. One well-drawn diagram may be a very convincing argument on its own, without being a formal proof. Diagrams and sketches can offer evidence that a conjecture may be true, and they can offer clues that lead to interesting discoveries.

For example, consider the diagram at the right, where the lengths  $a$  and  $b$  are unequal positive numbers.



1. Explain how the areas of the squares and rectangles in the diagram allow you to conclude that  $(a+b)^2 = (a-b)^2 + 4ab$ .
2. Explain how the saying "the whole is greater than its parts" allows you to conclude from the diagram that  $(a+b)^2 > 4ab$ .
3. By taking the square root of each side of the inequality above, what relationship do you discover between the arithmetic mean,  $\frac{a+b}{2}$ , and the geometric mean,  $\sqrt{ab}$ , of  $a$  and  $b$ ?

## Visual Thinking

Learn to communicate visually through a diagram or picture.

# Technology

“Technology enhances understanding, encourages exploration, and opens the door to many new and fascinating applications.”

## Technology Logos



indicates advanced technology suggested to get the full benefit of studying the lesson or exercise.

b.

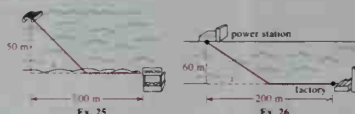
indicates specific exercises for which advanced technology is necessary.

24. **Manufacturing** A sheet of metal is 60 cm wide and 10 m long. It is bent along its width to form a gutter with a cross section that is an isosceles trapezoid with  $120^\circ$  angles, as shown at the right.
- Express the volume  $V$  of the gutter as a function of  $x$ , the length in centimeters of one of the equal sides. (Hint: Volume = area of trapezoid  $\times$  length of gutter)
  - For what value of  $x$  is the volume of the gutter a maximum?

**Part (b) of Exercises 25–29 requires the use of a computer calculator.**

25. From a raft 50 m offshore, a lifeguard wants to swim to shore, snack bar 100 m down the beach, as shown at the left below.
- If the lifeguard swims at 1 m/s and runs at 3 m/s, express the swimming and running time  $t$  as a function of the distance  $x$  in a diagram.

**Use a computer or graphing calculator to find the minimum time.**



26. **Engineering** A power station and a factory are on opposite sides of a river 60 m wide, as shown at the right above. A cable must be run from the power station to the factory. It costs \$25 per meter to run the cable in the river and \$20 per meter on land.

**Use a computer or graphing calculator to find the minimum cost.**

27. **Landscaping** A rectangular area of 60 m<sup>2</sup> has a wall as one of its sides, as shown. The sides perpendicular to the wall are made of fencing that costs \$6 per meter. The side parallel to the wall is made of decorative fencing that costs \$8 per meter.
- Express the total cost  $C$  of the fencing as a function of the length  $x$  of a side perpendicular to the wall.
  - Use a computer or graphing calculator to find the minimum cost.

28. A cylinder is inscribed in a sphere of radius 1.
- Express the volume  $V$  of the cylinder as a function of the base radius  $r$ .
  - Use a computer or graphing calculator to find the radius and height of the cylinder having maximum volume.

## Polynomial Equations

### 2-5 Using Technology to Approximate Roots of Polynomial Equations

**Objective** To use technology to approximate the real roots of a polynomial equation

You already know how to solve linear and quadratic equations. In this section we will discuss how to solve a polynomial equation of higher degree, such as

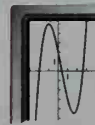
$$x^3 - x^2 - 5x + 2 = 0,$$

by using a computer, or a graphing or programmable calculator. In the next section we will see that some polynomial equations can be solved by factoring.

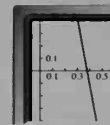
**Example 1** Solve  $x^3 - x^2 - 5x + 2 = 0$  by using a computer or graphing calculator.

**Solution** Draw the graph of the polynomial function  
 $y = x^3 - x^2 - 5x + 2$

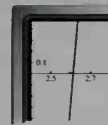
with a computer or graphing calculator. The  $x$ -intercepts of the graph are the roots of the equation. If estimating the  $x$ -intercepts is difficult because the graph is not labeled, you could use the TRACE feature, if available. To improve the accuracy of the approximations, you could rescale or use the ZOOM feature, if available.



Graph of function for  $-3 \leq x \leq 4$



Enlargement of graph for  $0 \leq x \leq 0.6$

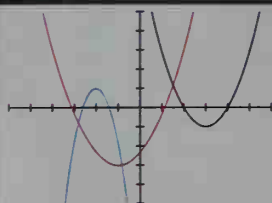


Enlargement of graph for  $2.4 \leq x \leq 2.8$

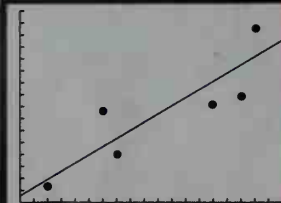
From the graphs of  $y = x^3 - x^2 - 5x + 2$  shown above, we estimate that the roots of  $x^3 - x^2 - 5x + 2 = 0$  are  $x = -2$ ,  $x = 0.4$ , and  $x = 2.6$ .

In the next example, we will solve the equation  $x^3 - x^2 - 5x + 2 = 0$  without drawing a graph. We will instead use the *location principle*, stated at the top of the next page.

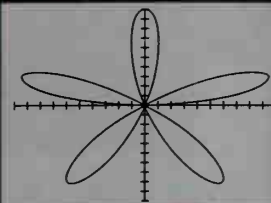
## Functions



## Statistics



## Polar Graphs



# Using Precalculus Plotter Plus

by David L. Myers

Includes User's Manual, Worked Examples,  
Enrichment Topics, Activity Sheets, Answers,  
Correlations to Advanced Mathematics, Introductory  
Analysis, and Trigonometry with Applications

## Advanced Mathematics

Precalculus with Discrete Mathematics  
and Data Analysis

Houghton Mifflin

Precalculus Plotter Plus  
Graphing • Quizzes • Explorations  
David L. Myers Houghton Mifflin Company  
(1 MB, system 0.5 or later)  
© 1995 by Houghton Mifflin Company. All rights reserved. Copies of the software may be made for personal use only. This computer program is a registered trademark of Houghton Mifflin Company. It is a registered trademark of Apple.

The diagram on the previous page illustrates the convention placed at the origin with the polar axis coinciding with the positive x-axis. Positive angles measured in the counterclockwise direction. The graph can be used as formulas for converting from one coordinate system to another. Many scientific and graphing calculators have built-in conversion of

**Example 2** a. Give polar coordinates for the point (3, 4).  
b. Give the rectangular coordinates for the point (3, 4).

**Solution** a.  $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$   
 $\tan \theta = \frac{y}{x} = \frac{4}{3}$ . Thus,  $\theta = 53.1^\circ$  is one possible value.  
(3, 4) is in Quadrant I. If  $\theta = 53.1^\circ$ , then the point is (5, 53.1°).  
ray, and so  $r = 5$ . Therefore, polar coordinates are (5, 53.1°).  
possible pair of polar coordinates of the same point is (5, 230.1°).  
b.  $x = r \cos 30^\circ = 3 \cos 30^\circ$  and  $y = r \sin 30^\circ = 3 \sin 30^\circ$ .  
Thus, rectangular coordinates of the point are  $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$ .

The graph of a polar equation can be drawn by a graphing calculator described below, provided the calculator has a polar mode or a parametric mode. Many calculators use  $t$  instead of  $\theta$ .

If the calculator has polar mode, enter the equation:  $r = f(t)$   
If the calculator has parametric mode, enter the equations:  $x = r \cos t = f(t)$   
 $y = r \sin t = f(t)$

The two equations for  $x$  and  $y$  are called *parametric equations*. We will study them in greater detail in Chapter 12, where  $t$  often represents time in a practical problem.

**Example 3** Sketch the polar graph of  $r = 2 \sin 2\theta$ .

**Solution** With a graphing calculator  
In polar mode, enter:  
 $r = 2 \sin 2t$

In parametric mode, enter:  
 $x = 2 \sin 2t \cos t$   
 $y = 2 \sin 2t \sin t$

The following values were used for the graph of  $r = 2 \sin 2t$  shown:

$0 \leq t \leq 2\pi$ , step:  $\frac{\pi}{30}$   
 $-2 \leq x \leq 2$ , scale: 1  
 $-2 \leq y \leq 2$ , scale: 1

(Solution continues on the next page.)

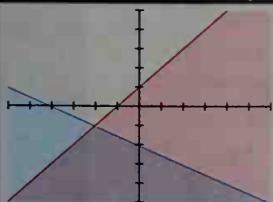
Polar Coordinates and Complex Numbers 397

## Precalculus Plotter Plus

Graphing software that lets you

- Enter equations and plot the graphs.
- Use data to create histograms, scatter plots, and regression lines of spreadsheet data.
- Test statistical hypotheses through sampling experiments.
- Explore sequences and series.
- Perform matrix operations.
- Check your understanding by determining equations for displayed graphs.

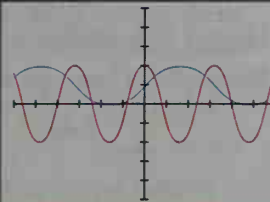
### Inequalities



### Matrices

$$T = \begin{bmatrix} 1 & 1 & 2 & -5 \\ 3 & 1 & -1 & -3 \\ 2 & 2 & 3 & 6 \end{bmatrix}$$

### Trigonometry







# Using a Graphing Calculator



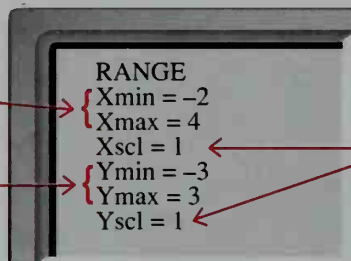
These pages will introduce you to the basic features of most graphing calculators. Because of the variety of graphing calculators available, specific keystrokes are not given. Refer to your calculator's instruction manual for details.

## Setting the RANGE Variables

When using a graphing calculator to display graphs, think of the screen as a "viewing window" that shows a portion of the coordinate plane. On many calculators, the "standard" viewing window uses values from  $-10$  to  $10$  on both axes. You adjust the viewing window by entering values for the *RANGE* variables, which appear on the screen when you press the RANGE key.

The x-axis will be shown for  $-2 \leq x \leq 4$ .

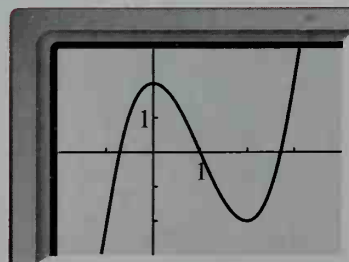
The y-axis will be shown for  $-3 \leq y \leq 3$ .



With scale variables set to equal 1, tick marks will be 1 unit apart on both axes.

## Graphing a Function

To graph a function, enter its equation and set the RANGE variables for an appropriate viewing window. (You may need to experiment to find the best viewing window.) Here, the graph of the cubic equation  $y = x^3 - 3x^2 + 2$  is shown using the viewing window described above. Note that the scale labels shown here, and on similar diagrams throughout the book, do not actually appear on a calculator display.



### Function form

An equation to be graphed must be entered in the form  $y = \dots$ , that is,  $y$  must be expressed as a *function* of  $x$ . For example, before graphing  $x^2y + y = 4$ , first solve the equation for  $y$ . Enter the equivalent equation  $y = \frac{4}{x^2 + 1}$ .

### Using parentheses

Be careful when you enter an equation like the one discussed above.

Enter  $Y = 4 \div (X^2 + 1)$ , not  $Y = 4 \div X^2 + 1$ .

The second equation will be interpreted as  $y = \frac{4}{x^2} + 1$ , not as  $y = \frac{4}{x^2 + 1}$ .

## Try This

1. Enter and graph each equation, using an appropriate viewing window. You may need to solve for  $y$  first.

a.  $y = x^2 + x - 5$

b.  $6 + 2y = x$

c.  $|x| - y = -8$

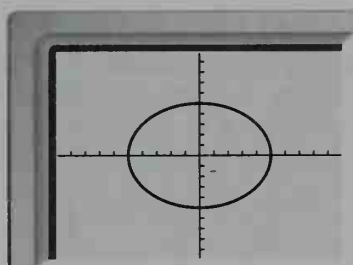
## Appearance of Graphs

Graphs displayed on a graphing calculator often look distorted. Adjusting some of the settings on your calculator may improve a graph's appearance.

### Squaring the screen

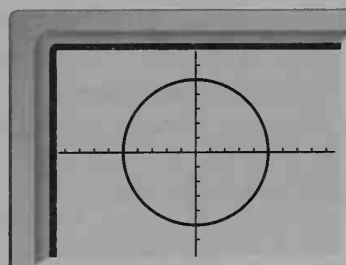
A *square screen* is a viewing window with equal unit spacing on the two axes.

Standard viewing window



Here, the circle  $x^2 + y^2 = 25$  appears stretched horizontally.

Square screen window



Displayed on a square screen, the same graph is undistorted.

On many graphing calculators, the ratio of the screen's height to its width is about 2 to 3. For a square screen, choose values for the RANGE variables that make the "length" of the  $y$ -axis about two-thirds the "length" of the  $x$ -axis:

$$(Y_{\max} - Y_{\min}) \approx \frac{2}{3}(X_{\max} - X_{\min})$$

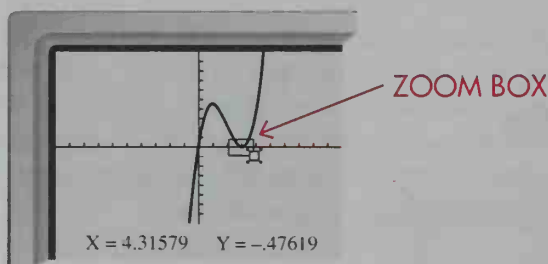
### Connected mode; graphs with asymptotes

When your calculator is in *connected mode*, the individual plotted points on a graph are joined by line segments. As a result, graphs often look jagged. Also, the separate pieces of a graph with vertical asymptotes may appear to be connected. If you take your calculator out of connected mode, only points on the graph will be plotted. This may give you a better sense of the true shape of the graph, although there can be large gaps between the points.

## ZOOM

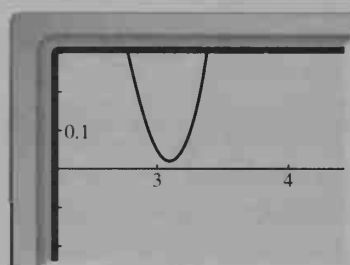
How many  $x$ -intercepts does the graph of  $y = x^3 - 6.2x^2 + 9.6x + 0.05$  (shown below) have between  $x = 2$  and  $x = 4$ ? To answer this question, use the calculator's *ZOOM* feature to enlarge the section of the graph near the point  $(3, 0)$ . On many calculators, you can do this by creating a "ZOOM BOX" around the point of interest. The contents of this box can then be drawn at full-screen size.

Standard viewing window



The coordinates of one corner of the ZOOM BOX are displayed.

ZOOM-BOX window



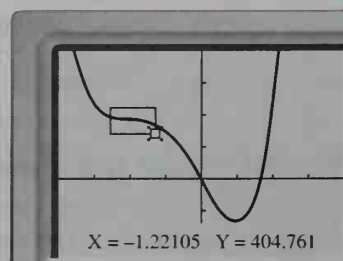
Now you can see that the graph has no  $x$ -intercepts near  $x = 3$ .

On most graphing calculators, the ZOOM feature offers several ways to adjust the viewing window. Many calculators will "zoom-in" (show a smaller portion of the coordinate plane), or "zoom-out" (show a larger portion of the coordinate plane) on a point you select, changing the RANGE variables by factors that you specify. Consult your calculator's instruction manual for specific details on ZOOM procedures.

## Try This

2. Use the fourth-degree equation  $y = 30x^4 + 122x^3 - 3x^2 - 492x$ .

- Graph the equation using a viewing window with  $-4 \leq x \leq 4$  and  $-500 \leq y \leq 1000$ .
- Use a ZOOM BOX to enlarge the graph's "flat" section. For an even more detailed view of this portion of the graph, set  $Y_{\min} = 450$  and  $Y_{\max} = 500$ . Describe the shape of the "flat" section of the graph.





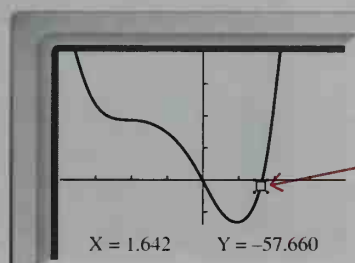


## TRACE

After a graph is displayed, you can use the calculator's *TRACE* feature. When you press the TRACE key, a flashing cursor appears on the graph. The  $x$ - and  $y$ -coordinates of the cursor's location are shown at the bottom of the screen. Press the left- and right-arrow keys to move the TRACE cursor along the graph.

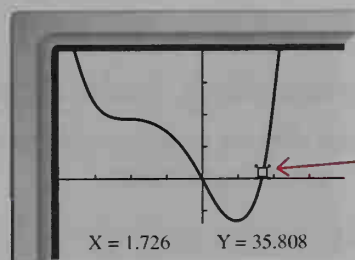
### Finding a point of interest

You can use the TRACE and ZOOM features to find the coordinates of a point of interest on a graph, such as an  $x$ -intercept or a high or low point. Consider the graph shown in Exercise 2, which has an  $x$ -intercept between 1 and 2. To find the coordinates of this  $x$ -intercept, begin by pressing the TRACE key.



Move the TRACE cursor to a point just below the  $x$ -axis.

The  $y$ -coordinate of this point is negative.

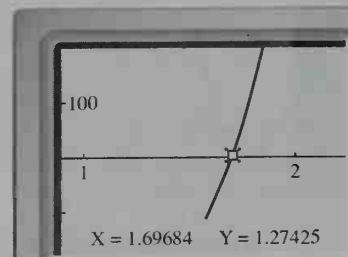


Now move the TRACE cursor to a point just above the  $x$ -axis.

The  $y$ -coordinate of this point is positive.

Somewhere between these two points, the graph must cross the  $x$ -axis (at a point where  $y = 0$ ). Therefore, the  $x$ -intercept is between 1.64 and 1.73.

"Zoom in" on a point near the graph's  $x$ -intercept. Move the TRACE cursor along the graph until it is just below the  $x$ -axis when  $x \approx 1.695$  and then just above the  $x$ -axis when  $x \approx 1.697$ .



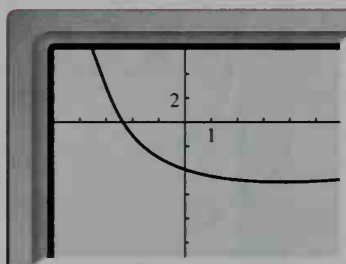
When rounded to the nearest hundredth, the two  $x$ -values mentioned above are the same. Thus, to the nearest hundredth, the  $x$ -intercept is  $x = 1.70$ . If you wish, you can increase the accuracy of this approximation by repeating this process.

## Finding an intersection point of two graphs

You can use a similar process to find the coordinates of an intersection point of two graphs. When the TRACE cursor is on one graph near the intersection point, note the value of the  $x$ -coordinate. Then press the up-arrow key. This moves the TRACE cursor to the point on the other graph that has the same  $x$ -coordinate. Compare the  $y$ -coordinates of the two points. You can "zoom-in" and repeat this process until the  $y$ -coordinates are the same to the desired degree of accuracy.

### Try This

3. a. Graph  $y = 2^{-x} - 5$ , as shown at the right. Find the  $x$ -intercept of the graph to the nearest tenth.
- b. Add the graph of  $y = 2x$  to the same set of axes. Find the coordinates of the intersection point of the two graphs to the nearest hundredth.

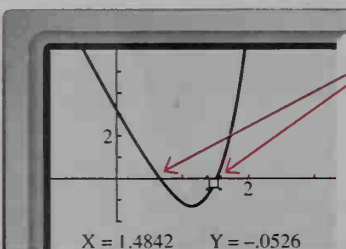


## Solving Equations with a Graphing Calculator

Finding approximate solutions to equations that are difficult or impossible to solve algebraically is a powerful and important use of a graphing calculator.

### Using an $x$ -intercept to solve an equation

To solve the fourth-degree equation  $x^4 - 6x + 4 = 0$ , use the graph of the related fourth-degree function  $y = x^4 - 6x + 4$ .



For any point on the  $x$ -axis,  $y = 0$ .  
Thus, each  $x$ -intercept of the function

$$y = x^4 - 6x + 4$$

is a solution of the equation

$$x^4 - 6x + 4 = 0.$$

To the nearest tenth, one solution of  $x^4 - 6x + 4 = 0$  is  $x = 1.5$ .

You can use the ZOOM and TRACE features to find both real solutions of the equation  $x^4 - 6x + 4 = 0$  to any desired degree of accuracy.

## Using an intersection point to solve an equation

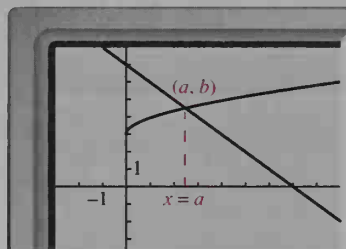
To solve the radical equation  $\sqrt{x} + 3 = 7 - x$ , use the graphs of the two functions  $y = \sqrt{x} + 3$  and  $y = 7 - x$ , drawn on the same set of axes.

The coordinates of the intersection point  $(a, b)$  must satisfy both equations:

$$b = \sqrt{a} + 3 \text{ and } b = 7 - a$$

Thus,  $\sqrt{a} + 3 = 7 - a$ .

Therefore,  $x = a$  is a solution of the equation  $\sqrt{x} + 3 = 7 - x$ .



Using ZOOM and TRACE, you can find that the solution is  $x \approx 2.44$ .

## Try This

4. For each equation, find all real solutions to the nearest tenth.

a.  $0 = x^5 - 3x^2 + 3$

b.  $\sqrt{x + 5} = |x|$

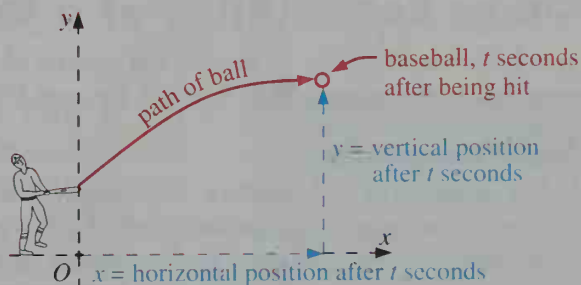
5. Use a graph to determine the *number* of real solutions of this equation.

$$x + 4 = -x^4 + 3x^3 - 1.5x + 5$$

## Graphing Parametric Equations

In the seconds after a baseball is hit, the ball has moved both horizontally and vertically, as shown in the diagram.

Instead of using one equation to describe the path of the ball, you can use *two* equations, one to express  $x$  in terms of  $t$  (the time in seconds) and one to express  $y$  in terms of  $t$ .



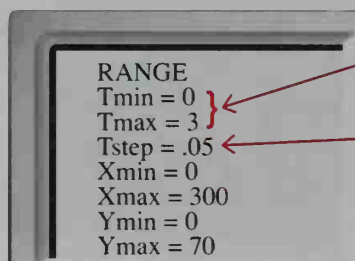
$$x = 100t \quad \text{and} \quad y = -16t^2 + 40t + 3$$

These two equations, used to express two variables ( $x$  and  $y$ ) in terms of a third variable ( $t$ ) are called *parametric equations*. The variable  $t$  is the *parameter* used to define the variables  $x$  and  $y$ .



## Parametric mode

If your graphing calculator has a built-in parametric mode, you can enter and graph parametric equations that express the variables  $x$  and  $y$  in terms of the variable  $t$ . The RANGE-variables screen will have three additional quantities for you to specify— $T_{\min}$ ,  $T_{\max}$ , and  $T_{\text{step}}$  (called “pitch” on some calculators).



The calculator will use  $t$ -values from  $t = 0$  to  $t = 3$ .

The difference between successive  $t$ -values will be 0.05.

*Note:* The  $x$ - and  $y$ -scale variables, omitted here, will also appear.

Each  $t$ -value is substituted in both the equation for  $x$  and the equation for  $y$ ; then the point  $(x, y)$  is plotted on the calculator screen.

You will learn more about parametric equations in Sections 11-1 and 12-3.

## Try This

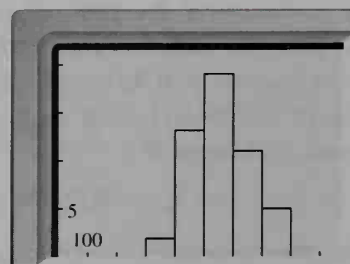
6. Graph the parametric equations given on page xxxi for the path of the ball. Set the RANGE variables as shown on the calculator screen above. Then use the TRACE feature to determine how many seconds it takes the ball to reach its maximum height.

## Other Capabilities of a Graphing Calculator

In addition to its ability to display the graphs of functions, your graphing calculator may have other capabilities that will be useful to you in this course.

### Statistical graphs

Many graphing calculators can display histograms, line graphs, or scatter plots of data you have entered. For example, the histogram at the right displays the data from the list that appears on page 639 of fifty scores on a standardized mathematics achievement test. The histogram shows the number of students whose scores fell in each 100-point interval. You can obtain statistics about the data, such as the mean and standard deviation (see Chapter 17), from a graphing calculator just as you would from a scientific calculator.

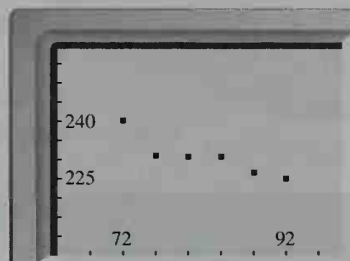




## Curve fitting

*Curve fitting* (see Chapter 18) is the process of finding an equation that describes a set of ordered pairs. Often, the first step is to graph the paired data in a scatter plot. For example, the chart shown below gives the winning times in the men's Olympic 400 m freestyle swimming race. The data can be entered in a graphing calculator and then displayed as plotted points  $(y, s)$ , where  $y$  is the number of years since 1900 and  $s$  is the time in seconds.

Year	Winning time (seconds)
1972	240.27
1976	231.93
1980	231.31
1984	231.23
1988	226.95
1992	225.00



The relationship between  $y$  and  $s$  can be approximated by a line. A graphing calculator will give an equation of the "line of best fit,"  $s = -0.653y + 285$ , and a correlation coefficient that reflects how well the equation models the data.

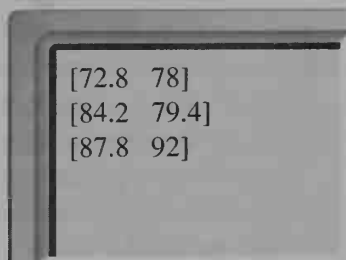
## Matrices

Many graphing calculators allow you to enter numerical information in *matrix form* (see Chapter 14) and can then perform a variety of *matrix operations*.

For example, the matrices  $A$  and  $B$  shown below display the quiz and homework averages of 3 precalculus students for each of two units.

Unit 1	Quiz	HW		Unit 2	Quiz	HW	
Mary	68	75	$= A$	Mary	76	80	$= B$
José	89	77		José	81	81	
Kasha	92	95		Kasha	85	90	

Unit 1 will be worth 40% and Unit 2 will be worth 60% of each student's midterm quiz and homework grades. A graphing calculator with matrix capabilities can calculate  $0.4a + 0.6b$  for each pair of corresponding elements in matrices  $A$  and  $B$ , and display the results in a new matrix that gives each student's quiz and homework averages.



## And more

Throughout the year, you are sure to find other topics that can be explored with a graphing calculator, and you will discover new methods and uses for this valuable tool. Be sure to share your discoveries with your classmates and your teacher.

## Careers in Mathematics and Science

Today's society is changing so rapidly that you cannot foresee the career opportunities that may be available to you within a few years. You can be sure, however, that many of the most exciting careers will involve the use of mathematics, science, and technology. Recent studies have shown that even now there is a shortage of scientists and engineers. By choosing to complete four years of high school mathematics you have already made one decision that will help you keep your options open. Continuing to study mathematics and related fields in college will prepare you to take advantage of a wide variety of career choices, including jobs in the social sciences and the arts, as well as engineering, medicine, and scientific research.

Now meet seven people whose careers use the tools and methods of mathematics, the language of science and technology.



Martha C. Zuniga

### IMMUNOLOGY

An immunologist investigates the body's immune response, the process by which the body identifies, reacts to, and fights off toxic or disease-causing agents. Research in immunology contributes to the medical profession's ability to enhance the body's capacity to fight cancer and other life-threatening diseases.

**Martha C. Zuniga**, who holds a Ph.D. in biology from Yale University, conducts research on the immune system's ability to distinguish alien virus and tumor cells from the body's healthy tissues. In 1989 the National Science Foundation honored her with its Presidential Young Investigator Award, a grant to provide her with funding to continue her research.

As a recipient of the NSF award, Dr. Zuniga maintains an active role in undergraduate education, helping to promote the importance of studying science. At University of California, Santa Cruz, where she teaches immunology, Dr. Zuniga encourages her students to enjoy the challenges of difficult academic work. "People get excited about heroism, and I think there can be heroism in intellectual pursuits," says Dr. Zuniga, who also works with students through the Society for the Advancement of Chicanos and Native Americans in Science.

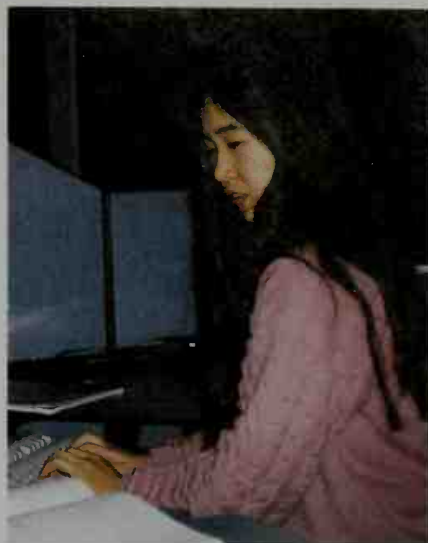


## COMPUTER GRAPHIC ARTS

Computer graphics have expanded the world of many artists. Computers help artists who work in traditional media to make decisions about composition, design, and color. For other artists, the end product of the artistic process is an image stored in the computer's memory and then reproduced on film, printers, or video.

**Midori Kitagawa De Leon**, a Ph.D. student at the Visualization Laboratory at Texas A & M University, became interested in computer graphics while majoring in oil painting at the Women's College of Fine Arts, in Tokyo. She went on to earn an M.A. in Computer Graphics and Animation from Ohio State University. There she wrote programs to generate three-dimensional "branching objects," such as trees and other plant-like forms, and to simulate the growth of the plants using genetically determined growth patterns as well as information about their environment.

Ms. Kitagawa De Leon's artistic work, which includes film animations of surrealistic plant life, has been featured in computer graphics art shows and magazines. In addition to its artistic value, her work will help landscape architects, who will use the programs to project the future appearance of their designs.



*Midori Kitagawa De Leon*



*Guion Bluford, Jr.*

## SPACE EXPLORATION

Any United States citizen holding a degree in mathematics, science, or engineering can apply to be an astronaut. Once selected, an astronaut candidate goes through a one-year training program, and then may spend many more years working on the ground before getting an opportunity to fly.

**Guion Bluford, Jr.** was a teenager when the space age began in 1957. He was fascinated by flying objects, from the model airplanes he built to the newspapers he tossed each day on his paper route, and he dreamed of entering the relatively new field of aerospace engineering. He graduated from Pennsylvania State University and the Air Force Institute of Technology, earning his Ph.D. in aerospace engineering in 1978. In the same year, NASA accepted him into its astronaut training program.

In August of 1983 Colonel Bluford flew on NASA's eighth shuttle mission, becoming the first African-American to travel into space. Two years later, his second space flight saw one Dutch, two West German, and five American astronauts fly together on a mission run by West Germany. In 1991 Colonel Bluford completed a third mission.



*Robert K. Whitman*

## ENGINEERING

Aerospace, biomedical, chemical, environmental, industrial—these are a few of the branches of the vast field of engineering. Engineers apply mathematics and science to the solution of practical problems. They design airplanes, buildings, highways, artificial limbs, lasers, and computers, among other things.

**Robert K. Whitman**, deputy director of The American Indian Science and Engineering Society, remembers his early interest in engineering. “I . . . took apart my parents’ radio, much to their dismay. I wanted to see . . . what made it pick up sounds. . . . I read . . . that engineering requires a lot of training in math and science. In high school, I took all the math and science courses I could get.” Bob Whitman went on to study electrical engineering at the University of New Mexico and at Colorado State University. He received a scholarship from NASA and an Outstanding Achievement Award from the Navajo Nation (1978). He has worked for IBM, where his projects included developing printed circuit boards used in electronic equipment, and designing computer software.

## COMPUTER PROGRAMMING

Computers help doctors make diagnoses, assist architects with their designs, and regulate the functioning of all kinds of machines from heart pacemakers to rocket engines. To carry out these and other tasks, computers require explicit instructions; thus, the power of computers to change and improve our lives can be realized only through the imagination and ingenuity of the people who program them.

**Martine Kempf** was still in her twenties when she succeeded in programming a computer to recognize and respond to the human voice. Using her program she invented the Katalavox, a small black box containing a voice-activated microcomputer. Installed on a wheelchair, the Katalavox gives quadriplegics unprecedented control of their lives. In addition, the Katalavox has become indispensable in the operating room, where surgeons’ hands are free to work while surgical microscopes are guided by voice.

Ms. Kempf, a native of France, was studying astronomy at the University of Bonn in 1982 when she first wrote the Katalavox program. She later moved to the United States to run her own business marketing her invention.



*Martine Kempf*

## PSYCHOLOGY

Psychologists study human and animal behavior, focusing on the mental functions involved in the emotional, intellectual, and physical development of individuals. Trained in the methods of scientific research and mathematical analysis of data, they formulate and test theories to explain and predict behavior. They also study the influence of the mind and body on each other.

**Patricia Cowings**, a psychophysicist for the Space Life Sciences Division of NASA, studies the body's response to the weightlessness experienced in space travel. She teaches astronauts to control, through mental techniques, such physical functions as blood pressure and heart rate, allowing them to counteract some of the negative effects of weightlessness.

Dr. Cowings first studied psychology at the State University of New York at Stony Brook, and then pursued graduate degrees at the University of California at Davis. While still a graduate student she began working for NASA at the Ames Research Center. She earned a Ph.D. in psychology in 1973.



*Patricia Cowings*



*Samuel Chao Chung Ting*

## PHYSICS

Physicists seek to understand all aspects of matter and energy, the fundamental components of our world. They pursue knowledge about subjects as varied as electromagnetism, optics, thermodynamics, acoustics, and quantum theory. All branches of physics require a thorough background in higher mathematics.

**Samuel Chao Chung Ting**, a researcher in the field of high-energy particle physics, studies the composition and behavior of subatomic particles. In 1974, while working at Brookhaven National Laboratory, Dr. Ting demonstrated the existence of the J particle, a discovery heralded as an important breakthrough in twentieth-century understanding of atomic structure. Also known as *psi*, the particle was independently observed by Dr. Burton Richter of the Stanford Linear Accelerator Center. In 1976 Dr. Ting and Dr.

Richter jointly received the Nobel Prize for their discovery.

American by birth, Dr. Ting grew up in China, returning to the United States to attend the University of Michigan, where he earned a Ph.D. in physics. In 1967 he joined the faculty of the Massachusetts Institute of Technology.



# 1

## *Linear and Quadratic Functions*



# Linear Functions

## 1-1 Points and Lines

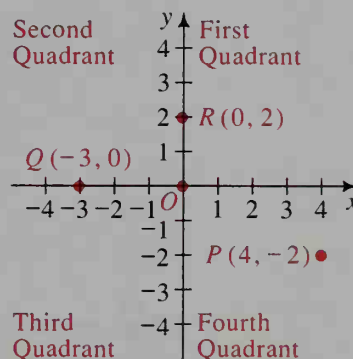
### Objective

To find the intersection of two lines and to find the length and the coordinates of the midpoint of a segment.

Each point in the plane can be associated with an ordered pair of numbers, called the **coordinates** of the point. Also, each ordered pair of numbers can be associated with a point in the plane. The association of points and ordered pairs is the basis of *coordinate geometry*, a branch of mathematics that connects geometric and algebraic ideas.

To set up a coordinate system, we can choose two perpendicular lines, one horizontal and the other vertical, as the **x-axis** and the **y-axis** and designate their point of intersection as the **origin**. Using a convenient unit of measure, we mark off the axes as number lines with zero located at the origin. The axes divide the plane into four **quadrants**.

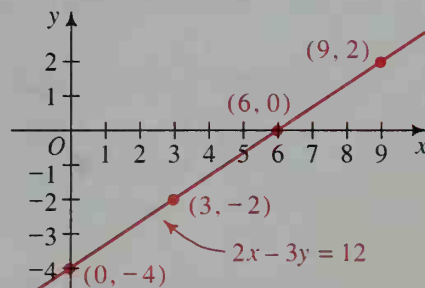
The diagram shows that  $P$  has **x-coordinate** 4 and **y-coordinate**  $-2$ . We write  $P(4, -2)$ . Points with x-coordinate 0, such as  $R$ , lie on the y-axis. Points with y-coordinate 0, such as  $Q$ , lie on the x-axis. The coordinates of the origin,  $O$ , are  $(0, 0)$ .



### Linear Equations

A **solution** of the equation  $2x - 3y = 12$  is an ordered pair of numbers that makes the equation true. For example,  $(0, -4)$  is a solution because  $2(0) - 3(-4) = 12$ . Several solutions are shown in the diagram. The set of all points in the plane corresponding to solutions of an equation is called the **graph** of the equation. The graph of  $2x - 3y = 12$  is the line shown in the diagram. We call  $-4$  the **y-intercept** of the graph because the line intersects the y-axis at  $(0, -4)$ . We call  $6$  the **x-intercept** of the graph because the line intersects the x-axis at  $(6, 0)$ .

Any equation of the form  $Ax + By = C$ , where  $A$  and  $B$  are not both 0, is called a **linear equation** because its graph is a line. Conversely, any line in the plane is the graph of a linear equation. The graph is often referred to as “the line  $Ax + By = C$ .” We call  $Ax + By = C$  the **general form** of a linear equation.



◀ In this view of Chicago from the air, the regular pattern of streets intersecting at right angles suggests a coordinate grid.

**Example 1** Sketch the graph of  $3x + 2y = 18$ .

**Solution** One way to sketch the graph is to find the intercepts.

**Step 1** To find the  $y$ -intercept, let  $x = 0$ .

$$3(0) + 2y = 18$$

$$y = 9$$

The line passes through  $(0, 9)$ , so the  $y$ -intercept is 9.

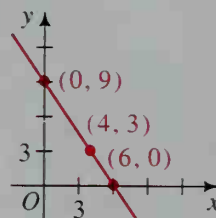
To find the  $x$ -intercept, let  $y = 0$ .

$$3x + 2(0) = 18$$

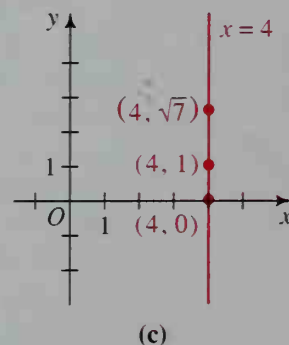
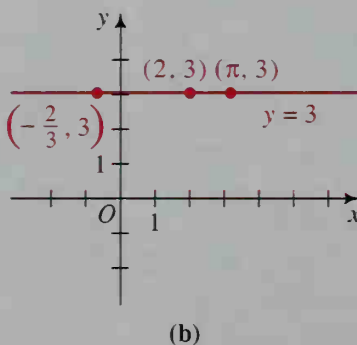
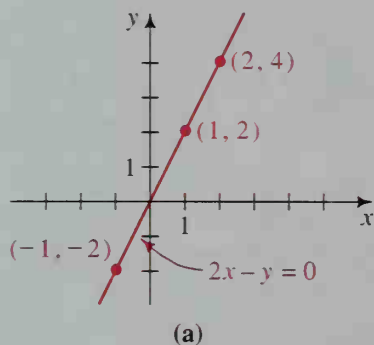
$$x = 6$$

The line passes through  $(6, 0)$ , so the  $x$ -intercept is 6.

**Step 2** Plot  $(0, 9)$  and  $(6, 0)$ . Draw a straight line through them. It is always a good idea to check that you have drawn the line correctly. Select a different point on the line and determine whether its coordinates satisfy the equation. In this case,  $(4, 3)$  does check since  $3(4) + 2(3) = 18$ .



When one of the constants  $A$ ,  $B$ , or  $C$  in  $Ax + By = C$  is 0, you can draw certain conclusions about the graph. In figure (a),  $C = 0$  and the line contains the origin. In figure (b),  $A = 0$  and the line is horizontal. In figure (c),  $B = 0$  and the line is vertical.



## Intersection of Lines

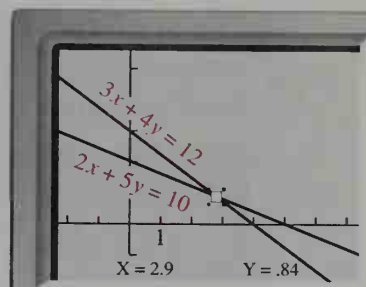
You can determine where two lines intersect by drawing their graphs or by solving their equations simultaneously. Consider the following pair of linear equations:

$$2x + 5y = 10 \quad (1)$$

$$3x + 4y = 12 \quad (2)$$



You can make hand-drawn sketches or you can use a graphing calculator or computer to obtain the graphs shown at the right. (If you use a calculator or computer, be sure to write  $y$  in terms of  $x$ .) From the figure it seems that at the point of intersection,  $x$  is a little less than 3 and  $y$  is a little less than 1. (With a calculator's trace feature or a computer's zoom feature, you can get better approximations to  $x$  and  $y$ .) Be aware that solutions found by graphing are not always exact. An algebraic solution yields the exact values.



To solve the equations simultaneously, you can multiply both sides of equation (1) by 3 and both sides of equation (2) by 2. Then subtract the second equation from the first equation.

$$\begin{aligned} 3(2x + 5y) &= 3(10) \longrightarrow 6x + 15y = 30 \\ 2(3x + 4y) &= 2(12) \longrightarrow 6x + 8y = 24 \\ \hline &7y = 6 \\ &y = \frac{6}{7} \end{aligned}$$

Now substitute  $\frac{6}{7}$  into equation (1) and solve for  $x$ :  $2x + 5\left(\frac{6}{7}\right) = 10$

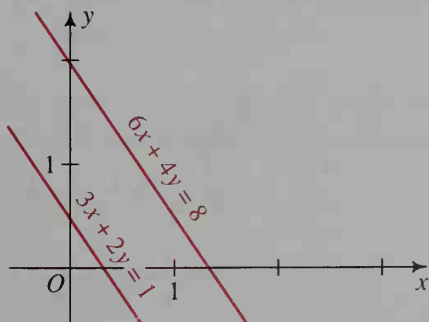
$$x = \frac{20}{7}$$

Thus  $\left(\frac{20}{7}, \frac{6}{7}\right)$  is the common solution of the two equations (or the intersection point of their graphs). Notice that the graphical estimate  $x \approx 3$  and  $y \approx 1$  is close to the exact answer. You can use a graphical estimate as a check on an algebraic solution.

When two linear equations have no common solution, their graphs are parallel lines. When two linear equations have infinitely many common solutions, the equations have the same graph.

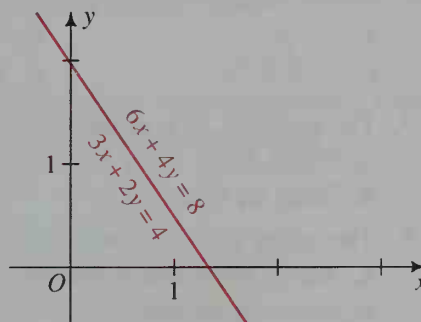
no common solution:

$$\begin{aligned} 6x + 4y &= 8 \\ 3x + 2y &= 1 \end{aligned}$$



infinitely many common solutions:

$$\begin{aligned} 6x + 4y &= 8 \\ 3x + 2y &= 4 \end{aligned}$$



We denote the line segment with endpoints  $A$  and  $B$  as  $\overline{AB}$  and its length as  $AB$ . You can use the formulas below to find  $AB$  and the midpoint of  $\overline{AB}$ . You are asked to derive these formulas in Written Exercises 34 and 35.

### The Distance and Midpoint Formulas

Let  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$ , and  $M$  be the midpoint of  $\overline{AB}$ . Then:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (\text{distance formula})$$

$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad (\text{midpoint formula})$$

**Example 2** If  $A = (-1, 9)$  and  $B = (4, -3)$ , find:

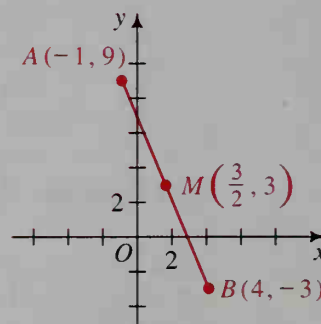
- the length of  $\overline{AB}$
- the coordinates of the midpoint of  $\overline{AB}$

**Solution**

- Find the distance between  $A$  and  $B$ .

$$\begin{aligned} AB &= \sqrt{(4 - (-1))^2 + (-3 - 9)^2} \\ &= \sqrt{25 + 144} \\ &= 13 \end{aligned}$$

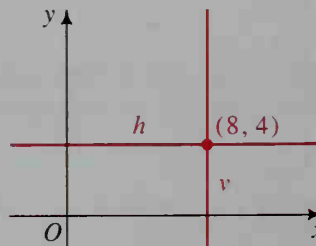
- $M = \left( \frac{-1 + 4}{2}, \frac{9 + (-3)}{2} \right) = \left( \frac{3}{2}, 3 \right)$



### CLASS EXERCISES

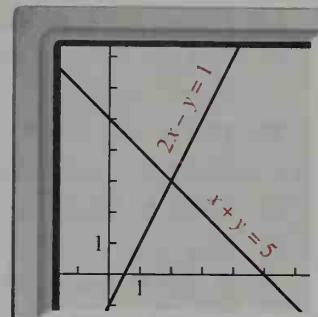
Find the length and the coordinates of the midpoint of  $\overline{CD}$ .

- $C(0, 0)$ ,  $D(8, 6)$
- $C(4, 2)$ ,  $D(6, 6)$
- $C(-3, 4)$ ,  $D(3, -2)$
- $C(7, -9)$ ,  $D(7, -1)$
- Let  $A = (2, 3)$ ,  $B = (6, 7)$ , and  $M$  be the midpoint of  $\overline{AB}$ . Find the coordinates of the midpoints of  $\overline{AM}$  and  $\overline{MB}$ .
- Which of the following points are on the line  $2x + 3y = 15$ ?
  - $(3, 3)$
  - $(9, -1)$
  - $(2.5, 3.5)$
  - $(-10.5, 12)$
- The point  $(8, 4)$  is on the horizontal line  $h$ .
  - Name three other points on  $h$ .
  - State an equation of  $h$ .
- The point  $(8, 4)$  is on the vertical line  $v$ .
  - Name three other points on  $v$ .
  - State an equation of  $v$ .



Exs. 7, 8

9. Find the coordinates of the points where the line  $4x - 3y = 18$  intersects the axes.
10. The diagram at the right shows the graphs of the equations  $x + y = 5$  and  $2x - y = 1$ .
- Estimate the coordinates of the point of intersection.
  - Find the exact coordinates of the point of intersection by solving the equations simultaneously.
  - Compare the solution with your estimate.



Ex. 10

## WRITTEN EXERCISES

Find the length of  $\overline{CD}$  and the coordinates of the midpoint of  $\overline{CD}$ .

- A**
- $C(1, 0), D(7, 8)$
  - $C(3, 3), D(15, 12)$
  - $C(-8, -3), D(7, 5)$
  - $C(-2, -1), D(4, 9)$
  - $C\left(\frac{1}{2}, \frac{9}{2}\right), D\left(-2, -\frac{3}{2}\right)$
  - $C\left(\frac{7}{2}, -1\right), D\left(-\frac{5}{2}, \frac{7}{2}\right)$
  - $C(4.8, 2.2), D(4.8, -2.8)$
  - $C(1.7, 5.7), D(-2.3, 5.7)$
9. Which of the following points are on the graph of  $3x - 2y = 15$ ?
- $(9, 6)$
  - $(8, 4)$
  - $\left(-\frac{4}{3}, -\frac{19}{2}\right)$
  - $(3.4, -3.2)$
  - $(-9, -22)$
10. Which of the following points are on the graph of  $-5x + 4y = 18$ ?
- $(-1.2, 3.0)$
  - $\left(3, -\frac{3}{4}\right)$
  - $(-18, 24)$
  - $(-6, -3)$
  - $(3.6, 9)$

In Exercises 11 and 12, graph each equation. Label the origin and the  $x$ - and  $y$ -intercepts as  $O$ ,  $P$ , and  $Q$ , respectively. Find the area of  $\triangle OPQ$ .

- $3x - 2y = 6$
  - $4x + 3y = 24$
13. On a single set of axes, sketch the horizontal line through  $(4, 3)$  and the vertical line through  $(5, -2)$ . What is the intersection of these lines? What are the equations of these lines?
14. Repeat Exercise 13 for the horizontal line through  $(-2, -1)$  and the vertical line through  $(-2, 3)$ .

In Exercises 15–18, solve the given pair of equations simultaneously. Then sketch the graphs of the equations and label the intersection point.

- $3x - 5y = 9$   
 $x + y = 3$
- $2x + 3y = 15$   
 $4x - 9y = 3$
- $x - 3y = 4$   
 $5x + y = -8$
- $-2x - 6y = 18$   
 $x - 3y = 6$



19. Plot  $A(1, 7)$ ,  $B(3, 5)$ ,  $C(4, -1)$ , and  $D(2, 1)$ . Use the distance formula to show that the opposite sides of quadrilateral  $ABCD$  are equal in length. What kind of figure is  $ABCD$ ?
20. Plot  $A(-6, 3)$ ,  $B(-1, 6)$ ,  $C(2, 1)$ , and  $D(-3, -2)$ . Use the distance formula to show that quadrilateral  $ABCD$  is a square. (*Hint*: Show that the four sides are equal in length and that the two diagonals are equal in length.)
21. Plot  $A(5, 1)$ ,  $B(7, -1)$ ,  $C(1, -3)$ , and  $D(-1, -1)$ . Use the midpoint formula to show that the diagonals of quadrilateral  $ABCD$  have the same midpoint. What kind of quadrilateral is  $ABCD$ ?
22. Plot  $A(2, 0)$ ,  $B(4, -6)$ ,  $C(9, 1)$ , and  $D(7, 7)$ . Show that  $\overline{AC}$  and  $\overline{BD}$  bisect each other. What kind of quadrilateral is  $ABCD$ ?
23. Given  $A(-3, 3)$ ,  $B(1, 11)$ , and  $C(3, 15)$ , show that  $B$  is on  $\overline{AC}$  by showing that  $AB + BC = AC$ .
24. Repeat Exercise 23 for  $A(-3, 7)$ ,  $B(-1, 4)$ , and  $C(3, -2)$ .

**B** 25. a. Show that  $P(4, 2)$  is equidistant from  $A(9, 2)$  and  $B(1, 6)$ .

b. If  $(2, k)$  is equidistant from  $A$  and  $B$ , find the value of  $k$ .

26. a. Show that  $P(1, 4)$  is equidistant from  $A(-5, -3)$  and  $B(-1, -5)$ .

b. If  $(3, k)$  is equidistant from  $A$  and  $B$ , find the value of  $k$ .

27.  $P$  is a point on the  $x$ -axis 13 units from the point  $(-3, 5)$ . Find all the possible coordinates for  $P$ .

28.  $Q$  is a point on the  $y$ -axis  $2\sqrt{10}$  units from the point  $(6, 1)$ . Find all the possible coordinates of  $Q$ .

29. Show that the three lines whose equations are  $x + 3y = 19$ ,  $2x - 5y = 5$ , and  $x - 2y = 4$  intersect in one point.

30. Determine whether the three lines whose equations are  $3x + 2y = 4$ ,  $5x - 2y = 0$ , and  $4x + 3y = 3$  intersect in one point.

31. a. Plot the points  $A(-6, 7)$ ,  $B(6, 3)$ , and  $C(-2, -1)$ . Then show that  $(BC)^2 + (AC)^2 = (AB)^2$ . What can you conclude about  $\angle C$ ?

b. Give the coordinates of the midpoint,  $M$ , of  $\overline{AB}$ . Verify these coordinates by showing that  $CM = \frac{1}{2}AB$ .

32. The area of a triangle with sides  $a$ ,  $b$ , and  $c$  units long can be found using *Hero's (or Heron's) formula*:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{a+b+c}{2}$$

Find to the nearest tenth the area of the triangle with vertices  $A(-13, 2)$ ,  $B(5, 17)$ , and  $C(22, -4)$ .



Parallel and intersecting lines create visual interest in the Pyramid at the Louvre Museum, Paris.

- C** 33. In  $\triangle ABC$ ,  $D(7, 3)$  is the midpoint of  $\overline{AB}$ ,  $E(10, 9)$  is the midpoint of  $\overline{BC}$ , and  $F(5, 5)$  is the midpoint of  $\overline{AC}$ . Find the coordinates of  $A$ ,  $B$ , and  $C$ .

34. In this proof, you may assume the following:  
The distance between two points on the same vertical line is the absolute value of the difference in  $y$ -coordinates; the distance between two points on the same horizontal line is the absolute value of the difference in  $x$ -coordinates. Note that the first quadrant is used for convenience.

Given:  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$

Prove:  $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

35. Given:  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$

$M$  is the midpoint of  $\overline{AB}$ .

$P$  is the midpoint of  $\overline{BC}$ .

$Q$  is the midpoint of  $\overline{AC}$ .

Prove: a.  $P = \left(x_2, \frac{y_1 + y_2}{2}\right)$

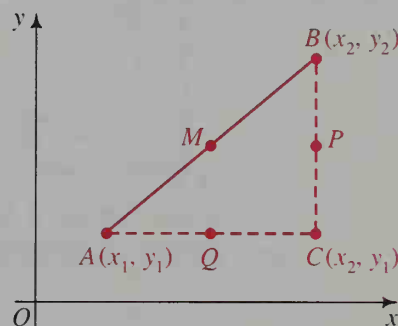
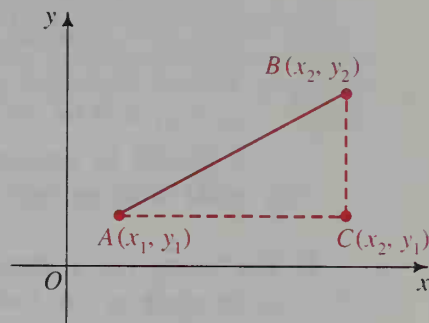
b.  $Q = \left(\frac{x_1 + x_2}{2}, y_1\right)$

c. Points  $M$  and  $P$  have the same  $y$ -coordinate.

d. Points  $M$  and  $Q$  have the same  $x$ -coordinate.

e.  $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

36. Three vertices of a parallelogram have coordinates  $(-3, 1)$ ,  $(1, 4)$ , and  $(4, 3)$ . Find the coordinates of the fourth vertex. How many possible answers are there?



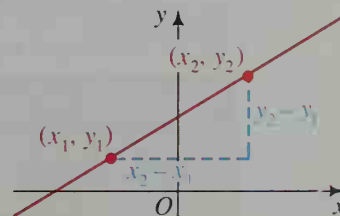
## 1-2 Slopes of Lines

### Objective

To find the slope of a line and to determine whether two lines are parallel, perpendicular, or neither.

The slope of a nonvertical line is a number measuring the steepness of the line relative to the  $x$ -axis. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be any two points on a line. The difference in the  $y$  values gives the *rise*, and the difference in the  $x$  values gives the *run*. The slope is the ratio of the rise to the run. That is, the slope  $m$  is defined by:

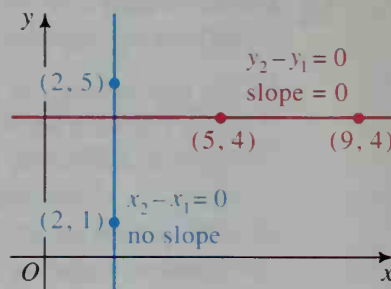
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



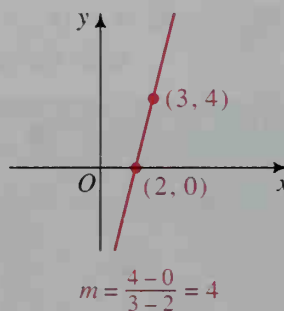
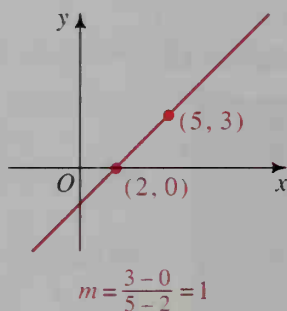
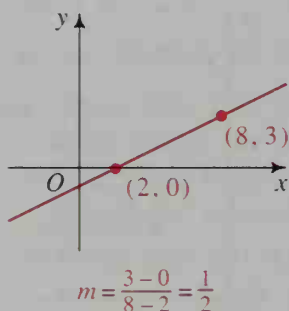
Some important facts about the slope of a line follow.

1. a. Horizontal lines have a slope of 0, because  $y_2 - y_1 = 0$  for all  $y$ .
- b. Vertical lines have no slope, because  $x_2 - x_1 = 0$  for all  $x$ .

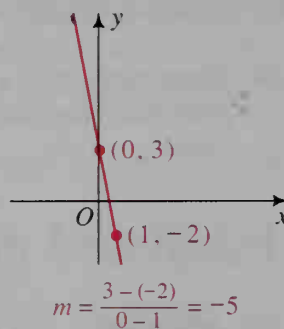
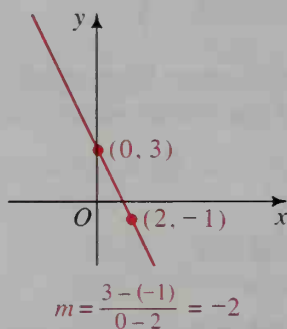
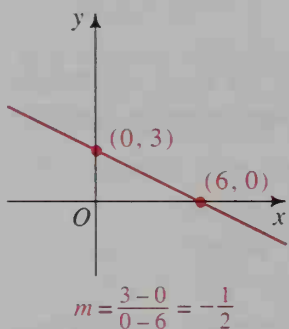
It should be apparent that having *no slope* is quite different from having *zero* slope.



2. The figures below show lines with *positive slope*. Lines with positive slope rise to the right as you look at points with greater and greater  $x$ -coordinates. The greater the slope, the more steeply the line rises.



3. The figures below show lines with *negative slope*. Lines with negative slope fall to the right as you look at points with greater and greater  $x$ -coordinates. The greater the absolute value of the slope, the more steeply the line falls.



The following theorem provides an easy way to find the slope of a line from its equation.

### Slope of a Line

The slope of  $y = mx + k$  is  $m$ .

Since  $m$  is a constant value, the slope of a line is constant.



The fact that the slope is constant tells us that any two points on a line will always give the same value for the slope. See Written Exercise 27 for an algebraic proof of the theorem on the previous page.

The numbers  $m$  and  $k$  in  $y = mx + k$  provide us with a mental picture of the line. We know that  $m$  is the slope of the line. The number  $k$  is the  $y$ -intercept, since  $y = m(0) + k = k$ . For these reasons, we refer to the form  $y = mx + k$  as the **slope-intercept form**.

### Example 1

What are the slope and  $y$ -intercept of the line  $7x + 13y = 26$ ?

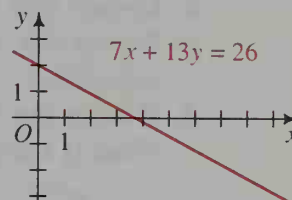
### Solution

Write the equation in slope-intercept form.

$$7x + 13y = 26$$

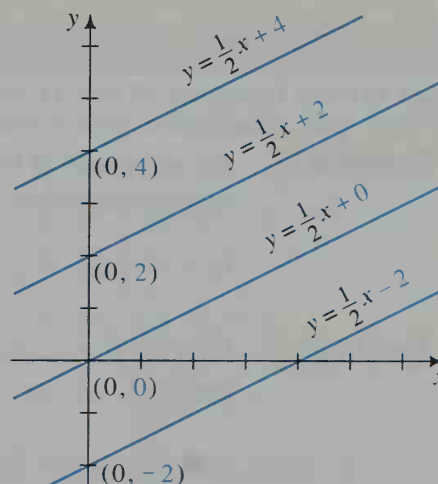
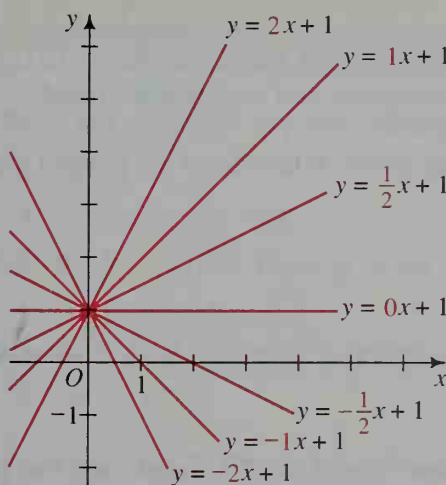
$$13y = -7x + 26$$

$$y = -\frac{7}{13}x + 2$$



The slope is  $-\frac{7}{13}$  and the  $y$ -intercept is 2.

The diagrams below illustrate the effect of  $m$  and  $k$  on the graph of an equation written in slope-intercept form. Notice that different lines with the same slope are parallel.



The two theorems that follow show how the concept of slope is related to the geometric ideas of parallel and perpendicular lines. See Written Exercises 28 and 29 for an algebraic proof and a geometric proof of the theorem below.

### Slopes of Parallel Lines

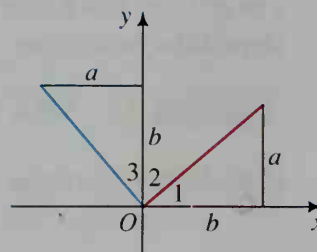
Two nonvertical lines are parallel if and only if they have the same slope.

As with the slopes of parallel lines, there is a special relationship between the slopes of perpendicular lines. The following activity, which leads us to the next theorem, demonstrates this relationship for line segments of equal length.

### Activity

A right triangle is placed along the  $x$ -axis. An identical triangle is placed along the  $y$ -axis.

- How are  $\angle 1$  and  $\angle 2$  related?  
How are  $\angle 1$  and  $\angle 3$  related?
- Deduce how  $\angle 2$  and  $\angle 3$  are related.
- Deduce the relationship between the red and blue segments.
- What is the slope of the red segment?  
What is the slope of the blue segment?



### Slopes of Perpendicular Lines

Two lines are perpendicular if and only if their slopes are *negative reciprocals* of each other. That is, if the slopes are  $m_1$  and  $m_2$ , then

$$m_1 = -\frac{1}{m_2}, \text{ or } m_1 \cdot m_2 = -1.$$

See Written Exercises 30 and 31 for a geometric proof and an algebraic proof.

### Example 2

The equations of three lines are given. Which lines are parallel and which are perpendicular?

$$l_1: y = \frac{3}{4}x - 7$$

$$l_2: 4x + 3y = 10$$

$$l_3: 3x - 4y = 11$$

### Solution

Find the slopes of the lines by rewriting their equations in slope-intercept form.

$$l_1: y = \frac{3}{4}x - 7 \qquad m_1 = \frac{3}{4}$$

$$l_2: y = -\frac{4}{3}x + \frac{10}{3} \qquad m_2 = -\frac{4}{3}$$

$$l_3: y = \frac{3}{4}x - \frac{11}{4} \qquad m_3 = \frac{3}{4}$$

Since both  $m_1$  and  $m_3$  are equal to  $\frac{3}{4}$ , lines  $l_1$  and  $l_3$  are parallel. Since

$m_2 \cdot m_1 = m_2 \cdot m_3 = -\frac{4}{3} \cdot \frac{3}{4} = -1$ , line  $l_2$  is perpendicular to lines  $l_1$  and  $l_3$ .

## CLASS EXERCISES

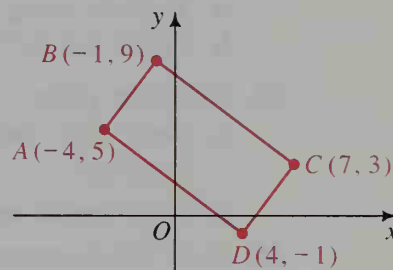
Find the slope of the line joining the given points.

1.  $(0, 0), (5, 7)$
2.  $(2, 0), (3, 6)$
3.  $(-2, 3), (1, 0)$
4.  $(2, -6), (8, -3)$
5.  $(-8, 2), (8, 2)$
6.  $(5, -3), (5, -7)$

Find the slope and y-intercept of the line whose equation is given.

7.  $y = 3x + 4$
8.  $y = \frac{3}{5}x - 3$
9.  $4x + 3y = 9$
10.  $y = -2$

11. Explain why a vertical line has no slope.
12. A line  $l$  has slope  $-2$ . What is the slope of a line perpendicular to  $l$ ?
13. A line  $l$  has equation  $3x + 2y = 10$ . What is the slope of a line (a) parallel to  $l$  and (b) perpendicular to  $l$ ?
14. a. Show that opposite sides of quadrilateral  $ABCD$  are parallel.  
b. Show that adjacent sides of quadrilateral  $ABCD$  are perpendicular.



## WRITTEN EXERCISES

Find the slope of the line joining the points whose coordinates are given.

- A**
1.  $(4, 2), (9, 5)$
  2.  $(0, 4), (12, 0)$
  3.  $(-4, -2), (2, -6)$
  4.  $(-2, 6), (2, -2)$
  5.  $(8, 5), (-7, 5)$
  6.  $(-3, 8), (-3, -2)$
  7.  $\left(-\frac{1}{3}, 2\right), \left(\frac{1}{4}, -\frac{1}{3}\right)$
  8.  $(0.25, 1.5), (0.5, 1)$
  9.  $(a, b), (b, a)$
  10.  $\left(a, \frac{a}{b}\right), \left(b, \frac{b}{a}\right)$

Find the slope and the y-intercept of the line whose equation is given.

11.  $y = 3x + 5$
12.  $y = 4 - 5x$
13.  $4x - 2y = 8$
14.  $3x + 9y = 7$
15.  $3y = 11x$
16.  $y = 5$

Tell which of the given equations have parallel line graphs and which have perpendicular line graphs.

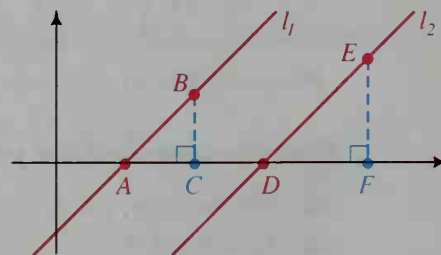
17. a.  $y = \frac{5}{2}x - 8$
- b.  $-15x + 6y - 10 = 0$
- c.  $4x + 10y = 15$
18. a.  $3y = 5x - 5$
- b.  $y = -\frac{3}{5}x + 4$
- c.  $10y = -6x - 7$



19. Show that the line through  $(2, -3)$  and  $(7, 2)$  is perpendicular to the line through  $(-3, 7)$  and  $(2, 2)$ .
  20. Show that the line through  $(2, 3)$  and  $(5, -2)$  is perpendicular to the line  $3x - 5y = 15$ .
  21. Find the value of  $k$  if the line joining  $(4, k)$  and  $(6, 8)$  and the line joining  $(-1, 4)$  and  $(0, 8)$  are (a) parallel, (b) perpendicular.
  22. Find the value of  $h$  if the line joining  $(3, h)$  and  $(5, 10)$  and the line  $y = 3x + 4$  are (a) parallel, (b) perpendicular.
- B**
23. Given the points  $A(-4, -6)$ ,  $B(2, 4)$ ,  $C(8, 6)$ , and  $D(2, -4)$ :
    - a. Show by using slopes that quadrilateral  $ABCD$  is a parallelogram.
    - b. Verify that both diagonals have the same midpoint.
  24. Given the points  $A(-4, 1)$ ,  $B(2, 3)$ ,  $C(4, 9)$ , and  $D(-2, 7)$ :
    - a. Show that quadrilateral  $ABCD$  is a parallelogram with perpendicular diagonals.
    - b. What special name is given to a quadrilateral like  $ABCD$ ?
  25. Show that the points  $(-4, -5)$ ,  $(-3, 0)$ ,  $(0, 2)$ , and  $(5, 1)$  are the vertices of an isosceles trapezoid.
  26. Show that  $(-1, -1)$ ,  $(9, 4)$ ,  $(20, 6)$ , and  $(10, 1)$  are the vertices of a rhombus, and then find the area of this rhombus.
  27. Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be any two points on the line  $y = Mx + k$ .

Show that  $M = \frac{y_2 - y_1}{x_2 - x_1}$ .

28. a. Suppose lines  $l_1$  and  $l_2$  have equations  $y = m_1x + b_1$  and  $y = m_2x + b_2$ , respectively. By solving these equations simultaneously, show that the lines intersect when  $x = \frac{b_2 - b_1}{m_1 - m_2}$ .
- b. What can you say about this value of  $x$  when  $m_1 = m_2$ ? What can you say about the intersection of the lines when  $m_1 = m_2$ ?
29. a. Express the slopes of lines  $l_1$  and  $l_2$  in terms of the lengths of the sides of the triangles shown.
- b. Prove that if the slopes of lines  $l_1$  and  $l_2$  are equal, then:
  - (1)  $\triangle ABC \sim \triangle DEF$
  - (2)  $\angle BAC \cong \angle EDF$
  - (3)  $l_1 \parallel l_2$
- c. The converse of part (b).  
Prove that if  $l_1 \parallel l_2$ , then:
  - (1)  $\angle BAC \cong \angle EDF$
  - (2)  $\triangle ABC \sim \triangle DEF$
  - (3) slope of  $l_1$  = slope of  $l_2$



Ex. 29

30. a. Express the slopes of lines  $l_1$  and  $l_2$  in terms of the lengths of the sides of the triangles shown.

- b. Prove that if the slopes of lines  $l_1$  and  $l_2$  are negative reciprocals of each other, then:

(1)  $\triangle OAB \sim \triangle OCD$

(2)  $\angle 1 \cong \angle 2$

(3)  $l_1 \perp l_2$

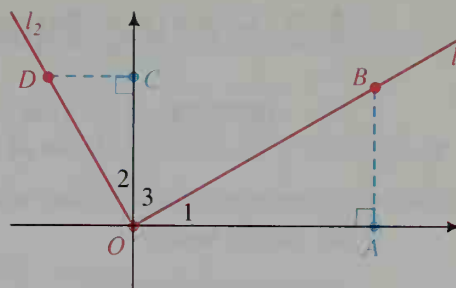
- c. The converse of part (b).

Prove that if  $l_1 \perp l_2$  then:

(1)  $\angle 1 \cong \angle 2$

(2)  $\triangle OAB \sim \triangle OCD$

(3) slope of  $l_1 \cdot$  slope of  $l_2 = -1$



Ex. 30

- C** 31. The diagram at the right shows lines  $l_1$  and  $l_2$ , chosen arbitrarily to intersect at  $O(0, 0)$ . These lines, whose equations are  $y = m_1x$  and  $y = m_2x$ , intersect the vertical line  $x = 1$  at  $A(1, m_1)$  and  $B(1, m_2)$ .

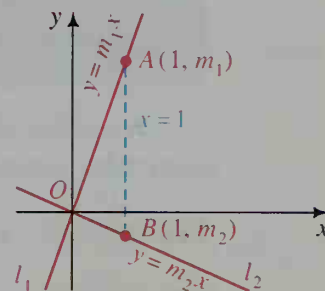
- a. Use the distance formula to find  $OA$ ,  $OB$ , and  $AB$ .

- b. Assume that  $l_1 \perp l_2$ . Use part (a) to show that if

$$(OA)^2 + (OB)^2 = (AB)^2,$$

then  $m_1 \cdot m_2 = -1$ .

- c. Prove the converse to the statement in part (b).



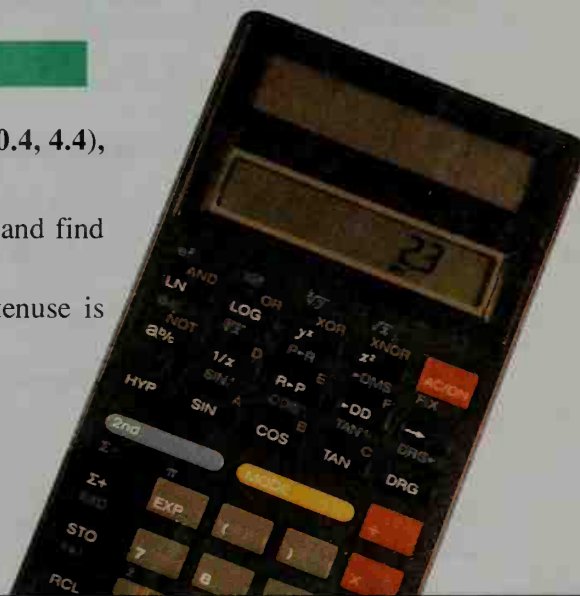
Ex. 31

32.  $ABCD$  is a square with vertices  $A(0, 0)$  and  $B(6, 8)$ . Give the coordinates of  $C$  and  $D$ . (Two answers are possible.)
33.  $PQRS$  is a rectangle with vertices  $P(-4, -1)$  and  $Q(-6, 5)$  and  $PQ = 2(QR)$ . Find the coordinates of  $R$  and  $S$ .

## ■■■■ CALCULATOR EXERCISES

Consider the points  $A(-1.8, 2.3)$ ,  $B(-0.4, 4.4)$ , and  $C(2.4, -0.5)$ .

- Verify that  $\triangle ABC$  is a right triangle and find its area.
- Show that the midpoint of the hypotenuse is equidistant from the three vertices.



# 1-3 Finding Equations of Lines

## Objective

To find an equation of a line given certain geometric properties of the line.

A linear equation in  $x$  and  $y$  can be written in several different forms, four of which are shown below.

The <b>general</b> form	$Ax + By = C$	
The <b>slope-intercept</b> form	$y = mx + k$	Line has slope $m$ and $y$ -intercept $k$ .
The <b>point-slope</b> form*	$\frac{y - y_1}{x - x_1} = m$	Line has slope $m$ and contains $(x_1, y_1)$ .
The <b>intercept</b> form	$\frac{x}{a} + \frac{y}{b} = 1$	Line has $x$ -intercept $a$ and $y$ -intercept $b$ .

When finding an equation of a line, sometimes it is easier to use one form rather than another. The following examples illustrate this.

### Example 1

Find an equation of the line with  $x$ -intercept 8 and  $y$ -intercept 4.

### Solution

Use the intercept form with  $a = 8$  and  $b = 4$ .

$$\frac{x}{8} + \frac{y}{4} = 1$$

The equation in general form is  $x + 2y = 8$ .

### Example 2

Find an equation of the line through  $(1, 4)$  and  $(5, -2)$ .

### Solution

**Step 1** First we find the slope of the line.

$$m = \frac{4 - (-2)}{1 - 5} = -\frac{3}{2}$$

**Step 2** Use the point-slope form with slope  $-\frac{3}{2}$  and a point on the line, choosing either  $(1, 4)$  or  $(5, -2)$ .

Using  $(1, 4)$ :

$$\frac{y - 4}{x - 1} = -\frac{3}{2}$$

Using  $(5, -2)$ :

$$\frac{y + 2}{x - 5} = -\frac{3}{2}$$

Each equation can be written in general form as  $3x + 2y = 11$ .

\*The point-slope equation is sometimes written in the form  $y - y_1 = m(x - x_1)$  to make it formally apparent that  $(x_1, y_1)$  satisfies the equation.



**Example 3**

Find an equation of the line with y-intercept  $-3$  and parallel to the line  $2x + 5y = 8$ .

**Solution**

Rewriting the equation  $2x + 5y = 8$  in slope-intercept form you get  $y = -\frac{2}{5}x + \frac{8}{5}$ . Thus, the slope of this line and the slope of the parallel line are both  $-\frac{2}{5}$ . Now use the slope-intercept form.

$$y = mx + k \longrightarrow y = -\frac{2}{5}x - 3$$

The equation in general form is  $2x + 5y = -15$ .

Example 4 illustrates two different ways of finding the perpendicular bisector of a line segment.

**Example 4**

Write an equation of the perpendicular bisector of the segment joining  $A(-2, 3)$  and  $B(4, -5)$ .

**Solution 1**

**Step 1** The slope of  $\overline{AB}$  is

$$\frac{-5 - 3}{4 - (-2)} = -\frac{8}{6} = -\frac{4}{3}.$$

The slope of the perpendicular bisector is the *negative reciprocal* of  $-\frac{4}{3}$ , namely  $\frac{3}{4}$ .

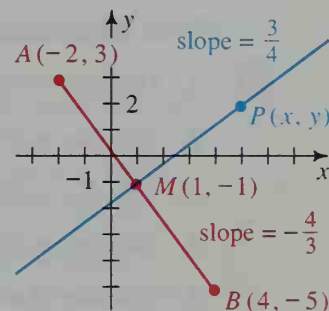
**Step 2** The perpendicular bisector contains the midpoint  $M$  of  $\overline{AB}$ :

$$M = \left( \frac{-2 + 4}{2}, \frac{3 + (-5)}{2} \right) = (1, -1)$$

**Step 3** The equation of the line through  $(1, -1)$  with slope  $\frac{3}{4}$  is

$$\frac{y - (-1)}{x - 1} = \frac{3}{4},$$

which can be written in general form as  $3x - 4y = 7$ .

**Solution 2**

This method uses the fact that  $P(x, y)$  is on the perpendicular bisector of  $\overline{AB}$  if and only if  $PA = PB$ . Using the distance formula, proceed as follows:

$$PA = PB$$

$$\sqrt{(x + 2)^2 + (y - 3)^2} = \sqrt{(x - 4)^2 + (y + 5)^2}$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = (x^2 - 8x + 16) + (y^2 + 10y + 25)$$

$$4x - 6y + 13 = -8x + 10y + 41$$

$$12x - 16y = 28$$

$$3x - 4y = 7$$

## CLASS EXERCISES

Find an equation in a convenient form of the line described.

1. The line with slope  $\frac{5}{3}$  and y-intercept  $-2$
2. The line through  $(-4, 6)$  and parallel to the line  $y = 3x + 8$
3. The line through  $(1, 2)$  and perpendicular to the line  $y = 4x - 3$
4. The line with x-intercept  $-1$  and y-intercept  $6$
5. The line through  $(8, 3)$  and  $(2, -1)$
6. In the special case of a nonvertical line passing through the origin, the general form of its equation,  $Ax + By = C$ , becomes  $Ax + By = 0$ . Write (a) the slope-intercept form, (b) the point-slope form, and (c) the intercept form of a nonvertical line passing through the origin.

## WRITTEN EXERCISES

Write an equation of the line described.

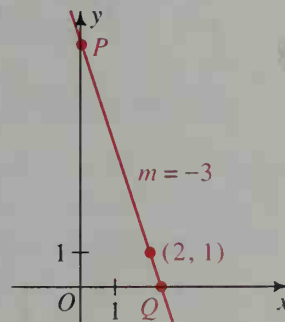
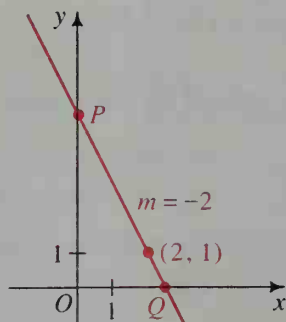
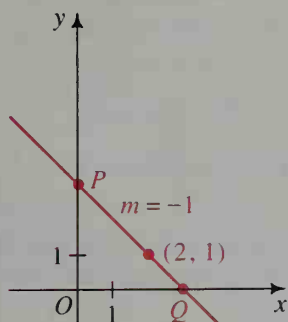
**A**

1. The line with slope  $-2$  and y-intercept  $8$
2. The line with slope  $\frac{3}{5}$  and passing through the origin
3. The line with y-intercept  $4$  and x-intercept  $-2$
4. The line with y-intercept  $-6$  and parallel to the line  $5x + 4y = 1$
5. The line through  $(-1, 4)$  and  $(5, 8)$
6. The line through  $(0, 5)$  and  $(6, 1)$
7. The horizontal line through  $(5, -7)$
8. The vertical line through  $(5, -7)$
9. The line through  $(2, -7)$  and  $(2, 3)$
10. The line through  $(5, -3)$  and  $(2, -3)$
11. The line parallel to the line  $0.3x - 1.2y = 6.4$  and with y-intercept  $1.8$
12. The line through  $(-2, 4)$  parallel to the line through  $(1, 1)$  and  $(5, 7)$
13. The line through  $(8, -2)$  perpendicular to the line  $y = 7 - 2x$
14. The line through the origin perpendicular to the line  $x - 3y = 9$
15. The perpendicular bisector of the segment joining  $(0, 3)$  and  $(-4, 5)$
16. The perpendicular bisector of the segment joining  $(2, 4)$  and  $(4, -4)$



The rows of crops suggest parallel lines.

- B** 17. Given  $A(2, 0)$  and  $B(8, 4)$ , show that  $P(3, 5)$  is on the perpendicular bisector of  $\overline{AB}$  by these two methods.  
*Method 1.* Show  $\overline{AB} \perp \overline{PM}$  where  $M$  is the midpoint of  $\overline{AB}$ .  
*Method 2.* Show  $PA = PB$ .
18. Repeat Exercise 17 using  $A(0, 7)$ ,  $B(2, -1)$ , and  $P(5, 4)$ .
19.  $\triangle PQR$  has vertices  $P(4, -1)$ ,  $Q(-2, 7)$ , and  $R(9, 9)$ .  
 a. Find an equation of the median from  $R$ .  
 b. Find an equation of the altitude from  $R$ .  
 c. Are your answers to (a) and (b) the same? Is  $\triangle PQR$  isosceles?
20.  $\triangle DEF$  has vertices  $D(-2, 5)$ ,  $E(6, -1)$ , and  $F(5, 6)$ .  
 a. Verify that  $\triangle DEF$  is isosceles.  
 b. Write an equation of the bisector of  $\angle F$ .
21.  $\triangle ABC$  has vertices  $A(-2, -1)$ ,  $B(0, 7)$ , and  $C(8, 3)$ .  
 a. Write equations for the three medians of the triangle.  
 b. Show that the three medians intersect at a single point  $G$ , called the *centroid*, or center of gravity, of the triangle.
22.  $\triangle PQR$  has vertices  $P(0, 4)$ ,  $Q(8, -2)$  and  $R(7, 5)$ .  
 a. Write equations for the perpendicular bisectors of the three sides of the triangle.  
 b. Show that the perpendicular bisectors intersect at a single point  $C$ , called the *circumcenter* of the triangle.  
 c. Note that  $C$  is the midpoint of  $\overline{PQ}$ . What kind of triangle is  $\triangle PQR$ ?
23.  $\triangle KLM$  has vertices  $K(0, 0)$ ,  $L(18, 0)$ , and  $M(6, 12)$ .  
 a. Write equations for the altitudes to the three sides of the triangle.  
 b. Show that the altitudes intersect at a single point  $O$ , called the *orthocenter* of the triangle.
24.  $\triangle RST$  has vertices  $R(1, 2)$ ,  $S(25, 2)$ , and  $T(10, 20)$ .  
 a. Find the centroid  $G$  of  $\triangle RST$  using the method of Exercise 21.  
 b. Find the circumcenter  $C$  of  $\triangle RST$  using the method of Exercise 22.  
 c. Find the orthocenter  $O$  of  $\triangle RST$  using the method of Exercise 23.  
 d. Show that  $G$ ,  $C$ , and  $O$  lie on the same line, which is called *Euler's line*.
25. Each of the lines shown below passes through  $(2, 1)$  and forms a triangle with the axes. Which of these three triangles has the least area?





26. Consider the set of all lines through  $(2, 6)$  with negative slope. Let  $P$  and  $Q$  be the points where the line intersects the  $y$ - and  $x$ -axes, respectively. Determine by experimentation the slope of the line that gives  $\triangle OPQ$  the least area.
27. The vertices of  $\triangle ABC$  are  $A(8, 5)$ ,  $B(0, 1)$ , and  $C(9, -2)$ .
- Find the length and an equation of  $\overline{BC}$ .
  - Find an equation of the altitude from  $A$  to  $\overline{BC}$ .
  - Find the point where the altitude from  $A$  intersects  $\overline{BC}$ .
  - Find the length of the altitude from  $A$  to  $\overline{BC}$ .
  - Find the area of  $\triangle ABC$ .
28. Find the distance from the point  $(9, 5)$  to the line  $4x - 3y = -4$ .



29. Given the point  $P(x_1, y_1)$ , and the line  $l$  with equation  $ax + by = c$ :
- Write an equation of the line  $j$  that passes through  $P$  and is perpendicular to  $l$ .
  - Show that  $l$  and  $j$  intersect at the point

$$Q\left(\frac{b^2x_1 - aby_1 + ac}{a^2 + b^2}, \frac{a^2y_1 - abx_1 + bc}{a^2 + b^2}\right).$$

- $PQ$  is the distance between the point  $P$  and the line  $l$ . Show that

$$PQ = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.$$

- Use the formula in part (c) to solve Exercise 28.
30.
  - Use the formula derived in Exercise 29 to verify that the point  $(7, 9)$  is 3 units from the lines  $12x - 5y = 0$  and  $3x - 4y = 0$ .
  - Tell why  $P(7, 9)$  is on the bisector of  $\angle DOE$ . Write an equation of this angle bisector and note that its slope is *not* the average of the slopes of lines  $OD$  and  $OE$ .
31.
  - Suppose  $P(x, y)$  is a point in the interior of  $\angle DOE$ , as shown in the diagram. Explain why  $P$  is on the bisector of  $\angle DOE$  if

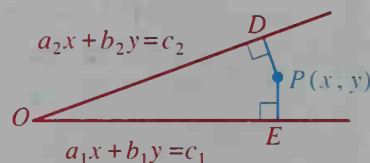
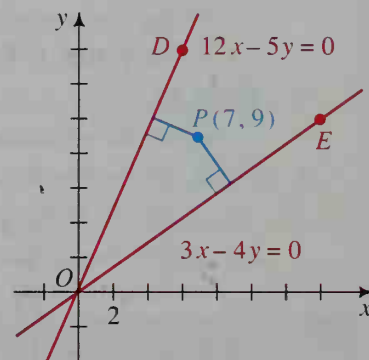
$$\frac{|a_1x + b_1y - c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y - c_2|}{\sqrt{a_2^2 + b_2^2}}.$$

- Lines  $l_1$  and  $l_2$  are defined by the equations below.

$$l_1: 4x - 3y = -6$$

$$l_2: 3x - 4y = -4$$

Use the equation of part (a) to write equations of the two lines that bisect the angles formed by  $l_1$  and  $l_2$ .



## 1-4 Linear Functions and Models

### Objective

To model real-world situations by means of linear functions.

A rail on a railroad track expands with heat and contracts with cold. Since its length depends on the temperature, we say that its length is a *function* of the temperature. If  $L$  stands for length in meters and  $T$  stands for temperature in degrees Celsius, then for a 10 m rail,  $L$  and  $T$  are related by this formula:

$$L = 0.0001T + 10$$



### Activity

- a. Complete the table of values below for  $L = 0.0001T + 10$ .

$T$ (in $^{\circ}\text{C}$ )	$0^{\circ}$	$10^{\circ}$	$20^{\circ}$	$30^{\circ}$	$40^{\circ}$
$L$ (in m)	10	?	?	?	?

- b. Graph the values found in part (a) using  $T$  as the horizontal axis and  $L$  as the vertical axis in a coordinate plane.
- c. The graph in part (b) will look very much like a horizontal line. Why?

A *function* describes a dependent relationship between quantities. For example, the value of the expression  $3x - 5$  depends on the value of  $x$ , so we say that  $3x - 5$  is a function of  $x$  and write  $f(x) = 3x - 5$ . The function notation  $f(x)$  is read “ $f$  of  $x$ .” It represents the value of the function  $f$  evaluated at  $x$ . Thus, if  $f(x) = 3x - 5$ , then  $f(2) = 1$ . Notice that  $f\left(\frac{5}{3}\right) = 0$ , so we say that  $\frac{5}{3}$  is a *zero* of the function  $f$ . In general, if  $f(a) = 0$ , then  $a$  is called a **zero** of the function  $f$ .

Some examples of linear functions are given below.

$$f(x) = 3x - 5$$

$f$  is a linear function of  $x$ .

$$L(T) = 0.0001T + 10$$

$L$  is a linear function of  $T$ .

$$g(s) = -1.2s + 4.7$$

$g$  is a linear function of  $s$ .

$$h(t) = 3$$

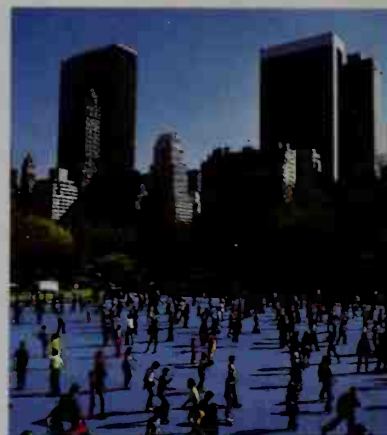
$h$  is a linear function of  $t$ .

Linear functions have the form  $f(x) = mx + k$ . The function  $h$  above is linear because it can be written as  $h(t) = 0t + 3$ . This special kind of linear function is called a *constant function*. The graphs of linear functions are straight lines or points on a line, as Example 1 shows on the next page.

**Example 1**

The senior class has paid \$200 to rent a roller skating rink for a fund raising party. Tickets for the party are \$5 each.

- Express the net income as a function of the number of tickets sold.
- Graph the function. Identify the point at which the class begins to make a profit.

**Solution**

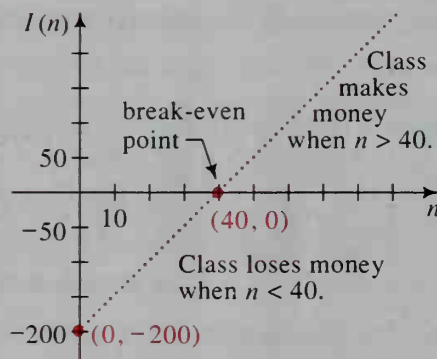
- Let  $I(n)$  = the net income when  $n$  tickets are sold.

Net income is the amount of money remaining after operating expenses are deducted from the amount of money earned.

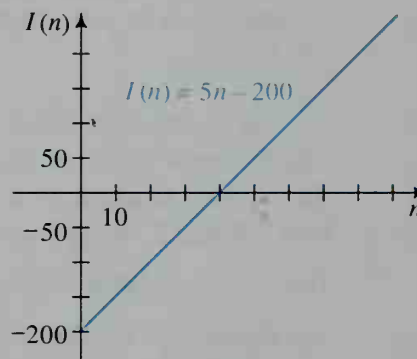
$$I(n) = \$5 \times \text{the number of tickets sold} - \$200 \text{ rental fee}$$

$$I(n) = 5n - 200$$

- The graph of  $I$  is shown in red below. Since  $n$  must be a nonnegative integer, the graph consists of discrete (isolated) points on a line. However, the line that contains the discrete points is often given as a sketch of the function. (See the graph in blue.) From these graphs you can see that  $n = 40$  is a zero of the income function. Therefore, the class must sell more than 40 tickets to make a profit.



Graph of function  $I$  as discrete points on a line



Graph of function  $I$  as a continuous portion of a line

The *domain* of a function is the set of values for which the function is defined. In the function  $I(n) = 5n - 200$  in Example 1,  $n$  represents the number of tickets sold, so the domain is the set of nonnegative integers. Substituting  $n = -3$  or  $n = 4.5$  does not make sense in this case. You can think of the domain of a function as the set of input values. The set of output values is called the *range* of the function. When you substitute the domain values  $n = 0, 1, 2, 3, \dots$ , you get the range values  $I(n) = -200, -195, -190, -185, \dots$ .

The function  $I(n) = 5n - 200$  is said to *model* income. A **mathematical model** is one or more functions, graphs, tables, equations, or inequalities that describe a real-world situation. Example 2 illustrates another model.



**Example 2**

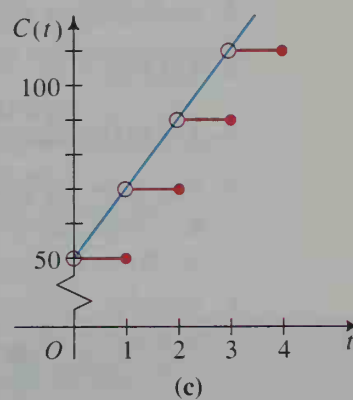
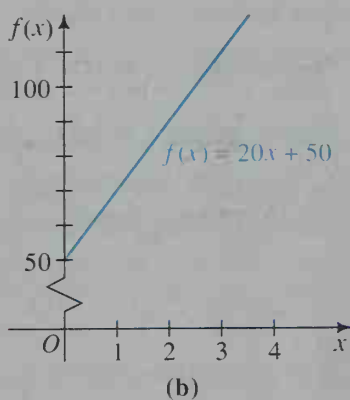
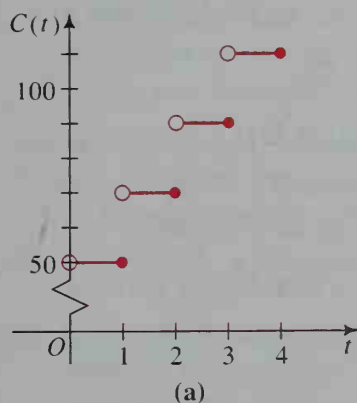
Suppose that it costs 50 cents for the first minute of a long distance telephone call and 20 cents for each additional minute or fraction thereof. Give a graphical model of the cost of a call lasting  $t$  minutes.

**Solution**

It is important to realize that the cost of a call lasting 2 min and 12 seconds (s) is the same as a 3 min call. Likewise, a  $3\frac{1}{2}$  min call costs the same as a 4 min call. The table below lists some of the input and output values for this cost function.

Input (time in minutes) $t$	Output (cost in cents) $C(t)$
$0 < t$ and $t \leq 1$	50
$1 < t$ and $t \leq 2$	$50 + 20(1) = 70$
$2 < t$ and $t \leq 3$	$50 + 20(2) = 90$
$3 < t$ and $t \leq 4$	$50 + 20(3) = 110$

The function that models this cost is a *step function*, so named because its graph has steps. See figure (a) below. Because the linear function  $f(x) = 50 + 20x$  gives good approximations to the costs, it can also be used to model the cost of long distance calls. See figure (b). Be aware that if you use the linear function, you will get overestimates of the cost, as you can see in figure (c), which shows the graphs of both functions on the same set of axes.

**CLASS EXERCISES**

1. a. If  $f(x) = 5x - 10$ , find  $f(3)$  and  $f(-3)$ .  
 b. Find the zero of  $f$ .  
 c. Graph the function. Where does the graph of  $f$  intersect the vertical axis?  
 the horizontal axis?  
 d. Where does the zero of  $f$  appear on the graph of  $f$ ?

2. Which of the following are linear functions?

a.  $f(t) = \frac{8-9t}{3}$

b.  $g(x) = \frac{x}{4} - 3$

c.  $h(x) = \frac{4}{x} - 3$

3. Using the equation  $s + 2t = 8$ , express  $t$  as a function of  $s$ .
4. It costs \$1.40 for the first minute of a phone call to Paris, France, and \$.80 for each additional minute or fraction thereof.
- Draw the graph of a step function that models this cost.
  - Sketch the graph of a linear function that approximates this cost.
5. **Writing** Suppose that  $f(x) = mx + k$ . Write an expression for  $f(a) - f(b)$  in terms of  $a$ ,  $b$ , and  $m$  only. In your own words, explain how this formula can be used to evaluate mentally  $f(79.6) - f(69.6)$  for the function  $f(x) = 50x + 387$ . Use the formula to find the value of  $f(12.8) - f(12.3)$ .



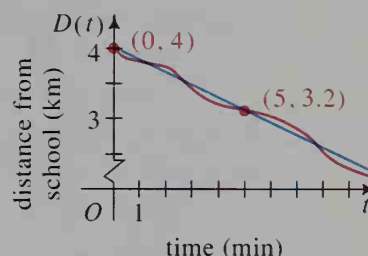
## WRITTEN EXERCISES

**A**

- If  $f(x) = \frac{3}{4}x - \frac{1}{2}$ , find  $f(2)$  and  $f(-2)$ .
  - Find the zero of  $f$ .
- If  $C(n) = 20 - \frac{5}{8}n$ , find  $C(0)$  and  $C(16)$ .
  - Find the zero of  $C$ .
- Let  $f(x) = 3x - 7$ . Decide whether  $f(2) + f(6) = f(8)$ .
- Let  $h(t) = \frac{9-4t}{2}$ . Decide whether  $h(4.5) - h(3.5) = h(1)$ .
- Consider the constant functions  $g(x) = 2$  and  $h(x) = -1$ .
  - Graph the two functions. What are the slopes of their graphs?
  - Write the functions  $g$  and  $h$  in the form  $f(x) = mx + k$ .
- Consider the constant function  $P(x) = -0.5$ .
  - Find  $P(1269.35)$ .
  - Does the function  $P$  have any zeros? Explain.
- What is the slope of the graph of  $f(x) = 1.5x - 2$ ?
  - What is the zero of the function?
  - What are the intercepts of the graph?
- What is the slope of the graph of  $C(t) = 80t + 5.2$ ?
  - Where does the graph of the function  $C$  intersect the vertical axis?
- The graph of the linear function  $f$  has slope  $-2$  and intersects the  $n$ -axis at  $n = 6$ . Find an equation for  $f(n)$ .
- The zero of the linear function  $S$  is 3. The graph of this function intersects the vertical axis at  $S(x) = -2$ . Find an equation for  $S(x)$ .

11. Let  $f$  be a linear function such that  $f(1) = 5$  and  $f(3) = 9$ .
- Sketch the graph of  $f$ .
  - Find an equation for  $f(x)$ .
12. Let  $g$  be a linear function such that  $g(-1) = -3$  and  $g(-4) = 12$ .
- Sketch the graph of  $g$ .
  - Find an equation for  $g(x)$ .

13. Bill bikes 4 km to school. After 5 min he is 3.2 km from the school. The graph of the function that models Bill's distance to school is shown in red. Because of hills and traffic conditions along the way, Bill's speed varies. Nevertheless, a linear function, whose graph is shown in blue, can also be used to model his distance as a function of his time spent bicycling.



- Give an equation of the linear function.
  - Use the equation to find approximately how long it takes Bill to bicycle to school.
14. Different round-trip fares from Boston to various cities are shown in the table below.

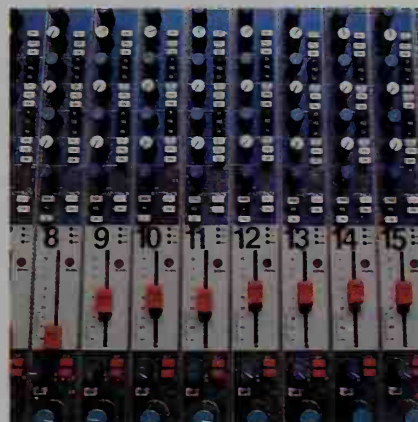
From Boston to:	Buffalo	Chicago	Houston	Los Angeles	Hawaii
distance (mi)	400	850	1600	2600	5100
cost (dollars)	118	198	298	350	400

- Using this data, sketch a graph of the cost as a function of distance.
- Does it appear that a linear function would give good approximations to this cost function? Explain.

- B** 15. Maria Correia's new car costs \$280 per month for car payments and insurance. She estimates that gas and maintenance cost \$0.15 per mile.
- Express her total monthly cost as a function of the miles driven during the month.
  - What is the slope of the graph of the cost function?

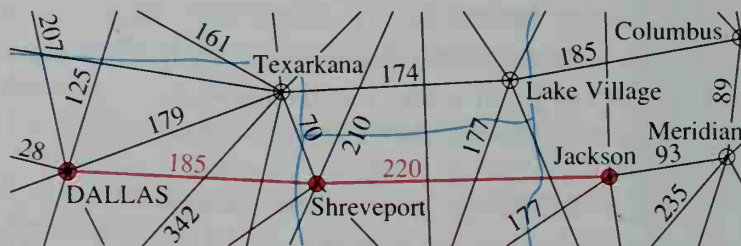
16. **Business** A recording studio invests \$24,000 to produce a master tape of a singing group. It costs \$1.50 to make each copy of the master and cover the operating costs.

- Express the cost of producing  $t$  tapes as a function  $C(t)$ .
- If each tape is sold for \$6.50, express the revenue (the total amount received from the sale) as a function  $R(t)$ .
- Sketch the graphs of the functions  $C$  and  $R$ .
- Find the coordinates of the break-even point (that is, where the graphs intersect).





17. A salesman whose home is in Dallas, Texas, leaves Shreveport, Louisiana, heading east for Jackson, Mississippi. He drives the 220 mi trip at a steady speed, arriving in Jackson exactly  $4\frac{1}{4}$  h after leaving Shreveport. Let  $t$  be the traveling time in hours after the salesman leaves Shreveport. The salesman's distance from Dallas,  $d(t)$ , is a linear function of  $t$ .



- Find to the nearest tenth the salesman's average driving speed after leaving Shreveport, and then find an equation for  $d(t)$ .
  - What is the domain of the function  $d$ ?
  - Sketch the graph of  $d$  and give its slope.
  - What is the relationship between the salesman's driving speed and the slope of the graph of the function?
18. **Accounting** A businesswoman buys a \$12,000 computer system. The value of the system is considered to depreciate by a fixed amount, which is 10% of the purchase price each year over a ten-year period. (This method of determining the value is called *straight-line depreciation*.)
- Express the value of the computer system as a function of the number of years since its purchase.
  - What is the domain of the function?
  - Sketch the graph of the function and give its slope.
  - What is the relationship between the amount of yearly depreciation of the computer system and the slope of the graph of the function?
19. **Writing** Write a sentence that uses the common meaning of the word slope, and then write a sentence that uses the mathematical meaning of the word slope. Write a paragraph that compares and contrasts the common and mathematical meanings.
20. At a city garage, it costs \$4 to park for the first hour and \$2 for each additional hour or fraction thereof. The fee is a function of the time parked.
- Sketch the graph of a function that models this parking fee.
  - Sketch the graph of a linear function that approximates the parking fee. Find an equation for this linear function.
21. A taxicab driver charges \$2.00 for the first half mile and \$.40 for each additional quarter mile. The fare is a function of the distance traveled.
- Sketch the graph of a function that models this fare.
  - Sketch the graph of a linear function that approximates the fare. Find an equation for this linear function.

22. The table below shows that the measure of each angle of a regular polygon is a function of the number of sides.

number of sides	3	4	5	6	...	$n$
angle measure	$60^\circ$	$90^\circ$	$108^\circ$	$120^\circ$	...	$A(n)$

- Explain how the data in the table tell us that the function is not linear.
  - What is the domain of the function?
  - Use your knowledge of geometry to write an equation for  $A(n)$ .
23. The last test that Mr. Clements gave was so hard that he decided to scale the grades upward. He decided to raise the lowest score of 47 to a 65 and the highest score of 78 to a 90. Find a linear function that would give a fair way to convert the other test scores.

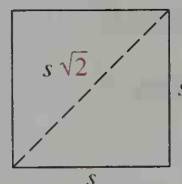
- C** 24. When David Arnold started out on his trip, his odometer read **45973** and his fuel gauge read  $\frac{7}{8}$  full. Exactly 1 h 48 min later, the odometer read **46081** and the fuel gauge read  $\frac{1}{2}$  full.
- Under what conditions does it seem reasonable that the odometer reading would be a linear function of the fuel gauge reading?
  - Find an equation for the linear function, and sketch its graph.
  - What is the real-world significance of the point where the graph of the function crosses the vertical axis?
  - When David sees the  $\frac{1}{8}$  full reading on his fuel gauge, he wonders if he has enough gas to get to Hartford, 40 mi away, driving at the same speed. Can he reach Hartford without refueling?
  - Express the odometer reading as a function of the time spent driving.

## ■ Quadratic Functions

### 1-5 The Complex Numbers

**Objective** To add, subtract, multiply, and divide complex numbers.

Throughout the history of mathematics, new kinds of numbers have been invented to fill deficiencies in the existing number system. In earliest times, there were only the *counting numbers*, 1, 2, 3, . . . The Egyptians and the ancient Greeks invented the *rational numbers*, so named because they are ratios of integers, to represent fractional parts of quantities. The Greeks also discovered that some numbers were not rational. For example, the ratio of the length of a diagonal of a square to the length of a side cannot be represented as the quotient of two integers. We know that this ratio is  $\sqrt{2}$ , an *irrational number*.



## Real Numbers

Zero was originally used by the Hindus to represent an empty column in a counting board similar to an abacus. The concept was brought to Europe by way of Arabia in the 9th century. The negative numbers were invented very much later in Renaissance Europe. One story tells that the plus and minus signs were first used in German warehouses to indicate whether crates of goods weighed more or less than some standard amount.

Numbers that are used for measurement and that can be represented on a continuous number line belong to the *real number system*. The real numbers consist of zero and all positive and negative integers, rational numbers, and irrational numbers. One of the basic properties of real numbers is that their squares are never negative.

## Complex Numbers

In the 16th century, a few mathematicians began to work with numbers whose squares *are* negative. The word “imaginary” gradually came to be used to describe such numbers as  $\sqrt{-1}$  and  $\sqrt{-15}$ . The use of the word imaginary reflects some of the original uneasiness many mathematicians had with nonreal numbers. Today, the phrase “imaginary numbers” seems a little unfortunate since these numbers are firmly established in mathematics. They are routinely used in advanced mathematics, electrical AC circuits, map making, and quantum mechanics, to name just a few fields.

We define the **imaginary unit**  $i$  with the following properties:

$$i = \sqrt{-1} \text{ and } i^2 = -1.$$

We then define the square root of any negative number as follows:

$$\text{If } a > 0, \sqrt{-a} = i\sqrt{a}.$$

Even though there is now a definition for the square root of a negative number, we cannot assume that all the square root properties of positive numbers will also be true for negative real numbers. For example, the property

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

is true when  $a$  and  $b$  are both positive real numbers, but not when  $a$  and  $b$  are both negative real numbers. (See Class Exercise 11.)

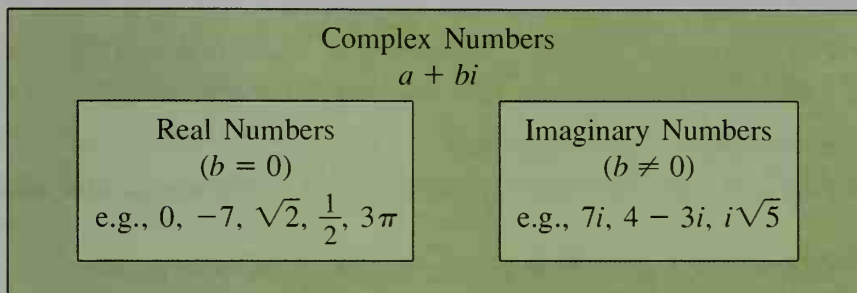
**Example 1** Simplify:   a.  $\sqrt{-25}$                       b.  $\sqrt{-7}$

**Solution**           a.  $\sqrt{-25} = i\sqrt{25} = 5i$                       b.  $\sqrt{-7} = i\sqrt{7}$

**Example 2** Simplify    $\sqrt{-9} - 2\sqrt{-25}$ .

**Solution**            $\sqrt{-9} - 2\sqrt{-25} = i\sqrt{9} - 2i\sqrt{25}$   
                              $= 3i - 2 \cdot 5i$   
                              $= -7i$

Any number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit, is called a **complex number**. For example,  $5i$ ,  $3 + 4i$ ,  $5.2 - i\sqrt{7}$ ,  $6$ , and  $0$  are all complex numbers. In  $a + bi$ ,  $a$  is called the **real part** and  $b$  is called the **imaginary part** of the complex number. If  $b \neq 0$ , the number is called an **imaginary number**.



As the Venn diagram illustrates, you can think of the complex numbers as consisting of two non-overlapping sets, the real numbers and the imaginary numbers. Those imaginary numbers in which  $a = 0$ , such as  $3i$ ,  $-i$ , and  $i\sqrt{7}$ , are called **pure imaginary** numbers.

Two complex numbers  $a + bi$  and  $c + di$  are **equal** if and only if  $a = c$  and  $b = d$ . The examples below show that you can add or multiply two complex numbers simply by treating  $i$  as if it were a variable and using the distributive laws.

**Example 3**  $(2 + 3i) + (4 + 5i) = 6 + 8i$

**Example 4**  $(2 + 3i)(4 + 5i) = 8 + 10i + 12i + 15i^2$   
 $= 8 + 22i + (15)(-1) \longleftarrow i^2 = -1$   
 $= -7 + 22i$

The complex numbers  $a + bi$  and  $a - bi$  are called **complex conjugates**. Their sum is a real number, and their product is a nonnegative real number. (See Written Exercises 35 and 36). The conjugate of the complex number  $z = a + bi$  is often denoted by  $\bar{z} = a - bi$ . In Example 5, we use the fact that the product of  $z$  and  $\bar{z}$  is a real number.

**Example 5** Express  $\frac{5 - 2i}{4 + 3i}$  in the form  $a + bi$ .

**Solution** Multiply the numerator and the denominator by the conjugate of the denominator  $4 + 3i$ .

$$\begin{aligned} \frac{5 - 2i}{4 + 3i} &= \frac{5 - 2i}{4 + 3i} \cdot \frac{4 - 3i}{4 - 3i} = \frac{20 - 15i - 8i + 6i^2}{16 - 9i^2} \\ &= \frac{20 - 23i - 6}{16 + 9} \\ &= \frac{14 - 23i}{25} = \frac{14}{25} - \frac{23}{25}i \end{aligned}$$



## CLASS EXERCISES

Simplify.

- $(2 + 6i) + (5 - 4i)$
- $(3 + i)(3 - i)$
- $(4 + i)(4 - i)$
- $(\sqrt{2} + i)(\sqrt{2} - i)$
- Find real numbers  $x$  and  $y$  such that  $4 - 5i = 2x + yi$ .
- Find  $i^4$ ,  $i^8$ ,  $i^{12}$ , and  $i^{16}$ . What can you say about  $i^n$  if  $n$  is a positive multiple of 4?
- The statement  $\sqrt{a}\sqrt{b} = \sqrt{ab}$  is true if  $a$  and  $b$  are both positive.
  - Verify that  $\sqrt{4} \cdot \sqrt{9} = \sqrt{4 \cdot 9}$ .
  - Verify that  $\sqrt{-4} \cdot \sqrt{9} = \sqrt{-4 \cdot 9}$ .
  - Compare the value of  $\sqrt{-4} \cdot \sqrt{-9}$  with the value of  $\sqrt{-4 \cdot (-9)}$ .
  - Is the statement  $\sqrt{a}\sqrt{b} = \sqrt{ab}$  true if both  $a$  and  $b$  are negative?

## WRITTEN EXERCISES

Simplify each expression.

- A**
- $\sqrt{-4} + \sqrt{-16} + \sqrt{-1}$
  - $\sqrt{-1}\sqrt{-9}$
  - $\frac{\sqrt{-12}}{\sqrt{-3}}$
  - $(4 - 3i) + (-6 + 8i)$
  - $4(3 + 5i) - 2(2 - 6i)$
  - $(6 - i)(6 + i)$
  - $(5 + i\sqrt{5})(5 - i\sqrt{5})$
  - $(8 + 3i)(2 - 5i)$
  - $(4 - 5i)^2$
  - $\sqrt{-49} - \sqrt{-9} + \sqrt{-36}$
  - $\sqrt{-2}\sqrt{-5}$
  - $\frac{\sqrt{-25}}{\sqrt{-50}}$
  - $(7 - 8i) - (6 + 2i)$
  - $\frac{1}{6}(7 - 2i) + \frac{2}{3}(5 - 5i)$
  - $(7 + 3i)(7 - 3i)$
  - $(\sqrt{3} + 4i\sqrt{2})(\sqrt{3} - 4i\sqrt{2})$
  - $(5 - 2i)(-1 + 3i)$
  - $(4 + 7i)^2$

Write each expression in the form  $a + bi$ .

- $\frac{1}{2 + 5i}$
- $\frac{5 + i}{5 - i}$
- $\frac{3 + i\sqrt{2}}{7 - i\sqrt{2}}$
- $\frac{1}{4 - 3i}$
- $\frac{3 - 2i}{3 + 2i}$
- $\frac{2 + i\sqrt{5}}{3 - i\sqrt{5}}$

- B** 25.  $\frac{5}{i}$
27.  $i + i^2 + i^3 + i^4 + i^5$
29.  $i^{-3}$
31.  $i^{-35}$
26.  $\frac{i^2 + 2i^3}{i}$
28.  $i^{46} + i^{47}$
30.  $i^{-6}$
32.  $(i^n)^4$ , where  $n$  is any integer
33. Find real numbers  $x$  and  $y$  such that  $(2x + y) + (3 - 5x)i = 1 - 7i$ .
34. Find real numbers  $x$  and  $y$  such that  $(3x - 4y) + (6x + 2y)i = 5i$ .
35. Show that the sum of  $a + bi$  and its conjugate is a real number.
36. Show that the product of  $a + bi$  and its conjugate is a nonnegative real number.
37. a. How could you show that 79 is a square root of 6241 without using a calculator?  
b. How could you show that  $3 - i$  is a square root of  $8 - 6i$ ?
38. Show that  $4 - 3i$  is a square root of  $7 - 24i$ .
39. Show that  $\frac{\sqrt{2}}{2}(1 + i)$  is a square root of  $i$ .
40. Find the square roots of  $3 + 4i$ .
- C** 41. If  $z = \bar{z}$ , what special kind of number is  $z$ ?
42. Show that  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ .
43. Show that  $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ .
44. Show that  $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$  if  $z_2 \neq 0$ .
45. Use Exercise 43 to show that  $\overline{z^2} = (\bar{z})^2$ .
46. a. Use Exercises 43 and 45 to show that  $\overline{z^3} = (\bar{z})^3$ .  
b. Make a generalization of Exercise 45 and part (a) above.



These windmills produce electricity. Complex numbers are used in the study of electricity.

### ■■■ CALCULATOR EXERCISE

Find  $(6 + 7i)^8$ . Use the fact that  $(x + yi)^2 = (x^2 - y^2) + (2xy)i$ .

### //// COMPUTER EXERCISE

Write a program that will compute  $(a + bi)^n$  when you input the positive integer  $n$  and the real numbers  $a$  and  $b$ .

## 1-6 Solving Quadratic Equations

### Objective

To solve quadratic equations using different methods.

Any equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where  $a \neq 0$ , is called a **quadratic equation**. A **root**, or **solution**, of a quadratic equation is a value of the variable that satisfies the equation. Three methods for solving quadratic equations are:

- (1) factoring   (2) completing the square   (3) the quadratic formula

### Factoring

Whenever the product of two factors is zero, at least one of the factors must be zero. For example, if  $(3x - 2)(x + 4) = 0$ , then  $3x - 2 = 0$  or  $x + 4 = 0$ . A quadratic equation must be written in the **standard form**  $ax^2 + bx + c = 0$  before it can be solved by factoring.

#### Example 1

Solve  $(3x - 2)(x + 4) = -11$ .

#### Solution

$$3x^2 + 10x - 8 = -11$$

$$3x^2 + 10x + 3 = 0$$

$$(3x + 1)(x + 3) = 0$$

Therefore,  $3x + 1 = 0$  or  $x + 3 = 0$

$$x = -\frac{1}{3} \text{ or } x = -3$$

The solutions are  $-3$  and  $-\frac{1}{3}$ .

### Completing the Square

The method of transforming a quadratic equation so that one side is a perfect square trinomial is called **completing the square**.

#### Example 2

Solve  $2x^2 - 12x - 7 = 0$ .

#### Solution

**Step 1** Divide both sides by the coefficient of  $x^2$  so that  $x^2$  will have a coefficient of 1.

$$x^2 - 6x - \frac{7}{2} = 0$$

**Step 2** Subtract the constant term from both sides.

$$x^2 - 6x = \frac{7}{2}$$

**Step 3** Complete the square. Add the square of one half the coefficient of  $x$  to both sides.

$$x^2 - 6x + (-3)^2 = \frac{7}{2} + (-3)^2$$

$$(x - 3)^2 = \frac{25}{2}$$

**Step 4** Take the square root of both sides and solve for  $x$ .

$$x - 3 = \pm \frac{5\sqrt{2}}{2}$$

$$x = 3 \pm \frac{5\sqrt{2}}{2}$$

The solutions are  $3 + \frac{5\sqrt{2}}{2}$  and  $3 - \frac{5\sqrt{2}}{2}$ .

## The Quadratic Formula

The *quadratic formula*, shown below, is derived by completing the square. (See Written Exercise 44.)

### The Quadratic Formula

The roots of the quadratic equation  $ax^2 + bx + c = 0$  are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0)$$

**Example 3** Solve  $2x^2 + 7 = 4x$ .

**Solution** Rewrite the equation in standard form:

$$2x^2 - 4x + 7 = 0$$

Substitute  $a = 2$ ,  $b = -4$ , and  $c = 7$  into the quadratic formula.

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 7}}{2 \cdot 2} \\ &= \frac{4 \pm \sqrt{-40}}{4} = \frac{4 \pm 2i\sqrt{10}}{4} \\ &= 1 \pm \frac{i\sqrt{10}}{2} \end{aligned}$$

The solutions are  $1 - \frac{i\sqrt{10}}{2}$  and  $1 + \frac{i\sqrt{10}}{2}$ .

## The Discriminant $b^2 - 4ac$

The quantity that appears beneath the radical sign in the quadratic formula,  $b^2 - 4ac$ , can tell you whether the roots of a quadratic equation are real or imaginary, as described on the next page. Because of this “discriminating ability,”  $b^2 - 4ac$  is called the **discriminant**.



## The Nature of the Discriminant

Given the quadratic equation  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers:

If  $b^2 - 4ac < 0$ , there are two conjugate imaginary roots.

If  $b^2 - 4ac = 0$ , there is one real root (called a *double root*).

If  $b^2 - 4ac > 0$ , there are two different real roots.

In the special case where the equation  $ax^2 + bx + c = 0$  has integral coefficients and  $b^2 - 4ac$  is the square of an integer, the equation has rational roots. That is, if  $b^2 - 4ac$  is the square of an integer, then  $ax^2 + bx + c$  has factors with integral coefficients.

## Choosing a Method of Solution

Although you can use the quadratic formula to solve *any* quadratic equation, it is often much easier to factor or complete the square. The list below suggests when to use which method.

Situation	Method to use
$a$ , $b$ , and $c$ are integers and $b^2 - 4ac$ is a perfect square.	factoring
The equation has the form $x^2 + (\text{even number})x + \text{constant} = 0$ .	completing the square

In the remaining situations the quadratic formula is usually the easiest to use, especially when solving an equation like  $px^2 + qx + r = 0$ , where the coefficients are letters. Also, the quadratic formula is the most efficient method to use when approximating real roots of a quadratic equation using a computer or calculator.

## Losing or Gaining a Root

If an equation contains variables on both sides or variables in a denominator, then you must carefully organize your method for solving in order not to lose a root or gain a root.

**Losing a Root.** It is possible to lose a root by dividing both sides of an equation by a common factor.

**Incorrect method**

$$4x(x - 1) = 3(x - 1)^2 \leftarrow \text{Divide both sides by } (x - 1).$$

$$4x = 3(x - 1)$$

$$4x = 3x - 3$$

$$x = -3$$

Actually, as shown below, the roots of  $4x(x - 1) = 3(x - 1)^2$  are 1 and  $-3$ . When we divided both sides of the equation by  $x - 1$ , we lost a root. Two ways to avoid this mistake are shown below. Both methods are correct.

### Method 1

If there is a factor common to both sides of the equation, remember to include as roots all values that make this factor zero.

$$4x(x - 1) = 3(x - 1)^2$$

If  $x - 1 = 0$ , then  $x = 1$  and both sides of the equation equal zero. Thus, 1 is a root.

If  $x - 1 \neq 0$ , we can divide both sides by  $x - 1$ , getting  $4x = 3(x - 1)$ . Thus,  $-3$  is a root.

### Method 2

Bring all terms to one side of the equation and then solve.

$$4x(x - 1) = 3(x - 1)^2$$

$$4x(x - 1) - 3(x - 1)^2 = 0$$

$$4x^2 - 4x - 3x^2 + 6x - 3 = 0$$

$$x^2 + 2x - 3 = 0$$

$$(x - 1)(x + 3) = 0$$

$$x = 1 \text{ or } x = -3$$

**Gaining a Root.** It is possible to gain a root by squaring both sides of an equation. (See Class Exercise 19.) Another possible way to gain a root is by multiplying both sides of an equation by an expression, as in the following example. Any gained root, called an **extraneous root**, satisfies the transformed equation but not the original equation. Always check your solutions in the *original equation* in order to eliminate extraneous roots.

**Example 4** Solve  $\frac{x+2}{x-2} + \frac{x-2}{x+2} = \frac{8-4x}{x^2-4}$ .

### Solution

Multiply both sides of the equation by  $(x + 2)(x - 2)$ .

$$(x + 2)^2 + (x - 2)^2 = 8 - 4x$$

$$(x^2 + 4x + 4) + (x^2 - 4x + 4) = 8 - 4x$$

$$2x^2 + 8 = 8 - 4x$$

$$2x^2 + 4x = 0$$

$$2x(x + 2) = 0$$

$$x = 0, -2$$

Check:  $x = 0$

$$\frac{0+2}{0-2} + \frac{0-2}{0+2} \stackrel{?}{=} \frac{8-0}{0^2-4}$$

$$-1 + (-1) = -2$$

Thus, 0 is a root.

$x = -2$

$$\frac{-2+2}{-2-2} + \frac{-2-2}{-2+2} \stackrel{?}{=} \frac{8-4(-2)}{(-2)^2-4}$$

Since two denominators are zero, the equation is meaningless. Thus,  $-2$  is *not* a root of the original equation.

Therefore, the solution is  $x = 0$ .

## CLASS EXERCISES

What must be added to the following expressions to complete the square?

1.  $x^2 + 8x + \underline{\quad ? \quad}$     2.  $x^2 - 10x + \underline{\quad ? \quad}$     3.  $x^2 + 7x + \underline{\quad ? \quad}$     4.  $x^2 + 2ax + \underline{\quad ? \quad}$

Does factoring, completing the square, or using the quadratic formula seem to you the easiest method for solving the following equations?

5.  $x^2 + 14x = 374$     6.  $4x^2 - 5x = 0$     7.  $3x^2 + 11x - 4 = 0$   
 8.  $px^2 + qx + r = 0$     9.  $4x^2 - x - 7 = 0$     10.  $x^2 - 14x - 736 = 0$

Exercises 11–13 refer to the quadratic equation  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers.

11. Explain why the equation has imaginary roots when  $b^2 - 4ac < 0$ .  
 12. Explain why the equation has only one root, and why this root is real, when  $b^2 - 4ac = 0$ . Does it seem reasonable to you to call this sort of root a *double root*?  
 13. If  $a$ ,  $b$ , and  $c$  are integers and  $b^2 - 4ac$  is the square of an integer, explain why the equation has rational roots.  
 14. The discriminant of the equation  $x^2 - \sqrt{5}x + 1 = 0$  is the square of an integer, but the roots of the equation are not rational. Does the equation offer a contradiction of the result of Exercise 13?  
 15. Can a quadratic equation with irrational coefficients be solved by using the quadratic formula? Explain your answer.  
 16. Comment on the following method for solving the equation  $9x^2 + 36x = x + 4$ .

$$\begin{aligned} 9x^2 + 36x &= x + 4 \\ 9x(x + 4) &= (x + 4) \\ 9x &= 1 \\ x &= \frac{1}{9} \end{aligned}$$

17. Solve  $(2x - 5)(x + 6) = 7(x + 6)$ .  
 18. Solve  $(3x - 5)(4x - 1) = (4x - 1)$ .  
 19. Comment on the following method for solving the equation  $\sqrt{x + 6} = x$ .

$$\begin{aligned} \sqrt{x + 6} &= x \\ x + 6 &= x^2 \\ 0 &= x^2 - x - 6 \\ 0 &= (x - 3)(x + 2) \\ x &= 3 \text{ or } -2 \end{aligned}$$

# WRITTEN EXERCISES

Solve by factoring.

**A**

1.  $3x^2 - 4x - 7 = 0$

2.  $4x^2 - 8x - 32 = 0$

3.  $(2x - 3)(x + 4) = 6$

4.  $(3y - 2)(y + 4) = 24$

Solve by completing the square. Give both real and imaginary roots.

5.  $x^2 - 10x = 1575$

6.  $x^2 - 6x = 391$

7.  $2z^2 - 16z - 1768 = 0$

8.  $x^2 - 8x - 20 = 0$

9.  $x^2 + 6x + 10 = 0$

10.  $y^2 + 10y + 35 = 0$

Solve by using the quadratic formula. Give your answers in simplest radical form. Give both real and imaginary roots.

11.  $5x^2 + 2x - 1 = 0$

12.  $4x^2 - 4x - 17 = 0$

13.  $3t^2 = 12t - 15$

14.  $5u^2 + 2 = 5u$

15.  $\frac{4}{v} = \frac{v-6}{v-4}$

16.  $\frac{4}{z} = \frac{3z}{z-3}$

Solve by whichever method seems easiest. Give both real and imaginary roots. Be sure not to lose or gain roots.

17.  $8x^2 = 7 - 10x$

18.  $4t = 1 + 15t^2$

19.  $(3x - 2)^2 = 121$

20.  $(4y + 4)^2 = -16$

21.  $(4x + 7)(x - 1) = 2(x - 1)$

22.  $(2x + 1)(4x - 3) = 3(4x - 3)^2$

23.  $2w(4w - 1) = w(1 - 4w)$

24.  $3(2x - 3)^2 = 4x(3 - 2x)$

25.  $\frac{x+3}{x-3} + \frac{x-3}{x+3} = \frac{18-6x}{x^2-9}$

26.  $\frac{r}{r-1} - \frac{r}{r+1} = \frac{2}{r^2-1}$

27.  $\frac{t^2+1}{t+2} = \frac{t}{3} + \frac{5}{t+2}$

28.  $\frac{x+2}{x^2-x-6} = 3 - \frac{4}{x-3}$

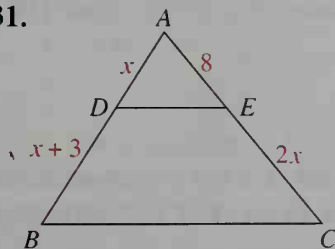
29.  $2\sqrt{x} = x - 8$

30.  $\sqrt{2x+5} = x + 1$

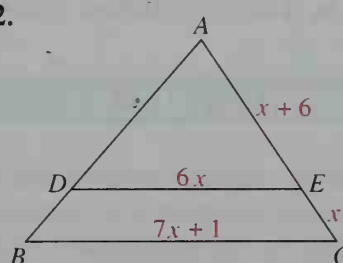
In Exercises 31 and 32,  $\overline{DE}$  is parallel to  $\overline{BC}$ . Find the value of  $x$ .

**B**

31.



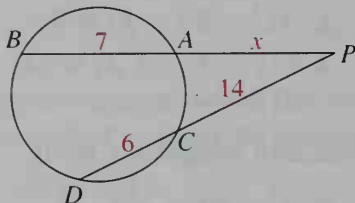
32.



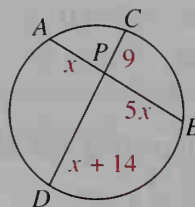


In Exercises 33 and 34,  $\overleftrightarrow{PB}$  intersects the circle at points  $A$  and  $B$ , and  $\overleftrightarrow{PD}$  intersects the circle at points  $C$  and  $D$ . By theorems from geometry,  $PA \cdot PB = PC \cdot PD$  in each case. Find the value of  $x$ .

33.



34.



35. a. What is the discriminant of the equation  $4x^2 + 8x + k = 0$ ?  
 b. For what value of  $k$  will the equation have a double root?  
 c. For what values of  $k$  will the equation have two real roots?  
 d. For what values of  $k$  will the equation have imaginary roots?  
 e. Name three values of  $k$  for which the given equation has rational roots.
36. Repeat Exercise 35 for the equation  $5x^2 + 8x + k = 0$ .

In Exercises 37–40, use the quadratic formula to solve each equation.

37.  $\sqrt{2}x^2 - 5x + \sqrt{8} = 0$

38.  $4x^2 - 2\sqrt{5}x - 1 = 0$

39.  $x^2 - 6ix - 9 = 0$

40.  $ix^2 - 3x - 2i = 0$

**C** Solve the equations in Exercises 41 and 42. You will need to square twice.

41.  $\sqrt{2x + 5} = 2\sqrt{2x} + 1$

42.  $\sqrt{y - 3} = 1 - \sqrt{2y - 4}$



Exercise 43 requires the use of a computer or a graphing calculator. Give answers to the nearest hundredth.

43. Use a computer or a graphing calculator to solve the equations in Exercises 41 and 42. For example, for Exercise 41, you could graph  $y = \sqrt{2x + 5}$  and  $y = 2\sqrt{2x} + 1$  on the same set of axes and then use the ZOOM feature to obtain approximations of the  $x$ -coordinates of any points of intersection. Do you encounter any extraneous roots using this method?
44. Derive the quadratic formula. (*Hint:* Solve the general quadratic equation  $ax^2 + bx + c = 0$ ,  $a \neq 0$ , by completing the square.)
45. Let  $a$  be a positive integer such that  $ax^2 + x - 6$  can be factored. Find:  
 a. The five smallest values of  $a$       b. Two values of  $a$  greater than 100

## /// COMPUTER EXERCISE

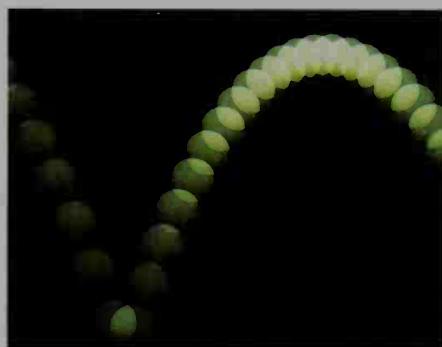
Write a computer program that will print the roots of  $ax^2 + bx + c = 0$  when you input real numbers  $a$ ,  $b$ , and  $c$ . Have the program print REAL ROOTS, DOUBLE ROOT, or IMAGINARY ROOTS as appropriate.

# 1-7 Quadratic Functions and Their Graphs

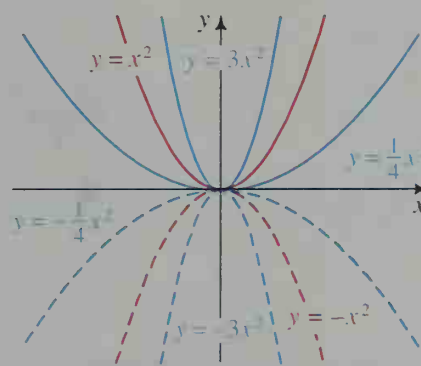
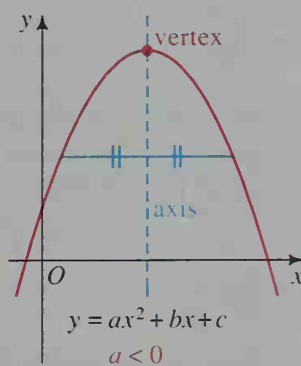
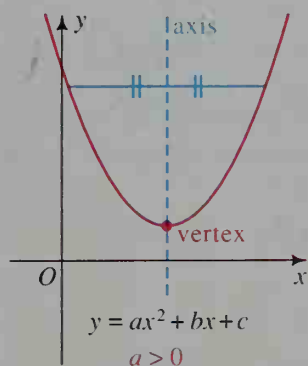
## Objective

To define and graph quadratic functions.

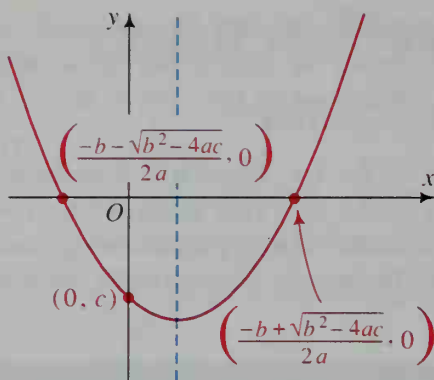
The graph of the **quadratic function**  $f(x) = ax^2 + bx + c$ , where  $a \neq 0$ , is the set of points  $(x, y)$  that satisfy the equation  $y = ax^2 + bx + c$ . This graph is a *parabola*, a curve that can be seen, for example, in the cables of a suspension bridge and in the path of a thrown ball. Parabolas can also be defined geometrically, as you will see in Chapter 6.



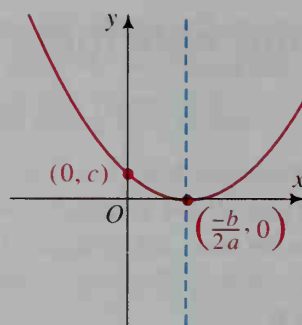
If a graph has an *axis of symmetry*, then when you fold the graph along this axis, the two halves of the graph coincide. The graph of a quadratic function has a vertical axis of symmetry, or **axis**. The **vertex** of the parabola is the point where the axis of symmetry intersects the parabola. If  $a > 0$ , the parabola opens upward, and the function has a minimum value. If  $a < 0$ , the parabola opens downward, and the function has a maximum value. The bigger  $|a|$  is, the narrower the parabola is. In the figure at the far right below, the graph of  $y = 3x^2$  is narrower than the graph of  $y = x^2$ .



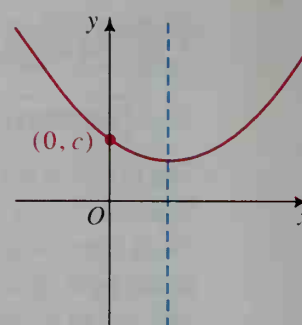
The  $y$ -intercept of a parabola with equation  $y = ax^2 + bx + c$  is  $c$ . The  $x$ -intercepts are the real roots of  $ax^2 + bx + c = 0$ . Since the quadratic equation may have two, one, or no real roots, depending on the value of the discriminant  $b^2 - 4ac$ , we have three possibilities regarding the  $x$ -intercepts of a parabola, as shown in the diagrams on the next page.



If  $b^2 - 4ac > 0$ , there are two  $x$ -intercepts.



If  $b^2 - 4ac = 0$ , there is one  $x$ -intercept (at a point where the parabola and  $x$ -axis are tangent).



If  $b^2 - 4ac < 0$ , there are no  $x$ -intercepts.

The axis of symmetry is a vertical line midway between the  $x$ -intercepts. The equation of the axis is  $x = -\frac{b}{2a}$ . (See Exercises 39 and 40.) In fact, this is the equation of the axis even if there are fewer than two  $x$ -intercepts.

### Example 1

Sketch the parabola  $y = 2x^2 - 8x + 5$ . Label the intercepts, axis of symmetry, and vertex.

### Solution

**Step 1** To find the  $y$ -intercept, you let  $x = 0$ . The  $y$ -intercept is 5.

**Step 2** To find the  $x$ -intercepts, solve  $2x^2 - 8x + 5 = 0$ .

$$x = 2 \pm \frac{\sqrt{6}}{2}$$

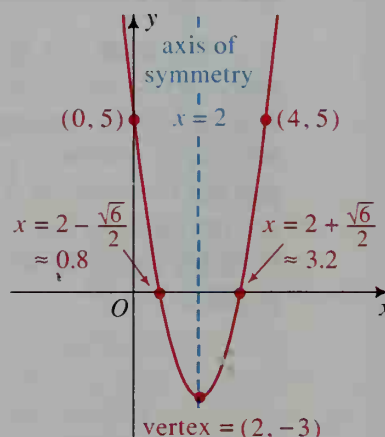
**Step 3** The axis of symmetry is the line

$$x = -\frac{b}{2a} = -\frac{-8}{2 \cdot 2} = 2.$$

**Step 4** Since the vertex is on the axis of symmetry, its  $x$ -coordinate is 2. To find the  $y$ -coordinate of the vertex, substitute  $x = 2$  into the equation of the parabola.

$$y = 2(2)^2 - 8(2) + 5 = -3$$

Thus, the vertex is  $(2, -3)$ .



If the equation of a parabola is written in the form

$$y = a(x - h)^2 + k,$$

then the vertex is  $(h, k)$ . For example:

The vertex of the parabola  $y = 2(x - 3)^2 + 7$  is  $(3, 7)$ .

The vertex of the parabola  $y = -4(x + 9)^2 + 2$  is  $(-9, 2)$ .

**Example 2**

- a. Find the vertex of the parabola  $y = -2x^2 + 12x + 4$  by completing the square.
- b. Find the  $x$ - and  $y$ -intercepts.

**Solution**

$$\begin{aligned}
 \text{a. } y &= -2x^2 + 12x + 4 \\
 &= -2(x^2 - 6x \quad \quad) + 4 \\
 &= -2(x^2 - 6x + 9) + 4 - (-2)(9) \\
 y &= -2(x - 3)^2 + 22
 \end{aligned}$$

The vertex is (3, 22).

- b. When  $x = 0$ ,  $y = 4$ . Thus, the  $y$ -intercept is 4, which is the constant term of the original equation.

To find the  $x$ -intercepts, let  $y = 0$ . Using the completed square form of the given equation:

$$\begin{aligned}
 -2(x - 3)^2 + 22 &= 0 \\
 -2(x - 3)^2 &= -22 \\
 (x - 3)^2 &= 11 \\
 x &= 3 \pm \sqrt{11}
 \end{aligned}$$

### Sketching the Graph of $y = ax^2 + bx + c$ , $a \neq 0$

1. Get a quick mental picture of the graph as follows:

- a. If  $a > 0$ , the graph opens upward.  
If  $a < 0$ , the graph opens downward.
- b. If  $b^2 - 4ac > 0$ , the graph has two  $x$ -intercepts.  
If  $b^2 - 4ac = 0$ , the graph has one  $x$ -intercept.  
If  $b^2 - 4ac < 0$ , the graph has no  $x$ -intercepts.

2. Find the axis and vertex of the parabola as follows:

*Method 1.* The equation of the axis is  $x = -\frac{b}{2a}$ . Substitute this value of  $x$  into  $y = ax^2 + bx + c$  to find the  $y$ -coordinate of the vertex.

*Method 2.* Rewrite the equation in the form

$$y = a(x - h)^2 + k.$$

The vertex is  $(h, k)$  and the axis is  $x = h$ .

3. Find the  $x$ - and  $y$ -intercepts.

- a. Using  $y = ax^2 + bx + c$ , the  $y$ -intercept is always  $c$ .
- b. To find the  $x$ -intercepts, solve  $ax^2 + bx + c = 0$  (from using Method 1) or solve  $a(x - h)^2 + k = 0$  (from using Method 2).





Example 3 presents two methods for finding the intersection points of a line and a parabola. The second method uses a computer or a graphing calculator.

**Example 3** Where does the line  $y = 2x + 5$  intersect the parabola  $y = 8 - x^2$ ?

**Solution**

**Method 1** Set  $2x + 5$  equal to  $8 - x^2$  and solve for  $x$ .

$$2x + 5 = 8 - x^2$$

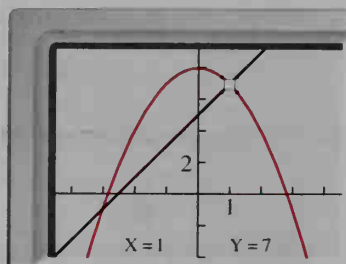
$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x = -3 \text{ or } x = 1$$

Substitute  $x = -3$  and  $x = 1$  into the equation  $y = 2x + 5$  to get  $y = -1$  and  $y = 7$ , respectively. The intersection points are  $(-3, -1)$  and  $(1, 7)$ .

**Method 2** Use a computer or a graphing calculator to graph the equations  $y = 2x + 5$  and  $y = 8 - x^2$  on a single set of axes. Using the TRACE feature, you can find the coordinates of the intersection points to be  $(-3, -1)$  and  $(1, 7)$ .



**Example 4** Find an equation of the function whose graph is a parabola with  $x$ -intercepts 1 and 4 and  $y$ -intercept  $-8$ .

**Solution**

Any parabola with  $x$ -intercepts 1 and 4 has an equation of the form  $y = a(x - 1)(x - 4)$  for some constant  $a$ . We use the fact that the parabola contains  $(0, -8)$  to find the value of  $a$ .

$$-8 = a(0 - 1)(0 - 4)$$

$$-2 = a$$

Then the equation is  $y = -2(x - 1)(x - 4)$ , or  $f(x) = -2x^2 + 10x - 8$ .

## CLASS EXERCISES

For each quadratic function, tell whether its graph (a) opens upward or downward, and (b) intersects the  $x$ -axis in two, one, or no points.

1.  $f(x) = x^2 - 2x + 3$

2.  $f(x) = -2x^2 - 12x - 18$

3.  $f(x) = 9 - x^2$

Without calculating, find the vertex of each parabola.

4.  $y = (x - 5)^2 + 4$

5.  $y = (x + 2)^2 - 3$

6.  $y = -x^2 - 10$

7. Explain in your own words the relationship between the zeros of a quadratic function and the  $x$ -intercepts of its graph.

## WRITTEN EXERCISES

In Exercises 1–20, sketch each parabola. Label the vertex, the axis of symmetry, and the  $x$ - and  $y$ -intercepts.

In Exercises 1–8, use the method of Example 1.

- A**
- |                         |                         |                        |
|-------------------------|-------------------------|------------------------|
| 1. a. $y = x^2 - 6x$    | b. $y = x^2 - 6x + 9$   | c. $y = x^2 - 6x + 10$ |
| 2. a. $y = -x^2 + 4x$   | b. $y = -x^2 + 4x - 4$  | c. $y = -x^2 + 4x - 8$ |
| 3. $y = (x + 5)(x + 3)$ | 4. $y = (x - 4)(x + 2)$ | 5. $y = 9 - x^2$       |
| 6. $y = x^2 - 16$       | 7. $y = x^2 - 2x - 15$  | 8. $y = x^2 + 3x - 10$ |

In Exercises 9–14, use the method of Example 2.

- |                           |                                   |                                    |
|---------------------------|-----------------------------------|------------------------------------|
| 9. $y = x^2 - 2x - 7$     | 10. $y = x^2 + 4x + 9$            | 11. $y = 4x^2 - 8x + 2$            |
| 12. $y = -3x^2 - 12x - 3$ | 13. $y = \frac{1}{2}x^2 + 4x + 8$ | 14. $y = -\frac{1}{3}x^2 + 2x + 1$ |

In Exercises 15–20, use the method that seems most appropriate to you.

- |                           |                            |                               |
|---------------------------|----------------------------|-------------------------------|
| 15. $y = 4x^2 - 16x + 15$ | 16. $y = -3(x + 3)^2 + 12$ | 17. $y = x^2 + 2x + 17$       |
| 18. $y = 2(x + 3)^2 + 6$  | 19. $y = (2x - 7)(2x - 1)$ | 20. $y = -\frac{1}{4}x^2 + 6$ |



In Exercises 21–24, sketch the graphs of the line and the parabola on a single set of axes. Find the coordinates of any points of intersection by solving algebraically or by using a computer or a graphing calculator.

- |                                       |                                   |
|---------------------------------------|-----------------------------------|
| 21. $y = 4 - 2x$ , $y = x^2 - 6x + 8$ | 22. $y = x + 3$ , $y = 4x - x^2$  |
| 23. $y + x = -6$ , $y = x^2 + 6x$     | 24. $2x + y = 10$ , $y = 9 - x^2$ |



You may find it helpful to have a computer or a graphing calculator to complete Exercises 25 and 26.

- B**
25. On a single set of axes, graph  $y = x^2 - 4$ ,  $y = 2x - 4$ ,  $y = 2x - 5$ , and  $y = 2x - 6$ . Which line appears to be tangent to the parabola? Show algebraically that this line intersects the parabola in only one point.
26. On a single set of axes, graph  $y = x^2 + 5x$  and  $y = x + k$  for various values of  $k$ . By experimenting, find the value of  $k$  that appears to make the line tangent to the parabola. Show algebraically that your answer is correct.

In Exercises 27–34, find an equation of the quadratic function described.

27. Its graph is a parabola with  $x$ -intercepts 2 and  $-1$  and  $y$ -intercept 6.
28. The function  $f$  has zeros 5 and 1 and  $f(0) = 1$ .
29. Its graph is a parabola with vertex  $(4, 8)$  and passing through the origin. (*Hint*: start with the form of the equation at the bottom of page 38.)

30. The graph is a parabola with vertex  $(3, -8)$  and passing through the origin.
31. The minimum value of  $f$  is  $f(3) = -5$ , and  $f(1) = 2$ .
32. The maximum value of  $g$  is  $g(-1) = 6$ , and  $g(-3) = 4$ .
33. Its graph is a parabola tangent to the  $x$ -axis at  $(4, 0)$  with  $y$ -intercept 6.
34. The function  $f$  has zeros  $-1$  and  $3$  and a maximum value of  $8$ .

35. **Sports** A baseball player tries to hit a ball over an outfield fence that is 4 m high and 110 m from home plate. The ball is hit 1 m above home plate and reaches its highest point 30 m above a point on the ground that is 60 m from home plate.



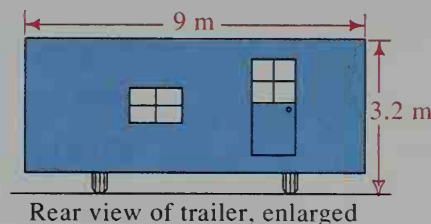
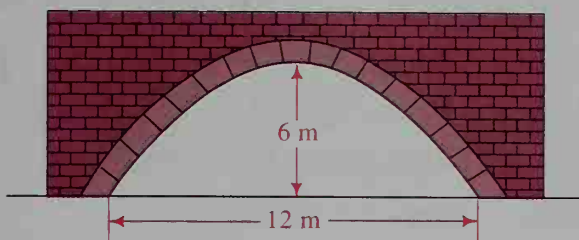
- Make a sketch showing the path of the baseball. If home plate is at the origin of a coordinate system, find an equation of the parabolic path of the baseball.
  - Will the ball go over the outfield fence?
36. **Sports** From the center of the 20 yd (60 ft) line, a football player attempts to make a field goal by kicking the ball directly toward the goal posts, which are 90 ft away. The goal-post crossbar is 10 ft above the ground. The ball reaches its highest altitude of 32 ft at a point 48 ft from where it was kicked.
- Make a sketch showing the path of the football. If the point from which the ball is kicked is the origin of a coordinate system, find an equation of the parabolic path of the football.
  - Will the kicker make the field goal?
37. a. If  $f(x) = 2x^2 - 4x + 7$ , show that  $f(1 + k) = f(1 - k)$  for all real  $k$ .  
b. How does part (a) show that the axis of symmetry is the line  $x = 1$ ?
38. a. If  $h(x) = 2x^2 - 12x$ , show that  $h(3) \leq h(3 + k)$  for all real  $k$ .  
What does this tell you about the graph of  $h(x)$ ?  
b. If  $f(x) = 9 + 8x - x^2$ , show that  $f(4) \geq f(4 + k)$  for all real  $k$ .  
What does this tell you about the graph of  $f(x)$ ?
39. Show that the average of the roots of  $ax^2 + bx + c = 0$  is  $-\frac{b}{2a}$ .
40. If  $ax^2 + bx + c = a(x - h)^2 + k$ , prove  $h = -\frac{b}{2a}$  and  $k = \frac{4ac - b^2}{4a}$ .



For part (a) of Exercise 41, you will need to use a computer or a graphing calculator.

- C** 41. a. Graph the parabola  $y = x^2$ . Using the ZOOM feature, enlarge a small section of the parabola that contains the point  $(1, 1)$  until the graph on the screen looks nearly linear. Use two points on this portion of the parabola to estimate the slope of a line tangent to the parabola at the point  $(1, 1)$ .  
b. Using your estimate from part (a), find an equation of a line tangent to the parabola  $y = x^2$  at the point  $(1, 1)$ . Show algebraically that this line intersects the parabola only at the point  $(1, 1)$ .

42. **Transportation** A hauling company needs to determine whether a large house trailer can be moved along a highway that passes under a bridge with an opening in the shape of a parabolic arch, 12 m wide at the base and 6 m high in the center. If the trailer is 9 m wide and 3.2 m tall (measured from the ground to the top of the trailer), will it fit under the bridge?



## 1-8 Quadratic Models

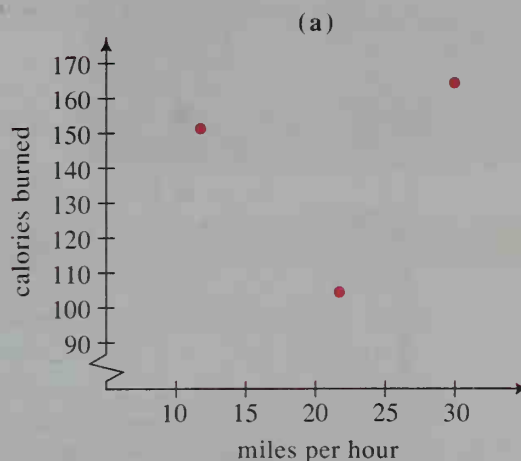
### Objective

To model real-world situations using quadratic functions.

Scientists are fascinated by the ability of birds to travel great distances without stopping. Using wind tunnels to monitor parakeets' oxygen consumption and carbon dioxide production in flight, they have investigated the rate at which parakeets expend energy in level flight. The data in the table below show the number of calories burned per gram hour (the number of calories needed to move each gram of body weight for one hour) for three different flight speeds.

Speed (in mi/h)	12	22	30
Calories burned	152	105	165

If we want to predict the bird's energy use at some other flight speed, we will want to find a good mathematical model based on these data. The information in the table is plotted on the graph in figure (a). Note that as the flight speed increases, the energy expended decreases and then increases. Clearly, a linear model would not be suitable to use. We can find many kinds of curves that pass through these data points. (This process is called *curve-fitting*, which we will discuss in more detail in Chapter 18.) One possible curve is a parabola with equation  $C(x) = ax^2 + bx + c$ . To find the values of  $a$ ,  $b$ , and  $c$ , substitute the data from the graph into this equation, as shown on the next page.





**Example 1**

Find a quadratic function in the form  $C(x) = ax^2 + bx + c$  given that  $C(12) = 152$ ,  $C(22) = 105$ , and  $C(30) = 165$ .

**Solution**

$$C(12) = a \cdot 12^2 + b \cdot 12 + c \qquad 144a + 12b + c = 152 \quad (1)$$

$$C(22) = a \cdot 22^2 + b \cdot 22 + c \qquad 484a + 22b + c = 105 \quad (2)$$

$$C(30) = a \cdot 30^2 + b \cdot 30 + c \qquad 900a + 30b + c = 165 \quad (3)$$

$$\text{Subtract (1) from (2):} \quad 340a + 10b = -47 \quad (4)$$

$$\text{Subtract (2) from (3):} \quad 416a + 8b = 60 \quad (5)$$

$$\text{Multiply (5) by } \frac{5}{4}: \quad 520a + 10b = 75 \quad (6)$$

$$\text{Subtract (4) from (6):} \quad 180a = 122$$

$$a \approx 0.678$$

Substituting  $a = 0.678$  into equation (4) gives  $b = -27.752$ .

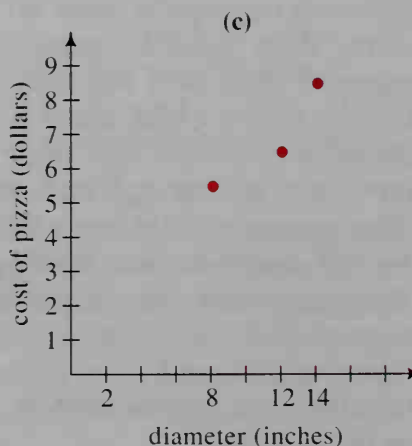
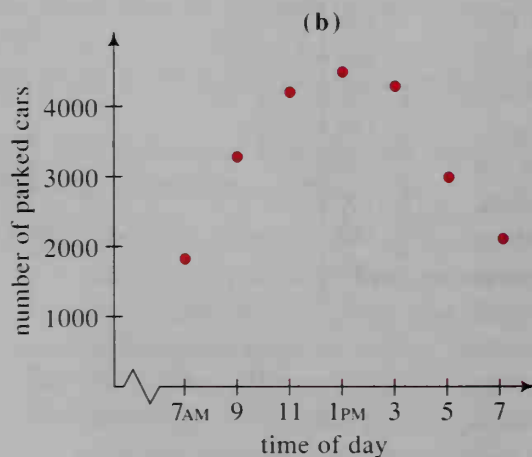
Substituting  $a = 0.678$  and  $b = -27.752$  into equation (3) gives  $c \approx 387.360$ . Thus,

$$C(x) = 0.678x^2 - 27.752x + 387.360.$$

We say that  $C(x)$  is a *quadratic model* of how the energy expenditure of parakeets depends on flight speed.

Generally speaking, quadratic models are often used to model real-world situations that have certain features, some of which are:

1. Values decrease and then increase, as in the example of the energy expenditure of parakeets in flight. See figure (a) on the previous page.
2. Values increase and then decrease. For example, the number of cars parked at a large office complex on a typical weekday is a quadratic function of the time of day. See figure (b) below.
3. Values depend on surface area. For example, the area  $A$  of a circular region is a quadratic function of the radius  $r$ ,  $A = \pi r^2$ . In particular, the cost of a plain pizza is a function of area and therefore a quadratic function of the radius (or diameter). See figure (c) below.



Once we have a model, we can determine its reliability by testing how well other experimental values satisfy the model. If after several experiments we find that the actual data agree with the model, then we can use the model to predict other data values with confidence.

**Example 2** Using the given model, determine the number of calories burned per gram hour by a parakeet flying level at 26 mi/h. Compare your answer with the actual laboratory result of 123.5 calories per gram hour.

**Solution** Substitute the speed of 26 mi/h into the quadratic equation:

$$C(x) = 0.678x^2 - 27.752x + 387.360$$

$$C(26) = 0.678(26)^2 - 27.752(26) + 387.360 = 124.136$$

Thus, the energy expenditure of a parakeet flying at 26 mi/h is about 124 calories per gram hour. Since this result is approximately within one unit of the actual energy expenditure of 123.5 calories per gram hour, we can assume that our model is probably a good one.

## CLASS EXERCISES

Discuss whether the relationship described can be modeled by (1) a linear function, (2) a quadratic function, or (3) some other kind of function.

1. The cost of paving a square lot depends on the length of a side.
2. The weekly pay of a salesperson who earns \$200 plus 10% of sales depends on total sales.
3. The altitude of a ball hit into the air depends on the time it is in flight.
4. A hot cup of coffee is poured and allowed to sit. Its temperature depends on the time since it was poured.
5. **Visual Thinking** What can you conclude about the flight efficiency of parakeets from the graph on page 43?



## WRITTEN EXERCISES

In Exercises 1–4, use the given values to find an equation of the form  $f(x) = ax^2 + bx + c$ .

- |  |  |
|--|--|
| <b>A</b> 1. $f(0) = 5, f(1) = 10, f(2) = 19$ | 2. $f(1) = 4, f(2) = 12, f(4) = 46$      |
| 3. $f(0) = 6, f(2) = 18, f(4) = 34$          | 4. $f(1) = 10.5, f(2) = 13, f(5) = 32.5$ |

5. **Consumer Economics** The table below shows the price in dollars,  $P(x)$ , for pizzas with diameter  $x$  inches. The quadratic function that fits the data is  $P(x) = 0.00625x^2 + 0.2625x + 2$ .

$x$	8	10	16
$P(x)$	4.50	5.25	7.80



- Verify that the data fit the function.
  - Does the model give a reasonable price for an 18 in. pizza?
  - Assuming that the pizza maker is willing to make 4 in. pizza snacks, does the model give a price you would be willing to pay?
6. **Physics** If you drive at  $x$  miles per hour and apply your brakes, your stopping distance in feet is approximately  $f(x) = x + \frac{x^2}{20}$ .
- By how much will your stopping distance increase if you increase your speed from 20 mi/h to 30 mi/h?
  - By how much will your stopping distance increase if you increase your speed from 50 mi/h to 60 mi/h?
  - An important driving guideline is that you leave one car length between your car and the car in front of you for every 10 mi/h of speed. Thus, you should leave three car lengths if you are driving at 30 mi/h and 5 car lengths if you are driving at 50 mi/h. Is this guideline consistent with the model  $f(x)$ ?
7. **Consumer Economics** Suppose your car contains just one gallon of gas. Driving at 20 mi/h you can go 26 mi. Likewise, you can go 34 mi driving at 40 mi/h and 32 mi driving at 50 mi/h.
- Find a quadratic function that models this data.
  - How far could you go if you drove at 65 mi/h?
  - The nearest gas station is 16 mi away. If the speed limit is 55 mi/h, at what maximum speed could you drive and still reach it?
8. **Agriculture** A farmer planting wheat has found that the yield per acre in bushels depends on the amount of seed sown. By experimenting, he has collected the data given in the following table.

number of bags of seed	3	4	5
number of bushels per acre	70	85	80

- Give reasons why the yield could decrease when the number of bags increases from 4 bags to 5 bags.
- Find a quadratic function that models the information in the table.
- What number of bags will maximize the yield per acre?

9. **Consumer Economics** A bakery sells cheesecakes having diameters 6 in., 8 in., and 10 in., all of the same height. Suppose that they cost \$8, \$12, and \$20, respectively.
- Which size gives you the most cheesecake per dollar?
  - Find a quadratic function that models how the cost of a cheesecake varies with diameter.
10. **Electronics** In an electric circuit, the available power in watts is  $P(I)$ , where  $I$  measures the current in amperes. In a 120-volt circuit with a resistance of 16 ohms, electric currents of 2, 4, and 5 amperes produce 176, 224, and 200 watts of power, respectively.
- Find a quadratic function that models how the power in the circuit varies with current.
  - Suppose you can set the current in the circuit to 1, 2, 3, 4, or 5 amperes. According to your model, which setting yields the most power? Which setting yields the most watts of power per ampere?

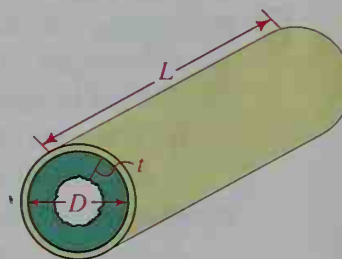
For exercises 11–14, use the following: Suppose that an object is thrown into the air with an initial upward velocity of  $v_0$  meters per second from a height  $h_0$  meters above the ground. Then,  $t$  seconds later, its height  $h(t)$  meters above the ground is modeled by the function  $h(t) = -4.9t^2 + v_0t + h_0$ . (This model doesn't account for air resistance.)

- B** 11. **Physics** A stone is thrown with an upward velocity of 14 m/s from a cliff 30 m high.
- Find its height above the ground  $t$  seconds later.
  - When will the stone reach its highest elevation?
  - When will the stone hit the ground?
12. **Physics** A ball is dropped out of a window of a tall building. If it hits the ground 3.5 s later, how high above the ground is the window?
13. **Physics** One half second after springing from a high diving board, a diver reaches her highest point above the water, 4.225 m. If the diving board is 3 m above the water, how long is the diver in the air?
14. **Physics** A stone is dropped into a deep, dark well. Although you don't see the stone hit the water, you hear it splash 4.0 s after it is dropped. What is the distance from the top of the well to the surface of the water? (*Hint:* Sound travels at about 343 m/s.)
15. **Business** A manufacturer charges \$24 for stereo headphones and has been selling about 1000 of them a week. He estimates that for every \$1 price reduction, 100 more headphones can be sold per week. (For example, he could sell 1100 headphones at \$23 each and 1200 headphones at \$22 each.)
- Let  $24 - x$  be the reduced price per set of headphones. Write a quadratic function that gives the total revenue received by the manufacturer in a week.
  - What price will maximize the total revenue?





16. **Business** An airplane with a 200-passenger capacity is to be chartered for a transatlantic flight. The airplane cannot be chartered unless there are at least 150 passengers. The cost of the trip is \$540 per passenger, except that the airline company agrees to reduce everyone's ticket price by \$2 for *each* ticket sold in excess of 150. Let  $x$  be the number of tickets sold in excess of 150 and let  $f(x)$  be the total ticket income that the company will receive. Draw a graph of  $f(x)$  and specify the domain, range, and maximum value of  $f(x)$ .
17. **Plumbing** A water pipe of fixed length  $L$  has a carrying capacity that depends on the inner diameter of the pipe. The pipe initially has inner diameter  $D$ , but over many years, as mineral deposits accumulate inside the pipe, its carrying capacity is reduced.
- Give a quadratic function  $f(t)$  that models the carrying capacity of the pipe as a function of the thickness  $t$  of mineral deposits. Your function will be in terms of the constants  $D$  and  $L$  and the variable  $t$ . (*Hint: The volume of a cylinder is  $\pi r^2 h$ .*)
  - Show that  $f(\frac{1}{4}D) = \frac{1}{4} \cdot f(0)$ , and then make a diagram illustrating this fact. Explain in your own words what your diagram says about the carrying capacity of the pipe.



## Chapter Summary

- Given any two points  $A(x_1, y_1)$  and  $B(x_2, y_2)$ :
  - The distance  $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ .
  - The *midpoint* of  $\overline{AB}$  is  $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ .
  - The slope of line  $AB$  is  $\frac{y_2 - y_1}{x_2 - x_1}$  if  $x_1 \neq x_2$ .
  - If  $x_1 = x_2$ , then line  $AB$  is a vertical line and has no slope.

2. The graph of a linear equation  $Ax + By = C$ , where  $A$  and  $B$  are not both zero, is a line. If the equation is written in the form  $y = mx + k$ , the slope is  $m$  and the  $y$ -intercept is  $k$ . The equation of a vertical line has the form  $x = C$ ; vertical lines have no slope.
3. Two different nonvertical lines are parallel if and only if they have the same slope. Two lines with slopes  $m_1$  and  $m_2$  are perpendicular if and only if

$$m_1 = -\frac{1}{m_2}, \text{ or } m_1 \cdot m_2 = -1.$$

4. a. A linear equation can be written in several different forms, as shown in the chart below.

The <b>general</b> form	$Ax + By = C$	
The <b>slope-intercept</b> form	$y = mx + k$	Line has slope $m$ and $y$ -intercept $k$ .
The <b>point-slope</b> form	$\frac{y - y_1}{x - x_1} = m$	Line has slope $m$ and contains $(x_1, y_1)$ .
The <b>intercept</b> form	$\frac{x}{a} + \frac{y}{b} = 1$	Line has $x$ -intercept $a$ and $y$ -intercept $b$ .

- b. To find an equation of a line, use whatever form is the most appropriate. See Examples 1 through 3 on pages 14–15.
5. A *function* describes a dependent relationship between quantities. A *linear function* has the form  $f(x) = mx + k$ . If  $f(a) = 0$ , then  $a$  is a *zero* of the function  $f$ . Certain real-world situations can be effectively modeled by linear functions.
6. a. The *imaginary unit*  $i$  is defined with the following properties:

$$i = \sqrt{-1} \text{ and } i^2 = -1.$$

If  $a > 0$ , then  $\sqrt{-a} = i\sqrt{a}$ .

- b. A complex number is a number of the form  $a + bi$  where  $a$  and  $b$  are real numbers. The *complex conjugate* of a complex number  $z = a + bi$  is  $\bar{z} = a - bi$ .
- c. The real numbers and the imaginary numbers are subsets of the complex numbers. See the Venn diagram on page 27.
7. A quadratic equation,  $ax^2 + bx + c = 0$ , can be solved by factoring, by completing the square, or by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the *discriminant*  $b^2 - 4ac > 0$ , there are two real roots.

If  $b^2 - 4ac = 0$ , there is one real root (called a double root).

If  $b^2 - 4ac < 0$ , there are two complex conjugate roots.

8. a. A quadratic function has the form  $f(x) = ax^2 + bx + c$ . Its graph is a parabola whose *axis of symmetry* is the line  $x = -\frac{b}{2a}$ , and whose *vertex* is the point where the parabola intersects this line.
- b. If the equation of a parabola is written in the form  $y = a(x - h)^2 + k$ , then the vertex of the parabola is the point  $(h, k)$ .
- c. The procedure for graphing a quadratic function is outlined on page 39.
9. Quadratic functions can be used effectively to model real-world situations, especially situations in which (1) values decrease and then increase, (2) values increase and then decrease, or (3) values depend on surface area.

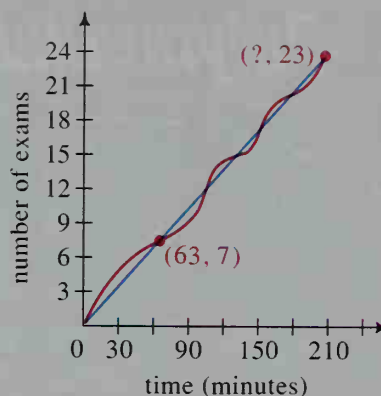
### Key vocabulary and ideas

linear equation (p. 1)	complex number (p. 27)
distance formula, midpoint formula (p. 4)	complex conjugate (p. 27)
slope of a line (pp. 7–8)	quadratic equation (p. 30)
slope-intercept form (p. 9)	completing the square (pp. 30–31)
parallel and perpendicular lines (pp. 9–10)	quadratic formula (p. 31)
function, zero of a function (p. 19)	discriminant (pp. 31–32)
linear function (p. 19)	quadratic function, parabola (p. 37)
imaginary unit $i$ (p. 26)	axis of symmetry, vertex (p. 37)

## Chapter Test

- Let  $A = (-2, -6)$  and  $B = (-4, 2)$ . Find: 1-1
  - The length of  $\overline{AB}$
  - The coordinates of the midpoint of  $\overline{AB}$
- Find the value of  $a$  if the point  $(4, -2)$  lies on the line  $2x + ay = 14$ .
- Solve the equations  $2x + 3y = 2$  and  $6x - y = -4$  simultaneously. Then sketch the graphs of the lines and label the intersection point.
- Find the slope and the  $y$ -intercept of the line  $4x - 2y = 7$ . 1-2
- Tell which of the following equations have parallel line graphs and which have perpendicular line graphs.
  - $2x + 3y = 1$
  - $y = \frac{3}{2}x + 3$
  - $6x - 4y - 10 = 0$
- Write an equation of the line through  $(6, -2)$  and  $(-3, 1)$ . 1-3
- Write an equation of the line through  $(5, 5)$  and parallel to the line  $4x + 3y = -2$ .
- Write an equation of the line with  $x$ -intercept  $-3$  and  $y$ -intercept  $-5$ .
- Write an equation of the vertical line through  $(4, -2)$ .
- Writing** Describe the steps of finding an equation of the perpendicular bisector of a line segment.
  - Write an equation of the perpendicular bisector of the segment joining the points  $(7, 0)$  and  $(1, 8)$ .

11. A teacher must grade 23 exam papers. Working for 63 min, she is able to grade exactly seven exams. The function that models the number of exams graded is shown in red at the right. Because some exams take longer to grade than others, the teacher does not work at a constant rate. Nevertheless, her rate is close enough to being constant that a linear function (whose graph is shown in blue) can be used effectively to model her progress.



1-4

- Give an equation of the linear function that models the number of exams as a function of time.
- Use the equation to find approximately how long it will take to grade all the exams.

Express each complex number in the form  $a + bi$ .

12.  $\sqrt{-50} - \sqrt{-8}$

13.  $(2 + 3i)^2$

1-5

14.  $\frac{1}{2 + 3i}$

15.  $\frac{\sqrt{3} + i}{\sqrt{3} - i}$

16.  $4(3 + 2i) - 5(1 - i)$

17.  $i^{17}$

18. Solve for  $x$ .

a.  $7x^2 - 2 = 5x$

b.  $x^2 - 4x = 9$

c.  $x^2 + 2x + 2 = 0$

1-6

19. Find the discriminant of  $3x^2 - 2x + 2 = 0$  and then solve by whichever method seems easiest.

20. Sketch the graph of each parabola. Label the vertex, the axis of symmetry, and the intercepts.

a.  $y = 8 - 2(x - 1)^2$

b.  $y = x^2 - 6x + 5$

1-7

21. Sketch the graphs of the line  $2x - y = -2$  and the parabola  $y = -x^2 + 4x + 1$ . Find the coordinates of any points of intersection.

22. A photo processing lab lists the following prices for reprints: 29¢ for a  $3 \times 5$  print; 39¢ for a  $4 \times 6$  print; and 55¢ for a  $5 \times 7$  print.

1-8

- Find a quadratic function of the form  $f(x) = ax^2 + bx + c$  that models the cost (in cents) per print in terms of the smaller dimension of the print size.
- Using this model, find the approximate price of a  $7 \times 9$  print.



# 2

## *Polynomial Functions*



# ■ Zeros and Factors of Polynomial Functions

## 2-1 Polynomials

### Objective

To identify a polynomial function, to evaluate it using synthetic substitution, and to determine its zeros.

In Chapter 1 we discussed linear functions like  $f(x) = 2x - 1$  and quadratic functions like  $g(x) = x^2 - 3x + 5$ . Linear and quadratic functions belong to a family of functions known as *polynomial functions*.

A **polynomial** in  $x$  is an expression that can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where  $n$  is a nonnegative integer. The expressions  $a_n x^n$ ,  $a_{n-1} x^{n-1}$ ,  $\dots$ ,  $a_2 x^2$ ,  $a_1 x$ ,  $a_0$  are called the **terms** of the polynomial, and the numbers  $a_n$ ,  $a_{n-1}$ ,  $\dots$ ,  $a_2$ ,  $a_1$ ,  $a_0$  are called the **coefficients** of the polynomial. In this book, the coefficients are real numbers, but the values of  $x$  may be real or imaginary numbers. (In more advanced courses, even the coefficients may be imaginary.)

Although the terms of a polynomial can be written in any order, we usually write them in descending powers of  $x$ . The term containing the highest power of  $x$  is called the **leading term**. The coefficient of the leading term is called the **leading coefficient**, and the power of  $x$  contained in the leading term is called the **degree** of the polynomial. The polynomial 0 has no degree. Polynomials of the first few degrees have special names, as indicated below.

Degree	Name	Example
0	constant	5
1	linear	$3x + 2$
2	quadratic	$x^2 - 4$
3	cubic	$x^3 + 2x + 1$
4	quartic	$-3x^4 + x$
5	quintic	$x^5 + \pi x^4 - 3.1x^3 + 11$

To identify a particular term of a polynomial, we use the name associated with the power of  $x$  contained in the term. For example, the polynomial  $x^2 - 4$  has a *quadratic* term of  $x^2$ , no *linear* term, and a *constant* term of  $-4$ . Note that the leading coefficient of  $x^2 - 4$  is understood to be 1.

Every polynomial defines a function, often called  $P$ . Any value of  $x$  for which  $P(x) = 0$  is a **root** of the equation and a **zero** of the function.

◀ The shape of this roller coaster, with its peaks and valleys, suggests the graph of a polynomial equation.

**Example 1**

State whether each function is a polynomial function. Give the zeros of each function, if they exist.

a.  $f(x) = 2x^3 - 32x$

b.  $g(x) = \frac{x+1}{x-1}$

**Solution**

- a. The function  $f$  is a polynomial function. To find its zeros, we solve  $f(x) = 0$ :

$$2x^3 - 32x = 0$$

$$2x(x^2 - 16) = 0$$

$$2x(x+4)(x-4) = 0$$

$$x = 0 \text{ or } x = -4 \text{ or } x = 4$$

The zeros of  $f$  are 0,  $-4$ , and  $4$ .

- b. The function  $g$  is *not* a polynomial function. To find its zeros, we solve  $g(x) = 0$ :

$$\frac{x+1}{x-1} = 0 \quad \leftarrow \begin{cases} \text{The fraction equals 0} \\ \text{when its numerator equals 0.} \end{cases}$$

$$x + 1 = 0$$

$$x = -1$$

The only zero of  $g$  is  $-1$ .

Given a polynomial  $P(x)$ , we can substitute not only numbers but also variable expressions for  $x$ , as shown in the following example.

**Example 2**

If  $P(x) = 3x^4 - 7x^3 - 5x^2 + 9x + 10$ , find:

a.  $P(2)$

b.  $P(-3n)$

**Solution**

$$\begin{aligned} \text{a. } P(2) &= 3 \cdot 2^4 - 7 \cdot 2^3 - 5 \cdot 2^2 + 9 \cdot 2 + 10 \\ &= 48 - 56 - 20 + 18 + 10 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{b. } P(-3n) &= 3(-3n)^4 - 7(-3n)^3 - 5(-3n)^2 + 9(-3n) + 10 \\ &= 3(81n^4) - 7(-27n^3) - 5(9n^2) + 9(-3n) + 10 \\ &= 243n^4 + 189n^3 - 45n^2 - 27n + 10 \end{aligned}$$

In Example 2,  $P(2)$  can be evaluated more simply using *synthetic substitution*. The basis for this technique involves rewriting the polynomial as follows:

$$\begin{aligned} P(x) &= 3x^4 - 7x^3 - 5x^2 + 9x + 10 \\ &= x(3x^3 - 7x^2 - 5x + 9) + 10 \\ &= x(x(3x^2 - 7x - 5) + 9) + 10 \\ &= x(x(x(3x - 7) - 5) + 9) + 10 \end{aligned}$$

Since  $P(x) = x(x(x(3x - 7) - 5) + 9) + 10$ , we can evaluate  $P(x)$  for any value of  $x$  by writing the coefficients in order,

3, -7, -5, 9, and 10,

and then applying the following algorithm, called **synthetic substitution**:

1. Start with the leading coefficient.
2. Repeat the following until the last coefficient is reached:
  - a. Multiply by the value of  $x$ .
  - b. Add the next coefficient.

The following example illustrates this algorithm.

### Example 3

- a. If  $P(x) = 3x^4 - 7x^3 - 5x^2 + 9x + 10$ , find  $P(2)$ .
- b. If  $S(x) = 3x^4 - 5x^2 + 9x + 10$ , find  $S(-2)$ .

### Solution

a.  $\underline{2} \mid$     3    -7    -5    9    10  
                   6    -2    -14    -10  
              3    -1    -7    -5    0     $\leftarrow P(2) = 0$

- b. Use 0 as the coefficient of the missing cubic term.

$\underline{-2} \mid$     3    0    -5    9    10  
                  -6    12    -14    10  
              3    -6    7    -5    20     $\leftarrow S(-2) = 20$

When doing synthetic substitution yourself, you do not need to write the red marks shown in the solution of Example 3.

## CLASS EXERCISES

Classify each polynomial function as linear, quadratic, cubic, quartic, or quintic. Also give its leading term, leading coefficient, and degree.

1.  $f(x) = 2 + 3x + 5x^2$
2.  $g(x) = 5x^4 - x^2$
3.  $h(x) = 7 - 4x^3$
4.  $P(x) = x^4 + 9x^2 - 1 - 2x^5$

If  $f(x) = 2x^2 + 5$ , find each of the following. (Remember:  $i = \sqrt{-1}$ .)

5.  $f(3)$
6.  $f(3i)$
7.  $f(3n)$
8.  $f(n + 3)$

In Exercises 9–12, give the zeros of each function.

9.  $f(x) = (x + 5)(3x - 9)$
10.  $g(x) = (x^2 - 16)(x^2 + 25)$
11.  $P(x) = x^2 - 6x + 9$
12.  $h(x) = ax^2 + bx + c$
13. If  $P(x) = 2x^3 - 9x^2 + 27$ , use synthetic substitution to find  $P(3)$  and  $P(-3)$ .



## WRITTEN EXERCISES

In Exercises 1–12, state whether the function is a polynomial function. Give the zeros of each function, if they exist.

**A**

1.  $f(x) = 17 - 3x$
2.  $g(x) = x^2 - 6x + 8$
3.  $h(x) = 9$
4.  $k(x) = x - \frac{1}{x}$
5.  $m(x) = \frac{x^2 - 3x - 4}{x^2 + 1}$
6.  $n(x) = \frac{x^2 + 2}{2}$
7.  $p(x) = x^3 - 9x$
8.  $q(x) = 1 - 5x + 6x^2$
9.  $f(x) = px^2 + r + qx$
10.  $r(x) = (x - 7)^2(x^2 + 7)$
11.  $v(x) = x^3 + 2x^2 + x$
12.  $w(x) = 2x^4 - x^3 - x^2$
13. a. Show that the expression  $\frac{x^3 + 2x^2 - x - 2}{x + 2}$  can be simplified to  $x^2 - 1$ .  
 b. Explain why the first expression is not a polynomial but the second is.  
 c. Give the values of  $x$ , if any, for which each expression is undefined.
14. a. For what values of  $x$  is  $f(x) = \frac{2x^3 - 3x^2 - 8x + 12}{x - 2}$  undefined?  
 b. Give the zeros of  $f$ .

In Exercises 15–18, find the values of the function. (Remember:  $i = \sqrt{-1}$ .) Simplify the answers.

15.  $h(x) = 2x^2 - 5x + 6$ 
  - a.  $h(-1)$
  - b.  $h(2i)$
  - c.  $h(1 + i)$
  - d.  $h(3a)$
16.  $P(x) = 8x - 4x^2$ 
  - a.  $P(2\sqrt{3})$
  - b.  $P(1 - \sqrt{2})$
  - c.  $P(1 + 2i)$
  - d.  $P\left(\frac{2}{x}\right)$
17.  $f(x) = x^3 - 9x$ 
  - a.  $f\left(-\frac{\sqrt{2}}{3}\right)$
  - b.  $f(i\sqrt{3})$
  - c.  $f\left(\frac{x}{3}\right)$
  - d.  $f(x - 3)$
18.  $k(x) = x^2(x^2 + 16)$ 
  - a.  $k(4 - \sqrt{2})$
  - b.  $k(1 + i)$
  - c.  $k\left(\frac{p}{q}\right)$
  - d.  $k(x^2)$
19. If  $P(x) = 4x^3 - 5x^2 + 7x - 9$ , use synthetic substitution to find:
  - a.  $P(2)$
  - b.  $P(3)$
  - c.  $P(-2)$
  - d.  $P(-3)$
20. If  $P(x) = x^4 - 3x^2 + 7x + 8$ , use synthetic substitution to find:
  - a.  $P(3)$
  - b.  $P(-3)$
  - c.  $P(-1)$
  - d.  $P(-2)$
21. If  $P(x) = 3x^3 - 7x^2 + 2x + 3$ , use synthetic substitution to find:
  - a.  $P\left(\frac{1}{3}\right)$
  - b.  $P\left(-\frac{2}{3}\right)$
22. If  $P(x) = 4x^4 - 3x^3 + 7x - 2$ , use synthetic substitution to find:
  - a.  $P\left(-\frac{1}{4}\right)$
  - b.  $P\left(\frac{3}{4}\right)$

- B** 23. If 4 is a zero of  $f(x) = 3x^3 + kx - 2$ , find the value of  $k$ .
24. If  $2i$  is a zero of  $f(x) = x^4 + x^2 + a$ , find the value of  $a$ .
25. A quadratic polynomial  $P$  has leading coefficient  $-2$ , a constant term of 6, and no linear term. Find the zeros of  $P$ .
26. The leading coefficient of a cubic polynomial  $P$  is 2, and the coefficient of the linear term is  $-5$ . If  $P(0) = 7$  and  $P(2) = 21$ , find  $P(3)$ .
27. a. What are the zeros of the quadratic function  $f(x) = 2(x - 1)(x - 4)$ ?  
 b. Find a quadratic function with zeros 2 and 3.  
 c. Find a cubic function with zeros 0, 2, and 3.
28. a. What are the zeros of the quadratic function  $f(x) = 3(x + 1)(x - 2)$ ?  
 b. Find a quadratic function with zeros 3 and  $-4$ .  
 c. Find a quartic function with zeros  $\pm 3$  and  $\pm 4$ .
29. If  $f(x) = 7x + 2$ , find: a.  $f(9.2) - f(8.2)$       b.  $f(x + 1) - f(x)$
30. If  $g(x) = 3 - 8x$ , find: a.  $g(6.25) - g(4.25)$       b.  $g(x + 2) - g(x)$
31. If  $f(x) = mx + k$ , show that the value of  $\frac{f(x + h) - f(x)}{h}$ , where  $h \neq 0$ , does not depend on  $x$  or  $h$ . Interpret this result graphically.
32. If  $f(x) = x^2$  and  $h \neq 0$ , find the value of  $\frac{f(x + h) - f(x)}{h}$ . Is the value of this expression independent of the values of  $x$  and  $h$ , as it was for the function of Exercise 31?
33. **Investigation** Multiply several pairs of nonzero polynomials. What is the relationship between the degree of the product and the degrees of the factors? How can you use this relationship to justify saying that the polynomial 0 has no degree, even though all other constant polynomials have degree 0?
- C** 34. a. Consider the following table of values for the quadratic function  $f(x) = x^2 + 2x + 3$ .

$x$	0	1	2	3	4	5
$f(x)$	3	6	11	18	27	38
differences		3	5	7	9	11

- What pattern do you observe in the differences?
- b. Make a difference table similar to the one in part (a) for the function  $g(x) = 2x^2 - 3x - 1$ . What pattern do you observe in the differences?
- c. Do the differences in the values of *all* quadratic functions form a pattern like those you observed in parts (a) and (b)? Justify your answer with a difference table.
35. **Investigation** Experiment with differences in the values of cubic functions. (See Exercise 34.) Can you detect any patterns?

## ////// COMPUTER EXERCISES

1. The two programs below evaluate  $P(x) = 2x^4 - 5x^3 + 7x^2 - 9x + 11$  for integral values of  $x$  from 1 to 100. In the first program  $P(x)$  is in its given form, while in the second program  $P(x)$  is in the form used for synthetic substitution:  $x(x(x(2x - 5) + 7) - 9) + 11$ . Determine the running time for each program. Which one is faster?

```
10 FOR X = 1 TO 100
20 LET P = 2 * X^4 - 5 * X^3 + 7 * X^2 - 9 * X + 11
30 NEXT X
40 PRINT 'DONE'
50 END
```

```
10 FOR X = 1 TO 100
20 LET P = X * (X * (X * (2 * X - 5) + 7) - 9) + 11
30 NEXT X
40 PRINT 'DONE'
50 END
```

2. Consider a third program, just like those in Exercise 1 except for line 20:

```
20 LET P = 2*X*X*X*X - 5*X*X*X + 7*X*X - 9 * X + 11
```

How does the running time for this program compare with the running times for the programs in Exercise 1?

## 2-2 Synthetic Division; The Remainder and Factor Theorems

### Objective

To use synthetic division and to apply the remainder and factor theorems.

In earlier algebra courses, you learned how to add, subtract, multiply, and divide polynomials. What follows is an example of finding the quotient and remainder when  $2x^4 - 15x^2 - 10x + 5$  is divided by  $x - 3$  using long division.

$$\begin{array}{r}
 2x^3 + 6x^2 + 3x - 1 \leftarrow \text{Quotient} \\
 \text{Divisor} \longrightarrow x - 3 \overline{) 2x^4 + 0x^3 - 15x^2 - 10x + 5} \leftarrow \text{Dividend} \\
 \underline{2x^4 - 6x^3} \phantom{+ 18x^2} \\
 6x^3 - 15x^2 \phantom{- 10x} \\
 \underline{6x^3 - 18x^2} \phantom{+ 5} \\
 3x^2 - 10x \phantom{+ 5} \\
 \underline{3x^2 - 9x} \phantom{+ 5} \\
 -1x + 5 \\
 \underline{-1x + 3} \\
 2 \leftarrow \text{Remainder}
 \end{array}$$

Now compare dividing  $P(x) = 2x^4 - 15x^2 - 10x + 5$  by  $x - 3$  with evaluating  $P(3)$  by synthetic substitution:

$$\begin{array}{r|rrrrrr}
 3 & 2 & 0 & -15 & -10 & 5 \\
 & & 6 & 18 & 9 & -3 \\
 \hline
 & 2 & 6 & 3 & -1 & \boxed{2}
 \end{array}
 \quad \leftarrow P(3) = 2$$

Notice that  $P(3)$ , shown in red, is the remainder when  $P(x)$  is divided by  $x - 3$ . Because this is true for any polynomial  $P(x)$  and any divisor  $x - a$ , we have the following theorem.

### The Remainder Theorem

When a polynomial  $P(x)$  is divided by  $x - a$ , the remainder is  $P(a)$ .

**Proof** When a polynomial  $P(x)$  is divided by  $x - a$ , the quotient  $Q(x)$  is a polynomial in  $x$  and the remainder  $R$  is a constant. These polynomials are related as follows:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

$$P(x) = (x - a) \cdot Q(x) + R$$

Substituting  $a$  for  $x$  in this equation, we get:

$$P(a) = (a - a) \cdot Q(a) + R$$

$$= 0 \cdot Q(a) + R$$

$$= R$$

Look again at the synthetic substitution shown above. Notice that the numbers shown in blue are the coefficients of the quotient when  $P(x)$  is divided by  $x - 3$ . Because synthetic substitution gives not only the remainder but also the quotient, we can use it in place of long division. When we do, we often refer to it as *synthetic division*.

**Example 1** Divide  $P(x) = x^3 + 5x^2 + 5x - 2$  by  $x + 2$ .

**Solution** After rewriting  $x + 2$  as  $x - (-2)$ , use synthetic division:

$$\begin{array}{r|rrrr}
 -2 & 1 & 5 & 5 & -2 \\
 & & -2 & -6 & 2 \\
 \hline
 & 1 & 3 & -1 & \boxed{0} \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 1x^2 + & 3x & -1 & 0
 \end{array}$$

The quotient is:

The remainder is:



Since the remainder in Example 1 is 0, we have:

$$P(x) = x^3 + 5x^2 + 5x - 2 = (x + 2)(x^2 + 3x - 1)$$

We see that  $P(x)$  is the product of two *factors*, one of which is  $x + 2$ . This leads us to another important theorem.

## The Factor Theorem

For a polynomial  $P(x)$ ,  $x - a$  is a factor if and only if  $P(a) = 0$ .

**Example 2** If  $P(x) = 2x^4 + 5x^3 - 8x^2 - 17x - 6$ , determine whether each of the following is a factor of  $P(x)$ :    **a.**  $x - 1$                       **b.**  $x - 2$

### Solution

**a.** Use direct substitution to find  $P(1)$ :

$$\begin{aligned} P(1) &= 2 \cdot 1^4 + 5 \cdot 1^3 - 8 \cdot 1^2 - 17 \cdot 1 - 6 \\ &= 2 + 5 - 8 - 17 - 6 = -24 \end{aligned}$$

Since  $P(1) \neq 0$ ,  $x - 1$  is not a factor of  $P(x)$ .

**b.** Use synthetic substitution to find  $P(2)$ :

$$\begin{array}{r|rrrrrr} 2 & 2 & 5 & -8 & -17 & -6 \\ & & 4 & 18 & 20 & 6 \\ \hline & 2 & 9 & 10 & 3 & 0 \end{array} \quad \leftarrow P(2) = 0$$

Since  $P(2) = 0$ ,  $x - 2$  is a factor of  $P(x)$ .

## CLASS EXERCISES

1. **a.** What is the remainder when  $P(x) = x^{15} + 3x^{10} + 2$  is divided by  $x - 1$ ?  
**b.** What is the remainder when  $P(x)$  is divided by  $x + 1$ ?

2. At the right is a division problem that has been done synthetically.

**a.** What is the dividend? What is the divisor?

**b.** What is the quotient? What is the remainder?

**c.** What is the value of the polynomial when  $x = -3$ ?

$$\begin{array}{r|rrrrrr} -3 & 1 & 0 & -8 & 5 & -1 \\ & & -3 & 9 & -3 & -6 \\ \hline & 1 & -3 & 1 & 2 & -7 \end{array}$$

3. Repeat parts (a) and (b) of Exercise 2 for the synthetic division problem shown at the right. Then give the value of the polynomial when  $x = \frac{1}{2}$ .

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & 8 & -4 \\ & & 1 & -2 & 3 \\ \hline & 2 & -4 & 6 & -1 \end{array}$$

4. Is  $x + 1$  a factor of  $P(x) = x^3 + 3x^2 + x - 1$ ? If so, how would you find the other factors of  $P(x)$ ?
5. Explain why the factor theorem is a special case of the remainder theorem.

## WRITTEN EXERCISES

- A**
- Find the remainder when  $x^5 - 2x^3 + x^2 - 4$  is divided by:
    - $x - 1$
    - $x + 1$
    - $x - 2$
    - $x + 2$
  - Find the remainder when  $x^3 - 3x^2 + 5$  is divided by:
    - $x - 2$
    - $x + 2$
    - $x - 3$
    - $x + 3$

In Exercises 3–10, find the quotient and the remainder when the first polynomial is divided by the second.

- $x^3 - 2x^2 + 5x + 1; x - 1$
- $2x^3 + x^2 + 3x + 7; x + 2$
- $x^4 - 2x^3 + 5x + 2; x + 1$
- $2x^4 - 3x^3 + 4x^2 - 5x + 2; x - 1$
- $x^5 + x^3 + x; x - 3$
- $x^2 - 3x^4; x + 2$
- $3x^4 - 2x^3 + 5x^2 + x + 1; x^2 + 2x$
- $x^5 + 3x^2 + 4; x^2 + 2x + 1$
- Determine whether  $x - 1$  or  $x + 1$  is a factor of  $x^{100} - 4x^{99} + 3$ .
- Determine whether  $x - 2$  or  $x + 2$  is a factor of  $x^{20} - 4x^{18} + 3x - 6$ .
- Which of the following are factors of  $P(x) = x^3 - 5x^2 + 3x + 9$ ?
  - $x - 1$
  - $x + 3$
  - $x - 3$
- Which of the following are factors of  $P(x) = x^4 - 3x^3 + 5x - 2$ ?
  - $x + 2$
  - $x - 2$
  - $x + 4$
- Show that  $x - a$  is a factor of  $x^n - a^n$  for any positive integer  $n$ .
- Show that  $x + a$  is a factor of  $x^n + a^n$  for any positive odd integer  $n$ .
- When a polynomial  $P(x)$  is divided by  $2x + 1$ , the quotient is  $x^2 - x + 4$  and the remainder is 3. Find  $P(x)$ .
- When a polynomial  $P(x)$  is divided by  $3x - 4$ , the quotient is  $x^3 + 2x + 2$  and the remainder is  $-1$ . Find  $P(x)$ .

In Exercises 19–24, you are given a polynomial equation and one or more of its roots. Find the remaining roots.

- $2x^3 - 5x^2 - 4x + 3 = 0$ ; root:  $x = 3$
- $6x^3 + 11x^2 - 4x - 4 = 0$ ; root:  $x = -2$
- B**  $2x^4 - 9x^3 + 2x^2 + 9x - 4 = 0$ ; roots:  $x = -1, x = 1$
- $4x^4 - 4x^3 - 25x^2 + x + 6 = 0$ ; roots:  $x = -2, x = 3$
- $x^4 + 3x^3 - 3x^2 + 3x - 4 = 0$ ; roots:  $x = -4, x = 1$
- $x^4 - 2x^3 + x^2 - 4 = 0$ ; roots:  $x = -1, x = 2$
- Use the factor theorem to show that  $x - a$  is a factor of  $x^2(a - b) + a^2(b - x) + b^2(x - a)$ .
- Use the factor theorem to show that  $x - c$  is a factor of  $(x - b)^3 + (b - c)^3 + (c - x)^3$ .

# ■ Graphs; Maximums and Minimums

## 2-3 Graphing Polynomial Functions

### Objective

To graph a polynomial function and to determine an equation for a polynomial graph.

### Graphs of Cubic and Quartic Functions

#### Activity 1

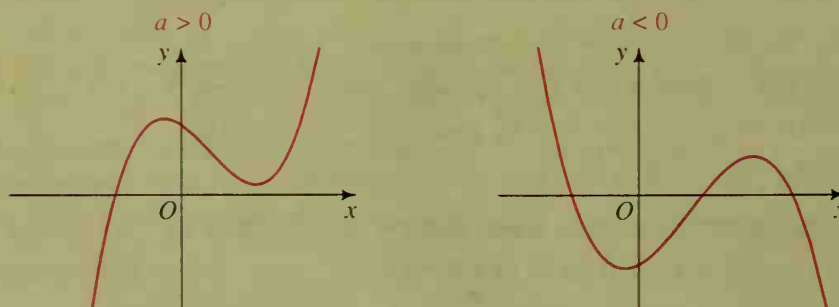


In this activity, you will use a computer or graphing calculator to explore the shape of the graphs of cubic functions.

- Graph  $f(x) = 2x^3 - 7x^2 + 4x + 20$ . Adjust the viewing window until you think the complete shape of the graph is visible. Then re-adjust the window, or use the TRACE feature, to determine what happens to the value of  $f(x)$  as the value of  $x$  increases or decreases without bound.
- Repeat part (a) using the function  $f(x) = -x^3 - 5x^2 - 12$ .
- Repeat part (a) using other cubic functions, some with positive leading coefficients and some with negative leading coefficients.
- Based on your work, describe the general shape of the graph of a cubic function. How does the sign of the leading coefficient affect the graph?

Generally speaking, the graph of  $y = ax^3 + bx^2 + cx + d$  is shaped like an unbroken “sideways S.” (Unbroken curves are said to be *continuous*, a concept that will be discussed in Section 19-1.) Although the graph has some bends in it, overall, the curve rises from left to right if  $a > 0$  and the curve falls from left to right if  $a < 0$ . The “generic” cubic graphs shown below illustrate these facts.

#### Graph of $f(x) = ax^3 + bx^2 + cx + d$



You can quickly sketch the graph of a cubic function by hand if the function is in factored form, as in the example on the next page.

**Example 1** Sketch the graph of the factored cubic function  
 $f(x) = (x + 1)(x - 1)(x - 2)$ .

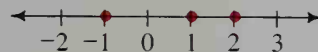
**Solution**

**Step 1** Find and plot the zeros of the function:

$$(x + 1)(x - 1)(x - 2) = 0$$

$$x = -1, x = 1, x = 2$$

The zeros of  $f$  are  $-1$ ,  $1$ , and  $2$ .

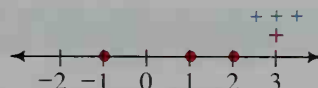


**Step 2** Perform a sign analysis of  $f(x)$  by testing one value of  $x$  from each of the intervals determined by the zeros.

Interval:  $x > 2$  Test:  $x = 3$

$$f(3) = (3 + 1)(3 - 1)(3 - 2)$$

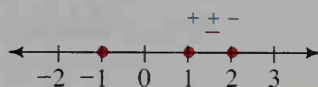
$$\underbrace{\begin{array}{ccc} + & + & + \\ \text{signs of} & & \text{sign of} \\ \text{factors} & & \text{product} \end{array}}_{= +}$$



Interval:  $1 < x < 2$  Test:  $x = 1.5$

$$f(1.5) = (1.5 + 1)(1.5 - 1)(1.5 - 2)$$

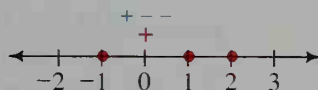
$$\underbrace{\begin{array}{ccc} + & + & - \\ \text{signs of} & & \text{sign of} \\ \text{factors} & & \text{product} \end{array}}_{= -}$$



Interval:  $-1 < x < 1$  Test:  $x = 0$

$$f(0) = (0 + 1)(0 - 1)(0 - 2)$$

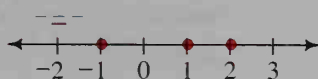
$$\underbrace{\begin{array}{ccc} + & - & - \\ \text{signs of} & & \text{sign of} \\ \text{factors} & & \text{product} \end{array}}_{= +}$$



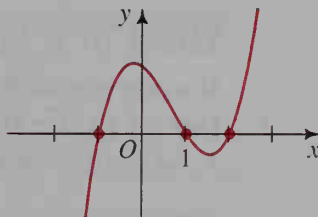
Interval:  $x < -1$  Test:  $x = -2$

$$f(-2) = (-2 + 1)(-2 - 1)(-2 - 2)$$

$$\underbrace{\begin{array}{ccc} - & - & - \\ \text{signs of} & & \text{sign of} \\ \text{factors} & & \text{product} \end{array}}_{= -}$$



**Step 3** Sketch the graph: Step 1 gives you the  $x$ -intercepts of the graph, and Step 2 tells you where the graph is above or below the  $x$ -axis.



**Activity 2**



Use the method of Example 1 to sketch the graphs of the following polynomial functions. Use a computer or graphing calculator to check your graphs.

a.  $y = x(x + 2)(x - 1)$

b.  $y = (x + 3)(x + 1)(x - 1)(x - 2)$

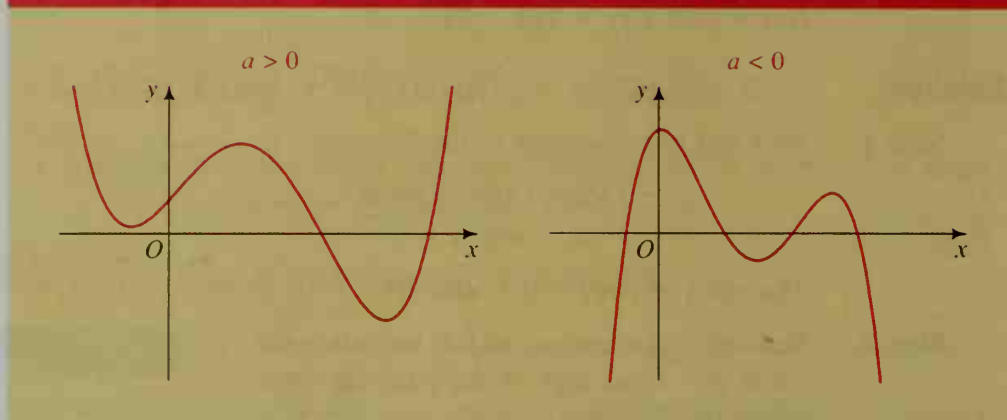
c.  $y = -x(x + 2)(x - 1)$

d.  $y = -(x + 3)(x + 1)(x - 1)(x - 2)$

The graphs of parts (b) and (d) of Activity 2 illustrate that the graph of a quartic function generally has a “W-shape” or “M-shape,” as shown on the next page.



## Graph of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$



### Activity 3



In part (a) of this activity, you may wish to use a computer or a graphing calculator either to draw the graphs or to check your hand-drawn graphs.

- a. In Example 1 we considered the graph of  $y = (x + 1)(x - 1)(x - 2)$ . Quickly sketch the three related graphs of

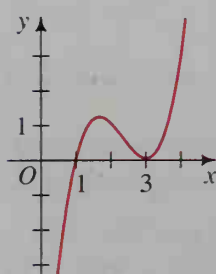
$$y = (x + 1)(x - 1)(x - k)$$

for  $k = 1.5$ ,  $k = 1.3$ , and  $k = 1.1$ . What happens to the graphs as the value of  $k$  approaches 1?

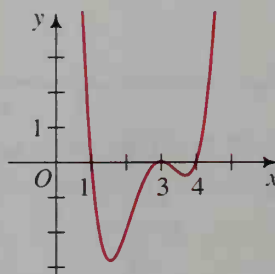
- b. In general, what effect does a squared factor, as in  $y = (x + 1)(x - 1)^2$ , have on the graph of a polynomial function? Use a sign analysis to explain your answer.

### Effect of a Squared or Cubed Factor

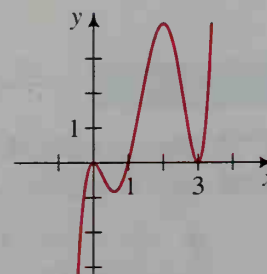
If a polynomial  $P(x)$  has a squared factor such as  $(x - c)^2$ , then  $x = c$  is a **double root** of  $P(x) = 0$ . In this case, the graph of  $y = P(x)$  is tangent to the  $x$ -axis at  $x = c$ , as shown in the graphs below.



Cubic  
 $y = (x - 1)(x - 3)^2$   
(1)

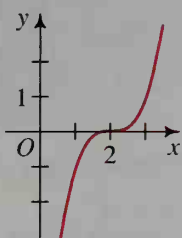


Quartic  
 $y = (x - 1)(x - 3)^2(x - 4)$   
(2)

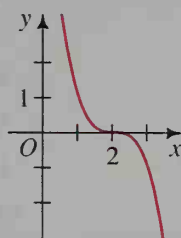


Quintic (5th degree)  
 $y = x^2(x - 1)(x - 3)^2$   
(3)

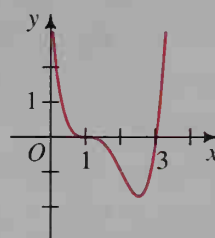
If a polynomial  $P(x)$  has a cubed factor such as  $(x - c)^3$ , then  $x = c$  is a **triple root** of  $P(x) = 0$ . In this case, the graph of  $y = P(x)$  flattens out around  $(c, 0)$  and crosses the  $x$ -axis at this point, as shown in the graphs below.



Cubic  
 $y = (x-2)^3$   
(4)



Cubic  
 $y = -(x-2)^3$   
(5)



Quartic  
 $y = (x-1)^3(x-3)$   
(6)

## CLASS EXERCISES

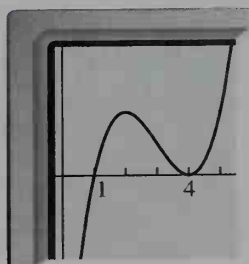
- Discussion** The general shape and direction of the graph of a polynomial function  $y = P(x)$  are determined by the “end-behavior” of the function, that is, the values  $P(x)$  takes on as the value of  $x$  increases or decreases without bound. Discuss how both the degree of  $P(x)$  and the sign of the leading coefficient affect the end-behavior and the shape of the graph of  $y = P(x)$ .
- Discussion** When the value of  $x$  is “close to”  $c$ , the value of  $(x - c)^3$  is close to 0. Use this fact to help explain why the graph of a polynomial with a cubed factor  $(x - c)^3$  flattens out around the  $x$ -intercept  $x = c$ .
- Study the graphs in figures (1)–(6) on this and the preceding page. For each polynomial function  $y = P(x)$ , tell how many distinct real roots the equation  $P(x) = 0$  will have. Indicate which real roots are double roots or triple roots.



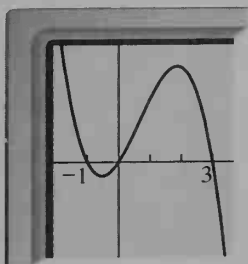
In Exercises 4 and 5, you may wish to use a computer or a graphing calculator to check your answers.

- Sketch the graph of each equation.
 

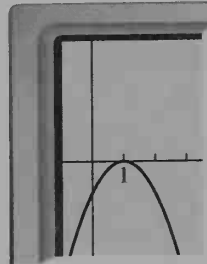
a. $y = (x + 2)(x - 2)(x - 4)$	b. $y = x(x + 2)(1 - x)$	c. $y = -x(x + 3)^2$
d. $y = -x^2(x + 4)(x - 4)$	e. $y = -x^3(x - 2)$	f. $y = x^3(2 + x)$
- In Exercise 4, you sketched the graph of each equation. Now, reverse the process and give an equation for each of the following polynomial graphs. Since the  $y$ -axis has no indicated scale, more than one equation can be correct.



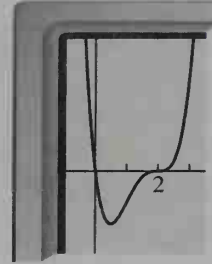
(a) Cubic



(b) Cubic



(c) Quadratic



(d) Quartic

## WRITTEN EXERCISES

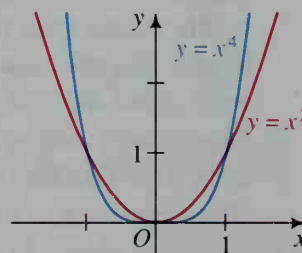
In Exercises 1–12, sketch the graph of each equation.

- A**
- |                                    |                                 |
|------------------------------------|---------------------------------|
| 1. $y = (x + 1)(x - 2)(x - 4)$     | 2. $y = -(x + 3)(x + 2)(x - 1)$ |
| 3. $y = -x(x + 5)(x + 3)$          | 4. $y = x(x - 1)(x - 4)$        |
| 5. $y = x^2(x + 2)$                | 6. $y = (x - 1)^3$              |
| 7. $y = x(1 - x)(1 + x)(2 + x)$    | 8. $y = x(x + 2)(x - 2)(x - 1)$ |
| 9. $y = (x + 1)^3(x - 2)$          | 10. $y = -x^2(2 - x)^2$         |
| 11. $y = x^2(x + 2)(x - 1)(x + 1)$ | 12. $y = x^2(1 - x)^2(2 + x)$   |

In Exercises 13–18, factor each polynomial function and sketch its graph.

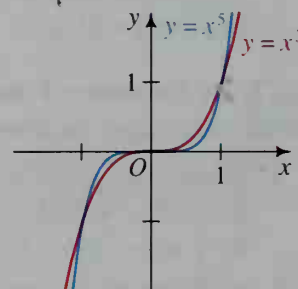
- |   |   |
|---|---|
| 13. $f(x) = x^3 - 4x$   | 14. $f(x) = x^3 - 4x^2 - 5x$  |
| 15. $f(x) = x^4 - x^2$  | 16. $f(x) = x^4 - 2x^3 + x^2$   |
| 17. $f(x) = x^4 - 2x^3 + 2x - 1$<br>(Hint: $x = 1$ is a triple root.) | 18. $f(x) = 4x^4 - 24x^3 + 35x^2 + 6x - 9$<br>(Hint: $x = 3$ is a double root.) |

19. The graphs of  $y = x^2$  and  $y = x^4$  are shown at the right.



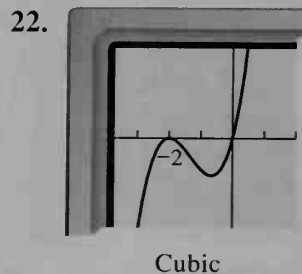
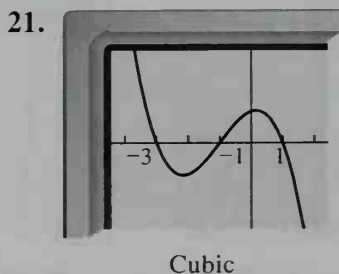
- a. Copy these graphs and then add the graph of  $y = x^6$ . (You may wish to use a computer or graphing calculator to check your graph.)
- b. What three points are common to all three graphs?

20. The graphs of  $y = x^3$  and  $y = x^5$  are shown at the right.

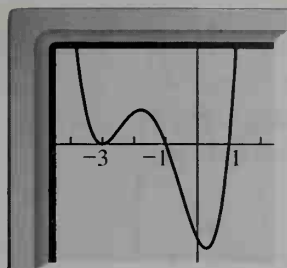


- a. Copy these graphs and then add the graph of  $y = x^7$ . (You may wish to use a computer or graphing calculator to check your graph.)
- b. What three points are common to all three graphs?

In Exercises 21–24, give an equation for each polynomial graph shown. Since the y-axis has no indicated scale, more than one answer is possible.

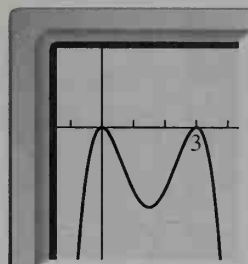


23.



Quartic

24.



Quartic



Use a computer or graphing calculator to complete part (b) of Exercises 25 and 26.

25. a. Sketch the graph of  $P(x) = 16x - x^3$ .  
 b. Evaluate  $P(2)$  and  $P(2.1)$  and conclude that the highest point of the graph in the interval  $0 \leq x \leq 4$  does *not* occur midway between the roots  $x = 0$  and  $x = 4$ . Find to the nearest tenth the  $x$ -coordinate of the highest point.  
 c. Show that  $P(-x) = -P(x)$ . (Because of this property, the origin is called a *point of symmetry* of the graph. This idea will be discussed in Section 4-3.)
26. a. Sketch the graph of  $P(x) = x^4 - 4x^2$ .  
 b. Evaluate  $P(1)$  and  $P(1.1)$  and conclude that the lowest point of the graph in the interval  $0 \leq x \leq 2$  does *not* occur midway between the roots  $x = 0$  and  $x = 2$ . Find to the nearest tenth the  $x$ -coordinate of the lowest point.  
 c. Show that  $P(-x) = P(x)$ . (Because of this property, the  $y$ -axis is called a *line of symmetry* of the graph. This idea will be discussed in Section 4-3.)

Sketch the graphs of the given equations on a single set of axes. Then determine the coordinates of any points of intersection. Although you may wish to use a computer or graphing calculator, the intersection points can be found just as easily by solving the system algebraically.

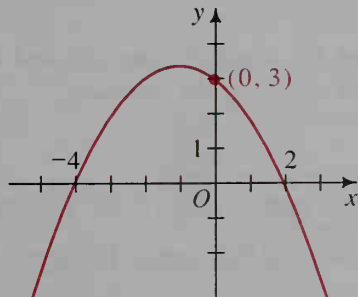
**B**

$$27. \begin{aligned} y &= x^3 - 4x \\ y &= -3x \end{aligned}$$

$$28. \begin{aligned} y &= -x(x - 2)^2 \\ y &= -x \end{aligned}$$

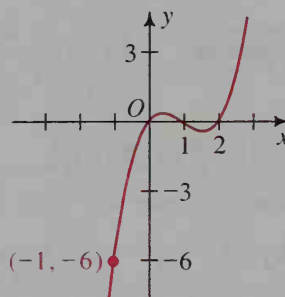
In Exercises 29–32, give an equation for each polynomial graph shown. Unlike Exercises 21–24, a scale on the  $y$ -axis is given.

29.



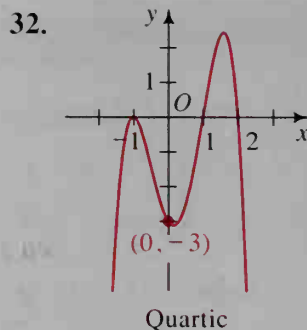
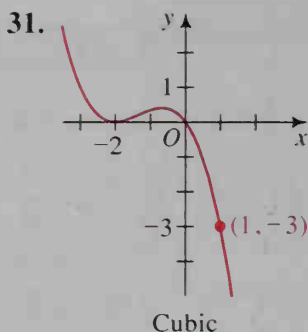
Quadratic

30.



Cubic





In Exercises 33 and 34, assume that the zeros of each polynomial function described are real and distinct. Sketch the graph of the function. If such a graph is impossible, say so.

33. A cubic function with (a) 3 zeros, (b) 2 zeros, (c) 1 zero, (d) no zeros.
34. A quartic function with (a) 4 zeros, (b) 3 zeros, (c) 2 zeros, (d) 1 zero, (e) no zeros.
35. A sixth degree polynomial function has  $r$  distinct real zeros. What are the possible values of  $r$ ?
36. A fifth degree polynomial function has  $r$  distinct real zeros. What are the possible values of  $r$ ?
37. Find an equation of the cubic function whose graph passes through the points  $(3, 0)$  and  $(1, 4)$ , and is tangent to the  $x$ -axis at the origin.
38. Find an equation of the quartic function whose graph passes through  $(0, -2)$  and is tangent to the  $x$ -axis at  $(-1, 0)$  and  $(2, 0)$ .
39. If  $P(x)$  is a cubic polynomial such that  $P(-3) = P(-1) = P(2) = 0$  and  $P(0) = 6$ , find  $P(x)$ .
40. If  $P(x)$  is a cubic polynomial such that  $P(0) = 0$ ,  $P(2) = -4$ , and  $P(x)$  is positive only when  $x > 4$ , find  $P(x)$ .

## 2-4 Finding Maximums and Minimums of Polynomial Functions

### Objective

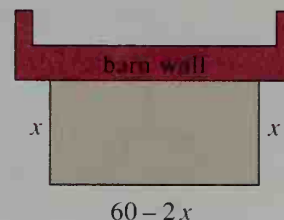
To write a polynomial function for a given situation and to find the maximum or minimum value of the function.

### Quadratic Functions

In Section 1-7, we saw that the maximum or minimum value of the quadratic function  $f(x) = ax^2 + bx + c$  occurs at  $x = -\frac{b}{2a}$ . We use this fact in Example 1 at the top of the next page.

**Example 1**

A rectangular dog pen is constructed using a barn wall as one side and 60 m of fencing for the other three sides. Find the dimensions of the pen that give the greatest area.

**Solution**

Let  $x$  = the length in meters of each side touching the barn. Then  $60 - 2x$  = the length in meters of the side parallel to the barn. If the lengths of these sides are to be positive, then  $x$  must be between 0 and 30. Letting  $A(x)$  represent the area of the pen, we have:

$$\begin{aligned} A(x) &= x(60 - 2x) \\ &= -2x^2 + 60x \end{aligned}$$

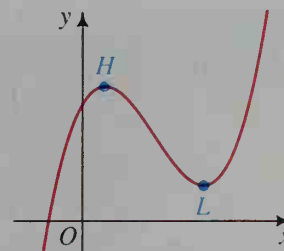
The maximum value of  $A(x)$  occurs at  $x = -\frac{60}{2(-2)} = 15$ . Thus, the area will be a maximum when the dimensions are 15 m by 30 m, with the longer side being parallel to the barn. The maximum area is  $A(15) = 450 \text{ m}^2$ .

**Cubic Functions**

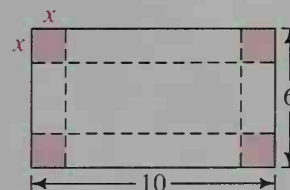
When the graph of a cubic function has a “peak” and a “valley” as shown at the right, we say that the function has a *local maximum* at the highest point of the peak (point  $H$ ) and a *local minimum* at the lowest point of the valley (point  $L$ ).



In the following example we use a computer or graphing calculator to approximate the local maximum of a cubic function.

**Example 2**

Squares with sides of length  $x$  are cut from the corners of a rectangular piece of sheet metal with dimensions of 6 in. and 10 in. The metal is then folded to make an open-top box. What is the maximum volume of such a box?

**Solution**

The dimensions of the box are:

$$\begin{aligned} \text{height} &= x \\ \text{length} &= 10 - 2x \\ \text{width} &= 6 - 2x \end{aligned}$$

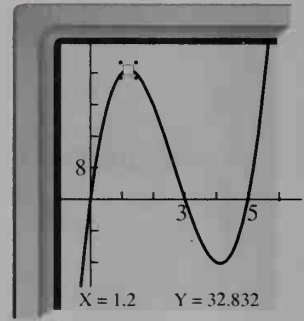
If these dimensions are to be positive, then  $x$  must be between 0 and 3.

(Solution continues on the next page.)

Letting  $V(x)$  represent the volume of the box, we have:

$$V(x) = x(10 - 2x)(6 - 2x)$$

One way to find the maximum value of  $V$  is to use a computer or graphing calculator to draw the graph of  $y = V(x)$ , as shown at the right. Then you can either trace the curve to its highest point between 0 and 3 or enlarge a region containing the highest point. Either way you can find that the highest point is approximately  $(1.2, 32.8)$ . Thus, the maximum volume of the box is approximately  $32.8 \text{ in.}^3$



Another way to find the maximum value of  $V(x)$  is to use a simple program like the one below on a computer or programmable calculator. The FOR-NEXT loop in lines 20–70 evaluates the volume  $V$  for values of  $X$  between 0 and 3. The variable  $M$ , introduced in line 10, will ultimately be the maximum value of  $V$ , but initially it is zero. Each time that a volume  $V$  is greater than the previous maximum  $M$ , line 50 replaces the previous value of  $M$  with the value of  $V$ , and line 60 replaces  $X1$  with the corresponding value of  $X$ .

```
10 LET M = 0
20 FOR X = 0 TO 3 STEP 0.01
30 LET V = X * (6 - 2 * X) * (10 - 2 * X)
40 IF V > M THEN
50 LET M = V
60 LET X1 = X
70 NEXT X
80 PRINT "MAXIMUM VOLUME IS APPROXIMATELY "; M;
  " AT X = "; X1
90 END
```

The program produces this output:

```
○ MAXIMUM VOLUME IS APPROXIMATELY 32.835 AT X = 1.21 ○
```

## CLASS EXERCISES

In Exercises 1–4, state (a) whether the quadratic function has a maximum or a minimum value and (b) the value of  $x$  at which the maximum or minimum occurs.

1.  $f(x) = (x - 1)(x - 7)$

2.  $g(x) = 8 - (x - 2)^2$

3.  $h(x) = 2x^2 - 6x + 9$

4.  $k(x) = 1 - 4x - 3x^2$

5. Does every cubic function have a local maximum and a local minimum? Explain.

6. **Reading** Why do you think the adjective “local” is used when describing a maximum or minimum value of a cubic function but not of a quadratic function?
7. Note that the graph of  $y = V(x)$  shown in the solution of Example 2 has a local minimum when  $x$  is between 3 and 5. Describe how you would modify the computer program to find this local minimum. Ignore the fact that  $V(x)$  does not represent a volume outside the interval  $0 < x < 3$ .

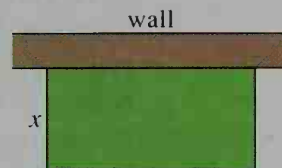
## WRITTEN EXERCISES

### Quadratic functions

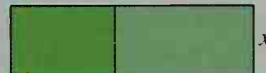
**A**

1. **Farming** A farmer wants to make a rectangular enclosure using a wall as one side and 120 m of fencing for the other three sides.

- Express the area in terms of  $x$  and state the domain of the area function.
- Find the value of  $x$  that gives the greatest area.



2. A rectangle has a perimeter of 80 cm. If its width is  $x$ , express its length and its area in terms of  $x$ . What is the maximum area of the rectangle?
3. Suppose you have 102 m of fencing to make two side-by-side rectangular enclosures, as shown. What is the maximum area that you can enclose?



4. Suppose you have to use exactly 200 m of fencing to make either one square enclosure or two separate square enclosures of any sizes you wish. What plan gives you the least area? the greatest area?
5. Suppose a scientist makes two measurements  $x_1$  and  $x_2$  of a quantity whose true measure  $x$  is not known. The errors are then  $|x - x_1|$  and  $|x - x_2|$ . What number can be assigned to  $x$  so that the sum of the squares of the errors is a minimum?
6. Repeat Exercise 5 if three measurements  $x_1$ ,  $x_2$ , and  $x_3$  are made of a quantity whose true measure  $x$  is not known.

**B**

7. Generalize the results of Exercises 5 and 6.
8. Show that of all rectangles having perimeter  $P$ , the square has the greatest area.
9. **Physics** If a ball is thrown vertically upward at 30 m/s, then its approximate height in meters  $t$  seconds later is given by  $h(t) = 30t - 5t^2$ .
- After how many seconds does the ball hit the ground?
  - What is the domain of  $h$ ?
  - How high does the ball go?





10. **Physics** If a ball is thrown upward from a building 30 m tall and the ball has a vertical velocity of 25 m/s, then its approximate height above the ground  $t$  seconds later is given by  $h(t) = 30 + 25t - 5t^2$ .

- After how many seconds does the ball hit the ground?
- What is the domain of  $h$ ?
- How high does the ball go?

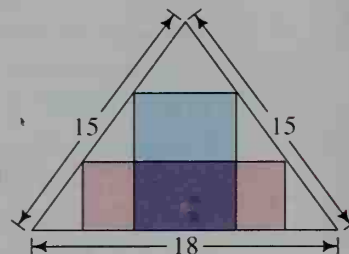
11. **Business** The publisher of a magazine that has a circulation of 80,000 and sells for \$1.60 a copy decides to raise the price of the magazine because of increased production and distribution costs. By surveying the readers of the magazine, the publisher finds that the magazine will lose 10,000 readers for each \$.40 increase in price. What price per copy maximizes the income?

12. **Business** An orange grower has 400 crates of fruit ready for market and will have 20 more for each day the grower waits. The present price is \$60 per crate and will drop an estimated \$2 per day for each day waited. In how many days should the grower ship the crop to maximize his income?



- C** 13. In the isosceles triangle shown at the right, two rectangles are inscribed. Many others could also be inscribed. What are the dimensions of the inscribed rectangle having the largest possible area?

14. Given the points  $A(0, 5)$ ,  $B(3, 7)$ , and  $C(6, 2)$ , find the point  $P$  on the  $x$ -axis that minimizes  $(PA)^2 + (PB)^2 + (PC)^2$ .



Ex. 13

### Cubic functions



In Exercises 1–9, you will need to use a computer or calculator as shown in Example 2. Give approximate values to the nearest hundredth unless otherwise instructed.

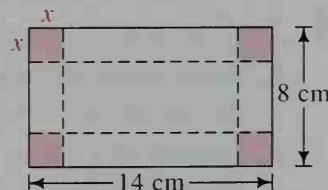
- A** 1. **Manufacturing** As the diagram indicates, a manufacturer cuts squares from the corners of an 8 cm by 14 cm piece of sheet metal and then folds the metal to make an open-top box.

- Show that the volume of the box is:

$$V(x) = x(8 - 2x)(14 - 2x)$$

- What is the domain of  $V$ ?

- Find the approximate value of  $x$  that maximizes the volume. Then give the approximate maximum volume.



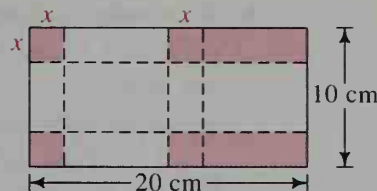
2. **Manufacturing** A 10 cm by 20 cm piece of sheet metal is cut and folded as indicated in the diagram to make a box with a top.

a. Show that the volume of the box is:

$$V(x) = x(10 - x)(10 - 2x)$$

b. What is the domain of  $V$ ?

c. Find the approximate value of  $x$  that maximizes the volume. Then give the approximate maximum volume.

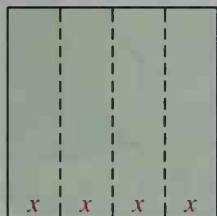


3. **Manufacturing** In a rectangular piece of cardboard with perimeter 20 ft, three parallel and equally spaced creases are made, as shown at the left below. The cardboard is then folded to make a rectangular box with open square ends.

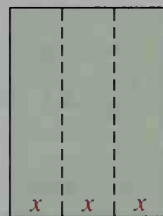
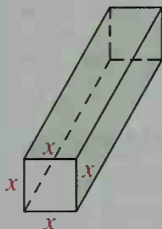
a. Show that the volume of the box is  $V(x) = x^2(10 - 4x)$ .

b. What is the domain of  $V$ ?

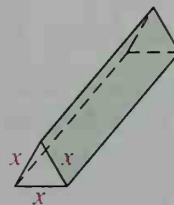
c. Find the approximate value of  $x$  that maximizes the volume. Then give the approximate maximum volume.



Ex. 3



Ex. 4



4. **Manufacturing** In a rectangular piece of cardboard with perimeter 30 in., two parallel and equally spaced creases are made, as shown at the right above. The cardboard is then folded to make a prism with open ends that are equilateral triangles.

a. Show that the volume of the prism is  $V(x) = \left(\frac{\sqrt{3}}{4}x^2\right)(15 - 3x)$ .

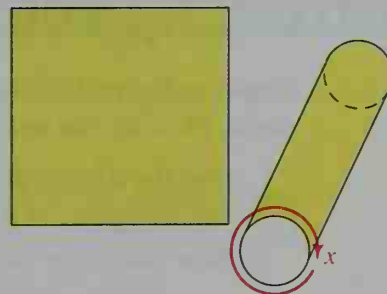
b. What is the domain of  $V$ ?

c. Find the approximate value of  $x$  that maximizes the volume. Then give the approximate maximum volume.

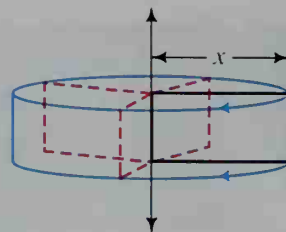
- B** 5. **Manufacturing** A rectangular piece of sheet metal with perimeter 50 cm is rolled into a cylinder with open ends, as shown at the right.

a. Express the volume of the cylinder as a function of  $x$ . Then give the domain of this function.

b. Find the approximate value of  $x$  that maximizes the volume. Then give the approximate maximum volume.

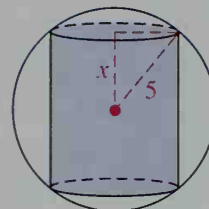


6. A cylinder is generated by rotating a rectangle with perimeter 12 in. about one of its sides, as shown at the right.



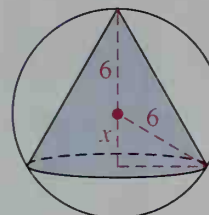
Ex. 6

7. A cylinder is inscribed in a sphere with radius 5, as shown at the right.



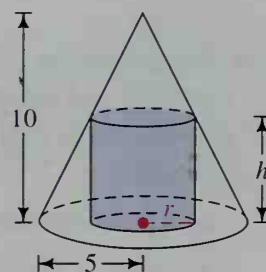
Ex. 7

8. A cone is inscribed in a sphere of radius 6, as shown at the right.



Ex. 8

- C** 9. A cylinder is inscribed in a cone with height 10 and a base of radius 5, as shown at the right. Find the approximate values of  $r$  and  $h$  for which the volume of the cylinder is a maximum. Then give the approximate maximum volume.



Ex. 9

## ■■■■ CALCULATOR EXERCISES

A piece of wire 40 cm long is to be cut into two pieces. One piece will be bent to form a circle; the other will be bent to form a square.

- Find the lengths of the two pieces that cause the sum of the areas of the circle and square to be a minimum.
- How could you make the total area of the circle and the square a maximum?

# Polynomial Equations

## 2-5 Using Technology to Approximate Roots of Polynomial Equations



### Objective

To use technology to approximate the real roots of a polynomial equation.

You already know how to solve linear and quadratic equations. In this section we will discuss how to solve a polynomial equation of higher degree, such as

$$x^3 - x^2 - 5x + 2 = 0,$$

by using a computer, or a graphing or programmable calculator. In the next section we will see that some polynomial equations can be solved by factoring.

### Example 1

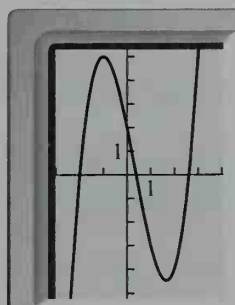
Solve  $x^3 - x^2 - 5x + 2 = 0$  by using a computer or graphing calculator.

### Solution

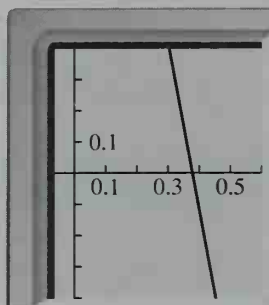
Draw the graph of the polynomial function

$$y = x^3 - x^2 - 5x + 2$$

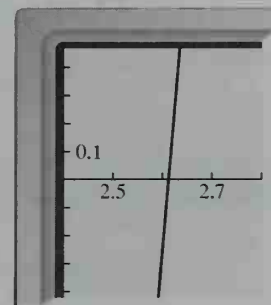
with a computer or graphing calculator. The  $x$ -intercepts of the graph are the roots of the equation. If estimating the  $x$ -intercepts is difficult because the graph is not labeled, you could use the TRACE feature, if available. To improve the accuracy of the approximations, you could rescale or use the ZOOM feature, if available.



Graph of function  
for  $-3 \leq x \leq 4$



Enlargement of graph  
for  $0 \leq x \leq 0.6$



Enlargement of graph  
for  $2.4 \leq x \leq 2.8$

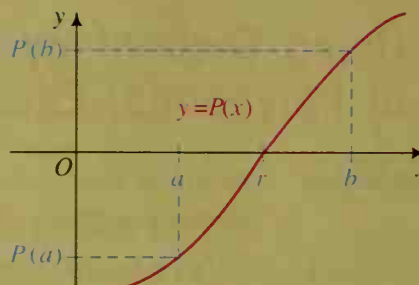
From the graphs of  $y = x^3 - x^2 - 5x + 2$  shown above, we estimate that the roots of  $x^3 - x^2 - 5x + 2 = 0$  are  $x = -2$ ,  $x = 0.4$ , and  $x = 2.6$ .

In the next example, we will solve the equation  $x^3 - x^2 - 5x + 2 = 0$  without drawing a graph. We will instead use the *location principle*, stated at the top of the next page.



## The Location Principle

If  $P(x)$  is a polynomial with real coefficients and  $a$  and  $b$  are real numbers such that  $P(a)$  and  $P(b)$  have opposite signs, then between  $a$  and  $b$  there is at least one real root  $r$  of the equation  $P(x) = 0$ .



Thus, if we want to find the roots of the polynomial equation  $P(x) = 0$ , the location principle tells us to look for changes of sign in a table of values for  $P(x)$ .

**Example 2** Solve  $x^3 - x^2 - 5x + 2 = 0$  without drawing a graph.

**Solution**

By using a computer or programmable calculator, we can obtain a table of values for the polynomial  $P(x) = x^3 - x^2 - 5x + 2$  from a simple program like the one below. Although the interval  $-4 \leq x \leq 4$  used in the program is wide enough for locating all the real roots of the given equation, wider intervals may be needed for other equations.

```
10 PRINT 'X','P(X)'
20 FOR X = -4 TO 4
30 PRINT X, X^3 - X^2 - 5 * X + 2
40 NEXT X
50 END
```

X	P(X)
-4	-58
-3	-19
-2	0
-1	5
0	2
1	-3
2	-4
3	5
4	30

← -2 is a root.

Change of sign indicates a root.

Change of sign indicates a root.

The table shows that  $-2$  is a root, that another root is between  $0$  and  $1$ , and that still another root is between  $2$  and  $3$ . To find more accurately the root between  $0$  and  $1$ , merely change line  $20$  to:

```
20 FOR X = 0 TO 1 STEP 0.1
```

The table at the top of the next page shows the results of this change in the program.

X	P(X)
0	2
0.1	1.491
0.2	0.968
0.3	0.437
0.4	-0.096
0.5	-0.625
0.6	-1.144
0.7	-1.647
0.8	-2.128
0.9	-2.581
1	-3

Change of sign indicates a root.

You can obtain greater accuracy by changing line 20 once again.

20 FOR X = 0.3 TO 0.4 STEP 0.01	
X	P(X)
0.3	0.437
0.31	0.383691
0.32	0.330368
0.33	0.277037
0.34	0.223704
0.35	0.170375
0.36	0.117056
0.37	0.063753
0.38	0.010472
0.39	-0.042781
0.4	-0.096

Change of sign indicates a root.

Of the two values 0.010472 and  $-0.042781$ , the first is closer to 0. We therefore take  $x = 0.38$  rather than  $x = 0.39$  as the better approximation of the root between 0 and 1.

## CLASS EXERCISES

In Exercises 1–4, mentally locate between two consecutive integers a real root of each equation.

1.  $x^3 + x - 5 = 0$

2.  $2x^3 - x - 3 = 0$

3.  $x^4 - 2x^3 - 3x - 3 = 0$

4.  $x^3 + 5x^2 - 3 = 0$

5. Using the table of values below, give the approximate roots of  $P(x) = 0$ .

x	-3	-2	-1	0	1	2	3	4
P(x)	4	-1	-5	-2	3	10	-3	-16

6. Explain why you cannot use the location principle to find a double root of a polynomial equation.

## WRITTEN EXERCISES



Many of these Written Exercises require the use of a computer or calculator.

Use the method of Example 1 to find the real roots to the nearest tenth.

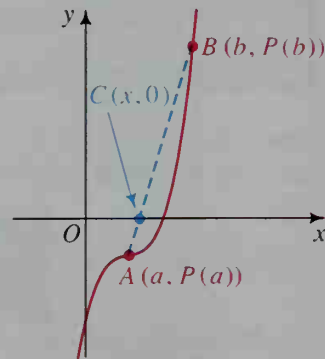
- |  |   |
|--|---|
| <p><b>A</b> 1. <math>2x^3 + x^2 + 4x - 15 = 0</math></p> <p>3. <math>x^3 - x^2 - 4x - 2 = 0</math></p> <p>5. <math>x^4 - 3x^2 = 9</math></p> | <p>2. <math>3x^3 + x^2 + 2x + 8 = 0</math></p> <p>4. <math>x^3 - 2x^2 - 7x + 2 = 0</math></p> <p>6. <math>x^4 - 5x^3 + x^2 - 10x - 2 = 0</math></p> |
|--|---|

Use the method of Example 2 to find the real roots to the nearest tenth.

- |  |  |
|--|--|
| <p>7. <math>x^3 - 5x^2 - 3x - 7 = 0</math></p> <p>9. <math>1.2x^4 - 0.7x^2 = 3.6x</math></p> | <p>8. <math>x^3 + 2 = 3x^2</math></p> <p>10. <math>0.23x^4 - 0.5x^3 = 4.7</math></p> |
|--|--|

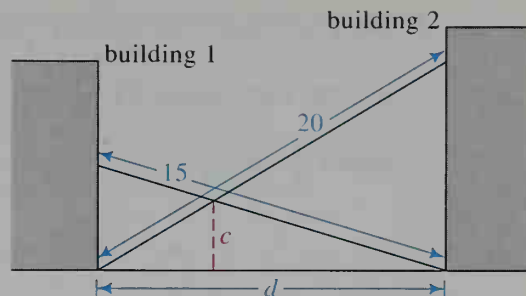
- B** 11. Example 2 shows that  $x = -2$  is a zero of  $P(x) = x^3 - x^2 - 5x + 2$  so that  $x + 2$  is a factor of  $P(x)$ . Find the quotient when  $P(x)$  is divided by  $x + 2$ . Then use the quadratic formula to find the two zeros of the quotient and compare them with the approximate answers given in Example 1.
12. Show that  $9x^4 - 8x^2 + 1 = 0$  has at least two real roots between 0 and 1.
13. Explain why the method of Example 2 does not reveal any real roots of the equation  $42x^2 - 13x + 1 = 0$  between 0 and 1 when the step size is 0.1, but does reveal real roots between 0.1 and 0.2 when the step size is 0.01.
14. **Writing** Compare the two methods of approximating roots presented in Examples 1 and 2. Write a paragraph or two in which you discuss any limitations that each method may have. Also identify which method you prefer to use and explain why.
15. A box has length 4, width 2, and height 1. If another box with twice the volume has dimensions  $4 + x$ ,  $2 + x$ , and  $1 + x$ , find the value of  $x$  to the nearest hundredth.
16. Suppose each face of a 3 cm  $\times$  5 cm  $\times$  4 cm block of wood is shaved (or planed) by  $x$  cm. If the shaved block has half the original volume, find the value of  $x$  to the nearest hundredth.

- C** 17. Given a polynomial function  $P$  such that  $P(a) < 0$  and  $P(b) > 0$ , we can use a method called *linear interpolation* to estimate the zero between  $a$  and  $b$ . As its name implies, this method assumes that the graph of  $P$  is approximately linear between  $a$  and  $b$ . For example, in the figure at the right, compare the dotted line and the graph of  $P$  between points  $A$  and  $B$ . The zero of  $P$  is approximately the  $x$ -intercept of the dotted line. Show that this  $x$ -intercept is:



$$x = a - \frac{P(a) \cdot (b - a)}{P(b) - P(a)}$$

18. Two buildings are  $d$  units apart. A ladder 20 units long has its foot resting against building 1 and its top against the side of building 2. A second ladder 15 units long has its foot against building 2 and its top against the side of building 1. The ladders touch each other at a point  $c$  units above the ground.



- a. Show that

$$\frac{1}{\sqrt{400 - d^2}} + \frac{1}{\sqrt{225 - d^2}} = \frac{1}{c}. \quad (\text{Hint: Use similar triangles.})$$

- b. If  $d = 12$ , use part (a) to find  $c$ .  
 c. If  $c = 8$ , find  $d$ . Since it is difficult to solve for  $d$  directly, use a computer or calculator to approximate  $d$  to the nearest hundredth by finding the value of  $d$  for which the expression below changes sign.

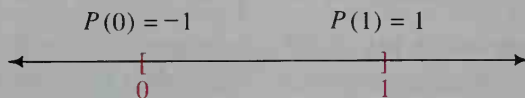
$$\frac{1}{\sqrt{400 - d^2}} + \frac{1}{\sqrt{225 - d^2}} - \frac{1}{8}$$

### //////COMPUTER EXERCISE

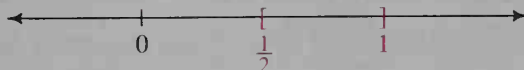
The *bracket-and-halving method* described below is particularly well suited to a computer. We illustrate the method with the equation:

$$P(x) = x^3 + x - 1 = 0$$

1. First, we mentally locate a root between two integers or have the computer do this. In this case, there is a root between  $x = 0$  and  $x = 1$  because  $P(0)$  is negative and  $P(1)$  is positive.



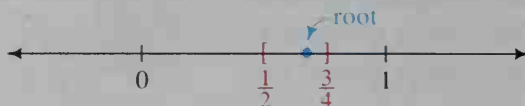
2. Now we divide the interval from 0 to 1 in half and determine in which half the root lies. Since  $P\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} - 1 < 0$  and  $P(1) > 0$ , the root is between  $\frac{1}{2}$  and 1.



(Computer Exercise continues on the next page.)



3. Next, we divide the interval from  $\frac{1}{2}$  to 1 in half and again find the half containing the root. Since  $P\left(\frac{3}{4}\right) = \frac{27}{64} + \frac{3}{4} - 1 > 0$  and  $P\left(\frac{1}{2}\right) < 0$ , the root is between  $\frac{1}{2}$  and  $\frac{3}{4}$ .



4. We continue to find intervals that bracket the root and then halve these intervals until we have the accuracy we desire.

Write a computer program that uses the bracket-and-halving-method, also called a *binary search* method, to approximate a zero of a polynomial function  $P$ . The following algorithm may be helpful.

1. Input the degree of  $P$ .
2. Input the coefficients of  $P$ .
3. Input numbers  $a$  and  $b$  for which  $P(a)$  and  $P(b)$  have opposite signs.
4. Calculate  $m = \frac{a+b}{2}$ . Then calculate  $P(m)$ ,  $P(a)$ , and  $P(b)$ .
5. If  $P(m)$  is sufficiently close to 0, then print that  $m$  is an approximation of the zero and stop. Otherwise go to step 6.
6. If  $P(a)$  and  $P(m)$  have opposite signs, let  $b = m$ , print that the zero is between  $a$  and  $b$ , and return to step 4. Otherwise let  $a = m$ , print that the zero is between  $a$  and  $b$ , and return to step 4.

## 2-6 Solving Polynomial Equations by Factoring

### Objective

To solve polynomial equations by various methods of factoring, including the use of the rational root theorem.

We have seen how a computer or calculator can be used to search for and approximate the real roots of a polynomial equation. Very often these roots are irrational, but if any are rational they can be found by various methods of factoring. The first example illustrates a method that involves grouping terms.

### Example 1

*Solving a polynomial equation by factoring after grouping terms*

Solve:  $x^3 + 5x^2 - 4x - 20 = 0$

### Solution

$$\begin{aligned}
 (x^3 + 5x^2) - (4x + 20) &= 0 \\
 x^2(x + 5) - 4(x + 5) &= 0 \\
 (x^2 - 4)(x + 5) &= 0 \\
 (x + 2)(x - 2)(x + 5) &= 0 \\
 x = -2, x = 2, x = -5
 \end{aligned}$$

Another method of solving higher-degree polynomial equations involves recognizing polynomials that have a *quadratic form*. For example,  $3x^4 + 2x^2 - 4$  has a quadratic form, because if we let  $y = x^2$ , we have

$$\begin{aligned} 3x^4 + 2x^2 - 4 &= 3(x^2)^2 + 2x^2 - 4 \\ &= 3y^2 + 2y - 4, \end{aligned}$$

which is quadratic in  $y$ .

**Example 2** *Solving a polynomial equation that has a quadratic form*

Solve:  $2x^4 - x^2 - 3 = 0$

**Solution**

Let  $y = x^2$  and proceed as follows:

$$\begin{aligned} 2x^4 - x^2 - 3 &= 0 \\ 2(x^2)^2 - x^2 - 3 &= 0 && \leftarrow \text{This is quadratic in } x^2. \\ 2y^2 - y - 3 &= 0 && \leftarrow \text{This is quadratic in } y. \\ (2y - 3)(y + 1) &= 0 \\ y &= \frac{3}{2} && \text{or} && y = -1 \\ x^2 &= \frac{3}{2} && && x^2 = -1 \\ x &= \pm \sqrt{\frac{3}{2}} = \pm \frac{\sqrt{6}}{2} && && x = \pm i \end{aligned}$$

A third method of solving higher-degree polynomial equations is based on generalizing the method of factoring. Consider a specific equation like  $12x^2 - 8x - 15 = 0$ . If you were using factoring to find the rational roots of this equation, you might begin by writing:

$$\begin{aligned} 12x^2 - 8x - 15 &= 0 \\ ( \quad )( \quad ) &= 0 \end{aligned}$$

For your first factor you might try  $12x - 5$  or  $6x - 5$  or many other possibilities. But you would not try  $7x - 5$  because 7 does not divide the leading coefficient 12. Nor would you try  $3x - 4$  because 4 does not divide the constant term  $-15$ .

Now the same reasoning holds if you were going to use factoring to solve the general polynomial equation:

$$\begin{aligned} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 &= 0 \\ (qx - p)( \quad ) &= 0 \end{aligned}$$

If  $qx - p$  is a factor (so that  $x = \frac{p}{q}$  is a root), then  $q$  must divide  $a_n$ , and  $p$  must divide  $a_0$ . This result, known as the *rational root theorem*, is formally stated at the top of the next page.

## The Rational Root Theorem

Let  $P(x)$  be a polynomial of degree  $n$  with integral coefficients and a nonzero constant term:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \text{ where } a_0 \neq 0$$

If one of the roots of the equation  $P(x) = 0$  is  $x = \frac{p}{q}$  where  $p$  and  $q$  are nonzero integers with no common factor other than 1, then  $p$  must be a factor of  $a_0$ , and  $q$  must be a factor of  $a_n$ .

### Example 3

- a. According to the rational root theorem, what are the possible rational roots of  $P(x) = 3x^4 + 13x^3 + 15x^2 - 4 = 0$ ?
- b. Determine whether any of the possible rational roots really *are* roots. Then find all other roots, real or imaginary.

### Solution

- a.  $x = \frac{p}{q}$  is a possible rational root if  $p$  divides  $-4$  and  $q$  divides 3. Thus  $p$  could equal  $\pm 1, \pm 2$ , or  $\pm 4$ , and  $q$  could equal  $\pm 1$  or  $\pm 3$ . Therefore,  $\frac{p}{q}$  could equal  $\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}$ , or  $\pm \frac{4}{3}$ .
- b. We use an abbreviated form of synthetic substitution to check the integral possibilities first.

		3	13	15	0	-4	← coefficients of $P(x)$
<u>1</u>	3	16	31	31	27	← $P(1) = 27$	
<u>-1</u>	3	10	5	-5	1	← $P(-1) = 1$	
<u>2</u>	3	19	53	106	208	← $P(2) = 208$	
<u>-2</u>	3	7	1	-2	0	← $P(-2) = 0$	

The last line shows that  $x = -2$  is a root, and we have:

$$\begin{aligned} P(x) &= 3x^4 + 13x^3 + 15x^2 - 4 \\ &= (x + 2)(3x^3 + 7x^2 + x - 2) \end{aligned}$$

To find the other roots of  $P(x) = 0$ , we now solve the associated *depressed equation*:

$$3x^3 + 7x^2 + x - 2 = 0$$

Picking up where we left off in our list of possible rational roots, we check  $x = -2$  again because it may be a double root:

		3	7	1	-2	
<u>-2</u>	3	1	-1	0	← $P(-2) = 0$	

So  $x = -2$  is indeed a double root, and the resulting depressed equation is quadratic, which we can solve using the quadratic formula:

$$3x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{13}}{6}$$

The roots are therefore  $x = -2$  (a double root) and  $x = \frac{-1 \pm \sqrt{13}}{6}$ .

## CLASS EXERCISES

- Solve  $x^3 + 4x^2 - 9x - 36 = 0$  by grouping terms.
- Explain why the equation  $x^4 - 5x^2 + 4 = 0$  has a quadratic form.
  - How would you solve this equation?
- Which of the following equations have a quadratic form?
  - $(2x - 1)^2 - 5(2x - 1) + 4 = 0$
  - $x^6 - 5x^3 + 4 = 0$
- Describe how you would solve  $x^3 + 2x^2 - 3x = 0$ .
  - In general, for what cubic equations can you use the method of part (a)?
- According to the rational root theorem, what are the possible rational roots of each equation?
  - $2x^3 - 3x^2 + 9x - 4 = 0$
  - $6x^4 - 2x^2 + 9x + 5 = 0$
- If  $P(x)$  is a polynomial such that  $P(1) = -1$  and  $P(2) = 3$ , must  $P(x) = 0$  have a rational root between  $x = 1$  and  $x = 2$ ? Justify your answer.
- If  $P(x) = x^3 + 2x^2 + x + 1$ , explain why the equation  $P(x) = 0$  has a real root between  $x = -2$  and  $x = -1$ .
  - Determine whether this real root is rational or irrational.
  - Explain why this equation has no positive roots.

## WRITTEN EXERCISES

For Exercises 1–10, tell whether each equation is: (a) a polynomial equation that can be solved by grouping terms, or (b) a polynomial equation that has a quadratic form. Then solve the equation.

**A**

- $x^4 - 4x^2 - 12 = 0$
- $x^3 + 6x^2 - 4x - 24 = 0$
- $3x^3 - 16x^2 - 12x + 64 = 0$
- $x^4 - 7x^2 - 8 = 0$
- $2x^4 = -7x^2 + 15$
- $2x^3 - x^2 - 2x + 1 = 0$
- $x^3 + 2x^2 - 6x = 12$
- $2x^3 - 3x^2 = 12 - 8x$
- $10x^3 + 5x = 6x^2 + 3$
- $2x^4 + 3x^2 - 20 = 0$
- Show that  $x^6 - 7x^3 - 8 = 0$  has a quadratic form. Then find the two real roots and the four imaginary roots of this equation.



12. a. Let  $y = x^2$ . Write the equation  $x^4 - x^2 - 1 = 0$  in terms of  $y$  and then solve for  $y$ , expressing solutions as decimals rounded to the nearest hundredth.
- b. Use your solutions from part (a) to approximate the two real roots and the two imaginary roots of the equation  $x^4 - x^2 - 1 = 0$ . Give each real root to the nearest hundredth, and for each imaginary root  $bi$ , give  $b$  to the nearest hundredth.

Use the rational root theorem to solve each equation, giving all real and imaginary roots.

13.  $x^3 - x^2 - x + 1 = 0$

15.  $x^4 - 10x^2 + 9 = 0$

17.  $3x^3 - 4x^2 - 5x + 2 = 0$

19.  $x^4 + 2x^3 - 2x^2 - 6x - 3 = 0$

21.  $2x^3 - 7x + 2 = 0$

14.  $x^3 + 2x^2 - x - 2 = 0$

16.  $2x^3 - 9x^2 + 3x + 4 = 0$

18.  $3x^4 + 2x^3 - 9x^2 - 12x - 4 = 0$

20.  $3x^3 - x^2 - 36x + 12 = 0$

22.  $2x^4 - x^3 - 7x^2 + x + 2 = 0$

In Exercises 23–28, factor each polynomial function and sketch its graph.

23.  $f(x) = x^3 + 2x^2 - 9x - 18$

25.  $h(x) = x^4 - 11x^2 + 28$

27.  $m(x) = 4x^3 - 13x - 6$

24.  $g(x) = 4x^3 + x^2 - 18x$

26.  $k(x) = -x^4 + 10x^2 - 24$

28.  $n(x) = 1 - 3x^2 - 2x^3$

29. Show that the equation  $x^3 + x^2 - 3 = 0$  has no rational roots, but that it does have an irrational root between  $x = 1$  and  $x = 2$ .
30. Show that the equation  $3x^3 - 4x^2 + 5x - 2 = 0$  has no rational roots, but that it does have an irrational root between  $x = 0$  and  $x = 1$ .

Sketch the graphs of the two given equations on a single set of axes. Then determine algebraically where the graphs intersect or are tangent.

**B** 31.  $y = x^3 - 3x$   
 $y = 2$

32.  $y = x^3 - x$   
 $y = 3x$

33.  $y = 4x^2 - x^4$   
 $y = 4$

34.  $y = x^4 - 6x^2$   
 $y = -9$

35.  $y = x^3 + 4x^2$   
 $y = 3x + 18$

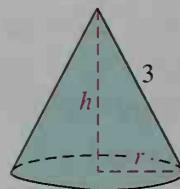
36.  $y = x^4 - 3x^3$   
 $y = 2x - 6$

37. Show that the positive real cube root of 12 is irrational. (*Hint*: What equation must the cube root of 12 satisfy?)
38. Show that the positive fifth root of 100 is irrational.
39. a. The slant height of a cone is 3 and its height is  $h$ . Show that its volume is:

$$V(h) = \frac{1}{3}\pi(9 - h^2)h$$

- b. Find the two values of  $h$  for which

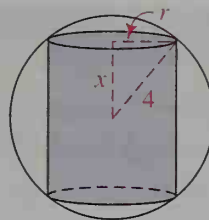
$$V(h) = \frac{10}{3}\pi.$$



40. a. A cylinder is inscribed in a sphere of radius 4. Show that the volume of the cylinder is:

$$V(x) = 2\pi x(16 - x^2)$$

- b. Find the two values of  $x$  for which  $V(x) = 42\pi$ .



- C** 41. A wooden block is in the shape of a rectangular prism with dimensions  $n$  cm,  $(n + 3)$  cm, and  $(n + 9)$  cm, for some integer  $n$ . The surface of the block is painted and the block is then cut into 1 cm cubes by cuts parallel to the faces. If exactly half of these cubes have no paint on them, find the dimensions of the original block.

## 2-7 General Results for Polynomial Equations

### Objective

To apply general theorems about polynomial equations.

In this section, we will state five general theorems about polynomial equations, some of which you may have already discovered yourself as you were studying this chapter. The first theorem, known as the *fundamental theorem of algebra*, is a cornerstone of much advanced work in mathematics. Its proof is beyond the scope of this book, but proofs of the other theorems in this section will be suggested in the exercises.

### Theorem 1. The Fundamental Theorem of Algebra

In the complex number system consisting of all real and imaginary numbers, if  $P(x)$  is a polynomial of degree  $n$  ( $n > 0$ ) with complex coefficients, then the equation  $P(x) = 0$  has exactly  $n$  roots (provided a double root is counted as 2 roots, a triple root is counted as 3 roots, and so on).

By the fundamental theorem of algebra, we can expect a cubic equation like  $2ix^3 + \sqrt{5}x^2 + (3 + 2i)x + 7 = 0$  to have 3 roots and a quartic equation like  $3x^4 - 11x^3 + 19x^2 + 25x - 36 = 0$  to have 4 roots. Sometimes these roots come in pairs, as the next two theorems indicate.

### Theorem 2. Complex Conjugates Theorem

If  $P(x)$  is a polynomial with real coefficients, and  $a + bi$  is an imaginary root of the equation  $P(x) = 0$ , then  $a - bi$  is also a root.

### Theorem 3

Suppose  $P(x)$  is a polynomial with rational coefficients, and  $a$  and  $b$  are rational numbers such that  $\sqrt{b}$  is irrational. If  $a + \sqrt{b}$  is a root of the equation  $P(x) = 0$ , then  $a - \sqrt{b}$  is also a root.

By the complex conjugates theorem, if we know that  $1 + i\sqrt{2}$  is a root of the equation  $x^3 - x^2 + x + 3 = 0$ , then we also know that  $1 - i\sqrt{2}$  is a root. Similarly, by Theorem 3, if we know that  $\frac{3}{2} + \sqrt{5}$  is a root of the equation  $4x^3 - 16x^2 + x + 11 = 0$ , then we also know that  $\frac{3}{2} - \sqrt{5}$  is a root.

If we apply the fundamental theorem of algebra and the complex conjugates theorem to a cubic polynomial with real coefficients, we can say that the polynomial has either three real roots or one real root and a pair of imaginary roots. In any event, the polynomial must have at least one real root. The next theorem generalizes this result.

### Theorem 4

If  $P(x)$  is a polynomial of odd degree with real coefficients, then the equation  $P(x) = 0$  has at least one real root.

The next theorem states an interesting relationship between the roots of a polynomial equation and the coefficients of the polynomial.

### Theorem 5

For the equation  $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_0 = 0$ , with  $a_n \neq 0$ :

the sum of the roots is  $-\frac{a_{n-1}}{a_n}$ ;

the product of the roots is  $\begin{cases} \frac{a_0}{a_n} & \text{if } n \text{ is even.} \\ -\frac{a_0}{a_n} & \text{if } n \text{ is odd.} \end{cases}$

By Theorem 5, we know that for the equation  $2x^3 - 5x^2 - 3x + 9 = 0$ , the sum of the roots is  $\frac{5}{2}$  and the product of the roots is  $-\frac{9}{2}$ . Similarly, for the equation  $x^4 + x^2 + x + 3 = 0$ , the sum of the roots is 0 (because there is no cubic term) and the product of the roots is 3.

Applying Theorem 5 to the general quadratic equation

$$ax^2 + bx + c = 0,$$

we know that the sum of the roots is  $-\frac{b}{a}$  and the product of the roots is  $\frac{c}{a}$ . Thus, after writing the general quadratic equation as

$$x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0,$$

we see that the equation has the form

$$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0.$$

This form gives us a pattern for obtaining a quadratic equation from its roots.

**Example 1** Find a quadratic equation with roots  $2 \pm 3i$ .

**Solution**

Find the sum of the roots:  $(2 + 3i) + (2 - 3i) = 4$

Find the product of the roots:  $(2 + 3i)(2 - 3i) = 13$

Write an equation:  $x^2 - (\text{sum})x + \text{product} = 0$

$$x^2 - 4x + 13 = 0$$

**Example 2** Find a cubic equation with integral coefficients that has no quadratic term and  $3 + i\sqrt{2}$  as one of the roots.

**Solution**

If a cubic equation of the form  $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$  has no quadratic term, then  $a_2 = 0$  and the sum of the roots is  $-\frac{a_2}{a_3} = 0$ . Since  $3 + i\sqrt{2}$  is a root,  $3 - i\sqrt{2}$  must be another root. Therefore, if  $r$  is the third root, we have:

$$\begin{aligned}(3 + i\sqrt{2}) + (3 - i\sqrt{2}) + r &= 0 \\ r &= -6\end{aligned}$$

Now that we know all the roots, one way to obtain a cubic equation is to write a quadratic equation with roots  $3 \pm i\sqrt{2}$  first:

$$\begin{aligned}x^2 - (\text{sum})x + \text{product} &= 0 \\ x^2 - 6x + 11 &= 0\end{aligned}$$

Then a cubic equation with roots  $-6$  and  $3 \pm i\sqrt{2}$  is:

$$\begin{aligned}(x + 6)(x^2 - 6x + 11) &= 0 \\ x^3 - 25x + 66 &= 0\end{aligned}$$

Another way to find such an equation is to write a product of factors in the form  $(x - \text{root})$ , with one factor for each root:

$$[x - (3 + i\sqrt{2})][x - (3 - i\sqrt{2})][x - (-6)] = 0$$



## Historical Development

The five theorems in this section have been proved in the last 300 years or so, but the study of polynomial equations has a long history, going back to the time of the ancient Greeks.

**Before the Renaissance** The Greeks were able to solve quadratic equations by a geometric method, but an algebraic method leading to the quadratic formula was not discovered until centuries later. The methods of equation solving of the Hindu Brahmagupta (about A.D. 628) or the Persian Omar Khayyam (about 1100) look complicated to us today because the notation used was cumbersome. Even as late as the 1500's, mathematicians were using abbreviated Latin words in their equations. For example,

$$4 \text{ Se.} - 5 \text{ Pri.} - 7 \text{ N.} = 0$$

was used instead of  $4x^2 - 5x - 7 = 0$ .

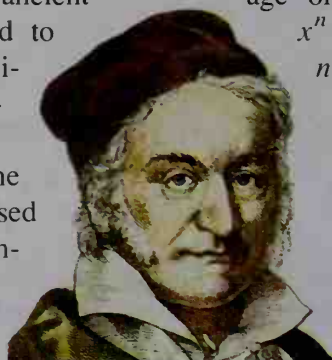
**The Renaissance** In a work published in 1545, the Italian mathematician Girolamo Cardano (1501–1576) stated a “cubic formula” and a “quartic formula.” These formulas gave the solutions of cubic and quartic equations using only radicals and the coefficients of the equations. Cardano's presentation of the formulas largely depended on the work of others. For example, Cardano solved cubic equations by first putting them in the form  $x^3 + mx^2 = n$  and then applying a method of solution that he obtained from the Italian mathematician Tartaglia (about 1499–1557). Similarly, Cardano solved quartic equations by a method that he learned from his own student Lodovico Ferrari (1522–1565).

**The Modern Age** The next goal of mathematicians was to discover a “quintic formula.” In 1824, however, a young Norwegian, Niels Henrik Abel (1802–1829), proved that such a formula does not exist. That is, he proved that it is impossible, in general, to express the roots of a fifth-degree equation in terms of radicals and coefficients.

The theory of polynomial equations has been used to answer a number of questions in geometry. For example, since the days of ancient Greece, people have tried to find a way to trisect an arbitrary angle with a straightedge and compass. Finally, in the 1800's, the theory of equations was used to prove that no such general method exists.

As another example, in 1796 the German mathematician *Carl Friedrich Gauss* (1777–1855), at the age of 19, used the equation  $x^n = 1$  to determine which  $n$ -sided regular polygons could be constructed with straightedge and compass. In short, the study of polynomial equations has helped link together algebra and geometry.

Carl Friedrich Gauss



## CLASS EXERCISES

1. A cubic equation with real coefficients has roots  $-1$  and  $\sqrt{3} + 2i$ . What is the third root?
2. A quartic equation with integral coefficients has roots  $-3 + \sqrt{2}$  and  $-2i$ . What are the other roots?
3. State the sum and product of the roots of each equation.
  - a.  $2x^2 + 5x - 3 = 0$
  - b.  $4x^3 - 2x^2 + 5x - 6 = 0$
  - c.  $x^4 + 2x^3 + 3x^2 = 0$
  - d.  $x^5 + 32 = 0$
4. **Visual Thinking** Use a graph to explain why a cubic equation with real coefficients must have at least one real root.
5. **Visual Thinking** Use a graph to explain why a quartic equation with real coefficients does not necessarily have any real roots.

## WRITTEN EXERCISES

Tell whether the statements in Exercises 1–8 are true or false. Justify your answers. (All equations mentioned have real coefficients.)

- A**
1. Some cubic equations have no real roots.
  2. Every polynomial equation has at least one real root.
  3. The roots of a certain quartic equation are  $\pm \frac{1}{2}$ ,  $0$ , and  $1 + i$ .
  4. The roots of a certain fifth-degree equation are  $-3$ ,  $4$ ,  $1 - \sqrt{2}$ , and  $\pm i$ .
  5. It is possible for the graph of a cubic function to be tangent to the  $x$ -axis at  $x = -2$ ,  $x = 1$ , and  $x = 6$ .
  6. No polynomial equation can have an odd number of imaginary roots.
  7. Suppose  $P(x)$  is a polynomial with rational coefficients, and  $b$  is rational but  $\sqrt{b}$  is irrational. If  $\sqrt{b}$  is a root of the equation  $P(x) = 0$ , then  $-\sqrt{b}$  is also a root.
  8. If  $a + bi$ ,  $b \neq 0$ , is a root of the polynomial equation  $P(x) = 0$ , then the equation must have an even number of roots.

In Exercises 9–12, find the sum and product of the roots of the given equation.

9.  $4x^2 - 3x + 6 = 0$
10.  $6x^3 - 9x^2 + x = 0$
11.  $3x^3 + 5x^2 - x - 2 = 0$
12.  $x^4 - 4x^2 = 5$

In Exercises 13–16, find a quadratic equation with integral coefficients that has the given roots.

13.  $1 \pm i$
14.  $4 \pm \sqrt{3}$
15.  $3 \pm \sqrt{2}$
16.  $\frac{1 \pm i\sqrt{2}}{3}$



The purpose of Exercises 31–38 is to prove the complex conjugates theorem, page 85, for a cubic polynomial. (These exercises should be done sequentially.)

Let  $x = r + si$  and  $z = u + vi$ . The conjugates are then  $\bar{x} = r - si$  and  $\bar{z} = u - vi$ .

31. Prove that  $\overline{x + z} = \bar{x} + \bar{z}$ . (The conjugate of a sum is the sum of the conjugates.)

32. Prove that  $\overline{xz} = \bar{x} \cdot \bar{z}$ . (The conjugate of a product is the product of the conjugates.)

33. Prove that  $\overline{x^2} = (\bar{x})^2$ . (The conjugate of a square is the square of the conjugate.)

34. Prove that  $\overline{x^3} = (\bar{x})^3$ . (The conjugate of a cube is the cube of the conjugate.)

35. a. Prove that the conjugate of a real number  $a$  is  $a$ .

b. Prove that  $\overline{ax^3} = a(\bar{x})^3$ .

36. Prove that  $\overline{bx^2} = b(\bar{x})^2$  and  $\overline{cx} = c\bar{x}$ .

37. Prove that  $\overline{ax^3 + bx^2 + cx + d} = a(\bar{x})^3 + b(\bar{x})^2 + c(\bar{x}) + d$ .

38. Prove that if  $P(x) = ax^3 + bx^2 + cx + d$  and  $P(x) = 0$ , then  $P(\bar{x}) = 0$ .

*Note:* The proof just given can be generalized by noting that  $\overline{ax^n} = a(\bar{x})^n$ . Moreover, a proof of Theorem 3, page 86, is similar to that just given if you interpret  $a + \sqrt{b}$  to mean  $a + \sqrt{b}i$ , where  $a$  and  $b$  are rational numbers and  $\sqrt{b}$  is an irrational number.

39. If  $P(x) = x^3 - x$ , the result of Exercise 37 tells you that  $\overline{P(1 + i)} = P(1 - i)$ .

Verify this result by expressing  $\overline{P(1 + i)}$  and  $P(1 - i)$  in the form  $a + bi$ .

**C** 40. **Reading** Consider the following statement:

Every polynomial equation with complex (that is, real or imaginary) coefficients and degree  $n$  ( $n > 0$ ) has at least one complex root.

Explain why this statement is equivalent to the fundamental theorem of algebra stated on page 85.

## Chapter Summary

1. A *polynomial* in  $x$  is an expression that can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where  $n$  is a nonnegative integer. *Synthetic substitution* can be used to evaluate a polynomial. If the value of a polynomial is 0 for some number, then the number is called a *zero* of the related polynomial function.



2. If you divide a polynomial  $P(x)$  by  $x - a$ , you obtain a quotient  $Q(x)$  and a remainder  $R$  where:

$$P(x) = (x - a) \cdot Q(x) + R$$

From this equation we have the *remainder theorem*:

When  $P(x)$  is divided by  $x - a$ , the remainder  $R = P(a)$ .

We also have the *factor theorem*:

$x - a$  is a factor of  $P(x)$  if and only if  $P(a) = 0$ .

3. Because synthetic substitution gives the quotient and remainder when a polynomial  $P(x)$  is divided by  $x - a$ , it is often called *synthetic division*.
4. a. The general shapes of the graphs of cubic and quartic functions are shown on pages 62 and 64.
- b. If a polynomial  $P(x)$  has a squared factor  $(x - c)^2$ , then the graph of  $y = P(x)$  is tangent to the  $x$ -axis at  $x = c$ . (See figures (1)–(3) on page 64.) If it has a cubed factor  $(x - c)^3$ , then the graph of  $y = P(x)$  flattens out at  $x = c$  and crosses the  $x$ -axis at this point. (See figures (4)–(6) on page 65.)
- c. Example 1 on page 63 demonstrates a useful technique for sketching the graph of a factored polynomial function.
5. The quadratic function  $f(x) = ax^2 + bx + c$  has a maximum or minimum value at  $x = -\frac{b}{2a}$ . A cubic function can have a local maximum and a local minimum, which can be approximated using a computer or calculator.
6. Sections 2-5 and 2-6 present several methods of solving polynomial equations.
- a. Technology is used in Section 2-5 to approximate the real roots of a polynomial equation. One approach is based on the *location principle*:

If  $P(x)$  is a polynomial with real coefficients and  $a$  and  $b$  are real numbers such that  $P(a)$  and  $P(b)$  have opposite signs, then between  $a$  and  $b$  there is at least one real root  $r$  of the equation  $P(x) = 0$ .

- b. Various methods of factoring are used in Section 2-6 to solve polynomial equations. One method is based on the *rational root theorem*:

If  $P(x) = a_n x^n + \cdots + a_0$  has integral coefficients with  $a_0 \neq 0$  and if one root of  $P(x) = 0$  is  $x = \frac{p}{q}$  where  $p$  and  $q$  are nonzero integers with no common factor other than 1, then  $p$  must be a factor of  $a_0$ , and  $q$  must be a factor of  $a_n$ .

7. Section 2-7 presents five theorems about polynomial equations and their roots. The most important of these is the *fundamental theorem of algebra*:

In the complex number system consisting of all real and imaginary numbers, if  $P(x)$  is a polynomial of degree  $n$  ( $n > 0$ ) with complex coefficients, then the equation  $P(x) = 0$  has exactly  $n$  roots (provided multiple roots are counted individually).

### Key vocabulary and ideas

polynomial (p. 53)

synthetic substitution (p. 55)

remainder theorem (p. 59)

synthetic division (p. 59)

factor theorem (p. 60)

double root (p. 64)

triple root (p. 65)

local maximum, local minimum (p. 69)

location principle (p. 76)

rational root theorem (p. 82)

fundamental theorem of algebra (p. 85)

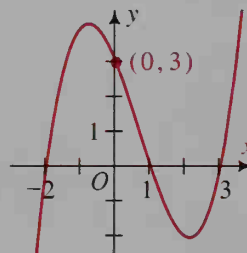
complex conjugates theorem (p. 85)


sum and product of the roots of a

polynomial equation (p. 86)

## Chapter Test

1. If  $P(x) = 4x^3 - 5x - 2$ , use synthetic substitution to find:  
a.  $P(2)$                       b.  $P(-0.5)$                       c.  $P(i)$  2-1
2. If 2 is a zero of  $P(x) = 3x^2 + kx - 8$ , find the value of  $k$ .
3. When a polynomial  $P(x)$  is divided by  $x + 3$ , the quotient is  $x^2 - x + 5$  and the remainder is 2. Find  $P(x)$ . 2-2
4. Two roots of the equation  $x^4 - 3x^3 - 14x^2 + 12x + 40 = 0$  are  $x = 2$  and  $x = 5$ . Find the remaining roots.
5. Sketch the graph of  $y = x(x + 1)(x - 2)^2$ . 2-3
6. Give an equation of the cubic graph shown at the right. 2-4
7. A rectangular enclosure is to be made using a barn as one side and 80 m of fencing to form the other three sides. What is the maximum area of such an enclosure?



 **Part (b) of Exercise 8 requires the use of a computer or graphing calculator.**

8. a. If  $P(x) = 2x^4 - 3x^3 - 1$ , explain how you know that the equation  $P(x) = 0$  has at least one real root between  $x = 1$  and  $x = 2$ . 2-5  
b. Find each real root of  $P(x) = 0$  to the nearest tenth.
9. Solve  $2x^3 - 3x^2 = 6x - 9$ . 2-6
10. Find all real and imaginary roots of the polynomial equation  $3x^4 - x^3 + 4x^2 - 2x - 4 = 0$ .
11. **Writing** Write a paragraph or two in which you compare and contrast the two basic methods for solving polynomial equations presented in this chapter: using technology and factoring.
12. Find the sum and product of the roots of  $2x^3 - 5x^2 - x + 8 = 0$ . 2-7
13. Find a cubic equation with integral coefficients and roots  $3i$  and  $-5$ .
14.  $P(x)$  is a cubic polynomial with real coefficients. Explain why an imaginary number cannot be a double root of the equation  $P(x) = 0$ .

3

# *Inequalities*





# Inequalities in One Variable

## 3-1 Linear Inequalities; Absolute Value

### Objective

To solve and graph linear inequalities in one variable.

In this chapter you will study linear and polynomial inequalities and their graphs. Inequalities can be written in one or more variables.

$$\text{Linear Inequalities:} \quad |3x - 1| < 4 \quad 2x + 3y > 6$$

$$\text{Polynomial Inequalities:} \quad x^3 - x > 0 \quad y < 3x^2 - 4x + 1$$

The properties of real numbers used in solving linear inequalities are similar to the properties used in solving linear equations.

1. You can add the same number to (or subtract the same number from) both sides of an inequality. That is, if  $a < b$ , then  $a + c < b + c$ .
2. You can multiply (or divide) both sides of an inequality by the same positive number. That is, if  $a < b$  and  $c > 0$ , then  $ac < bc$ .
3. You can multiply (or divide) both sides of an inequality by the same negative number *if you reverse the inequality sign*. That is, if  $a < b$  and  $c < 0$ , then  $ac > bc$ .

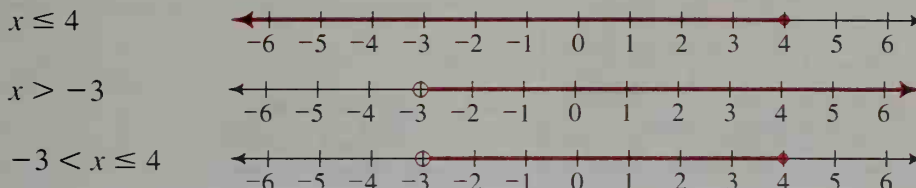
### Example 1

$$\begin{aligned} \text{a. } 3x - 4 &\leq 10 + x \\ 3x &\leq 14 + x \\ 2x &\leq 14 \\ x &\leq 7 \end{aligned}$$

$$\begin{aligned} \text{b. } \frac{8 - 2x}{5} &> 6 \\ 8 - 2x &> 30 \\ -2x &> 22 \\ x &< -11 \end{aligned}$$

In the solution to part (b), the inequality sign *is reversed* because we divided both sides of the inequality  $-2x > 22$  by  $-2$ .

The graphs of several linear inequalities are shown below. If an endpoint is to be included in the graph, a solid dot ( $\bullet$ ) is used. Otherwise, an open dot ( $\circ$ ) indicates that the endpoint is not to be included in the graph. The last inequality means  $-3 < x$  and  $x \leq 4$ . The graph of this *combined inequality* is the intersection of the first two graphs.



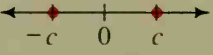
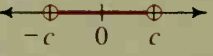
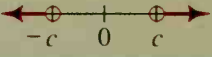
Many contemporary works of art owe their existence to mathematics. "Redbal," pictured at the left, was created entirely with the computer by twentieth-century artist David Em.



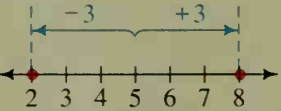
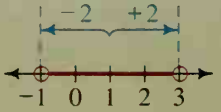
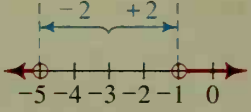
**Warning.** Care must be taken when inequalities are combined. For example, the statement  $6 < x < 1$  says that  $6 < x$  and  $x < 1$ . Since no real number  $x$  satisfies both these inequalities, the combined inequality represents the empty set.

## Absolute Value

The absolute value of a number  $x$ , denoted  $|x|$ , can be interpreted geometrically as the distance from  $x$  to zero in either direction on the number line. Thinking in this way, you can solve linear equations and inequalities involving  $|x|$ . In all three cases below,  $c$  represents a distance, so  $c \geq 0$ .

Sentences Involving $ x $			
Sentence	Meaning	Graph	Solution
$ x  = c$	The distance from $x$ to 0 is <i>exactly</i> $c$ units.		$x = c$ or $x = -c$
$ x  < c$	The distance from $x$ to 0 is <i>less than</i> $c$ units.		$-c < x < c$
$ x  > c$	The distance from $x$ to 0 is <i>greater than</i> $c$ units.		$x < -c$ or $x > c$

Equations and inequalities involving  $|x - k|$ , where  $k$  is a constant, can be solved by interpreting  $|x - k|$  as the distance from  $x$  to  $k$  on the number line.

Sentences Involving $ x - k $			
Sentence	Meaning	Graph	Solution
$ x - 5  = 3$	The distance from $x$ to 5 is 3 units.		$x = 2$ or $x = 8$
$ x - 1  < 2$	The distance from $x$ to 1 is less than 2 units.		$-1 < x < 3$
$ x + 3  > 2$ or $ x - (-3)  > 2$	The distance from $x$ to $-3$ is greater than 2 units.		$x < -5$ or $x > -1$

Absolute-value sentences can be solved not only by the geometric method shown on the previous page but also algebraically by using the three sets of equivalent sentences listed below. The equivalent sentences are obtained by replacing  $x$  by  $ax + b$  in the first chart given on page 96. In all three cases,  $c \geq 0$ .

Sentence	Equivalent Sentence
$ ax + b  = c$	$ax + b = \pm c$
$ ax + b  < c$	$-c < ax + b < c$
$ ax + b  > c$	$ax + b < -c$ or $ax + b > c$

### Example 2

Solve: a.  $|3x - 9| > 4$       b.  $|2x + 5| \leq 7$

### Solution

#### Algebraic Method

a.  $|3x - 9| > 4$  means  $3x - 9 < -4$  or  $3x - 9 > 4$ .  
 $3x < 5$  or  $3x > 13$

Thus,  $x < \frac{5}{3}$  or  $x > \frac{13}{3}$ .

b.  $|2x + 5| \leq 7$  means  $-7 \leq 2x + 5 \leq 7$ .  
 $-7 \leq 2x + 5$  and  $2x + 5 \leq 7$   
 $-12 \leq 2x$  and  $2x \leq 2$   
 $-6 \leq x$  and  $x \leq 1$

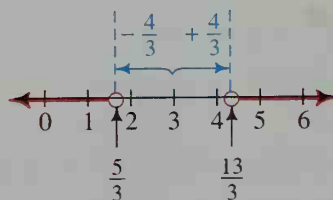
Thus,  $-6 \leq x \leq 1$ .

**Geometric Method:** A geometric solution can also be given to these inequalities. To do so, we must use the fact that

$$|ab| = |a| \cdot |b|.$$

(See Exercise 35 on page 99.) This allows us to conclude, for example, that  $|3x - 9| = |3| \cdot |x - 3| = 3|x - 3|$ .

a.  $|3x - 9| > 4$   
 $3|x - 3| > 4$   
 $|x - 3| > \frac{4}{3}$



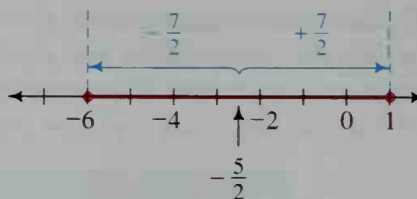
This means the distance from  $x$  to 3 is greater than  $\frac{4}{3}$ , which gives the picture above. Thus,  $x < \frac{5}{3}$  or  $x > \frac{13}{3}$ .

(Solution continues on the next page.)

$$\text{b. } |2x + 5| \leq 7$$

$$2 \left| x + \left( \frac{5}{2} \right) \right| \leq 7$$

$$\left| x - \left( -\frac{5}{2} \right) \right| \leq \frac{7}{2}$$



This means the distance from  $x$  to  $-\frac{5}{2}$  is less than or equal to  $\frac{7}{2}$ , as illustrated above. Thus,  $-6 \leq x \leq 1$ .

## CLASS EXERCISES

Solve each inequality.

$$1. 3x > -12$$

$$2. -3x > -12$$

$$3. 1 - 2x < 11$$

$$4. |x| < 2$$

$$5. |x - 5| < 2$$

$$6. |x + 3| > 1$$

In Exercises 7–12, match the inequalities with graphs a–f.

$$7. 1 < x \leq 4$$

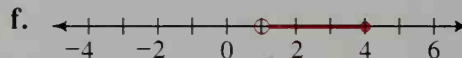
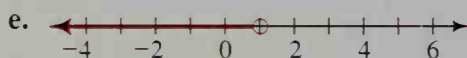
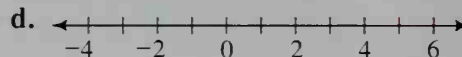
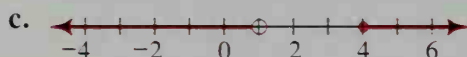
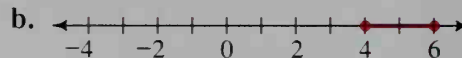
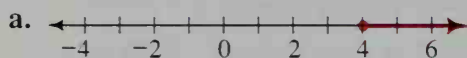
$$8. 4 \leq x$$

$$9. x < 1$$

$$10. x < 1 \text{ or } x \geq 4$$

$$11. 6 \geq x \geq 4$$

$$12. 6 \leq x \leq 4$$



## WRITTEN EXERCISES

In Exercises 1–24, solve the given equation or inequality and graph its solution. If there is no solution, say so.

**A**

$$1. 7x - 12 < 9$$

$$2. 8x + 6 > 30$$

$$3. \frac{15 - 6x}{3} > 5$$

$$4. \frac{8 - 11x}{4} \leq 13$$

$$5. \frac{1}{4}(x - 1) \leq \frac{x + 4}{6}$$

$$6. \frac{2 - x}{3} < \frac{3 - 2x}{5}$$

$$7. 8x + 3(x + 1) > 5x - (9x - 6)$$

$$8. 2x(6x - 1) \geq (3x - 2)(4x + 3)$$

$$9. \frac{x + 2}{4} - \frac{2 - x}{3} + \frac{4x - 5}{6} < 4$$

$$10. \frac{4}{3}\left(x - \frac{1}{2}\right) + \frac{1}{2}x \geq \frac{2}{3}\left(2x - \frac{5}{2}\right)$$

$$11. |x| < 3$$

$$12. |x| \geq 5$$

$$13. |x| < -8$$

$$14. |x| > -2$$

15.  $|x - 4| < 3$       16.  $|x - 7| > 3$   
 17.  $|x + 7| \geq 3$       18.  $|x + 2| < -1$   
 19.  $|x - 8| = 4$       20.  $|x + 3| = 7$   
 21.  $|2x - 4| \leq 5$       22.  $|3x - 9| \geq 9$   
 23.  $|4x + 8| \leq 9$       24.  $|6 - 3x| < 12$

- B** 25. **Horticulture** Plant experts advise that gardenias kept indoors must have high humidity, plenty of sunlight during the day, and cool temperatures at night. The recommended nighttime temperature range in degrees Fahrenheit is  $60^\circ \leq F \leq 65^\circ$ . Given that  $C = \frac{5}{9}(F - 32)$ , express the corresponding temperature range in degrees Celsius.



26. a. **Writing** Use the geometric definition of  $|x|$  and the definition of a combined inequality to write a sentence that gives the meaning of the expression  $2 < |x| < 4$ .  
 b. Solve the inequality  $2 < |x| < 4$  and graph its solution.  
 c. Solve the inequality  $1 \leq |x| < 5$  and graph its solution.

**In Exercises 27–32, solve the given inequality and graph its solution.**

27.  $1 \leq |x - 4| \leq 3$       28.  $2 < |x - 6| \leq 5$   
 29.  $0 \leq |x - 7| < 2$       30.  $0 < |x + 3| < 1$   
 31.  $\frac{1}{x} < 2$  (*Hint: Consider two cases.*)      32.  $\frac{1}{x - 1} > 4$

**Use the following definition of  $|x|$  to complete Exercises 33–37:**

$$|x| = \begin{cases} x & \text{if } x \geq 0. \\ -x & \text{if } x < 0. \end{cases}$$

**Solve.**

- C** 33.  $|x| + |x - 2| = 2$       34.  $|x| + |x - 2| > 5$   
 35. Show that  $|ab| = |a| \cdot |b|$  when (1)  $a > 0$  and  $b > 0$ ; (2)  $a < 0$  and  $b > 0$ ; (3)  $a > 0$  and  $b < 0$ ; and (4)  $a < 0$  and  $b < 0$ .  
 36. a. Give three examples illustrating the triangle inequality:  

$$|a + b| \leq |a| + |b|$$
  
 b. **Investigation** In Chapter 11 we will define the absolute value of a complex number as follows:  $|x + yi| = \sqrt{x^2 + y^2}$ . Decide whether the triangle inequality holds if  $a$  and  $b$  are complex numbers.  
 37. Use the triangle inequality stated in Exercise 36(a) to prove:  
 a.  $|a - b| \leq |a| + |b|$       b.  $|a| - |b| \leq |a - b|$



## 3-2 Polynomial Inequalities in One Variable

**Objective** To solve and graph polynomial inequalities in one variable.

Let  $P(x)$  be any polynomial. Then  $P(x) < 0$  and  $P(x) > 0$  are called **polynomial inequalities**. There are two ways to solve a polynomial inequality.

**Method 1** Use a *sign graph* of  $P(x)$ .

This method was first introduced in Section 2-3 and is illustrated below in Examples 1–3.

**Method 2** Analyze a graph of  $P(x)$ .

Note that  $P(x) > 0$  when the graph is *above* the  $x$ -axis, and  
 $P(x) < 0$  when the graph is *below* the  $x$ -axis.

Using a sign graph is an easy way to solve a polynomial inequality if the polynomial is factorable. Recall that to perform a sign analysis of a polynomial  $P(x)$ , you test one value of  $x$  from each of the intervals determined by the zeros of  $P(x)$ . Then you determine the sign of  $P(x)$  in each of these intervals.

**Example 1** Solve  $x^3 - 2x^2 - 3x < 0$  by using a sign graph.

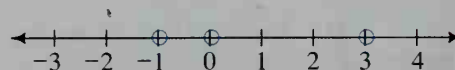
**Solution**

**Step 1** Find zeros of the polynomial:  $P(x) = x^3 - 2x^2 - 3x$   
 $P(x) = x(x^2 - 2x - 3)$   
 $P(x) = x(x + 1)(x - 3)$

The zeros are  $-1$ ,  $0$ , and  $3$ .

Plot the zeros on a number line.

Use open dots since  $P(x)$  is strictly less than zero.



**Step 2** Test  $x = -2$ ,  $x = -\frac{1}{2}$ ,  $x = 1$ , and  $x = 4$ , for example, to find the sign of  $P(x)$  in each interval.

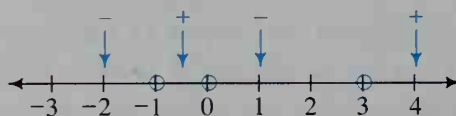
$$P(x) = x(x + 1)(x - 3)$$

$$P(-2) = -2(-2 + 1)(-2 - 3) = (-)(-)(-) = -$$

$$P\left(-\frac{1}{2}\right) = -\frac{1}{2}\left(-\frac{1}{2} + 1\right)\left(-\frac{1}{2} - 3\right) = (-)(+)(-) = +$$

$$P(1) = 1(1 + 1)(1 - 3) = (+)(+)(-) = -$$

$$P(4) = 4(4 + 1)(4 - 3) = (+)(+)(+) = +$$



The solution of  $P(x) < 0$  is  $x < -1$  or  $0 < x < 3$ .

Notice that the polynomial in Example 1 changes signs at each of its zeros. Is this fact true for all polynomials? The following activity explores this question.

### Activity

- What is the sign of  $(x - 1)^2$  for  $x < 1$ ? for  $x > 1$ ?
- Explain why  $(x - 1)^2$  does not change sign at  $x = 1$ .
- What is the sign of the product  $-2(x + 3)(x - 1)^2$  for  $-3 < x < 1$ ?  
What is the sign of the product  $-2(x + 3)(x - 1)^2$  for  $x > 1$ ?  
Explain why this product should not change sign at  $x = 1$ .

Not all polynomials change sign at a zero. A polynomial  $P(x)$  will not change sign at a zero  $c$  if  $c$  corresponds to the squared factor  $(x - c)^2$ . Example 2 illustrates such a polynomial.

**Example 2** Solve  $(x^2 - 1)(x - 4)^2 \geq 0$ .

### Solution

**Step 1** Find the zeros of the polynomial:  $P(x) = (x^2 - 1)(x - 4)^2$   
 $P(x) = (x + 1)(x - 1)(x - 4)^2$

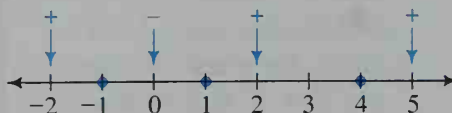
The zeros are  $-1$ ,  $1$ , and  $4$ .

Since  $P(x)$  is greater than or equal to zero, use solid dots to plot the zeros.



**Step 2** Test  $x = -2$ ,  $x = 0$ ,  $x = 2$ , and  $x = 5$ , for example, to find the sign of  $P(x)$  in each interval.

$$\begin{aligned}
 P(x) &= (x + 1)(x - 1)(x - 4)^2 \\
 P(-2) &= (-)(-)(-)^2 = + \\
 P(0) &= (+)(-)(-)^2 = - \\
 P(2) &= (+)(+)(-)^2 = + \\
 P(5) &= (+)(+)(+)^2 = +
 \end{aligned}$$



The solution of  $P(x) \geq 0$  is  $x \leq -1$  or  $x \geq 1$ .

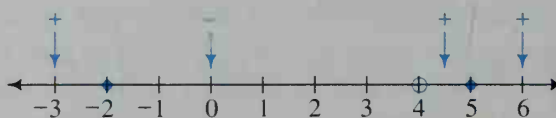
The technique used in Example 2 can be used to solve inequalities involving functions of the form  $\frac{P(x)}{Q(x)}$  where  $P(x)$  and  $Q(x)$  are polynomials or products of polynomials.

**Example 3** Solve  $\frac{(x+2)(x-5)^2}{x-4} \leq 0$ .


**Solution**

**Step 1** Plot the zeros of all linear factors occurring in a numerator or a denominator. In this case there are zeros at  $-2$ ,  $4$ , and  $5$ . Since  $f(x)$  can equal  $0$ , use solid dots for  $-2$  and  $5$ . Since the denominator can never be zero, use an open dot for  $4$  to remind yourself to omit it from the solution.

**Step 2** Do a sign analysis of  $f(x) = \frac{(x+2)(x-5)^2}{x-4}$  by testing convenient values in each of the intervals determined by the zeros above. For example,  $f(0) = \frac{(+)(-)^2}{(-)} = -$ .



The solution of  $f(x) \leq 0$  is  $-2 \leq x < 4$  or  $x = 5$ .

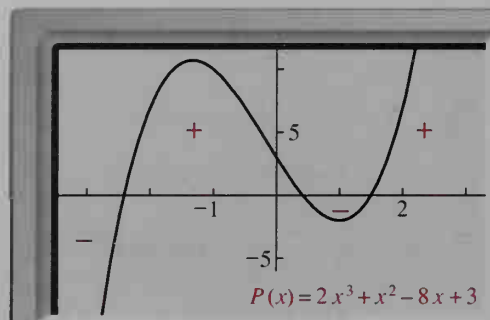
 A quicker method for solving a polynomial inequality is to use a computer or a graphing calculator to graph the polynomial, and then analyze the graph, as shown in Example 4.

**Example 4** Solve  $2x^3 + x^2 - 8x + 3 > 0$  by using a computer or graphing calculator to sketch a graph.

**Solution**

A graph of  $P(x) = 2x^3 + x^2 - 8x + 3$  is shown below. Note that the zeros of  $P$  are about  $-2.4$ ,  $0.4$ , and  $1.5$ . (Zeros can be found more accurately using a zoom feature.) The exact zeros, which can be found by using the techniques of Chapter 2, are  $1.5$  and  $-1 \pm \sqrt{2}$ .

$P(x) > 0$  when the graph is above the  $x$ -axis. Looking at the graph, we can see that  $P(x) > 0$  when  $-2.4 < x < 0.4$  or when  $x > 1.5$ .

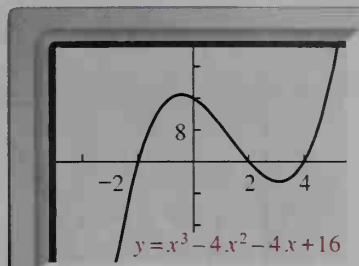


The solution is  $-2.4 < x < 0.4$  or  $x > 1.5$ .

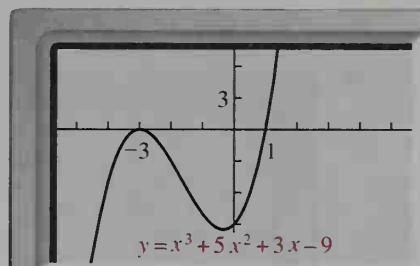
## CLASS EXERCISES

In Exercises 1 and 2, use the given graph to solve each inequality.

1.  $x^3 - 4x^2 - 4x + 16 > 0$



2.  $x^3 + 5x^2 + 3x - 9 \leq 0$



Use a sign graph to solve each inequality.

3.  $(x - 1)(x - 2)(x - 3)^2(x - 4) > 0$

4.  $\frac{(x - 2)(x - 7)}{x - 4} < 0$

5.  $(x^2 + 1)(x - 5) \leq 0$

6.  $|x^2 - 2x| < 0$

## WRITTEN EXERCISES

Solve each inequality.

**A**

1.  $(x - 3)(x + 4) > 0$

3.  $(x - 1)(x - 2)(x - 4) > 0$

5.  $(x - 4)(x - 2)^2(x - 3)^2 < 0$

7.  $x^2 - 2x - 15 < 0$

9.  $2x^2 - x - 3 \geq 0$

11.  $2x^2 + 5x - 7 \leq 0$

13.  $x^4 - 3x^2 - 10 > 0$

15.  $a^3 + 2a^2 - 4a - 8 > 0$

17.  $n^3 - 7n + 6 < 0$

19.  $2x^3 + x^2 - 5x < -2$

21.  $y^4 + y^3 < 4(y + 4)$

23.  $4x^4 - 4x^3 - 3x^2 + 4x - 1 > 0$

**B**

25.  $\frac{(x - 3)(x - 4)}{(x - 5)(x - 6)^2} < 0$

27.  $\frac{(2x - 5)^3}{x^2 - 3x - 28} \geq 0$

29.  $\frac{2x^2 + 7x + 8}{x^2 + 1} > 0$

2.  $(x + 7)(x + 9) > 0$

4.  $(1 - x)(x - 3)(x - 5) > 0$

6.  $(2x - 5)^2(x + 3)(x + 2) < 0$

8.  $x^2 + 3x - 18 > 0$

10.  $1 - 2x - 3x^2 < 0$

12.  $x^2 - 8x + 16 \leq 0$

14.  $3x^3 + 7x^2 - 6x \leq 0$

16.  $b^4 - 16 < 0$

18.  $2y^3 + 3y^2 - 1 \geq 0$

20.  $r^3 - 9r > 8r^2$

22.  $x^4 + 6 < 5x^2$

24.  $2y^3 + y^2 - 12y + 9 < 0$

26.  $\frac{(x + 1)(x - 3)^2}{(x - 5)^2} > 0$

28.  $\frac{(3n - 12)^2}{3n^2 - 12} \leq 0$

30.  $\frac{n^2 + 4n + 4}{n^2 + 4n} > 0$





For Exercises 31–34, solve each inequality by using a computer or a graphing calculator.

31.  $x^3 + 2x^2 - 3x - 6 > 0$

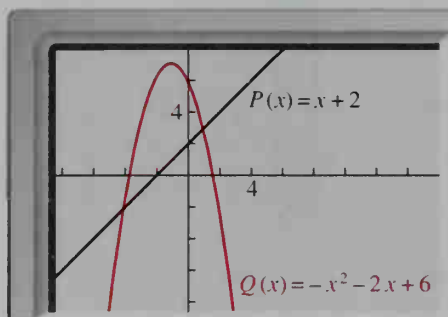
32.  $4x^3 - 3x^2 - 9x - 2 \leq 0$

33.  $2x^3 - 5x^2 + 1 \geq 0$

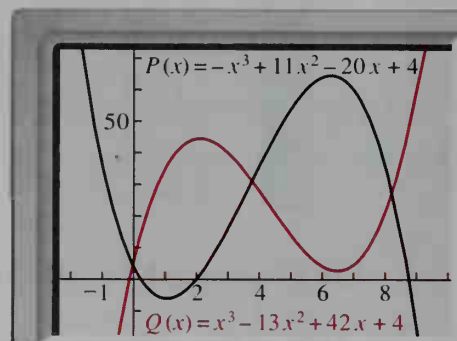
34.  $6x^4 - x^3 - 32x^2 + 5x + 10 < 0$

For Exercises 35 and 36, use the graphs to solve the inequality  $P(x) > Q(x)$ .

35.



36.



For Exercises 37–42, use a computer or a graphing calculator, and the method suggested by Exercises 35 and 36, to solve each inequality.

37.  $x^2 + x + 14 > -x + 10$

38.  $x^3 - 2x^2 - 2x - 3 \leq |x - 1|$

39.  $2 - x \leq \sqrt{x + 1}$

40.  $x^2 < \sqrt{4x^2 + 5}$

41.  $\sqrt{x^2 - 9} \leq \sqrt{25 - x^2}$

42.  $\sqrt{3x + 7} > 3\sqrt{x} - 1$

Solve each inequality.



43.  $\frac{|2x - 5|}{x^2 + 4} < 0$

44.  $\frac{6x^2 + 13x - 5}{|5x^3 - 10|} < 0$

## Inequalities in Two Variables

### 3-3 Polynomial Inequalities in Two Variables

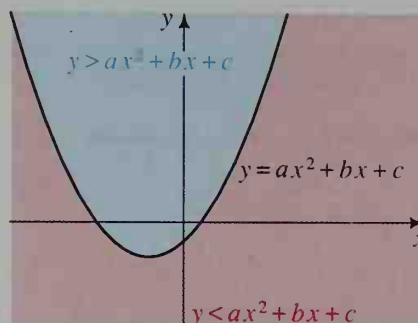
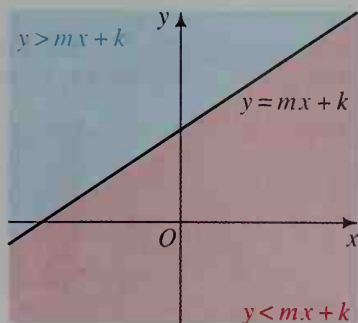
#### Objective

To graph polynomial inequalities in two variables and to graph the solution set of a system of inequalities.

Like solutions to equations in two variables, solutions to inequalities in two variables are ordered pairs of numbers. Thus, graphs of inequalities in two variables consist of points in the plane.

The graphs of linear and quadratic inequalities in two variables are closely related to the graphs of lines and parabolas, as illustrated on the next page. If a line has equation  $y = mx + k$ , any point  $(x, y)$  above the line satisfies the inequality  $y > mx + k$ . Any point  $(x, y)$  below the line  $y = mx + k$  satisfies the inequality

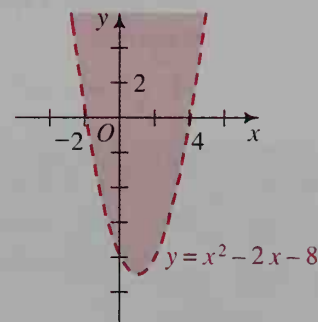
$y < mx + k$ . A similar statement can be made for points  $(x, y)$  above and below a parabola.



**Example 1** Sketch the graph of  $y > x^2 - 2x - 8$ .

**Solution**

**Method 1** Follow the approach suggested above. If a parabola has equation  $y = x^2 - 2x - 8$ , then any point  $(x, y)$  above the parabola satisfies the inequality. We show the parabola as a dashed curve since points on the parabola are not included in the solution.



**Method 2** Draw the graph of  $y = x^2 - 2x - 8$ , which separates the plane into two regions. Choose a point in one region and check to see whether it satisfies the inequality. For example, the point  $(0, 0)$  satisfies the inequality. Thus the region containing  $(0, 0)$  is shaded as the graph of the inequality.

If you want to graph the solution of a system of inequalities, you find the intersection of the graphs of the individual inequalities.

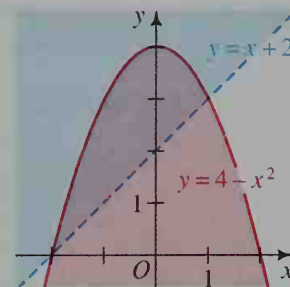
**Example 2** Graph the solution set of the inequalities  $y \leq 4 - x^2$  and  $y > x + 2$ .

**Solution**

**Step 1** The graph of  $y \leq 4 - x^2$  consists of points on or below the parabola  $y = 4 - x^2$ . This region is shaded in red.

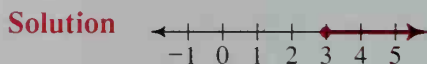
**Step 2** The graph of  $y > x + 2$ , shaded in blue, consists of points above the line  $y = x + 2$ . Because the points of the line are not included in the graph, you use a dashed line.

**Step 3** The intersection of the blue and red regions is the graph of the solution set of the system of inequalities.

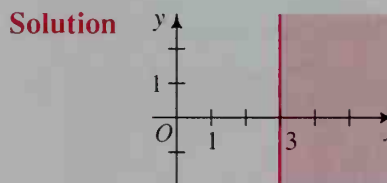


If an inequality contains just one variable, you must be aware of whether you are working just on the number line or in the whole coordinate plane. The two possibilities for the inequality  $x \geq 3$  are shown below.

**Problem** Find all  $x$  such that  $x \geq 3$ .



**Problem** Find all  $(x, y)$  such that  $x \geq 3$ .

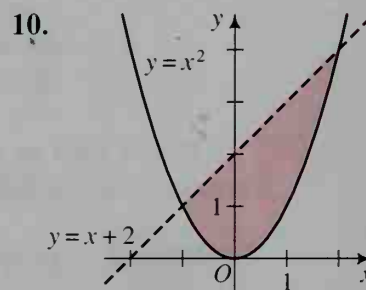
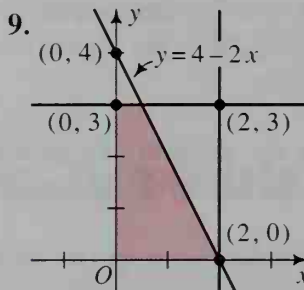
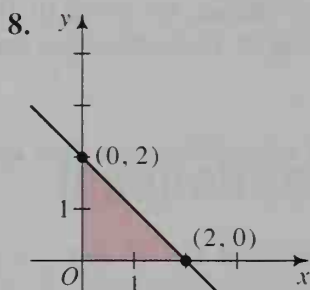


## CLASS EXERCISES

In each of the following exercises, the coordinates of a point and an equation are given. Tell whether the point is above, on, or below the graph of the equation.

1.  $(3, 4)$ ;  $y = x$
2.  $(3, 6)$ ;  $y = 2x + 1$
3.  $(1, -1)$ ;  $y = 3x - 4$
4.  $(0, 9)$ ;  $y = x^2 - 8x + 9$
5.  $(-1, 12)$ ;  $y = 6 - 5x - x^2$
6.  $(2, 10)$ ;  $y = x^3 + x^2 + x + 1$
7. a. If a point  $(x, y)$  is in the first quadrant, what inequalities must be satisfied?  
b. What if  $(x, y)$  is in the second quadrant? third quadrant? fourth quadrant?

Give a set of inequalities that defines the shaded regions.



## WRITTEN EXERCISES

In Exercises 1–16, sketch the graph of the given inequality.

- A**
1.  $y \geq x$
  2.  $y \leq 2x - 1$
  3.  $3x + 4y < 12$
  4.  $4 - 2x < 3y$
  5.  $y \leq x^2$
  6.  $y \geq x^2 + 4x + 8$
  7.  $y < 2x^2 - 4x + 1$
  8.  $y > 3x^2 + 6x - 2$
  9.  $0 \leq x \leq 2$
  10.  $-1 \leq y \leq 3$

11.  $|y| > 1$

12.  $|x| \leq 2$

13.  $|x - 3| < 2$

14.  $2 < |x + 4| < 3$

15.  $y > x^3 - 9x$

16.  $y < x^4 - 5x^2 + 4$

In Exercises 17–30, graph the solution set of the given system of inequalities.

17.  $x \geq 0$

$x + 2y \leq 4$

18.  $y \leq 0$

$2x + y \leq 4$

19.  $x < 0$

$3x - 2y \leq -6$

20.  $y < 0$

$x - y > -1$

21.  $y \geq x^2 - 2$

$y < x$

22.  $y \leq 6 - x^2$

$2x - y \leq -3$

**B**

23.  $y \leq x^3 + 4$

$y > x^2 + x - 6$

24.  $y \leq x^2 + x - 2$

$y \leq -x^2 + x + 12$

25.  $0 \leq x \leq 3$

$0 \leq y \leq 2$

$y \leq 2 - x$

26.  $-1 \leq x \leq 4$

$-2 \leq y \leq 5$

$5y \geq x + 11$

27.  $|x| < 3$

$|y| < 1$

28.  $|x| \geq 2$

$|y| \leq 4$

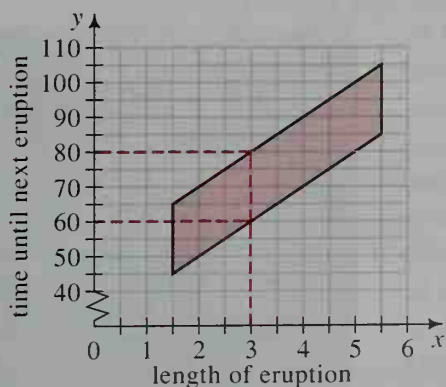
29.  $1 \leq |x - 4| \leq 3$

$1 \leq |y - 4| \leq 3$

30.  $1 \leq |x + 3| \leq 3$

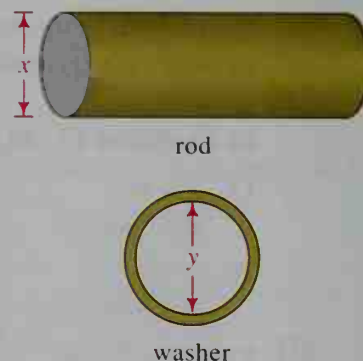
$1 \leq |y + 4| \leq 4$

31. **Geology** Over the years, Yellowstone National Park rangers have compiled data on the eruption of the Old Faithful geyser. The graph below shows the results of the data. For example, if the eruption lasts 3 min, the next eruption will most likely occur sometime between 60 min and 80 min later. Eruptions lasting less than 1.5 min or more than 5.5 min are rare. Describe the shaded region in the graph by a set of inequalities.





32. **Manufacturing** A manufacturer makes rods and washers for machines. The rods have an outside diameter of  $x = 2.00 \pm 0.10$  cm. The washers have an inside diameter of  $y = 2.08 \pm 0.05$  cm.
- Make a graph showing the points  $(x, y)$  that give the possible dimensions for  $x$  and  $y$ .
  - In order for a washer to fit on a rod,  $y$  must be larger than  $x$ . Shade a subset of your graph in part (a) to show the points  $(x, y)$  that satisfy this condition.



33. On one set of axes, graph each of the following inequalities. Then shade the region that satisfies all the inequalities. Label each corner point of this region with its coordinates.
- (1)  $y \geq x$       (2)  $x \geq 3$       (3)  $y \leq 8$       (4)  $x + 2y \geq 16$
34. Repeat Exercise 33 for the following inequalities:
- (1)  $x \geq 0$       (2)  $y \geq 3$       (3)  $x + y \leq 12$       (4)  $3y - x \leq 12$

Sketch the graph of each equation or inequality.

**C** 35.  $|x| + |y| = 2$

36.  $|2x - 3y| < 6$

## 3-4 Linear Programming

### Objective

To solve certain applied problems using linear programming.

Running a profitable business requires a careful balancing of resources (for example, people's time, materials, and machine availability). A manager must choose the best use of these resources. Often the range of possible choices can be described by a set of linear inequalities called *constraints*. In most situations the number of alternative solutions to the constraints is so great that it is hard to find the best one. *Linear programming*, a relatively new branch of mathematics, provides mathematical methods for finding the best solution.

Linear programming was developed by mathematician George Dantzig during World War II to improve large-scale military planning. Around 1950, industry began using these methods for practical planning and scheduling. Today, by using linear programming methods on computers, many otherwise impossible decision-making problems can be solved, such as the routing of millions of long-distance calls over an immense network of telephone lines.



Suppose that a small TV manufacturing company produces console and portable TV's using three different machines, A, B, and C. The table below shows how many hours are required on each machine per day in order to produce a console TV or a portable TV.

Machine	Console	Portable	Hours available
A	1 h	2 h	16
B	1 h	1 h	9
C	4 h	1 h	24



These requirements for the different machines can be described by the following inequalities, where  $x$  represents the number of console TV's and  $y$  represents the number of portable TV's:

a.  $x \geq 0$

b.  $y \geq 0$

{ The number of TV's cannot be negative.

c.  $x + 2y \leq 16$

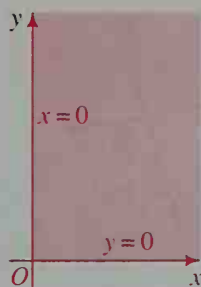
{ Machine A needs 1 hour for each console TV and 2 hours for each portable. Thus, for  $x$  console TV's and  $y$  portable TV's, Machine A needs  $x + 2y$  hours. Since this machine is available for at most 16 hours a day,  $x + 2y \leq 16$ .

d.  $x + y \leq 9$

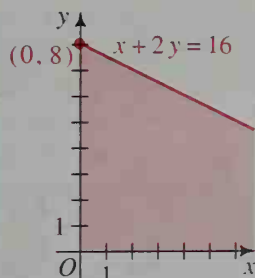
e.  $4x + y \leq 24$

{ The last two inequalities are similar to the one above. Machines B and C are available for at most 9 hours and 24 hours, respectively.

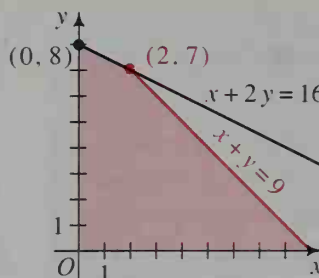
The possible solutions to any problem subject to these constraints would have to satisfy all five inequalities. The easiest way to find these values is to find the region that is formed by the graphs of the system of inequalities. A step-by-step procedure is shown below.



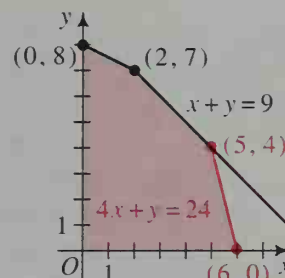
(a) - (b)



(c)



(d)



(e)

Any point  $(x, y)$  in graph (e) is called a *feasible solution*. The set of these points is called the *feasible region*.

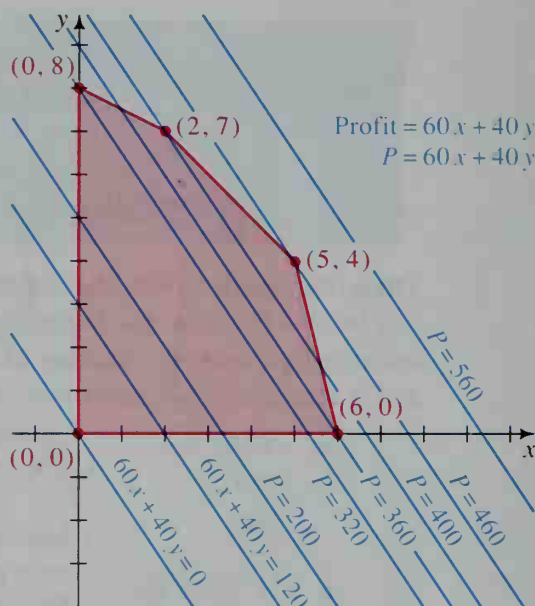
**Maximizing a Profit.** Suppose that the manufacturing company described on the previous page makes a \$60 profit on each console TV and a \$40 profit on each portable TV. How many console TV's and how many portable TV's should be produced each day to maximize the profit? The profit for  $x$  console TV's and  $y$  portable TV's is:

$$P = 60x + 40y \text{ (dollars)}$$

Since there are so many feasible solutions, it would be undesirable to evaluate  $P = 60x + 40y$  at all feasible solutions in order to determine which one gives the greatest profit. However, we can narrow the search by observing what happens when we plot several *profit lines* on the graph of the feasible region, as shown at the right. We can see that as the amount of profit increases, the corresponding profit lines move farther away from the origin. We have to exclude any profit value greater than 460 since these profit lines contain no points that lie in the feasible region. Thus, the “greatest-profit line” containing points in the feasible region is the line  $60x + 40y = 460$ . The only feasible

solution on this line occurs at  $(5, 4)$ , which is a vertex, or *corner point*, of the feasible region. Therefore, the company will make the maximum profit of \$460 when 5 console TV's and 4 portable TV's are produced each day.

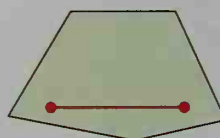
Notice that the “least-profit line” goes through the corner point  $(0, 0)$ . In general, the greatest or least value of a linear expression (profit or cost, to name two) will always occur at a corner point of the feasible region.



### The Corner-Point Principle

A maximum or minimum value of a linear expression  $P = Ax + By$ , if it exists, will occur at a corner point of the feasible region.

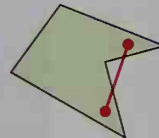
The corner-point principle applies only to *convex* polygonal regions. A polygonal region is convex if every line segment joining two points of the region is contained in the region. Regions (a) and (b) are convex, but region (c) is not.



(a)



(b)



(c)

Because of the corner-point principle, we need to evaluate a specific expression only at the corner points of the feasible region in order to find an optimal value, as the example below shows.

### Example

**Minimizing a Cost.** Every day Rhonda Miller needs a dietary supplement of 4 mg of vitamin A, 11 mg of vitamin B, and 100 mg of vitamin C. Either of two brands of vitamin pills can be used: Brand X at 6¢ a pill or Brand Y at 8¢ a pill. The chart below shows that a Brand X pill supplies 2 mg of vitamin A, 3 mg of vitamin B, and 25 mg of vitamin C. Likewise, a Brand Y pill supplies 1, 4, and 50 mg of vitamins A, B, and C, respectively. How many pills of each brand should she take each day in order to satisfy the minimum daily need most economically?

	Brand X	Brand Y	Minimum daily need
Vitamin A	2 mg	1 mg	4 mg
Vitamin B	3 mg	4 mg	11 mg
Vitamin C	25 mg	50 mg	100 mg
Cost per pill	6¢	8¢	

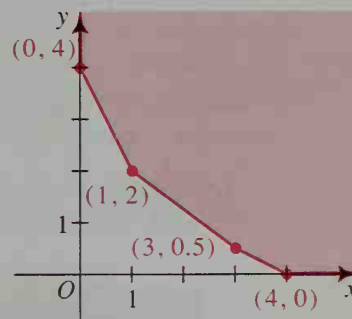
### Solution

Let  $x$  = number of Brand X pills, and  $y$  = number of Brand Y pills.

**Step 1** Set up a system of inequalities that describe the constraints.

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\2x + y &\geq 4 \\3x + 4y &\geq 11 \\x + 2y &\geq 4\end{aligned}$$

**Step 2** Determine the feasible region by graphing the system of inequalities, as shown at the right above.



**Step 3** The cost (in cents) can be expressed by the equation  $C = 6x + 8y$ . Evaluate the cost at each corner point.

Corner point	Value of $6x + 8y$
$(0, 4)$	$6(0) + 8(4) = 32¢$
$(1, 2)$	$6(1) + 8(2) = 22¢$
$(3, 0.5)$	$6(3) + 8(0.5) = 22¢$
$(4, 0)$	$6(4) + 8(0) = 24¢$

} Minimum cost

The minimum cost occurs at both  $(1, 2)$  and  $(3, 0.5)$ . It is, however, inconvenient to take half a pill. Therefore, the best choice is 1 Brand X pill and 2 Brand Y pills.



## WRITTEN EXERCISES

- A** 1. a. Refer to the example on page 110 and suppose each console TV manufactured produces an \$80 profit and each portable TV manufactured produces a \$55 profit. How many of each kind of TV should be manufactured to maximize the profit?
- b. Suppose the profit on a console TV is \$90 but is only \$20 on a portable TV. Show that the maximum profit is achieved by producing only console TV's.
2. a. **Writing** Refer to the example on page 111. For each inequality listed in Step 1 of the solution, write a sentence that explains in your own words what the inequality means.
- b. Suppose each Brand X pill costs 3¢ and each Brand Y pill costs 9¢. What combination of pills should Rhonda take to minimize the cost?
- c. Suppose each Brand X pill costs 10¢ and each Brand Y pill costs 4¢. Show that the cost is minimized by taking only Brand Y pills.
3. Suppose the data of Exercise 1 are changed as shown below. How many of each kind of TV should be produced to maximize the profit?

	Console	Portable	Total hours available per day
Machine A	1	3	18
Machine B	1	1	8
Machine C	3	1	18
Profit per TV	\$70	\$40	

4. Suppose the data of Exercise 2 are changed as shown below. How many of each type of pill should be taken daily to keep the cost at a minimum?

	Brand X	Brand Y	Minimum daily need
Vitamin A	4 mg	2 mg	10 mg
Vitamin B	6 mg	6 mg	24 mg
Vitamin C	25 mg	50 mg	125 mg
Cost per pill	12¢	15¢	

In Exercises 5–8, find the point or points  $(x, y)$  that satisfy the given constraints and give the stated objective.

5. Given:  $x + 2y \leq 4$

$x + y \leq 3$

$x \geq 0, y \geq 0$

Maximize  $P = x + 3y$ .

6. Given:  $x + 2y \geq 6$

$x + 3y \geq 7$

$x \geq 0, y \geq 0$

Minimize  $C = 3x + 4y$ .



7. Given:  $2x + y \geq 12$

$x + y \geq 10$

$x + 3y \geq 16$

$x \geq 0, y \geq 0$

Minimize: a.  $C = 5x + 6y$

b.  $C = 5x + 5y$

8. Given:  $3x + 4y \leq 36$

$3x + 2y \leq 24$

$x \leq 6$

$x \geq 0, y \geq 0$

Maximize: a.  $P = 3x + 3y$

b.  $P = 3x + 4y$

- B** 9. **Agriculture** A farmer can make a profit of \$250 for every ton of alfalfa harvested and \$350 for every ton of corn harvested. Corn requires 3 h per ton to harvest while alfalfa requires only 2 h per ton. Each crop requires 1 h per ton for planting. The planting time available is 500 h and the harvesting time available is 1200 h. Organize the given information in a table.

	Alfalfa	Corn	Hours available
Planting time (hours/ton)	?	?	?
Harvest time (hours/ton)	?	?	?
Profit (dollars/ton)	?	?	

Let  $x$  and  $y$  represent the tons of alfalfa and corn to be grown, and write the inequalities that must be satisfied. How much of each crop should the farmer grow in order to maximize the profit?

10. **Manufacturing** A manufacturer of stereo speakers makes two kinds of speakers, an economy model that sells for \$50 and a deluxe model that sells for \$200. The deluxe model uses 1 woofer (a low frequency range speaker), 2 tweeters (high frequency range speakers) and 1 mid-frequency range speaker. The economy model uses 1 tweeter, 1 mid-frequency range speaker, and no woofers. The manufacturer's current inventory consists of 20 woofers, 45 tweeters, and 35 mid-frequency range speakers. Organize the given information in a table.

	Economy	Deluxe	Number available
Number of woofers	?	?	?
Number of tweeters	?	?	?
Number of mid-ranges	?	?	?
Income per model	?	?	

Let  $x$  and  $y$  represent the number of economy and deluxe models to be produced, and write the inequalities that must be satisfied. How many of each model should be manufactured to maximize the income from their sale?

11. a. **Manufacturing** A sporting goods company manufactures two types of skis: a racing model and a free-style model. Each pair of racing skis requires 3 h of labor, and the company can produce at most 20 pairs of racing skis per day. Each pair of free-style skis requires 2 h of labor, and the company can produce at most 30 pairs of free-style skis per day. The maximum number of hours of labor available for ski production is 96. The profit on each pair of racing skis is \$30 and the profit on each pair of free-style skis is \$40. Find the number of each that should be manufactured in order to maximize profits.
- b. Suppose the profit on each pair of free-style skis is \$20. Show that there is more than one way to maximize profits.



12. **Manufacturing** A game company is developing two new games, a board game and a mechanical game. Each board game requires  $\frac{1}{2}$  h to manufacture,  $\frac{1}{2}$  h to assemble, and  $\frac{1}{4}$  h to inspect and package. Each mechanical game requires 1 h to manufacture,  $\frac{1}{2}$  h to assemble, and  $\frac{1}{2}$  h to inspect and package. In a given week there are 40 h available for manufacturing, 32 h for assembly, and 18 h for inspection and packaging. Suppose the profit on each board game is \$10 and the profit on each mechanical game is \$15. How many of each game should be produced each week for maximum profit?
- C** 13. Let  $x$  and  $y$  be positive integers such that  $x + y \leq 13$  and  $3x + y \leq 24$ . Find the maximum value of: a.  $x + y$       b.  $4x + y$       c.  $2x + y$

## Chapter Summary

1. The rules for solving *linear inequalities* are similar to the rules for solving linear equations and are summarized on page 95. The graph of a linear inequality in one variable consists of points on a number line.
2. Equations and inequalities involving  $|x - k|$  can be solved in two ways:
  - a. By interpreting  $|x - k|$  as the distance from  $x$  to  $k$  on the number line. (See the examples given in the chart on page 96.)
  - b. By using an equivalent sentence and then solving it algebraically. (See the set of equivalent sentences given in the chart on page 97.)
3. The solution to a *polynomial inequality*  $P(x) > 0$  (or  $P(x) < 0$ ) consists of those values of  $x$  for which the graph of the polynomial  $y = P(x)$  is above (or below) the  $x$ -axis. As an alternative to analyzing the graph of the polynomial  $P(x)$ , you can analyze a sign graph of the factors of  $P(x)$  to solve the polynomial inequality. These methods are illustrated in the examples of Section 3–2.

4. Solutions to *inequalities in two variables* are ordered pairs of real numbers. Their graphs consist of points in the coordinate plane. (See Example 1 on page 105.) The graph of the solution of a system of inequalities is the intersection of the graphs of the individual inequalities. (See Example 2 on page 105.)
5. *Linear programming* is a method for solving certain decision-making problems where a linear expression is to be maximized or minimized. This method uses the *corner-point principle*, which is stated on page 110.

### Key vocabulary and ideas

linear inequality (p. 95)

absolute value (p. 96)

polynomial inequality (p. 100)

sign graph (p. 100)

inequality in two variables (p. 104)

system of inequalities (p. 105)

linear programming (p. 108)

feasible region (p. 109)

## Chapter Test

Solve and graph each inequality on a number line.

1. a.  $3(8 - 4x) < 9 - 7x$

b.  $\frac{1}{3}(x - 3) \leq \frac{x - 2}{4}$  **3-1**

2. a.  $|x - 3| < 5$

b.  $|x + 6| > 4$

c.  $|3x + 4| = 4$

d.  $|7 - 5x| \leq 2$

3. Solve each inequality.

a.  $(2x - 1)(x + 4)(x - 3)^2 \geq 0$

b.  $4x^2 + 5x - 6 < 0$

c.  $2x^3 - x^2 > x$

d.  $\frac{(x - 6)(x + 1)}{(x - 3)^2} \leq 0$  **3-2**

4. **Writing** Describe the set of points in the coordinate plane that satisfies the inequality  $y < 2x - 5$ . **3-3**

5. Graph each inequality in the coordinate plane.

a.  $2 - 3y \geq 3x$

b.  $|x| < 3$

6. Graph the solution set of each system.

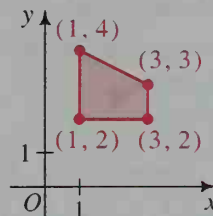
a.  $x > 0$

b.  $y \geq x^2 - 2x$

$3x + 4y < 2$

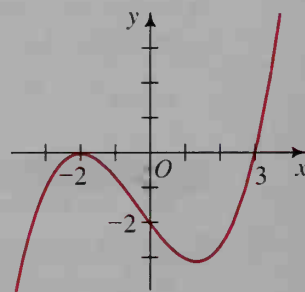
$y \leq 6 - x$

7. Give a set of inequalities that defines the shaded region shown at the right.



8. **3-4** Cars and trucks are made in a factory that is divided into two shops. Shop 1 performs the basic assembly operation, working 6 person-days on each truck and only 3 person-days on each car. Shop 2 performs finishing operations, working 4 person-days on each car or truck that it produces. Shop 1 has 150 person-days per week available, while Shop 2 has 120 person-days per week. The manufacturer makes a profit of \$500 on each truck and \$350 on each car. How many of each should be produced each week to maximize the profit?

1. Let  $A = (-4, 5)$  and  $B = (-7, -2)$ . Find the length of  $\overline{AB}$  and the coordinates of the midpoint of  $\overline{AB}$ .
2. Write an equation of the line described.
  - a. The line through  $(4, -2)$  and  $(2, -4)$
  - b. The line with y-intercept 5 and parallel to  $2x - 3y = 3$
3. Each mailing sent out by a charitable organization costs \$1027. The average donation received as a result of the mailing is \$12.50.
  - a. Express the organization's profit as a function of the number of donations received.
  - b. How many donations are needed to realize a profit of \$500?
4. Simplify each expression.
  - a.  $\sqrt{-1} + \sqrt{-25} - \sqrt{-64}$
  - b.  $\sqrt{-12}\sqrt{-36}$
  - c.  $\frac{3-i}{6+2i}$
5. Solve each equation. Give both real and imaginary roots.
  - a.  $x^2 - 6x = -18$
  - b.  $4x^2 - 12x + 7 = 0$
  - c.  $\sqrt{2x+1} = \sqrt{x-3} + 2$
6. Sketch the graph of the line  $y = 2x + 4$  and the parabola  $y = x^2 - 4$ . Find the coordinates of any points of intersection.
7. Write an equation of the quadratic function  $f$  whose graph has  $x$ -intercepts 3 and 7 and  $f(5) = 8$ .
8. Write an equation of the form  $f(x) = ax^2 + bx + c$  in which  $f(0) = -6$ ,  $f(2) = -3$ , and  $f(4) = 2$ .
9. If  $P(x) = 2x^3 - x^2 + 3$ , use synthetic substitution to find:
  - a.  $P(2)$
  - b.  $P(-1)$
  - c.  $P\left(-\frac{3}{2}\right)$
  - d.  $P(i)$
10. a. One root of  $P(x) = x^3 + 2x^2 - 5x - 6$  is  $x = -1$ . Find the other roots.  
 b. Sketch the graph of  $y = x^3 + 2x^2 - 5x - 6$ .
11. Write an equation of the cubic function whose graph is shown at the right.
12. Find the maximum area of a rectangle whose perimeter is 100cm.
13. Use a computer or a graphing calculator to find the real roots of  $x^3 - 3x - 1 = 0$  to the nearest tenth.
14. Find all real and imaginary roots of  $x^3 - 3x^2 + 4x - 12 = 0$ .
15. Find a cubic equation with integral coefficients and roots  $-\frac{1}{3}$  and  $2 + 2i$ .
16. Solve and graph each inequality on a number line.
  - a.  $|x - 5| > 12$
  - b.  $(x - 2)^2(x + 3)(x - 1) \geq 0$
17. Graph the solution set of the system of inequalities  $x \geq 0$  and  $y \leq 1 - x^2$ .



Ex. 11



# Careers in Architecture and Urban Planning

A city is a complex and endlessly evolving world. People who design the buildings and plan the layout of communities shape the character of individual cities, and have a significant influence on the quality of urban life.

## ARCHITECTURE

An architect must have a unique blend of talents, a strong sense of visual aesthetics, and the technical expertise needed to create well-constructed, functional buildings. With a strong background in mathematics an architect is able to design beautiful and efficient uses of three-dimensional space.

**Minoru Yamasaki** (1912–1986) began to think of a career in architecture when he saw his uncle's plans for the United States Embassy in Tokyo. Twenty-five years later, Yamasaki also designed a building for Japan. His commission for the American Consulate in Kobe helped build his growing reputation as an innovative architect.

Yamasaki, famous for designing New York City's World Trade Center, moved away from the starkness that characterized much of modern architecture, believing that good buildings have "a joyful quality," and "provide the delight of change and surprise." Many of his buildings won honors and awards.



*Minoru Yamasaki*



*Yvonne Goldsberry*

## URBAN PLANNING

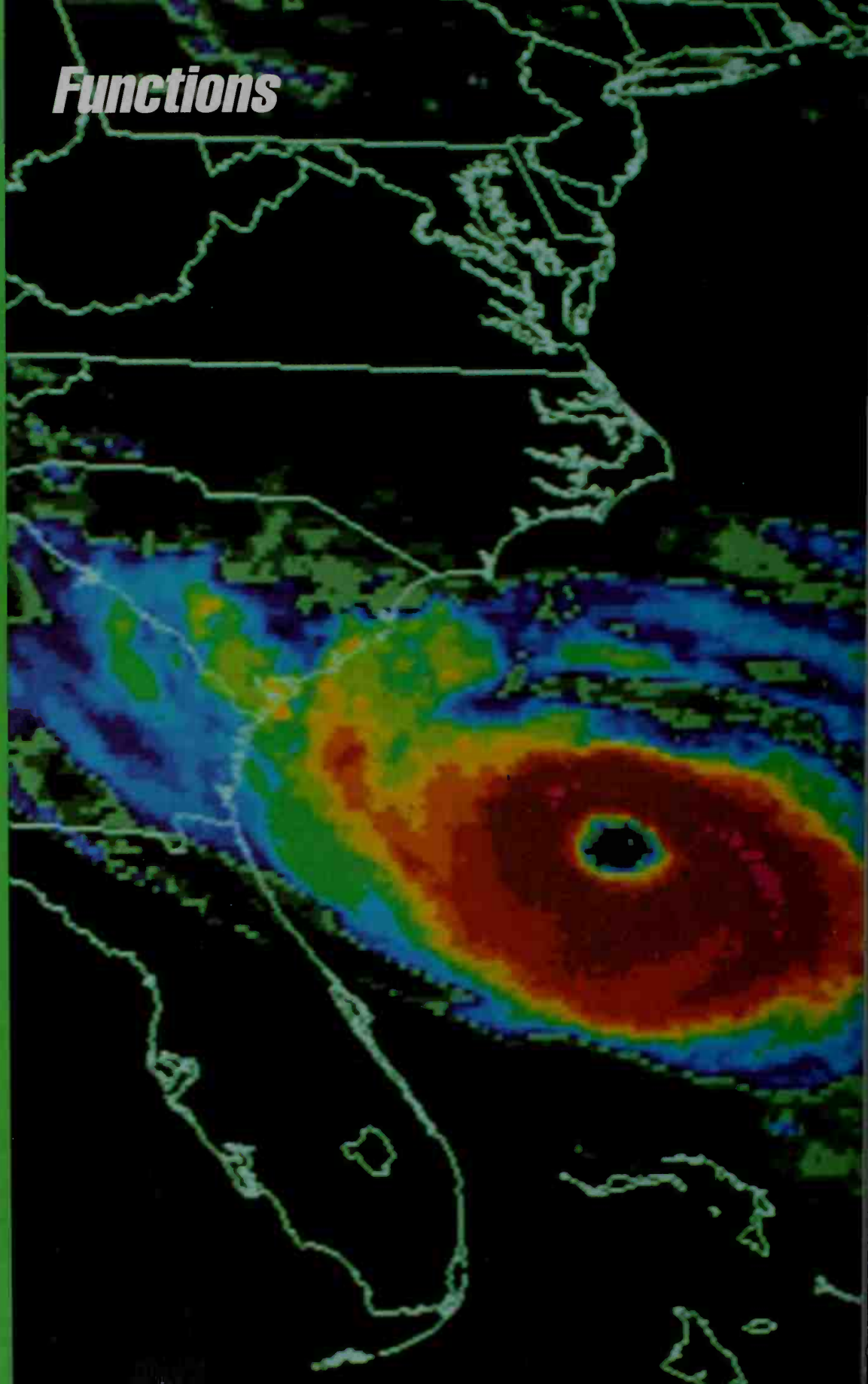
Architects often collaborate with urban planners who decide the types and locations of new buildings in a city. Urban planners, whose jobs may also include designing traffic control systems or helping businesses with management and marketing issues, base their recommendations on extensive analysis of population growth, economic trends, environmental impact, and transportation requirements.

**Yvonne Goldsberry** holds master's degrees in Urban Planning and Public Health from Columbia University. In 1987, using her experience as a planner of health care for the elderly, Ms. Goldsberry founded Aureus, Inc., a company designed to provide the skills of an urban planner to a variety of health and human service organizations, including hospitals, mental health centers, adoption agencies, nursing homes, and child care facilities.



# 4

## *Functions*



# ■ Properties of Functions

## 4-1 Functions

### Objective

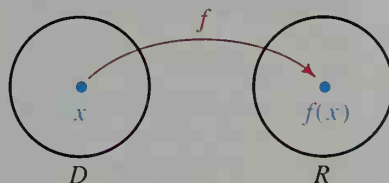
To identify a function, to determine the domain, range, and zeros of a function, and to graph a function.

You have already studied linear functions, quadratic functions, and polynomial functions. The purpose of this chapter is to consider functions in general. You will see how functions can be combined to produce new functions and how simple changes in a function's rule will change its graph. These ideas will be used later in the book when you study exponential functions, logarithmic functions, and trigonometric functions.

### Definition of a Function

A **function** is a correspondence or rule that assigns to every element in a set  $D$  exactly one element in a set  $R$ . The set  $D$  is called the **domain** of the function, and the set  $R$  is called the **range**.

The diagram at the right shows a function  $f$  mapping, or pairing, a domain element  $x$  to a range element  $f(x)$ , read “the value of  $f$  at  $x$ ” or “ $f$  of  $x$ .” Although  $f$  names the function and  $f(x)$  gives its value at  $x$ , we sometimes refer to the function  $f(x)$ , thereby indicating both the function  $f$  and the variable  $x$  of its domain.



In this book the domain and range of a function are typically sets of numbers, but this need not be the case, as the following illustrations show.

1. Birthday function  $B$  maps a person to his or her birthday. For example,

$$B(\text{George Washington}) = \text{February 22}$$

and  $B(\text{Martin Luther King, Jr.}) = \text{January 15}.$

2. Area function  $A$  maps a geometric figure to its area. For example, if  $PQRS$  is a square with sides of length 5, then  $A(PQRS) = 5^2$ .

We can treat a function  $f$  as a set of ordered pairs  $(x, y)$  such that  $x$  is an element of the domain of  $f$  and  $y$  is the corresponding element of the range. This is written formally as  $\{(x, y) \mid y = f(x)\}$  or more simply as

$$y = f(x).$$

Although the letters  $f$ ,  $x$ , and  $y$  are commonly used in general discussions of functions, other letters can be used. For example,  $v = g(u)$  is a function  $g$  that assigns a domain element  $u$  to a range element  $v$ .

◀ Weather maps often involve the visual representation of functions of more than one variable. Each colored band in the satellite photo at the left represents a particular range of wind velocity plotted as a function of latitude and longitude.

A function is frequently given in terms of a rule and a domain. If the domain of a function is not specified, then it is understood to consist of those real numbers for which the function produces real values.

**Example 1** Give the domain of each function.

a.  $g(x) = \frac{1}{x-7}$

b.  $h(r) = \sqrt{1-r^2}$

**Solution**

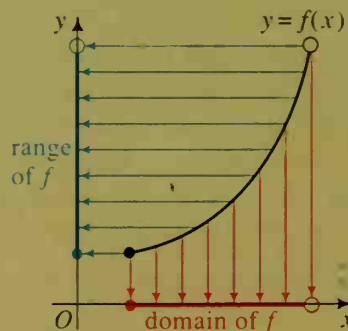
- a. The domain of  $g$  is the set of all real numbers except 7, since  $\frac{1}{x-7}$  is not defined when the denominator is 0.
- b.  $\sqrt{1-r^2}$  is a real number only if  $1-r^2 \geq 0$ . Therefore, the domain of  $h$  is  $\{r \mid -1 \leq r \leq 1\}$ .

## The Graph of a Function

The graph of a function  $y = f(x)$  consists of all points  $(x, f(x))$  in an  $xy$ -plane. We can obtain the domain, range, and zeros of a function from its graph, as indicated below.

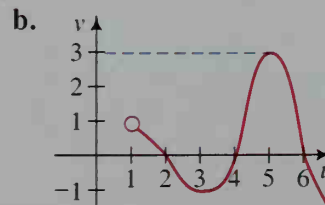
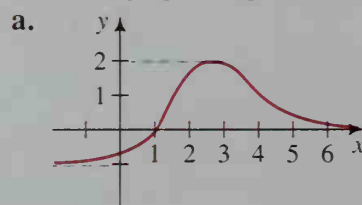
### Analyzing the Graph of a Function

1. The **domain** of  $y = f(x)$  is the set of  $x$ -coordinates of points on the graph of  $f$ . Projecting the graph of  $f$  on the  $x$ -axis gives a graph of the domain.
2. The **range** of  $y = f(x)$  is the set of  $y$ -coordinates of points on the graph of  $f$ . Projecting the graph of  $f$  on the  $y$ -axis gives a graph of the range.
3. The zeros of  $y = f(x)$  are the  $x$ -intercepts of the graph.



**Example 2**

Use the graph to give the domain, range, and zeros of each function.



**Solution**

a. Domain = {all real numbers}  
Range =  $\{y \mid -1 < y \leq 2\}$   
Zero: 1

b. Domain =  $\{u \mid u > 1\}$   
Range =  $\{v \mid v \leq 3\}$   
Zeros: 2, 4, and 6

In applied settings, we often work with equations relating two or more measured quantities. For example, if gasoline is priced at \$1.25 per gallon at a certain gas station, then a pump registers the cost  $C$  of  $g$  gallons of gas by using the equation

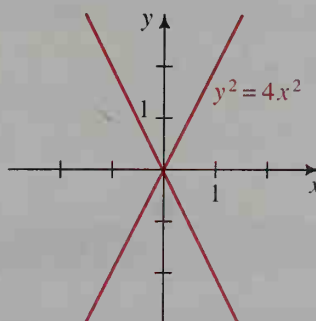
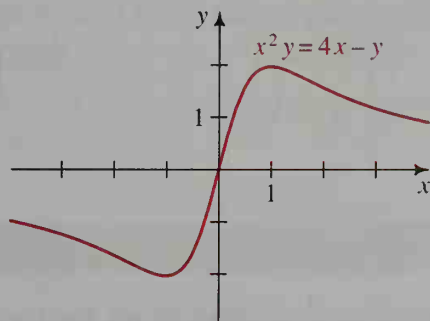
$$C = 1.25g.$$

Alternatively, the equation

$$C(g) = 1.25g$$

emphasizes that  $C$  is a function of  $g$ . Because the cost depends on the number of gallons,  $C$  and  $g$  are called **dependent** and **independent variables**, respectively.

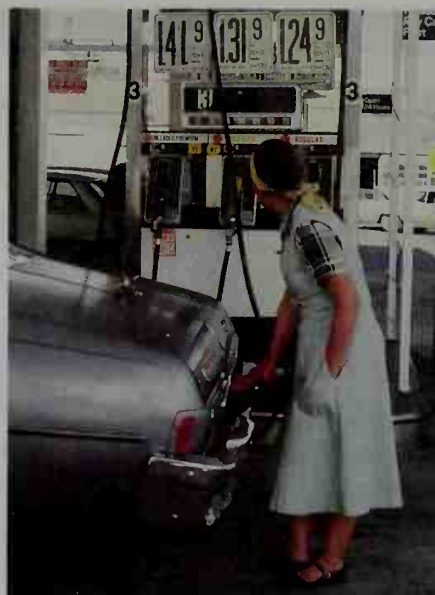
Functions are a subset of the more general class of correspondences called **relations**. A **relation** is *any* correspondence or rule that pairs the members of two sets (the domain and the range). In general, a relation expressed as an equation in two variables, say  $x$  and  $y$ , defines  $y$  as a function of  $x$  provided there is exactly one  $y$ -value for each  $x$ -value. For example, the relation described by the equation  $x^2y = 4x - y$  defines  $y$  as a function of  $x$ , but the relation described by the equation  $y^2 = 4x^2$  does not. One way to determine this is to solve each equation for  $y$ , if possible. The first equation becomes  $y = \frac{4x}{x^2 + 1}$ . Because there is only one  $y$ -value for each  $x$ -value, this relation is a function. The second equation becomes  $y = \pm 2x$ . Because there are two  $y$ -values for each nonzero  $x$ -value, this relation is not a function. The graphs of these two equations are shown below.



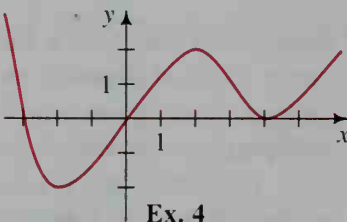
Another way to determine whether an equation in  $x$  and  $y$  defines  $y$  as a function of  $x$  is to apply the *vertical-line test* to the graph of the equation.

### The Vertical-Line Test

If no vertical line intersects a given graph in more than one point, then the graph is the graph of a function.



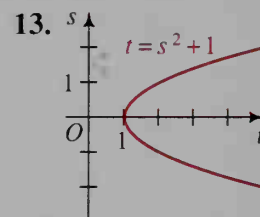
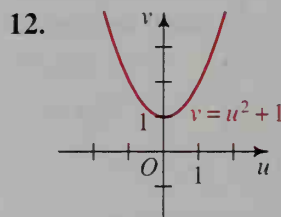
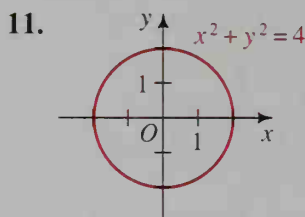
## CLASS EXERCISES

- What are the domain and range of the cost-of-gas function described on the preceding page?
  - Sketch the graph of the cost-of-gas function in a  $gC$ -plane.
- Give the domain of each function.
  - $f(x) = \frac{4}{x-2}$
  - $g(t) = \sqrt{t}$
  - $h(s) = \sqrt{s-4}$
- Show that 2 is a zero of the function  $f(x) = 3x^2 - 12$ .
  - What is another zero?
  - Find the range of the function.
- Give the domain, range, and zeros of the function whose graph is shown at the right.
 
- Apply the vertical-line test to the graphs of  $x^2y = 4x - y$  and  $y^2 = 4x^2$  shown on the preceding page.
- Which of the relations described by the two equations  $x^2 - y = 1$  and  $y^2 - x = 1$  does *not* define  $y$  as a function of  $x$ ? Explain.

Tell whether the relation described by each equation defines  $y$  as a function of  $x$ . If it does, sketch the graph.

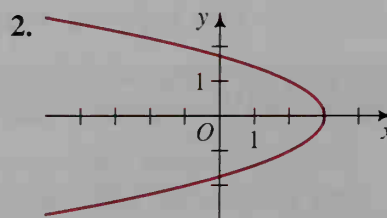
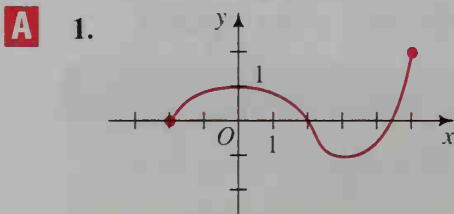
- $y = 2x + 5$
- $y = |x|$
- $x = |y|$
- $y^2 = x^2$

Tell whether the graph of each relation is the graph of a function. Explain your answer.

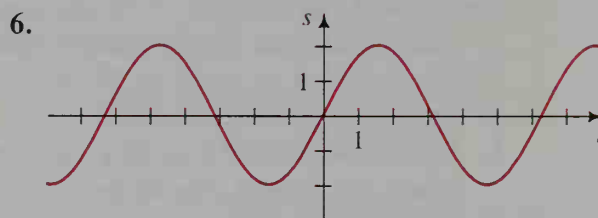
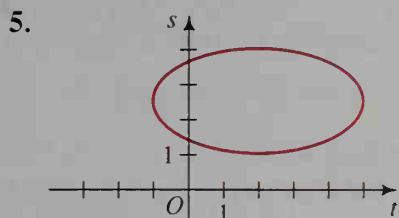
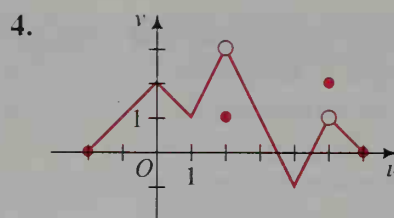
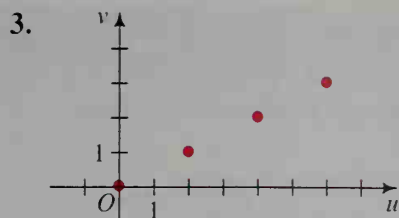


## WRITTEN EXERCISES

In Exercises 1–6, tell whether the graph of each relation is the graph of a function. If it is, give the domain and range of the function.







7. Explain why the equation  $x^2 + y^2 = 1$  does not define  $y$  as a function of  $x$ .

8. Explain why the equation  $x^3 + y^3 = 1$  defines  $y$  as a function of  $x$ .

Give the domain of each function.

9. a.  $f(x) = \frac{1}{x}$

b.  $g(x) = \frac{1}{x-9}$

c.  $h(x) = \frac{3x}{x^2-4}$

10. a.  $f(t) = \frac{1}{t+3}$

b.  $g(t) = \frac{t+2}{t^2+5t+6}$

c.  $h(t) = \frac{2t^2-3}{t^3-9t}$

Give the domain, range, and zeros of each function.

11. a.  $f(x) = |x|$

b.  $g(x) = |x-2|$

c.  $h(x) = |x| - 2$

12. a.  $f(t) = \sqrt{t}$

b.  $g(t) = \sqrt{9-t}$

c.  $h(t) = \sqrt{9-t^2}$

In Exercises 13–18, sketch the graph of each function. Use the graph to find the range and zeros of the function.

13.  $f(x) = x^2 - 6x + 8$

14.  $g(x) = 4 - (x-3)^2$

15.  $f(t) = (t-2)^3$

16.  $g(t) = t^3 + 4t^2 - t - 4$

17.  $h(u) = \begin{cases} u^2 & \text{if } -2 \leq u < 1 \\ 2-u & \text{if } 1 \leq u < 4 \end{cases}$

18.  $g(u) = \begin{cases} u-1 & \text{if } u < 0 \\ u^2-2u-3 & \text{if } 0 \leq u \leq 3 \\ 0 & \text{if } u > 3 \end{cases}$

19. a. Let  $V$  be the function that assigns to each solid its volume. If  $C$  is a cylinder with radius 3 and height 4, find  $V(C)$ .

b. Give the domain and range of  $V$ .

20. a. A formula from geometry states that  $S = (n-2)180^\circ$ . Give the meaning of this formula.

b. Is  $S$  a function of  $n$ ? If so, give the domain and range of this function.

- B** 21. The **greatest integer function** assigns to each number the greatest integer less than or equal to the number. If we denote the greatest integer in  $x$  by  $\lfloor x \rfloor$ , then we have  $\lfloor 5.28 \rfloor = 5$ ,  $\lfloor 5 \rfloor = 5$ ,  $\lfloor \pi \rfloor = 3$ , and  $\lfloor -1.7 \rfloor = -2$ .
- Sketch the graph of  $y = \lfloor x \rfloor$ .
  - Give the domain and range of the greatest integer function.
22. The greatest integer function  $f(x) = \lfloor x \rfloor$ , described in Exercise 21, is sometimes called “the floor of  $x$ .” By contrast,  $c(x) = \lceil x \rceil$  is called “the ceiling of  $x$ ” and is the least integer greater than or equal to  $x$ . Thus  $\lceil 5.28 \rceil = 6$ ,  $\lceil 5 \rceil = 5$ ,  $\lceil \pi \rceil = 4$ , and  $\lceil -1.7 \rceil = -1$ .
- Sketch the graph of  $y = \lceil x \rceil$ .
  - The cost of parking a car in a municipal parking lot is \$3 for the first hour or any part thereof, plus \$2 for each additional hour or part thereof. Sketch the graph of this cost function and find a rule for the cost  $C$  as a function of time  $t$ . Your rule should use the ceiling function.
23. **Writing** Think about what it means for two functions to be *equal*. Would you say that the functions  $f(x) = |x|$  and  $g(x) = \sqrt{x^2}$  are equal? Are the functions  $f(x) = |x|$  and  $h(x) = (\sqrt{x})^2$  equal? Write a brief defense of your conclusions.
24. **Research** Use a mathematics dictionary to find the meaning of the phrase *implicit function*. Then determine what implicit functions, if any, are defined by each of the following equations.
- $x = y^2$
  - $x^2 + y^2 = 1$
  - $x^2 - y^2 = 0$
25. For which of the following functions does  $f(a + b) = f(a) + f(b)$ ?
- $f(x) = x^2$
  - $f(x) = \frac{1}{x}$
  - $f(x) = 4x + 1$
  - $f(x) = 4x$
26. For which of the functions in Exercise 25 does  $f(ab) = f(a) \cdot f(b)$ ?
27. If  $f(a + b) = f(a) + f(b)$  for some function  $f$ , prove that  $f(0) = 0$ .
28. If  $f(ab) = f(a) \cdot f(b)$  for some function  $f$ , prove that  $f(1) = 1$ .

## 4-2 Operations on Functions

### Objective

To perform operations on functions and to determine the domains of the resulting functions.

Suppose a company manufactures and sells a certain product. If the cost of manufacturing  $x$  items of the product is given by the function  $C(x)$  and the revenue generated by the sale of the  $x$  items is given by the function  $R(x)$ , then the company's profit is given by the function  $P(x)$  where

$$P(x) = R(x) - C(x).$$

That is, the profit function is the *difference* between the revenue and cost functions. As this example from economics suggests, it is possible to combine two given functions to produce a new function.

## The Sum, Difference, Product, and Quotient of Functions

Each function listed below is defined for all  $x$  in the domains of both  $f$  and  $g$ .

1. **Sum** of  $f$  and  $g$ :  $(f + g)(x) = f(x) + g(x)$
2. **Difference** of  $f$  and  $g$ :  $(f - g)(x) = f(x) - g(x)$
3. **Product** of  $f$  and  $g$ :  $(f \cdot g)(x) = f(x) \cdot g(x)$
4. **Quotient** of  $f$  and  $g$ :  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ , provided  $g(x) \neq 0$

**Example 1** Let  $f(x) = x + 1$  and  $g(x) = x^2 - 1$ . Find a rule for each of the following functions.

a.  $(f + g)(x)$

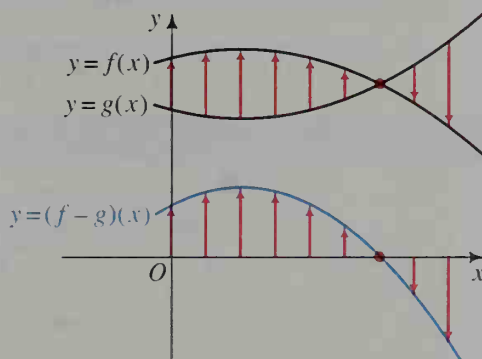
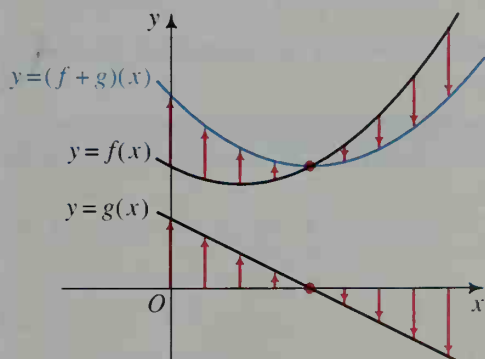
b.  $\left(\frac{f}{g}\right)(x)$

## Solution

**a.**  $(f + g)(x) = f(x) + g(x) = (x + 1) + (x^2 - 1)$   
 $= x^2 + x$

$$\begin{aligned}\text{b. } \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} = \frac{x+1}{x^2-1} \\ &= \frac{x+1}{(x+1)(x-1)} \\ &= \frac{1}{x-1}, \text{ provided } x \neq \pm 1\end{aligned}$$

The graph of the sum function,  $f + g$ , can be obtained directly from the graphs of  $f$  and  $g$ . As the diagram at the left below shows, vertical arrows from the  $x$ -axis to the graph of  $g$  are “added” to the graph of  $f$  to obtain the graph of  $f + g$ .



The diagram at the right above shows that the graph of the difference function,  $f - g$ , gives the vertical distance from the graph of  $g$  to the graph of  $f$ . Wherever the graph of  $f$  lies above the graph of  $g$ ,  $f - g$  is positive, and wherever the graph of  $f$  lies below the graph of  $g$ ,  $f - g$  is negative.

A fifth way of combining functions can be illustrated by the sport of cycling. When an 18-speed touring bicycle is in sixth gear, the gear ratio is 3 : 2, which means that the wheels of the bicycle revolve 3 times for every 2 revolutions of the pedals. This relationship can be expressed as

$$w = \frac{3}{2}p$$

where  $w$  and  $p$  represent wheel and pedal revolutions, respectively. Since the wheels of a touring bicycle have a diameter of 27 in.,  $w$  revolutions of the wheels move the bicycle a distance  $d$ , in inches, given by:

$$d = 27\pi w$$

Notice that  $d = 27\pi w$  gives distance as a function of wheel revolutions and that  $w = \frac{3}{2}p$  gives wheel revolutions as a function of pedal revolutions. By substituting  $\frac{3}{2}p$  for  $w$  in  $d = 27\pi w$ , we get

$$d = 27\pi\left(\frac{3}{2}p\right) = 40.5\pi p,$$

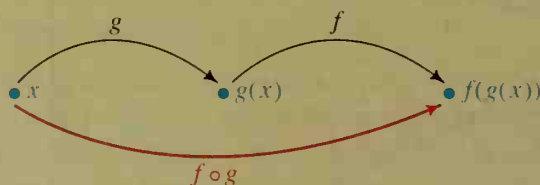
which gives distance as a function of pedal revolutions. This function  $d(p)$  is said to be the *composite* of the functions  $d(w)$  and  $w(p)$ .



## The Composite of Functions

The **composite** of  $f$  and  $g$ , denoted  $f \circ g$ , is defined by two conditions:

1.  $(f \circ g)(x) = f(g(x))$ , which is read “ $f$  circle  $g$  of  $x$  equals  $f$  of  $g$  of  $x$ ”;
2.  $x$  is in the domain of  $g$  and  $g(x)$  is in the domain of  $f$ .



The domain of  $f \circ g$  is the set of  $x$  satisfying condition (2) above. The operation that combines  $f$  and  $g$  to produce their composite is called the **composition** of functions.



As Example 2 shows, using a computer or graphing calculator to graph a composite function helps you see that its domain may need to be restricted.

### Example 2

Let  $f(x) = x^4 - 3x^2$  and  $g(x) = \sqrt{x-2}$ . Find a rule for  $(f \circ g)(x)$  and give the domain of the composite function. Confirm the domain by using a computer or graphing calculator to graph  $y = (f \circ g)(x)$ .

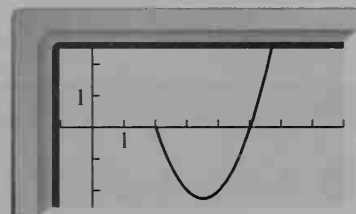
### Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(\sqrt{x-2}) \leftarrow g(x) \text{ is defined for } x \geq 2. \\ &= (\sqrt{x-2})^4 - 3(\sqrt{x-2})^2 \leftarrow f(g(x)) \text{ is defined for } x \geq 2. \\ &= x^2 - 7x + 10\end{aligned}$$

The domain of the composite function  $f \circ g$  is  $\{x \mid x \geq 2\}$  even though the expression  $x^2 - 7x + 10$  is also defined for  $x < 2$ . To use a graphing calculator to confirm the domain, enter the equation of  $f \circ g$  in its unsimplified form:

$$y = (\sqrt{x-2})^4 - 3(\sqrt{x-2})^2$$

Then the graph of  $y = (f \circ g)(x)$  is the portion of the parabola  $y = x^2 - 7x + 10$  for which  $x \geq 2$ .



### Example 3

Let  $f(x) = \frac{1}{x}$  and  $g(x) = x + 1$ . Find rules for  $(f \circ g)(x)$  and  $(g \circ f)(x)$  and give the domain of each composite function.

### Solution

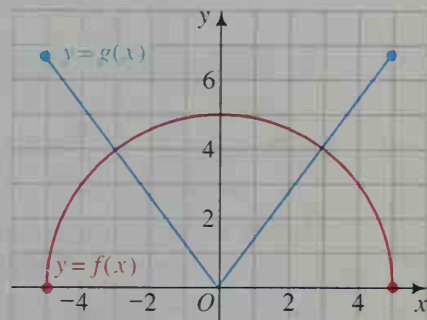
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(x+1) = \frac{1}{x+1} \quad \text{Domain} = \{x \mid x \neq -1\} \\ (g \circ f)(x) &= g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 1 \quad \text{Domain} = \{x \mid x \neq 0\}\end{aligned}$$

Notice in Example 3 that  $(f \circ g)(x) \neq (g \circ f)(x)$ . Since composition is not necessarily commutative, the *order* of two functions being composed is important.

## CLASS EXERCISES

For Exercises 1–4, use the graphs of  $f$  and  $g$  shown at the right.

- Find each of the following.
  - $(f+g)(0)$
  - $(f+g)(3)$
- For what values of  $x$  is:
  - $(f-g)(x) = 0$ ?
  - $(f-g)(x) > 0$ ?
- Sketch the graph of each of the following.
  - $y = (f+g)(x)$
  - $y = (f-g)(x)$
- Sketch the graph of each of the following.
  - $y = f(x) + 1$
  - $y = g(x) - 1$





In Exercises 5–10, let  $f(x) = x^2 + x$  and  $g(x) = x + 1$ . Find each of the following.

5.  $(f + g)(x)$       6.  $(f - g)(x)$       7.  $(f \cdot g)(x)$       8.  $\left(\frac{f}{g}\right)(x)$

9. a.  $f(g(2))$       b.  $(f \circ g)(x)$       10. a.  $g(f(2))$       b.  $(g \circ f)(x)$

11. **Visual Thinking** On a single set of axes, sketch the graphs of  $f(x) = \sqrt{x}$  and  $g(x) = x + 1$ . Demonstrate how the graphs of  $f$  and  $g$  can be used to find the value of (a)  $f(g(3))$  and (b)  $g(f(4))$ .

12. If  $F(X)$  is the father of  $X$  and  $M(X)$  is the mother of  $X$ , what expression represents the maternal grandfather of  $X$ ? the paternal grandmother of  $X$ ?

13. If  $B$  is the birthday function defined on page 119 and if  $F$  is the father function defined in Exercise 12, which of the composite functions  $B \circ F$  and  $F \circ B$  is defined?

## WRITTEN EXERCISES

**A**

1. Copy the graph of  $y = f(x)$  shown at the right. On a single set of axes, draw the graphs of  $y = f(x) + 2$  and  $y = f(x) - 3$ .

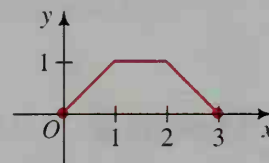
2. On a single set of axes, graph  $y = |x|$ ,  $y = |x| + 5$ , and  $y = |x| - 4$ .

3. The graphs of  $y = f(x)$  and  $y = g(x)$  are shown at the left below.

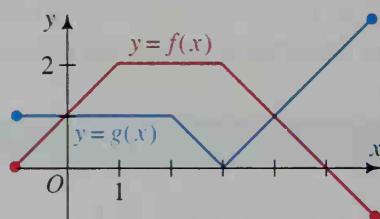
a. Find  $f(1) - g(1)$ .

b. For what values of  $x$  is  $f(x) - g(x)$  positive? negative? zero?

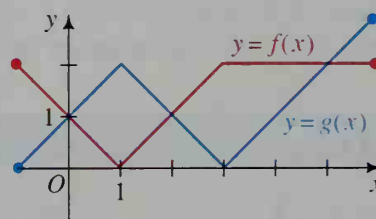
c. What is the maximum value of  $f(x) - g(x)$ ?



Ex. 1



Ex. 3



Ex. 4

4. Repeat Exercise 3 using the graphs of  $y = f(x)$  and  $y = g(x)$  shown at the right above.

5–10. Let  $f(x) = x^3 - 1$  and  $g(x) = x - 1$ . Evaluate the expressions in Class Exercises 5–10.

11. On a single set of axes, graph  $f(x) = x$  in one color, graph  $g(x) = |x|$  in a second color, and graph  $f + g$  in a third color.

12. Using the graphs given in Exercise 4, graph  $f + g$ .

13. **Visual Thinking** Given two functions  $f$  and  $g$ , one way to obtain the real solutions of the equation  $f(x) = g(x)$  is to graph the equations  $y = f(x)$  and  $y = g(x)$  in an  $xy$ -plane and then find the  $x$ -coordinates of any points of intersection. Describe another way to solve  $f(x) = g(x)$  that also involves graphing in an  $xy$ -plane but that is based on the difference function  $f - g$ .



Use a computer or graphing calculator and one of the methods from Exercise 13 to find the real solutions of each of the following equations. Give answers to the nearest hundredth.

14.  $x^3 = x + 1$

15.  $\sqrt{x+1} = 2x$

16.  $\sqrt{1-x^2} = |x|$

**B**

17. Let  $f(x) = 2x - 3$ ,  $g(x) = \frac{x+3}{2}$ , and  $h(x) = 3x + 2$ .

a. Show that  $f(g(x)) = g(f(x))$  for all  $x$ .

b. Show that  $f(h(x)) \neq h(f(x))$  for any  $x$ .

18. Let  $f(x) = x^3$ ,  $g(x) = \sqrt[3]{x}$ , and  $h(x) = 3x$ .

a. Show that  $f(g(x)) = g(f(x))$  for all  $x$ .

b. Show that  $f(h(x)) = h(f(x))$  for only one value of  $x$ .

Let  $f(x) = \sqrt{x}$ ,  $g(x) = 6x - 3$ , and  $h(x) = \frac{x}{3}$ . Find each of the following.

19. a.  $f(g(h(6)))$

b.  $f(g(h(x)))$

20. a.  $h(g(f(4)))$

b.  $h(g(f(x)))$

21. a.  $h\left(f\left(g\left(\frac{1}{2}\right)\right)\right)$

b.  $h(f(g(x)))$

22. a.  $g(h(f(9)))$

b.  $g(h(f(x)))$

Let  $f(x) = x^3$ ,  $g(x) = \sqrt{x}$ ,  $h(x) = x - 4$ , and  $j(x) = 2x$ . Express each function  $k$  as a composite of three of these four functions.

23.  $k(x) = 2(x - 4)^3$

24.  $k(x) = \sqrt{(x - 4)^3}$

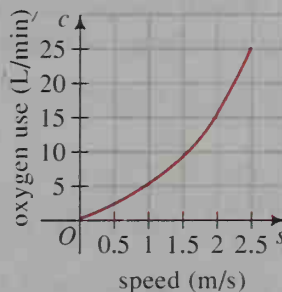
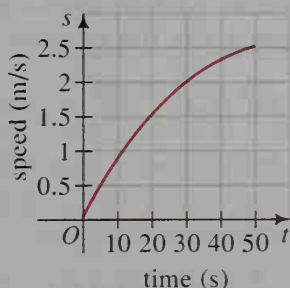
25.  $k(x) = (2x - 8)^3$

26.  $k(x) = \sqrt{x^3 - 4}$

27. **Physiology** The graph at the left below shows a swimmer's speed  $s$  as a function of time  $t$ . The graph at the right below shows the swimmer's oxygen consumption  $c$  as a function of  $s$ . Time is measured in seconds, speed in meters per second, and oxygen consumption in liters per minute.

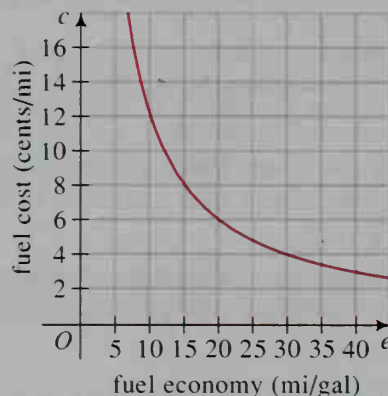
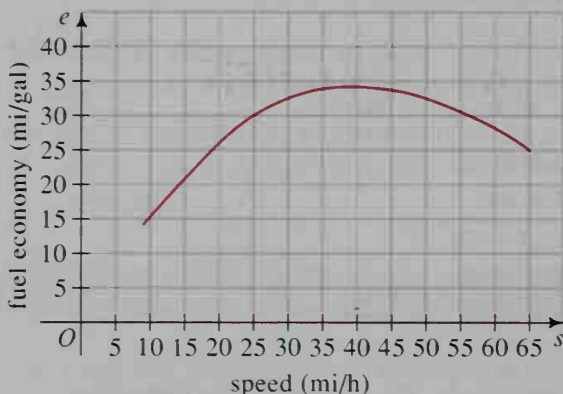
a. What are the speed and oxygen consumption after 20 s of swimming?

b. How many seconds have elapsed if the swimmer's oxygen consumption is 15 L/min?



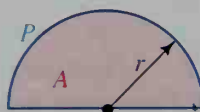
**28. Consumer Economics** The graph at the left below shows a car's fuel economy  $e$  as a function of the speed  $s$  at which the car is driven. The graph at the right below shows the per-mile fuel cost  $c$  as a function of  $e$ . Fuel economy is measured in miles per gallon, speed in miles per hour, and fuel cost in cents per mile.

- If the car is driven at 55 mi/h, what is the fuel cost?
- If the fuel cost is to be kept at or below 4 cents per mile, at what speeds should the car be driven?



- Express the radius  $r$  of a circle as a function of the circumference  $C$ .
- Express the area  $A$  of the circle as a function of  $C$ .

- Express the area  $A$  and perimeter  $P$  of a semicircular region in terms of the radius  $r$ .



- Express  $A$  as a function of  $P$ .

- Physics** The speed  $s$  of sound in air is given by the formula  $s = 331 + 0.6C$  where  $s$  is measured in meters per second and  $C$  is the Celsius temperature. If  $C = \frac{5}{9}(F - 32)$ , express  $s$  as a function of  $F$ , the Fahrenheit temperature.

- The surface area and volume of a sphere are given in terms of the radius by the following formulas:  $A = 4\pi r^2$  and  $V = \frac{4}{3}\pi r^3$ .

- Express  $r$  as a function of  $A$ .
- Express  $V$  as a function of  $A$ .



In Exercises 33–36, find rules for  $(f \circ g)(x)$  and  $(g \circ f)(x)$  and give the domain of each composite function. You may wish to use a computer or graphing calculator to confirm your answer for the domain.

33.  $f(x) = 2x$ ,  $g(x) = \sqrt{16 - x^2}$

34.  $f(x) = \sqrt{x}$ ,  $g(x) = \frac{1}{x-4}$

35.  $f(x) = x^2$ ,  $g(x) = \sqrt{1 - x}$

36.  $f(x) = x^2$ ,  $g(x) = \sqrt{16 - x^2}$

37. If  $g(x) = \frac{x+3}{2}$ , find  $g(g(1))$ ,  $g(g(g(1)))$ , and  $g(g(g(g(1))))$ .

38. If  $f(x) = 2x - 1$ , show that  $f(f(x)) = 4x - 3$ . Find  $f(f(f(x)))$ .

- C 39. Physics** The luminous intensity  $I$ , measured in candela (cd), of a 100 watt light bulb is 130 cd. The law of illumination states that  $E = \frac{I}{d^2}$ , where  $E$  is the illumination and  $d$  is the distance in meters to the light bulb. Suppose you hold a book 1 m away from a 100 watt bulb and begin walking away from the bulb at a rate of 1 m/s.
- Express  $E$  in terms of the time  $t$  in seconds after you begin walking.
  - When will the illumination on the book be 1% of its original value?

## ■ Graphs and Inverses of Functions

### 4-3 Reflecting Graphs; Symmetry

#### Objective

To reflect graphs and to use symmetry to sketch graphs.

In this section and the next we will see how the graph of an equation is transformed when the equation is altered. This will allow us to graph a simple equation and—by reflecting it, stretching or shrinking it, or sliding it—to obtain the graph of a related, more complicated equation.

We begin by considering the *reflection* of a graph in a line. The *line of reflection* acts like a mirror and is located halfway between a point and its reflection. (If the point being reflected is on the line, then the point is its own reflection.)

#### Activities



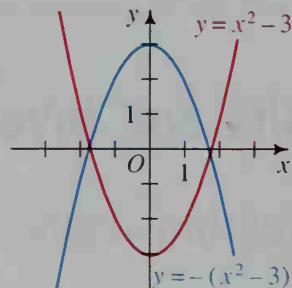
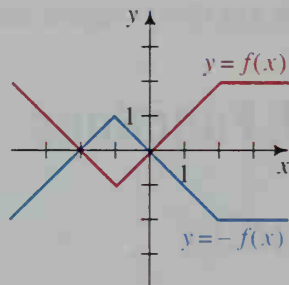
By completing the following activities, you will see how a change in an equation results in a reflection of its graph in some line. For parts (a) and (b) of each activity, graph each pair of equations on a single set of axes. You may wish to use a computer or graphing calculator to help you with your graphs.

- Graph  $y = x^2$  and  $y = -x^2$ .
  - Graph  $y = x^3 + 2x^2$  and  $y = -(x^3 + 2x^2)$ .
  - In general, how are the graphs of  $y = f(x)$  and  $y = -f(x)$  related?
- Graph  $y = x^2 - 1$  and  $y = |x^2 - 1|$ .
  - Graph  $y = x(x - 1)(x - 3)$  and  $y = |x(x - 1)(x - 3)|$ .
  - In general, how are the graphs of  $y = f(x)$  and  $y = |f(x)|$  related?
- Graph  $y = 2x - 1$  and  $y = 2(-x) - 1$ .
  - Graph  $y = \sqrt{x}$  and  $y = \sqrt{-x}$ .
  - In general, how are the graphs of  $y = f(x)$  and  $y = f(-x)$  related?
- Graph  $y = 2x + 1$  and  $x = 2y + 1$ .
  - Graph  $y = x^2$  and  $x = y^2$ .
  - In general, how is the graph of an equation affected when you interchange the variables in the equation?

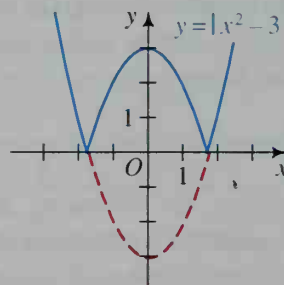
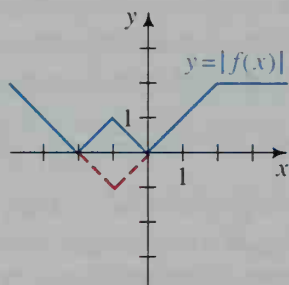
As the activities on the preceding page show, simple changes in an equation can produce reflections of its graph in the  $x$ -axis, the  $y$ -axis, and the line  $y = x$ .

## Reflection in the $x$ -axis

The graph of  $y = -f(x)$  is obtained by reflecting the graph of  $y = f(x)$  in the  $x$ -axis. In the graphs below, notice that each point  $(x, y)$  on the original (red) graph becomes the point  $(x, -y)$  on the reflected (blue) graph.

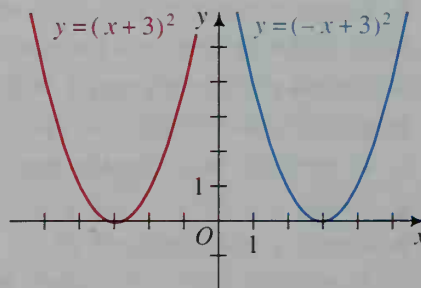
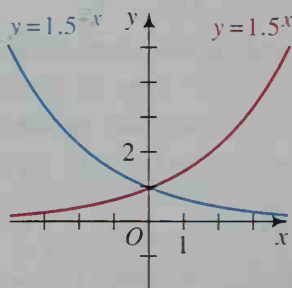


The graph of  $y = |f(x)|$  is identical to the graph of  $y = f(x)$  when  $f(x) \geq 0$  and is identical to the graph of  $y = -f(x)$  when  $f(x) < 0$ . This principle is applied to the graphs shown above to produce the following graphs. Notice that the graphs do not dip below the  $x$ -axis.



## Reflection in the $y$ -axis

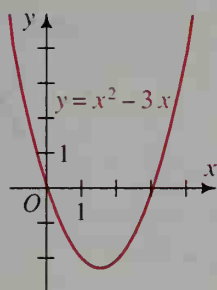
The graph of  $y = f(-x)$  is obtained by reflecting the graph of  $y = f(x)$  in the  $y$ -axis. In the graphs below, notice that each point  $(x, y)$  on the original (red) graph becomes the point  $(-x, y)$  on the reflected (blue) graph.



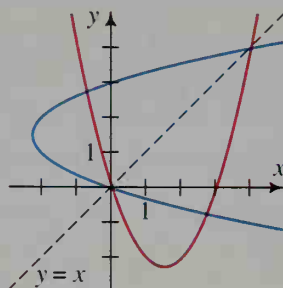


## Reflection in the Line $y = x$

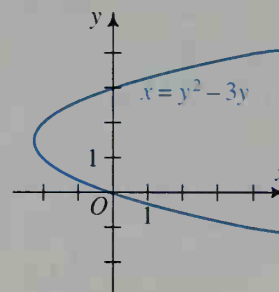
Reflecting the graph of an equation in the line  $y = x$  is equivalent to interchanging  $x$  and  $y$  in the equation.



Original graph  
and equation



Reflection in  $y = x$

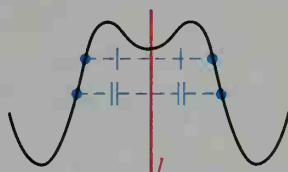


Reflected graph  
and altered equation

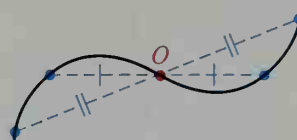
## Symmetry

A line  $l$  is called an **axis of symmetry** of a graph if it is possible to pair the points of the graph in such a way that  $l$  is the perpendicular bisector of the segment joining each pair. (See the figure at the left below.)

A point  $O$  is called a **point of symmetry** of a graph if it is possible to pair the points of the graph in such a way that  $O$  is the midpoint of the segment joining each pair. (See the figure at the right below.)



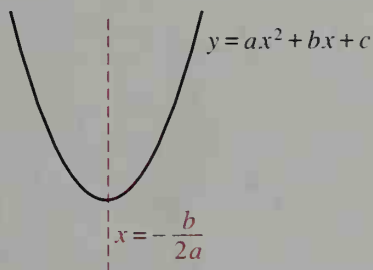
$l$  = axis of symmetry



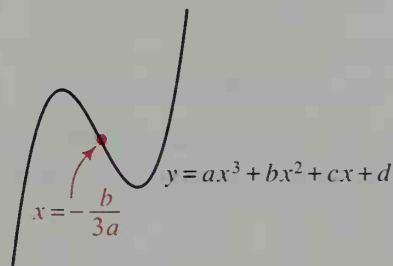
$O$  = point of symmetry

As you know, the graph of  $y = ax^2 + bx + c$  has an axis of symmetry with equation  $x = -\frac{b}{2a}$ . In Exercise 16 on page 145, we will show that the graph of

$y = ax^3 + bx^2 + cx + d$  has a point of symmetry at  $x = -\frac{b}{3a}$ .

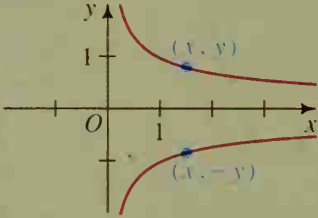
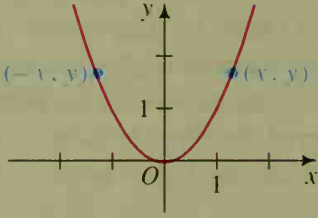
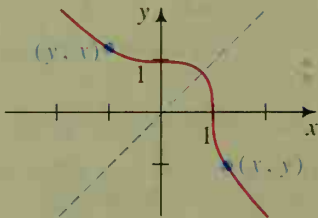
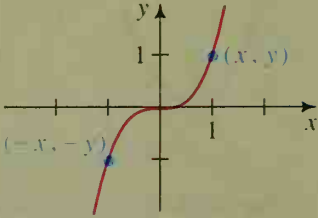


The graph of every quadratic  
function has a line of symmetry.



The graph of every cubic function  
has a point of symmetry.

## Special Tests for the Symmetry of a Graph

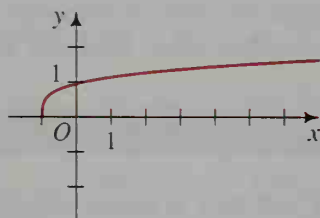
Type of Symmetry	Example
<p><b>Symmetry in the <math>x</math>-axis</b></p> <p><i>Meaning:</i>  <math>(x, -y)</math> is on the graph whenever <math>(x, y)</math> is.</p> <p><i>Testing an equation of a graph:</i>            In the equation, leave <math>x</math> alone and substitute <math>-y</math> for <math>y</math>. Does an equivalent equation result?</p>	<p> <math>y^2x = 1</math>  <math>(-y)^2x = 1</math> </p> <p style="text-align: right;">equivalent</p> 
<p><b>Symmetry in the <math>y</math>-axis</b></p> <p><i>Meaning:</i>  <math>(-x, y)</math> is on the graph whenever <math>(x, y)</math> is.</p> <p><i>Testing an equation of a graph:</i>            In the equation, substitute <math>-x</math> for <math>x</math> and leave <math>y</math> alone. Does an equivalent equation result?</p>	<p> <math>y = x^2</math>  <math>y = (-x)^2</math> </p> <p style="text-align: right;">equivalent</p> 
<p><b>Symmetry in the line <math>y = x</math></b></p> <p><i>Meaning:</i>  <math>(y, x)</math> is on the graph whenever <math>(x, y)</math> is.</p> <p><i>Testing an equation of a graph:</i>            In the equation, interchange <math>x</math> and <math>y</math>. Does an equivalent equation result?</p>	<p> <math>x^3 + y^3 = 1</math>  <math>y^3 + x^3 = 1</math> </p> <p style="text-align: right;">equivalent</p> 
<p><b>Symmetry in the origin</b></p> <p><i>Meaning:</i>  <math>(-x, -y)</math> is on the graph whenever <math>(x, y)</math> is.</p> <p><i>Testing an equation of a graph:</i>            In the equation, substitute <math>-x</math> for <math>x</math> and <math>-y</math> for <math>y</math>. Does an equivalent equation result?</p>	<p> <math>y = x^3</math>  <math>-y = (-x)^3</math> </p> <p style="text-align: right;">equivalent</p> 

**Example**

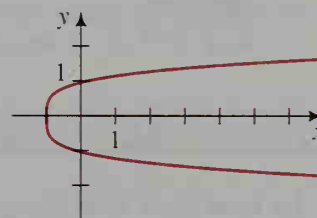
Use symmetry to sketch the graph of  $y^4 = x + 1$ .

**Solution**

The equation  $y^4 = x + 1$  can be rewritten as  $y = \pm\sqrt[4]{x+1}$ . You can see from either of these equations that the graph has symmetry in the  $x$ -axis. Therefore, you only need to graph  $y = \sqrt[4]{x+1}$  (by plotting a few points or by using a computer or graphing calculator). You can then reflect the graph in the  $x$ -axis to obtain the graph of  $y = -\sqrt[4]{x+1}$ . The two pieces together comprise the complete graph of  $y = \pm\sqrt[4]{x+1}$ .



Graph of  $y = \sqrt[4]{x+1}$

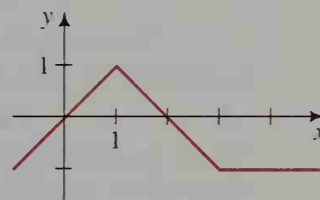


Graph of  $y = \pm\sqrt[4]{x+1}$   
or  $y^4 = x + 1$

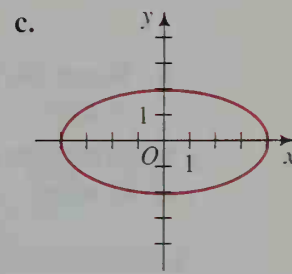
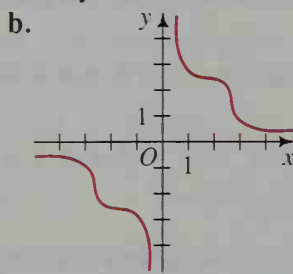
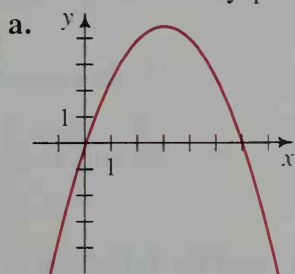
**CLASS EXERCISES**

1. The graph of  $y = f(x)$  is shown at the right. Sketch the graph of each of the following equations.

- $y = -f(x)$
- $y = |f(x)|$
- $y = f(-x)$



2. For each graph, give the equation(s) of any line(s) of symmetry and give the coordinates of any point of symmetry.



3. Tell whether the graph of each equation has symmetry in:

(i) the  $x$ -axis, (ii) the  $y$ -axis, (iii) the line  $y = x$ , and (iv) the origin.

a.  $x^4 + y^4 = 1$

b.  $xy^3 = 1$

c.  $x(x + y) = 1$

4. **Visual Thinking** If a graph has symmetry in both the  $x$ - and  $y$ -axes, what other symmetry must it have? Explain.

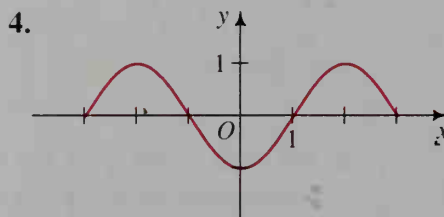
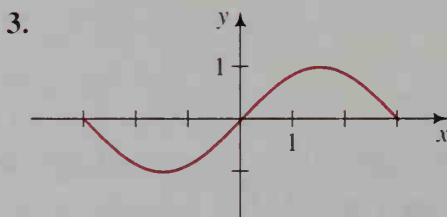
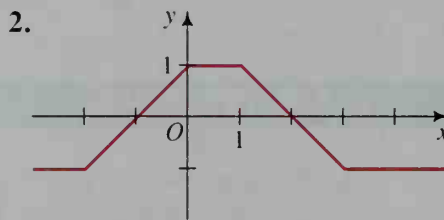
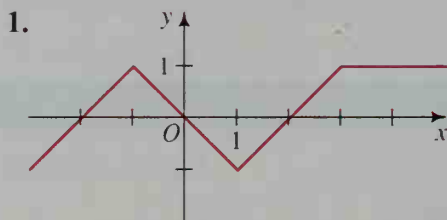
5. **Visual Thinking** Can a graph that has symmetry in the  $x$ -axis be the graph of a function? Explain.
6. **Visual Thinking** Describe how to obtain the graph of  $y = \sqrt{|x|}$  from the graph of  $y = \sqrt{x}$ .
7. Give the equation of the axis of symmetry for the graph of each quadratic function.
  - a.  $f(x) = x^2 - 8x - 7$
  - b.  $g(x) = 8x - 4x^2$
  - c.  $h(x) = x^2 + 3$
8. Give the coordinates of the point of symmetry for the graph of each cubic function.
  - a.  $f(x) = x^3 - 6x^2 + 5x + 7$
  - b.  $g(x) = 9x + 6x^2 + 2x^3$
  - c.  $h(x) = 3x^3 - 3x + 7$

## WRITTEN EXERCISES

In Exercises 1–4, the graph of  $y = f(x)$  is given. Sketch the graphs of:

- a.  $y = -f(x)$
- b.  $y = |f(x)|$
- c.  $y = f(-x)$

**A**



5. Sketch the graphs of  $y = x^2 - 9$ ,  $y = 9 - x^2$ , and  $y = |9 - x^2|$  on a single set of axes.
6. Sketch the graphs of  $y = |x| - 2$ ,  $y = 2 - |x|$ , and  $y = |2 - |x||$  on a single set of axes.

In Exercises 7–14, sketch the graph of each equation and the reflection of the graph in the line  $y = x$ . Then give an equation of the reflected graph.

7.  $y = 3x - 4$
8.  $y = \frac{1}{2}x + 1$
9.  $y = x^2 - 2x$
10.  $y = x^2 + 3x$
11.  $y = x^3$
12.  $y = \sqrt{x}$
13.  $y = |x| + 2$
14.  $y = |x| - 3$
15. Test each equation to see if its graph has symmetry in:
  - (i) the  $x$ -axis, (ii) the  $y$ -axis, (iii) the line  $y = x$ , and (iv) the origin.
  - a.  $y^2 - xy = 2$
  - b.  $x^2 + y^2 = 1$
  - c.  $y = x|x|$

16. Repeat Exercise 15 for each of the following equations.

a.  $x^2 + xy = 4$

b.  $|x| + |y| = 1$

c.  $y = \frac{x}{|x|}$

Use symmetry to sketch the graph of each equation.

**B** 17.  $|x| + |y| = 2$

18.  $|x|^{1/2} + |y|^{1/2} = 2$

19.  $x^2y = 1$

20.  $|xy| = 12$

In Exercises 21–26, graph each parabola, showing the vertex with its coordinates and the axis of symmetry with its equation. The pairs of graphs in Exercises 21–23 should be done on a single set of axes.

21. a.  $y = (x - 3)^2 + 5$

22. a.  $y = 2(x + 1)^2 + 3$

23. a.  $y = 3 - (x - 4)^2$

b.  $x = (y - 3)^2 + 5$

b.  $x = 2(y + 1)^2 + 3$

b.  $x = 3 - (y - 4)^2$

24.  $x = 2(y + 1)^2 + 4$

25.  $x = y^2 + 6y + 8$

26.  $x = y^2 + 2y - 3$

27. The graph of a cubic function has a local minimum at  $(5, -3)$  and a point of symmetry at  $(0, 4)$ . At what point does a local maximum occur?

28. a. Find the point of symmetry of the graph of the cubic function  $f(x) = -x^3 + 15x^2 - 48x + 45$ .

b. The function has a local minimum at  $(2, 1)$ . At what point does a local maximum occur?

29. a. Graph  $y = 3x^2 - x^3$ . At what point does a local minimum occur?

b. Find the point of symmetry of the graph and then deduce the coordinates of the point where a local maximum occurs.

30. a. Graph  $y = -x^3 - 6x^2 - 9x$ . At what point does a local minimum occur?

b. Find the point of symmetry and then deduce the coordinates of the point where a local maximum occurs.

Use the following definitions to complete Exercises 31–36.

$f$  is an **even** function if  $f(-x) = f(x)$ .

$f$  is an **odd** function if  $f(-x) = -f(x)$ .

31. Classify each function as even, odd, or neither.

a.  $f(x) = x^2$

b.  $f(x) = x^3$

c.  $f(x) = x^2 - x$

d.  $f(x) = x^4 + 2x^2$

e.  $f(x) = x^3 + 3x^2$

f.  $f(x) = x^5 - 4x^3$

32. Use the results of Exercise 31 to guess the reasons for using the terms “even” and “odd” as they are applied to polynomial functions.

33. a. What kind of symmetry does the graph of an even function have?

b. What kind of symmetry does the graph of an odd function have?

34. Study the graphs shown in Exercises 3 and 4. Then tell whether each function graphed is even or odd.

35. If  $f$  and  $g$  are both odd functions and  $h(x) = f(x) \cdot g(x)$ , prove that  $h$  is even.

36. If  $f$  is an even function and  $g$  is an odd function, prove that  $h(x) = f(x) \cdot g(x)$  is odd.





Use the following definition to complete Exercises 37–39. For Exercise 38, you will need to use a computer or graphing calculator.

Suppose  $x_1$  and  $x_2$  are any two domain elements of a function  $f$ . We say that  $f$  is **increasing** in its domain if  $f(x_2) > f(x_1)$  whenever  $x_2 > x_1$ , and  $f$  is **decreasing** in its domain if  $f(x_2) < f(x_1)$  whenever  $x_2 > x_1$ .

37. a. For what value(s) of  $m$  is the linear function  $f(x) = mx$  an increasing function? a decreasing function? neither?

b. **Visual Thinking** Describe what the definitions of increasing and decreasing functions imply about the graphs of the functions.

38. Graph each function using a computer or a graphing calculator. Then use the graph to tell whether the function is increasing or decreasing in its domain.

a.  $f(x) = x^3 + x - 1$       b.  $f(x) = \sqrt[3]{1 - x^3}$       c.  $f(x) = \frac{10}{1 + 2^x}$

39. **Visual Thinking** Suppose the graph of an increasing function is reflected in (a) the  $x$ -axis, (b) the  $y$ -axis, and (c) the line  $y = x$ . In each case, tell whether the reflected graph represents an increasing or a decreasing function.

40. On one set of axes, graph  $y = |x - 2|$ ,  $y = |x - 2| - 2$ , and  $y = ||x - 2| - 2|$ .

**C** 41. Use symmetry to sketch the graph of  $x^{2/3} + y^{2/3} = 1$ .

## 4-4 Periodic Functions; Stretching and Translating Graphs

### Objective

To determine periodicity and amplitude from graphs, to stretch and shrink graphs both vertically and horizontally, and to translate graphs.

### Periodic Functions

The world is full of periodic phenomena. The tides come in and go out again and again, each *cycle* or *period* lasting about 12.4 h. The amount of daylight increases and decreases with a period of one year. The functions that describe periodic behavior are called *periodic* functions.

A function  $f$  is **periodic** if there is a positive number  $p$ , called a **period** of  $f$ , such that

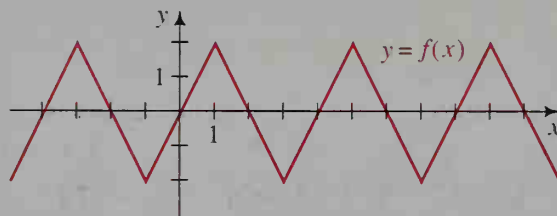
$$f(x + p) = f(x)$$

for all  $x$  in the domain of  $f$ . The smallest period of a periodic function is called the **fundamental period** of the function.



The definition of a periodic function implies that if  $f$  is a periodic function with period  $p$ , then  $f(x) = f(x + mp)$  for all  $x$  and any integer  $m$ .

**Example 1** The graph of a periodic function  $f$  is shown below. Find:  
**a.** the fundamental period of  $f$       **b.**  $f(99)$



**Solution**

- a.** If we start at the origin and follow the graph to the right, the graph takes 4 units to complete one up-and-down cycle; another such cycle then begins. Thus,  $f$  is a periodic function with fundamental period 4.  
**b.** Since the fundamental period of  $f$  is 4 and  $99 \div 4 = 24$  with remainder 3, we have:

$$f(99) = f(99 - 24 \cdot 4) = f(3) = -2$$

If a periodic function has a maximum value  $M$  and a minimum value  $m$ , then the **amplitude**  $A$  of the function is given by:

$$A = \frac{M - m}{2}$$

**Example 2** Find the amplitude of the function  $f$  described in Example 1.

**Solution**

The function's maximum value is  $M = 2$ ; its minimum value is  $m = -2$ . The amplitude is half the difference between  $M$  and  $m$ :

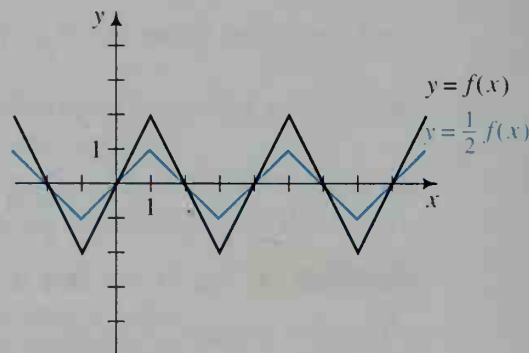
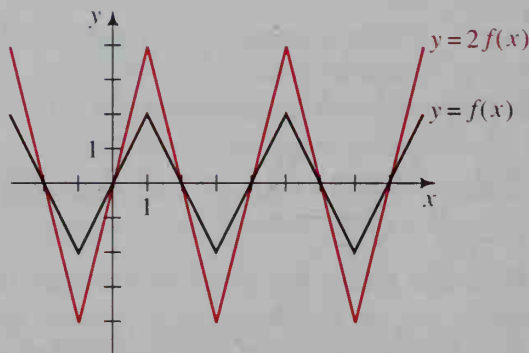
$$A = \frac{2 - (-2)}{2} = 2$$

## Stretching and Shrinking Graphs

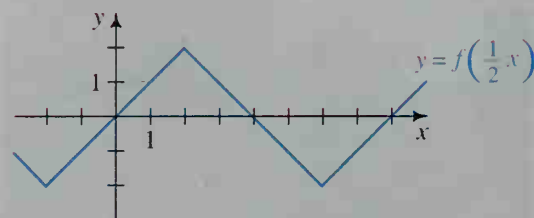
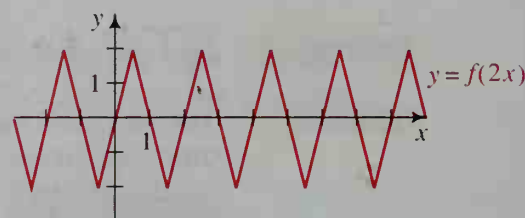
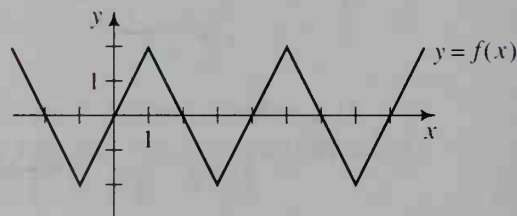
If you go into a house of mirrors at a circus or carnival, you may see your reflection distorted by some of the mirrors. For example, one mirror might make you look tall and thin, while another might make you look short and broad. Just as mirrors can stretch or shrink your reflection both vertically and horizontally, it is possible to stretch or shrink the graph of an equation both vertically and horizontally.



The graph of  $y = cf(x)$  where  $c$  is positive (and not equal to 1) is obtained by *vertically* stretching or shrinking the graph of  $y = f(x)$ . For example, in the graphs below, notice that points on the  $x$ -axis remain fixed, while all other points move away from the  $x$ -axis for  $c > 1$  (a vertical stretch) or toward the  $x$ -axis for  $0 < c < 1$  (a vertical shrink).



The graph of  $y = f(cx)$  where  $c$  is positive (and not equal to 1) is obtained by *horizontally* stretching or shrinking the graph of  $y = f(x)$ . For example, in the graphs at the right, notice that points on the  $y$ -axis remain fixed, while all other points move toward the  $y$ -axis for  $c > 1$  (a horizontal shrink) or away from the  $y$ -axis for  $0 < c < 1$  (a horizontal stretch).



The graphs shown above are all based on a periodic function  $f$  with fundamental period 4 and amplitude 2. Notice that a vertical stretching or shrinking of the graph of  $f$  affects the amplitude but not the period, and a horizontal stretching or shrinking of the graph affects the period but not the amplitude. A more formal statement of these results is given at the top of the next page.

## Changing the Period and Amplitude of a Periodic Function

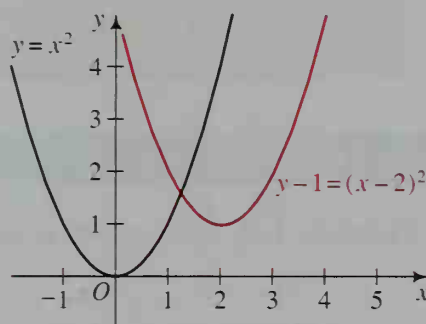
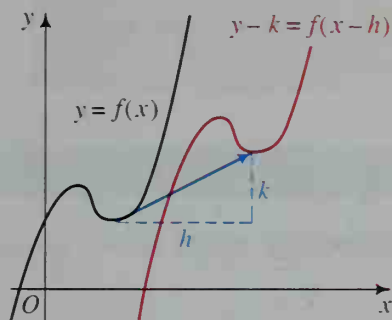
If a periodic function  $f$  has period  $p$  and amplitude  $A$ , then:

$y = cf(x)$  has period  $p$  and amplitude  $cA$ , and

$y = f(cx)$  has period  $\frac{p}{c}$  and amplitude  $A$ .

## Translating Graphs

The graph of  $y - k = f(x - h)$  is obtained by translating the graph of  $y = f(x)$  horizontally  $h$  units and vertically  $k$  units. For example, as shown at the right below, the graph of  $y - 1 = (x - 2)^2$  is the graph of  $y = x^2$  translated 2 units horizontally and 1 unit vertically.



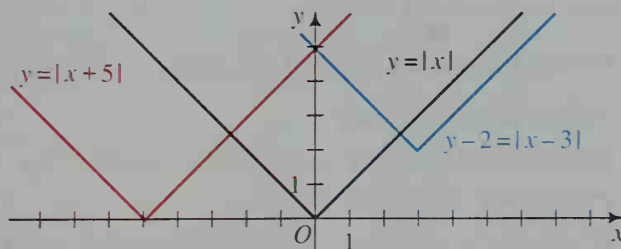
### Example 3

Sketch the graphs of the following equations:

$$y = |x|, \quad y - 2 = |x - 3|, \quad \text{and} \quad y = |x + 5|$$

### Solution

Once we have graphed  $y = |x|$ , we can slide the graph 3 units horizontally and 2 units vertically to obtain the graph of  $y - 2 = |x - 3|$ . Since the equation  $y = |x + 5|$  is equivalent to  $y = |x - (-5)|$ , we slide the original graph  $-5$  units horizontally and 0 units vertically to obtain the graph of  $y = |x + 5|$ .

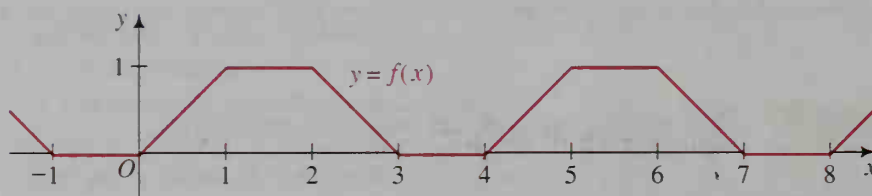


In this chapter we have seen how certain simple changes in the equation of a curve can stretch or shrink the curve, reflect it, or translate it. These results are summarized at the top of the next page.

If the equation $y = f(x)$ is changed to:	Then the graph of $y = f(x)$ is:
$y = -f(x)$	reflected in the $x$ -axis.
$y =  f(x) $	unchanged when $f(x) \geq 0$ and reflected in the $x$ -axis when $f(x) < 0$ .
$y = f(-x)$	reflected in the $y$ -axis.
$x = f(y)$	reflected in the line $y = x$ .
$y = cf(x), c > 1$	stretched vertically.
$y = cf(x), 0 < c < 1$	shrunk vertically.
$y = f(cx), c > 1$	shrunk horizontally.
$y = f(cx), 0 < c < 1$	stretched horizontally.
$y - k = f(x - h)$	translated $h$ units horizontally and $k$ units vertically.

## CLASS EXERCISES

For Exercises 1–4, refer to the graph of a function  $y = f(x)$  shown below.



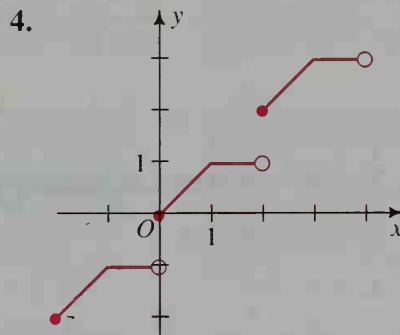
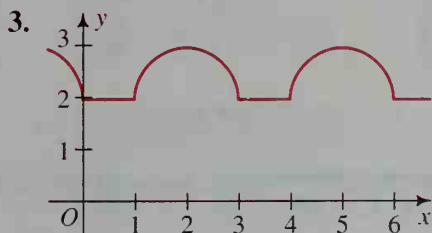
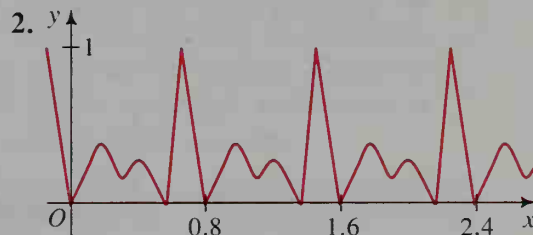
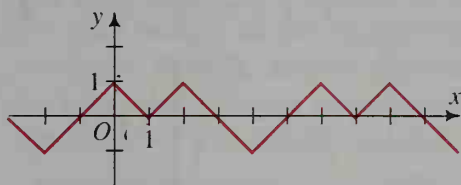
- What is the fundamental period of  $f$ ?
  - What is the amplitude?
  - Find  $f(25)$  and  $f(-25)$ .
- Sketch the graphs of  $y = 2f(x)$  and  $y = \frac{1}{2}f(x)$ . Give the fundamental period and amplitude of each.
- Sketch the graphs of  $y = f(2x)$  and  $y = f\left(\frac{1}{2}x\right)$ . Give the fundamental period and amplitude of each.
- Sketch the graphs of  $y = f(x) - 2$  and  $y = f(x - 2)$ . Give the fundamental period and amplitude of each.
- If a graph is vertically stretched or shrunk, explain why points on the  $x$ -axis remain fixed.
- If a graph is horizontally stretched or shrunk, explain why points on the  $y$ -axis remain fixed.



# WRITTEN EXERCISES

In Exercises 1–4, the graph of a function  $f$  is given. Tell whether  $f$  appears to be periodic. If so, give its fundamental period and its amplitude, and then find  $f(1000)$  and  $f(-1000)$ .

**A** 1.



5. Use the graph of  $y = f(x)$ , shown at the right, to sketch the graph of each of the following.

a.  $y = 2f(x)$

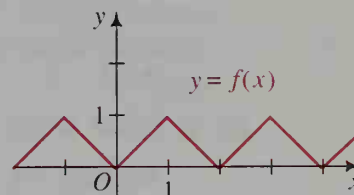
b.  $y = -\frac{1}{2}f(x)$

c.  $y = f(-2x)$

d.  $y = f\left(\frac{1}{2}x\right)$

e.  $y = f\left(x - \frac{1}{2}\right)$

f.  $y = f(-x) + 1$



6. The greatest integer function  $y = \lfloor x \rfloor$  gives the greatest integer less than or equal to  $x$ . Thus,  $\lfloor 2.1 \rfloor = 2$  and  $\lfloor -3.1 \rfloor = -4$ . Use the graph of this function, shown at the right, to sketch the graph of each of the following.

a.  $y = \frac{1}{2}\lfloor x \rfloor$

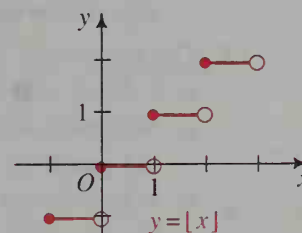
b.  $y = -2\lfloor x \rfloor$

c.  $y = \left\lfloor -\frac{1}{2}x \right\rfloor$

d.  $y = \lfloor 2x \rfloor$

e.  $y = \lfloor x - 1 \rfloor$

f.  $y = 2\lfloor x \rfloor + 1$



7. Sketch the graph of each of the following.

a.  $y + 2 = |x|$

b.  $y = |x - 3|$

d.  $y = 2|x + 1|$

e.  $y + 1 = -|x|$

c.  $y - 4 = |x + 5|$

f.  $y - 3 = |2x|$

8. Sketch the graph of each of the following.

a.  $y - 1 = \sqrt{x}$

b.  $y = \sqrt{x + 4}$

c.  $y + 2 = \sqrt{x - 5}$

d.  $y = 2\sqrt{x - 3}$

e.  $y - 2 = \sqrt{-x}$

f.  $y - 4 = \sqrt{4x}$

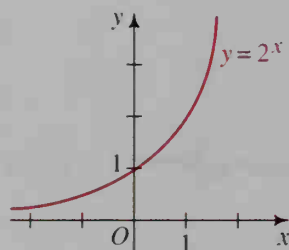
**B** 9. Use the graph of  $y = 2^x$ , shown at the left below, to sketch the graph of each of the following.

a.  $y = 2^{-x}$

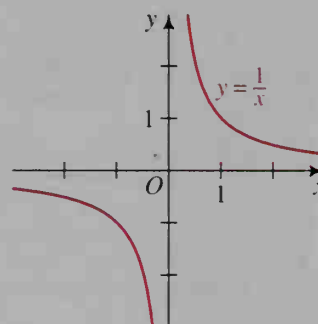
b.  $y = 2^{x-1}$

c.  $y = 3 - 2^x$

d.  $x = 2^y$



Ex. 9



Ex. 10

10. Use the graph of  $y = \frac{1}{x}$ , shown at the right above, to sketch the graph of each of the following.

a.  $y = -\frac{1}{x}$

b.  $y = \frac{1}{x-2}$

c.  $y = 1 + \frac{1}{x}$

d.  $x = \frac{1}{y}$

11. Given that the equation of a circle with radius 3 and center  $(0, 0)$  is  $x^2 + y^2 = 9$  (graph at right), deduce the equation of the circle if it is translated so that its center is  $(8, 4)$ .

12. a. Refer to the circle with equation  $x^2 + y^2 = 9$  in Exercise 11. Sketch the graph of  $\left(\frac{x}{2}\right)^2 + y^2 = 9$ .

b. In Exercise 11, the area of the circular region is  $\pi r^2 = \pi \cdot 3^2 = 9\pi$ . Make a conjecture about the area of the region enclosed by the graph in part (a).

13. The graph of  $y = f(x)$ , shown at the right, has  $x$ -intercepts at 0 and 6 and a local maximum at  $(4, 32)$ .

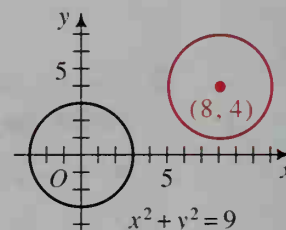
a. Where do the  $x$ -intercepts and local maximum occur on the graph of  $y = f(2x)$ ?

b. Where do the  $x$ -intercepts and local maximum occur on the graph of  $y = 2f(x)$ ?

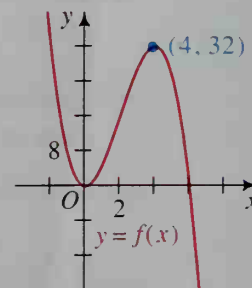
c. Where do the  $x$ -intercepts and local maximum occur on the graph of  $y = f(x - 2)$ ?

d. Where do the  $x$ -intercepts and local maximum occur on the graph of  $y = f(x + 2)$ ?

e. If  $f$  is a cubic polynomial, find a rule for  $f(x)$ .



Exs. 11, 12

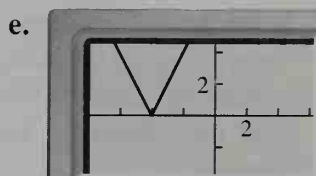
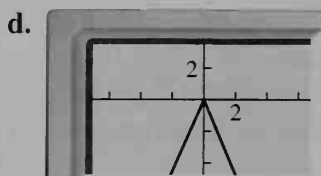
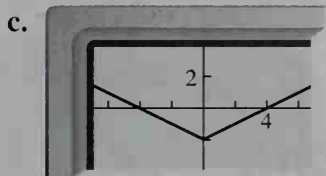
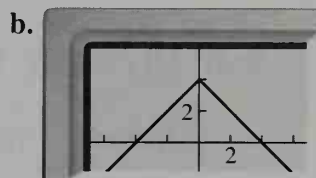
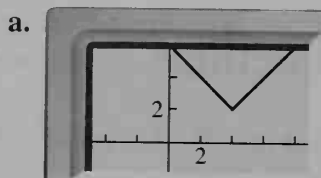
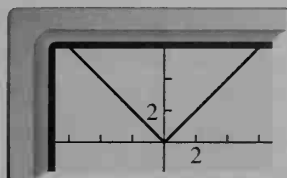


14. a. Sketch the graph of  $y = x^3 - 3x^2 + 2x$  and label all intercepts.  
 b. Sketch the graph of  $y = \left(\frac{1}{2}x\right)^3 - 3\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right)$  by using the graph of part (a). Label all intercepts.



The first figure in Exercise 15 shows the calculator-drawn graph of  $y = |x|$ . Changes in the equation  $y = |x|$  cause its graph to be reflected, stretched, shrunk, and/or translated to produce the graphs shown in parts (a)–(e). Give an equation for each graph. You may wish to use a computer or a graphing calculator to confirm your answers.

15.



16. In this exercise, you will show that the graph of

$$y = ax^3 + bx^2 + cx + d$$

has a point of symmetry at  $x = -\frac{b}{3a}$ .

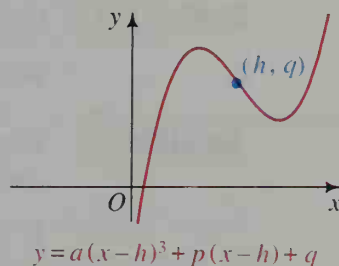
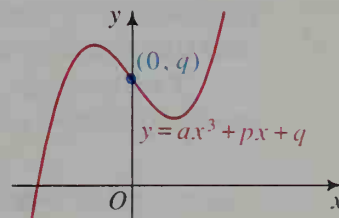
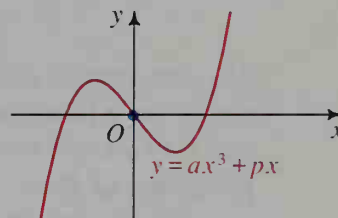
- If  $y = f(x) = ax^3 + px$ , show that  $f(-x) = -f(x)$ .
- Explain how part (a) shows that the origin is a symmetry point of the graph of  $y = ax^3 + px$ .
- Explain why  $(0, q)$  is a symmetry point for the graph of  $y = ax^3 + px + q$ .
- Explain why  $(h, q)$  is a symmetry point for the graph of  $y = a(x - h)^3 + p(x - h) + q$ .
- Suppose the equation

$$y = ax^3 + bx^2 + cx + d$$

is rewritten in the equivalent form

$$y = a(x - h)^3 + p(x - h) + q.$$

By comparing the coefficients of the  $x^2$  terms, show that  $h = -\frac{b}{3a}$ . Then use the results of part (d) to conclude that  $y = ax^3 + bx^2 + cx + d$  has a point of symmetry at  $x = -\frac{b}{3a}$ .



17. If the periodic functions  $f$  and  $g$  both have fundamental period  $p$  and  $h(x) = (f + g)(x)$ , show that  $h(x + p) = h(x)$  for all  $x$  in the domain of  $h$ .
18. If  $f$  has fundamental period 2 and  $g$  has fundamental period 3, what is the fundamental period of  $f + g$ ?
- C** 19. Is a constant function like  $f(x) = 5$  periodic? If so, does it have a fundamental period? Explain.
20. A function  $f$  is defined for all real numbers as follows:  $f(x) = 1$  if  $x$  is rational, and  $f(x) = 0$  if  $x$  is irrational. Show that if  $p$  is a rational number, then  $f(x + p) = f(x)$  for all  $x$ . Does  $f$  have a fundamental period? Explain.

## 4-5 Inverse Functions

**Objective** To find the inverse of a function, if the inverse exists.

Conversion formulas, such as those used in converting between U.S. customary and metric measurements, come in pairs. For example, consider the conversion formulas for the Fahrenheit and Celsius temperature scales:

$$F = \frac{9}{5}C + 32 \text{ and } C = \frac{5}{9}(F - 32)$$

The first formula gives a Fahrenheit temperature  $F$  as a function of a Celsius temperature  $C$ , while the second formula gives  $C$  as a function of  $F$ .

Notice that in the first formula,  $F = 32$  when  $C = 0$ , and in the second formula,  $C = 0$  when  $F = 32$ . Because each formula undoes what the other one does, the formulas are examples of *inverses*.

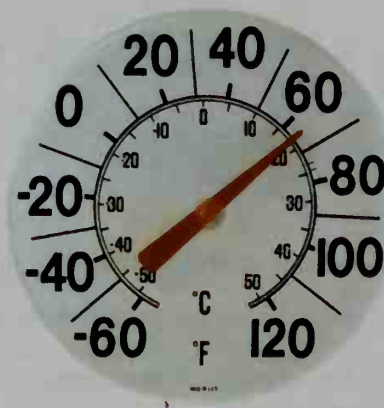
Two functions  $f$  and  $g$  are called **inverse functions** if the following statements are true:

1.  $g(f(x)) = x$  for all  $x$  in the domain of  $f$ ;
2.  $f(g(x)) = x$  for all  $x$  in the domain of  $g$ .

**Example 1** If  $f(x) = \frac{x-1}{2}$  and  $g(x) = 2x + 1$ , show that  $f$  and  $g$  are inverses of each other.

**Solution** We must check that conditions (1) and (2) stated in the definition above are satisfied:

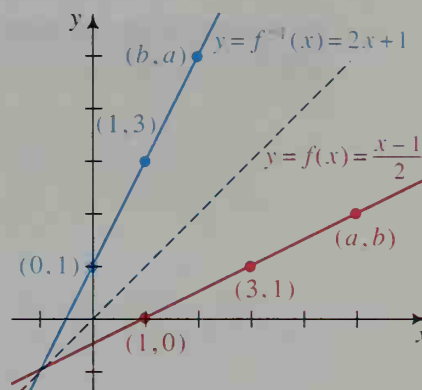
1.  $g(f(x)) = g\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$  for all  $x$
2.  $f(g(x)) = f(2x + 1) = \frac{(2x + 1) - 1}{2} = x$  for all  $x$



We denote the inverse of a function  $f$  by  $f^{-1}$ , read “ $f$  inverse.” The symbol  $f^{-1}(x)$ , read “ $f$  inverse of  $x$ ,” is the value of  $f^{-1}$  at  $x$ . Note that  $f^{-1}(x)$  does *not* mean  $\frac{1}{f(x)}$ .

The graphs of  $f$  and  $f^{-1}$  in Example 1 are shown at the right. Notice that for every point  $(a, b)$  on the graph of  $y = f(x)$ , the point  $(b, a)$  is on the graph of  $y = f^{-1}(x)$ . This means that the graph of  $f^{-1}$  is the reflection of the graph of  $f$  in the line  $y = x$ .

When a function  $f$  has an inverse, the graph of  $f^{-1}$  can be obtained from the graph of  $f$  by changing every point  $(x, y)$  on the graph to the point  $(y, x)$ . Similarly, the rule for  $f^{-1}(x)$  can be obtained by interchanging  $x$  and  $y$  in the equation  $y = f(x)$ , as shown in the following example.



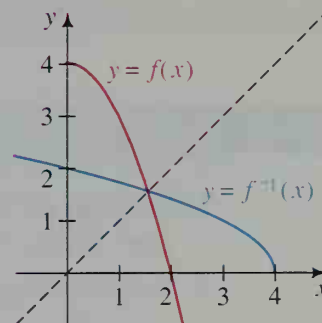
### Example 2

Let  $f(x) = 4 - x^2$  for  $x \geq 0$ .

- Sketch the graph of  $y = f^{-1}(x)$ .
- Find a rule for  $f^{-1}(x)$ .

### Solution

- Since the domain of  $f(x) = 4 - x^2$  is  $x \geq 0$ , the graph of  $y = f(x)$  is half of a parabola, as shown in red at the right. By reflecting the graph of  $y = f(x)$  in the line  $y = x$ , we obtain the graph of  $y = f^{-1}(x)$ , as shown in blue at the right.



1. Set  $y = f(x)$ .  $y = 4 - x^2, x \geq 0$  ← describes  $y = f(x)$   
 $\downarrow \quad \downarrow \quad \downarrow$   
 2. Interchange  $x = 4 - y^2, y \geq 0$  ← describes  $y = f^{-1}(x)$   
 $x$  and  $y$ .  
 3. Solve for  $y$ .  $y = \pm\sqrt{4 - x}, y \geq 0$   
 $y = \sqrt{4 - x}$

We therefore have the rule  $f^{-1}(x) = \sqrt{4 - x}$ . The graph shows that the domain of  $f^{-1}$  is  $\{x \mid x \leq 4\}$ .

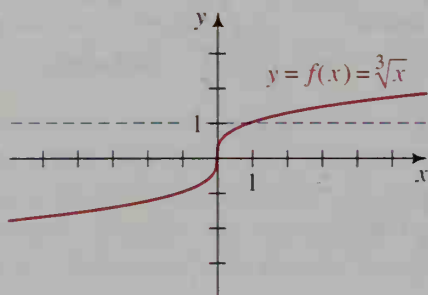
Note that in Example 2, if the domain of  $f(x) = 4 - x^2$  were the set of all real numbers instead of nonnegative real numbers, the reflection of the graph of  $y = f(x)$  in the line  $y = x$  would not be the graph of a function. Thus, not all functions have inverses.



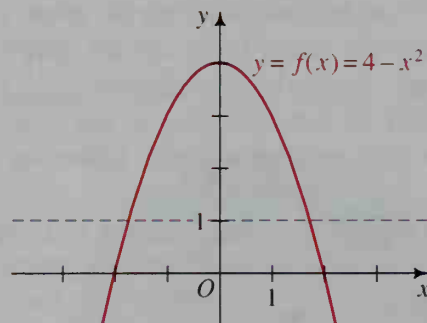
A function  $y = f(x)$  that has an inverse is called a **one-to-one function**, because not only does each  $x$ -value correspond to exactly one  $y$ -value, but also each  $y$ -value corresponds to exactly one  $x$ -value. We can determine whether a function is one-to-one by applying the *horizontal-line test* to its graph.

### The Horizontal-Line Test

If the graph of the function  $y = f(x)$  is such that no horizontal line intersects the graph in more than one point, then  $f$  is one-to-one and has an inverse.



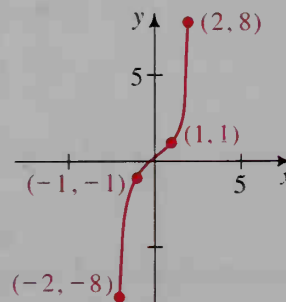
$f$  is one-to-one  
and has an inverse.



$f$  is not one-to-one and has no inverse.  
(Compare with  $f$  in Example 2.)

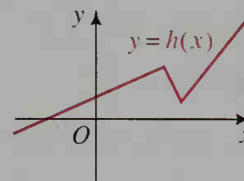
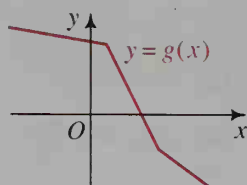
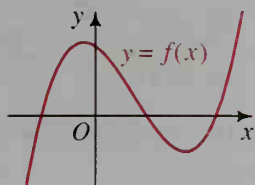
### CLASS EXERCISES

- Consider the birthday function described on page 119. Is this function one-to-one? Does it have an inverse?
- Suppose a function  $f$  has an inverse. If  $f(2) = 3$ , find each of the following.
  - $f^{-1}(3)$
  - $f(f^{-1}(3))$
  - $f^{-1}(f(2))$
- Find a rule for  $g^{-1}(x)$  if:
  - $g(x) = 4x$
  - $g(x) = 3x + 2$
  - $g(x) = 2x - 1$
  - $g(x) = 4 - 5x$
- The graph of  $f(x) = x^3$  is shown at the right.
  - Name several points on the graph of  $y = f^{-1}(x)$  and then sketch the graph.
  - Find a rule for  $f^{-1}(x)$ .
- Is the function  $f(x) = x^3 - 1$  a one-to-one function? Explain.
  - Is the function  $g(x) = x^3 - x$  a one-to-one function? Explain.
- Consider the two temperature conversion formulas given on page 146. Explain how one formula can be obtained from the other.



Ex. 4

7. The graphs of  $f$ ,  $g$ , and  $h$  are shown below. Which functions are one-to-one? Which functions have inverses?



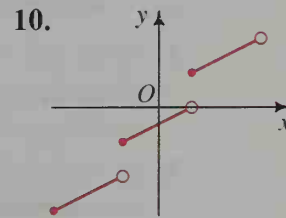
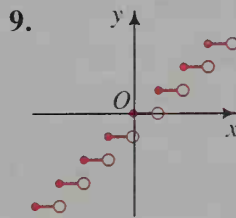
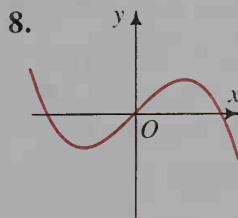
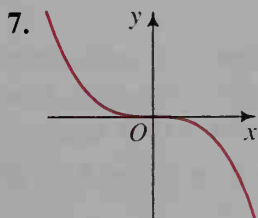
8. Which of the following functions have inverses?
- $f(x) = |x|$
  - $f(x) = x^3$
  - $f(x) = x^4$
  - $f(x) = x^4, x \leq 0$
9. On the dial or the buttons of a telephone, a telephone function  $T$  pairs letters of the alphabet with the digits 2–9. For example,  $T(A) = 2$  and  $T(D) = 3$ . Does  $T$  have an inverse? Explain.
10. Explain how the vertical-line test (given on page 121) can be used to justify the horizontal-line test (given on page 148).
11. Explain why the domain of a one-to-one function  $f$  is the range of  $f^{-1}$  and why the domain of  $f^{-1}$  is the range of  $f$ .

## WRITTEN EXERCISES

**A**

- Suppose a function  $f$  has an inverse. If  $f(2) = 6$  and  $f(3) = 7$ , find:
  - $f^{-1}(6)$
  - $f^{-1}(f(3))$
  - $f(f^{-1}(7))$
- Suppose a function  $f$  has an inverse. If  $f(0) = -1$  and  $f(-1) = 2$ , find:
  - $f^{-1}(-1)$
  - $f^{-1}(f(0))$
  - $f(f^{-1}(2))$
- If  $g(3) = 5$  and  $g(-1) = 5$ , explain why  $g$  has no inverse.
- Explain why  $f(x) = x^3 + x^2$  has no inverse.
- Let  $h(x) = 4x - 3$ .
  - Sketch the graphs of  $h$  and  $h^{-1}$ .
  - Find a rule for  $h^{-1}(x)$ .
- Let  $L(x) = \frac{1}{2}x - 4$ .
  - Sketch the graphs of  $L$  and  $L^{-1}$ .
  - Find a rule for  $L^{-1}(x)$ .

In Exercises 7–10, the graph of a function is given. State whether the function has an inverse.



State whether the function  $f$  has an inverse. If  $f^{-1}$  exists, find a rule for  $f^{-1}(x)$  and show that  $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ .

11.  $f(x) = 3x - 5$

12.  $f(x) = |x| - 2$

13.  $f(x) = \sqrt[4]{x}$

14.  $f(x) = \frac{1}{x}$

15.  $f(x) = \frac{1}{x^2}$

16.  $f(x) = \sqrt{5 - x}$

17.  $f(x) = \sqrt{4 - x^2}$

18.  $f(x) = \sqrt{5 - x^2}$

19.  $f(x) = \sqrt[3]{1 + x^3}$

Sketch the graphs of  $g$  and  $g^{-1}$ . Then find a rule for  $g^{-1}(x)$ .

**B** 20.  $g(x) = x^2 + 2, x \geq 0$

21.  $g(x) = 9 - x^2, x \leq 0$

22.  $g(x) = (x - 1)^2 + 1, x \leq 1$

23.  $g(x) = (x - 4)^2 - 1, x \geq 4$



In Exercises 24–26, show that  $h^{-1}(x) = h(x)$ . Then sketch the graph of  $h$ . You may wish to use a computer or graphing calculator.

24.  $h(x) = \sqrt[3]{1 - x^3}$

25.  $h(x) = \frac{x}{x - 1}$

26.  $h(x) = \sqrt{1 - x^2}, x \geq 0$

27. a. Using the results of Exercises 24–26, state how the graph of  $h$  is related to the line  $y = x$  when  $h^{-1}(x) = h(x)$ .

b. Find a function  $h$ , different from those in Exercises 24–26, such that  $h^{-1}(x) = h(x)$ .

28. Refer to the definition of an increasing function given on page 138.

a. Explain why an increasing function must have an inverse.

b. Suppose  $f$  is an increasing function. Is  $f^{-1}$  also an increasing function? Explain your answer and support it with at least two examples.

**C** 29. Which statement below is true? Prove it.

(i)  $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$

(ii)  $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

30. If  $f$  is a linear function such that  $f(x + 2) - f(x) = 6$ , find the value of  $f^{-1}(x + 2) - f^{-1}(x)$ .

31. Suppose  $a$ ,  $b$ , and  $c$  are constants such that  $a \neq 0$ . Let  $P(x) = ax^2 + bx + c$  for  $x \leq -\frac{b}{2a}$ . Find a rule for  $P^{-1}(x)$ .

## IIII CALCULATOR EXERCISES

In the next chapter we will define the functions  $f(x) = e^x$  and  $g(x) = \ln x$ . You can use a calculator to learn something about these functions.

- Enter any number. Press the  $e^x$  and  $\ln x$  keys alternately several times. What do you notice? Repeat this process for several other numbers. How would you describe the relationship between  $f(x) = e^x$  and  $g(x) = \ln x$ ?
- By entering various numbers, determine whether  $f(x) = e^x$  is defined for all real numbers.
- By experimenting, determine the domain of  $g(x) = \ln x$ .

# Applications of Functions

## 4-6 Functions of Two Variables

### Objective

To graph functions of two variables in a two-dimensional coordinate system and to read such graphs.

The formula for the perimeter of a rectangle,  $P = 2l + 2w$ , tells us that the perimeter  $P$  depends on both the length  $l$  and the width  $w$  of the rectangle. In other words,  $P$  is a function of two variables,  $l$  and  $w$ , and we write

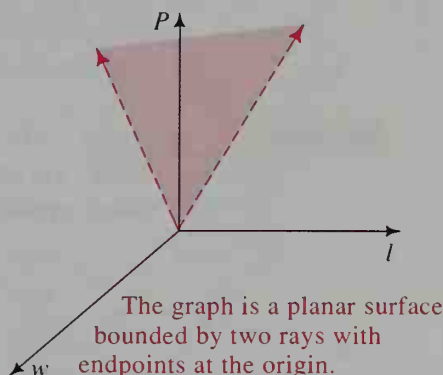
$$P(l, w) = 2l + 2w$$

to emphasize the functional relationship.

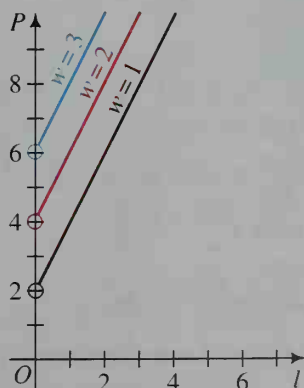
Graphing a function of two variables requires a three-dimensional coordinate system as shown at the right. Notice that there are three axes, each perpendicular to the other two. We will study such coordinate systems in Chapter 12, but for now let us consider two ways to describe  $P$  in a two-dimensional coordinate system.

One way is to draw curves for various constant values of either  $w$  or  $l$ . For example, by using constant values of  $w$ , we get such equations as  $P(l, 1) = 2l + 2$ ,  $P(l, 2) = 2l + 4$ , and  $P(l, 3) = 2l + 6$ . The graphs of these equations, called curves of constant width, are shown in the  $lP$ -plane at the left below.

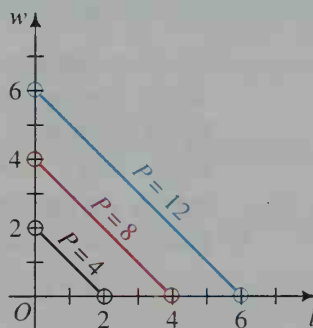
Another way to describe  $P$  in a two-dimensional plane is to draw curves along which the value of the perimeter function is constant. For example,  $P(l, w) = 8$  when  $(l, w) = (1, 3)$ ,  $(2, 2)$ , and  $(3, 1)$ . These points and others for which  $P(l, w) = 8$  are shown on the red curve in the  $lw$ -plane at the right below. Other curves of constant perimeter are also shown.



Graph of  $P(l, w) = 2l + 2w$   
(where  $l > 0$  and  $w > 0$ )  
in a 3-dimensional coordinate system



Curves of constant width



Curves of constant perimeter

## Example

The volume of a cylinder is given by the formula  $V = \pi r^2 h$ .

- Sketch the graphs of  $V$  versus  $r$  for  $h = 1$ ,  $h = 2$ , and  $h = 3$ . (These graphs are called curves of constant height.)
- Sketch the graphs of  $h$  versus  $r$  for  $V = 36\pi$  and  $V = 72\pi$ . (These graphs are called curves of constant volume.)
- If  $h$  is held constant, what happens to  $V$  when  $r$  is doubled?



## Solution

- Since  $V(r, h) = \pi r^2 h$ , the three curves to be graphed have equations:

$$V(r, 1) = \pi r^2,$$

$$V(r, 2) = 2\pi r^2,$$

$$\text{and } V(r, 3) = 3\pi r^2.$$

With both  $r$  and  $V$  being positive quantities, we use only the first quadrant when graphing the equations. Each curve is half of a parabola, as shown.

- If  $V = 36\pi$ , we substitute  $\pi r^2 h$  for  $V$  and solve for  $h$ :

$$\pi r^2 h = 36\pi$$

$$h = \frac{36}{r^2}$$

We use the last equation to make a table of values:

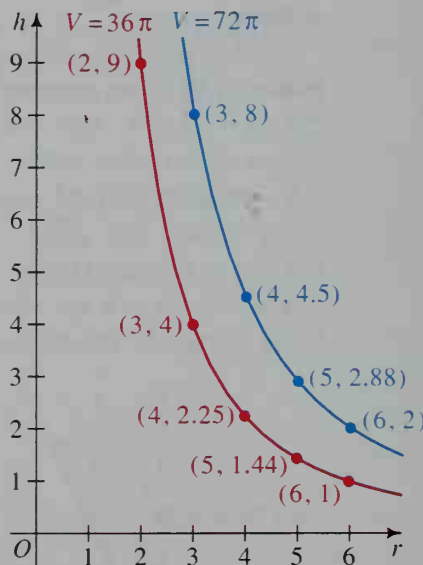
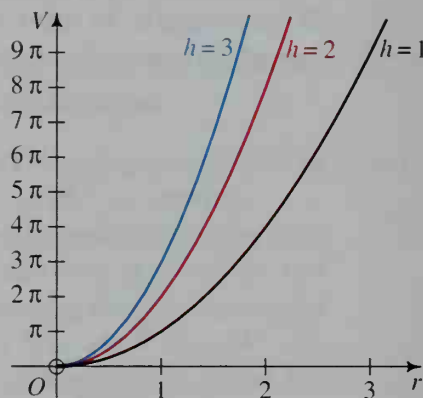
$r$	1	2	3	4	5	6
$h = \frac{36}{r^2}$	36	9	4	2.25	1.44	1

We then plot the pairs  $(r, h)$  to obtain the curve for  $V = 36\pi$ . In the same way, we sketch the curve for  $V = 72\pi$ . Both curves are shown at the right.

- Holding  $h$  constant and doubling  $r$ , we have:

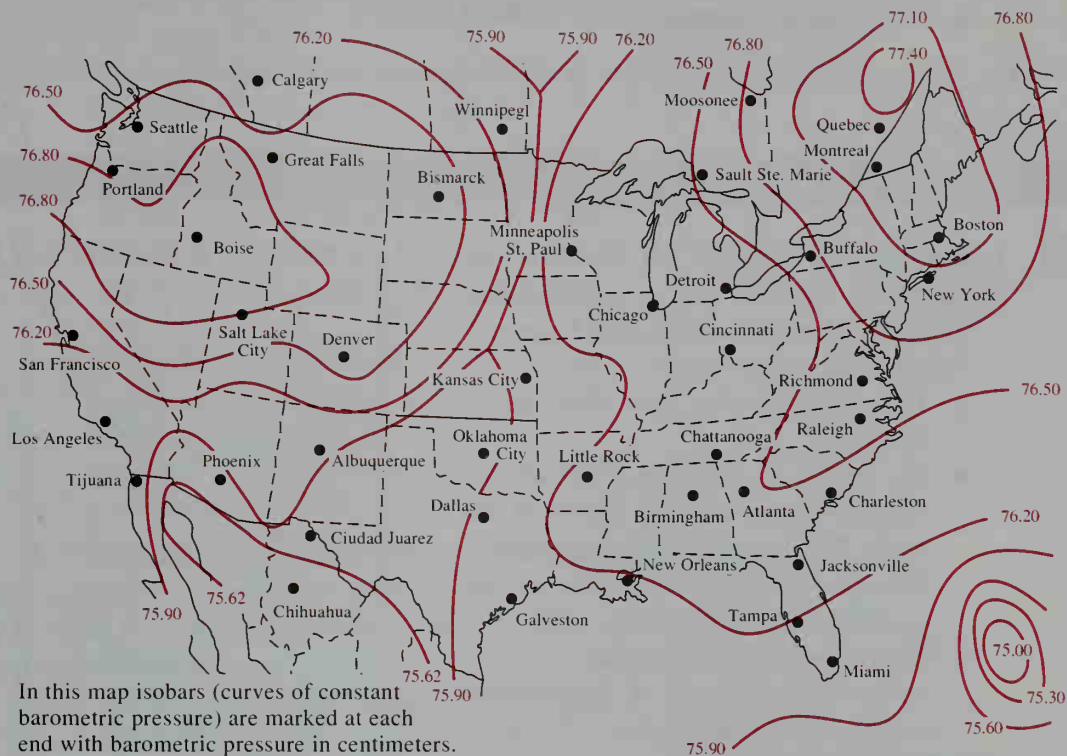
$$V(2r, h) = \pi(2r)^2 h = 4 \cdot \pi r^2 h = 4 \cdot V(r, h)$$

Thus, when  $h$  is held constant and  $r$  is doubled,  $V$  is quadrupled.





Real-world functions of more than one variable are often graphed in two dimensions. For example, meteorologists produce maps like the one below that show curves of constant barometric pressure.



## CLASS EXERCISES

Each of the following formulas from mathematics and science expresses some quantity as a function of two variables. Describe this functional relationship in words.

1.  $A = lw$

2.  $d = rt$

3.  $V = \frac{1}{3}\pi r^2 h$

4.  $F = ma$

5.  $D = \frac{m}{V}$

6.  $A = 2\pi rh + 2\pi r^2$

7. **Visual Thinking** Given the formula  $\text{density} = \text{mass} \div \text{volume}$  (see Exercise 5), what do curves of constant density look like in an  $mV$ -plane?

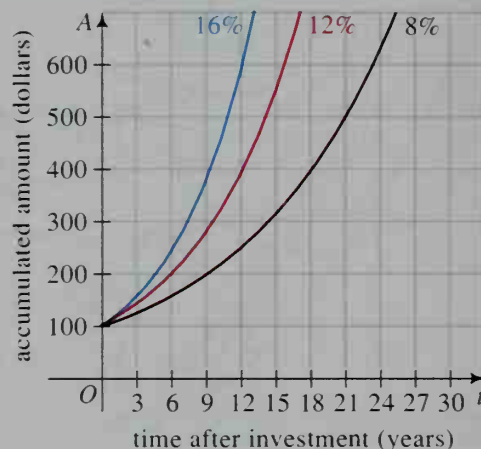
8. Refer to the barometric pressure graph shown above.

- Name two other cities with approximately the same barometric pressure as New Orleans.
- Bad weather usually accompanies low pressure. In which cities is the weather apt to be poorest?
- Good weather usually accompanies high pressure. In which cities is the weather apt to be best?

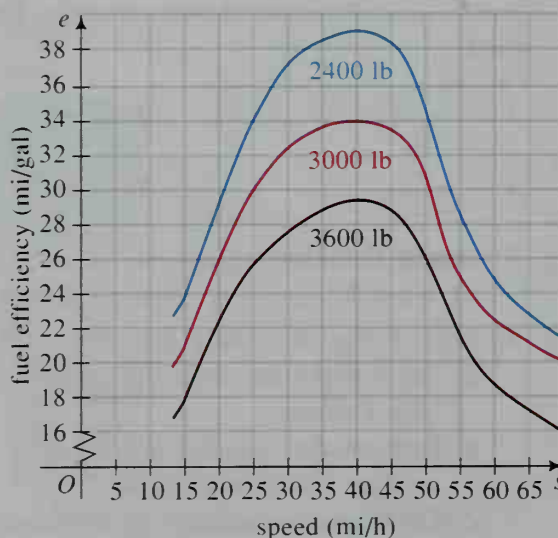
9. **Discussion** The amount that a person pays for auto insurance is a function of many variables. Discuss what some of these variables are.
10. a. If  $f(x, y) = \sqrt{x^2 + y^2}$ , find  $f(3, 4)$ ,  $f(-4, 3)$ , and  $f(0, 5)$ .  
b. Sketch some constant curves of the function  $f$ .
11. Name several pairs  $(a, b)$  not in the domain of  $f(a, b) = \frac{a + b}{a - b}$ .

## WRITTEN EXERCISES

- A** 1. **Consumer Economics** If \$100 is invested at interest rate  $r$  compounded annually, then the accumulated amount  $t$  years later is  $A = 100(1 + r)^t$ .
- a. The formula shows that  $A$  is a function of ? and ?.
- b. The graphs of  $A$  versus  $t$  for several constant values of  $r$  are shown at the right. The equation of the rightmost curve is  $A = 100(1.08)^t$ . What are the equations of the other two curves?
- c. About how many years does it take to double your money at 8% interest compounded annually? at 12% interest? at 16% interest?

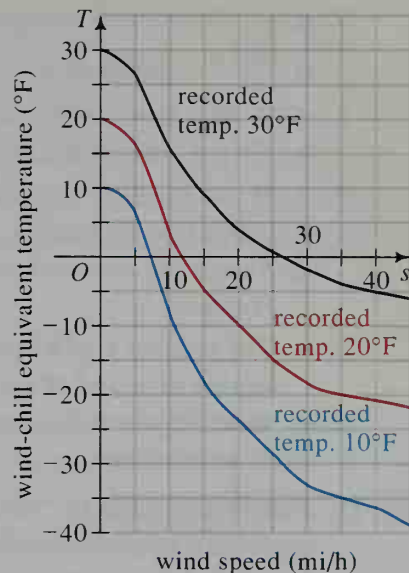


2. **Consumer Economics** The graph at the right below shows the fuel efficiency at various speeds for 2400 lb, 3000 lb, and 3600 lb cars. The fuel efficiency is given in miles per gallon (mi/gal).
- a. Fuel efficiency is a function of ? and ?.
- b. At what other speed will a 3000 lb car have approximately the same fuel efficiency as it has when traveling at 25 mi/h?
- c. At what speeds does a 3600 lb car have approximately the same fuel efficiency as a 2400 lb car traveling at 55 mi/h?
- d. Regardless of what a car weighs, at approximately what speed is maximum fuel efficiency reached?



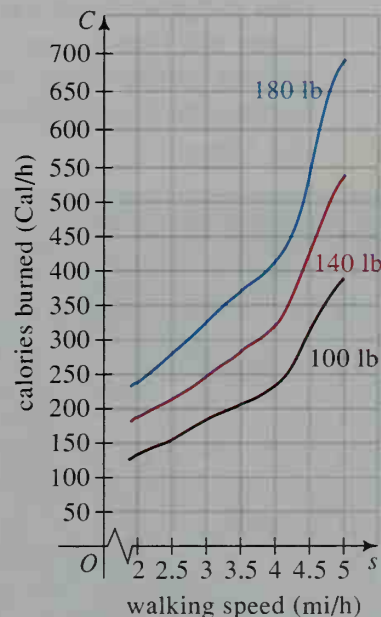
3. **Meteorology** The graph shows the wind-chill equivalent temperatures for recorded temperatures of  $30^{\circ}\text{F}$ ,  $20^{\circ}\text{F}$ , and  $10^{\circ}\text{F}$ .

- Describe the functional relationship that the graph depicts.
- If the recorded temperature is  $10^{\circ}\text{F}$ , approximately what wind speed will produce a wind-chill equivalent temperature of  $-20^{\circ}\text{F}$ ?
- Would you rather be in a place where the recorded temperature is  $30^{\circ}\text{F}$  with a 30 mi/h wind or  $20^{\circ}\text{F}$  with a 10 mi/h wind? Why?
- Regardless of the recorded temperature, is there a greater change in the wind-chill equivalent temperature when the wind speed is between 5 mi/h and 10 mi/h or when it is between 35 mi/h and 40 mi/h?



4. **Physiology** The graph shows the number of calories burned each hour when 100 lb, 140 lb, and 180 lb people walk at various speeds.

- Describe the functional relationship that the graph depicts.
- At approximately what speed must a 140 lb person walk in order to burn 450 Cal/h?
- At approximately what speed must a 100 lb person walk in order to burn as many calories per hour as a 180 lb person walking at 2 mi/h?
- Regardless of what a person weighs, is there a greater change in the number of calories burned per hour when the person increases his or her walking speed from 3 mi/h to 3.5 mi/h or from 4 mi/h to 4.5 mi/h?



- If  $A(b, h) = \frac{1}{2}bh$ , find  $A(8, 3)$  and  $A(16, 6)$ .
  - Give a geometric interpretation of the function  $A$ .
  - Give several pairs  $(b, h)$  for which  $A(b, h) = 5$ .
- If  $V(r, h) = \frac{1}{3}\pi r^2 h$ , find  $V(2, 6)$  and  $V(4, 12)$ .
  - Give a geometric interpretation of the function  $V$ .
  - Give several pairs  $(r, h)$  for which  $V(r, h) = 75\pi$ .

7. a. If  $A(b, h) = \frac{1}{2}bh$ , show that  $A(3b, 3h) = 9 \cdot A(b, h)$ .  
 b. Give a verbal description of the equation  $A(3b, 3h) = 9 \cdot A(b, h)$ .
8. a. If  $V(r, h) = \pi r^2 h$ , show that  $V(2r, 2h) = 8 \cdot V(r, h)$ .  
 b. Give a verbal description of the equation  $V(2r, 2h) = 8 \cdot V(r, h)$ .
9. a. If you travel at a constant rate  $r$  for  $t$  hours, the distance  $d$  that you travel is a function of  $r$  and  $t$ . Give a rule for this function.  
 b. Sketch the constant distance curves  $d(r, t) = 200$  and  $d(r, t) = 400$  in an  $rt$ -plane.
10. a. The volume  $V$  of a square prism is a function of the length  $s$  of a side of the square base and of the height  $h$  of the prism. Give a rule for this function.  
 b. Sketch two curves of constant volume in an  $sh$ -plane.

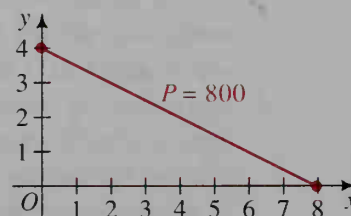
- B** 11. **Research** Look up “contour map” in a dictionary or encyclopedia. Draw a contour map of the butte shown at the right.
12. **Research** Most daily newspapers show a weather map on which curves of constant temperature are drawn. Obtain such a map and from it read the approximate forecast temperatures in Los Angeles, Seattle, Chicago, Dallas, New York City, and Miami.



13. Describe the domain of  $f(x, y) = \frac{xy}{x^2 - y^2}$ .

14. Describe the domain of  $g(x, y) = \frac{4 + y^2}{4 - x^2}$ .

15. **Retailing** The profit in selling  $x$  black-and-white television sets and  $y$  color television sets is  $P(x, y) = 100x + 200y$  dollars, where  $x \geq 0$  and  $y \geq 0$ . All points  $(x, y)$  on the line segment labeled  $P = 800$  will give a profit of \$800. This line is a *constant profit line*. Copy this graph and draw the constant profit lines  $P(x, y) = 1200$  and  $P(x, y) = 1600$ .



16. **Manufacturing** The cost of producing  $x$  wooden tennis rackets and  $y$  graphite tennis rackets is  $C(x, y) = 24x + 60y$ . Draw several lines of constant cost in an  $xy$ -plane.
17. a. If  $A(l, w, h) = 2lw + 2wh + 2lh$ , find  $A(4, 3, 5)$  and  $A(6, 4, 7)$ .  
 b. Give a geometric interpretation of the function  $A$ .
18. a. If  $A(b_1, b_2, h) = \frac{1}{2}(b_1 + b_2)h$ , find  $A(3, 4, 2)$  and  $A(5, 7, 3)$ .  
 b. Give a geometric interpretation of the function  $A$ .

For Exercises 19 and 20, refer to textbooks in the given fields to find several examples of functional relationships involving three or more variables. Be prepared to discuss these relationships in class.

19. **Research** Physics, chemistry, or biology

20. **Research** Medicine, psychology, or economics

- C** 21. **Sports** For some international sailing competitions, the rating  $R$  of a yacht is a function of several variables:

$$R(A, L, V) = 0.9 \left( \frac{L\sqrt{A}}{12\sqrt[3]{V}} + \frac{L + \sqrt{A}}{4} \right)$$

where  $A$  = surface area of the sails in square meters,  $L$  = length of the yacht in meters, and  $V$  = volume of water yacht displaces in cubic meters. To be in the  $R$ -5.5 class, a yacht must have a rating less than 5.5. Does a yacht that has  $37 \text{ m}^2$  of sail, is 10 m long, and displaces  $8.5 \text{ m}^3$  of water qualify?

## 4-7 Forming Functions from Verbal Descriptions

### Objective

To form a function of one variable from a verbal description and, when appropriate, to determine the minimum or maximum value of the function.

An important concern of mathematics is finding the minimum or maximum value of a function. You have already seen quadratic and cubic examples that involve minimizing costs and maximizing profits. Other such applications use mathematics to minimize the structural stress on a girder or to maximize the volume of a container made from a given amount of material.

Minimum and maximum values are often referred to as *extreme values*. Approximate extreme values of a function can be found using a computer or graphing calculator. Exact extreme values are most often found using calculus.

Whether technology or calculus is used, we almost always need to write a rule for the function to be minimized or maximized. If the rule depends on two or more variables, then we also need to find a relationship among the variables so that the function can be written in terms of only one variable. Developing these skills is the goal of this section.





**Example 1**

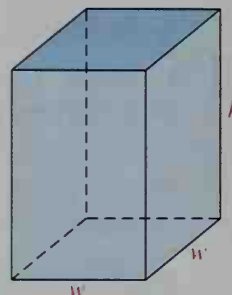
An open-top box with a square base is to be constructed from sheet metal in such a way that the completed box is made of  $2 \text{ m}^2$  of sheet metal. Express the volume of the box as a function of the base width.

**Solution**

1. Sketch the box. Let  $h$  be the height of the box, and let  $w$  be the width. The volume of the box is a function of  $h$  and  $w$ :

$$V(w, h) = w^2 h$$

2. The equation above gives  $V$  in terms of  $w$  and  $h$ . To get  $V$  in terms of  $w$  alone, we must replace  $h$  in the above equation with a function of  $w$ . We use the fact that the box is made of  $2 \text{ m}^2$  of sheet metal.

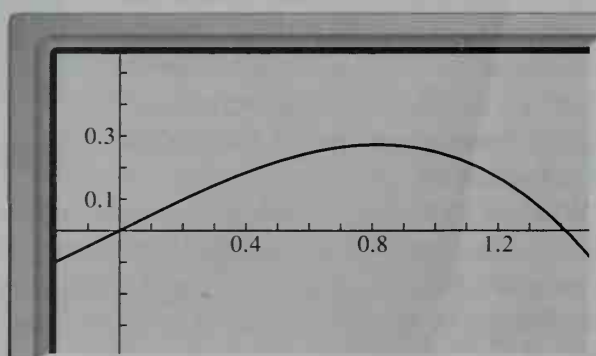
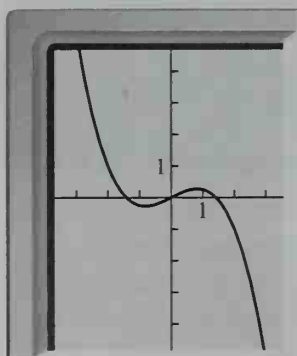


$$\begin{aligned} \text{Area of sheeting used} &= 2 \\ \text{Area of the base} + 4 \cdot (\text{Area of a side}) &= 2 \\ w^2 + 4wh &= 2 \\ h &= \frac{2 - w^2}{4w} \end{aligned}$$

3. To find  $V$  in terms of  $w$  alone, substitute  $\frac{2 - w^2}{4w}$  for  $h$ .

$$\begin{aligned} V(w, h) &= w^2 h \\ V(w) &= w^2 \left( \frac{2 - w^2}{4w} \right) = \frac{2w - w^3}{4} \end{aligned}$$

In Example 1, the volume of the box was expressed as a function of its width:  $V(w) = \frac{2w - w^3}{4}$ . To find the maximum volume, you can use a computer or graphing calculator to obtain the graph at the left below. Notice that this graph includes some values of  $w$  not in the domain of the function. By zooming in on the high point of the graph in the first quadrant, you can obtain the “blowup” shown at the right below. This shows that the maximum volume is approximately  $V(0.8) = 0.3 \text{ m}^3$ .



### Example 2

A north-south bridle path intersects an east-west river at point  $O$ . At noon, a horse and rider leave  $O$  traveling north at 12 km/h. At the same time, a boat is 25 km east of  $O$  traveling west at 16 km/h. Express the distance  $d$  between the horse and the boat as a function of the time  $t$  in hours after noon.

### Solution

1. Make a sketch showing the horse and boat at some time  $t$ . Let  $h$  be the horse's distance from  $O$ , and let  $b$  be the boat's distance east of  $O$ . Since the horse is traveling from  $O$  at 12 km/h,

$$h = 12t.$$

Since the boat is 25 km from  $O$  and traveling toward  $O$  at 16 km/h,

$$b = 25 - 16t.$$

2. By the Pythagorean theorem,

$$d(h, b) = \sqrt{h^2 + b^2}.$$

3. To find  $d$  in terms of  $t$ , substitute  $12t$  for  $h$  and  $25 - 16t$  for  $b$ .

$$\begin{aligned} d(t) &= \sqrt{(12t)^2 + (25 - 16t)^2} \\ &= \sqrt{144t^2 + 625 - 800t + 256t^2} \\ &= \sqrt{400t^2 - 800t + 625} \end{aligned}$$

In Example 2, it is possible to determine when the horse and boat are closest to each other (that is, when the distance between them is a minimum) and what the minimum distance is. Since  $d(t)$  is the square root of the function

$$f(t) = 400t^2 - 800t + 625,$$

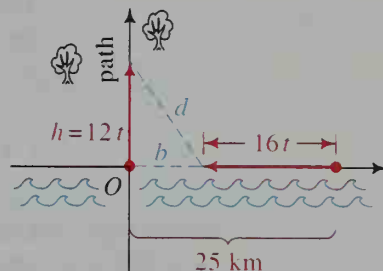
$d(t)$  will be a minimum when  $f(t)$  is a minimum. Since the minimum or maximum value of  $f(x) = ax^2 + bx + c$  occurs at  $x = -\frac{b}{2a}$ , the minimum value of  $f(t) = 400t^2 - 800t + 625$  occurs at:

$$t = -\frac{-800}{2(400)} = 1$$

Therefore, the horse and boat are closest 1 h after noon. The minimum distance between them is  $d(1)$ .

$$\begin{aligned} d(1) &= \sqrt{400 \cdot 1^2 - 800 \cdot 1 + 625} \\ &= \sqrt{225} = 15 \end{aligned}$$

Thus, the minimum distance between the horse and boat is 15 km.



**Example 3**

Water flows into a conical tank 100 cm wide and 250 cm deep at a rate of  $40 \text{ cm}^3/\text{s}$ . Find the volume  $V$  of the water in the tank as a function of the height  $h$  of the water. Represent  $h$  as a function of the time  $t$  that the water has been flowing into the empty tank.

**Solution**

1. Make a sketch. Let  $r$  be the radius of the water surface, and let  $h$  be the height of the water in the tank. The volume of water in the tank is a function of  $r$  and  $h$ :

$$V(r, h) = \frac{1}{3}\pi r^2 h$$

2. Find an expression for  $r$  in terms of  $h$  by using the similar triangles from a cross section of the tank.

$$\frac{r}{h} = \frac{50}{250}$$

$$r = \frac{h}{5}$$

3. To find  $V$  in terms of  $h$  alone, substitute  $\frac{h}{5}$  for  $r$ .

$$V(r, h) = \frac{1}{3}\pi r^2 h$$

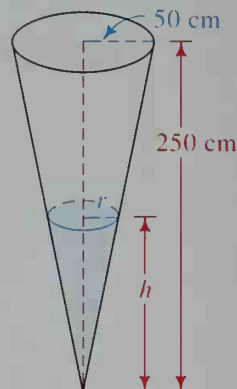
$$V(h) = \frac{1}{3}\pi \left(\frac{h}{5}\right)^2 h = \frac{\pi}{75}h^3$$

4. To represent  $h$  as a function of the time  $t$ , note that the volume of water in the tank at  $t$  seconds is  $V = 40t$ . Therefore, substitute  $40t$  for  $V$  and solve for  $h$ :

$$40t = \frac{\pi}{75}h^3$$

$$h = \sqrt[3]{\frac{3000t}{\pi}}$$

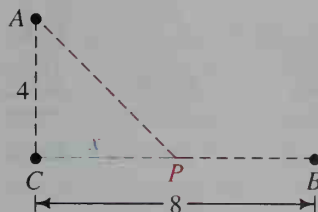
$$\text{Thus, } h(t) = \sqrt[3]{\frac{3000t}{\pi}}.$$

**CLASS EXERCISES**

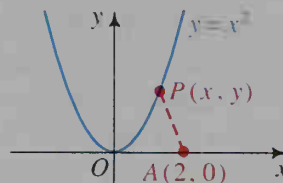
1. A square has a side of length  $s$  and a diagonal of length  $d$ , as shown.
  - a. Express  $d$  as a function of  $s$ .
  - b. Express  $s$  as a function of  $d$ .
  - c. Express the area  $A$  of the square as a function of  $d$ .
2. a. Express the volume  $V$  of a cube as a function of the length  $e$  of an edge.
  - b. The length of a diagonal of the cube is  $d = e\sqrt{3}$ . Express  $V$  as a function of  $d$ .



3. In the figure at the left below, point  $A$  is 4 km north of point  $C$ , point  $B$  is 8 km east of  $C$ , and  $P$  is a point on  $\overline{BC}$  at a distance of  $x$  km from  $C$ .
- Express  $AP + PB$  as a function of  $x$ .
  - What is the domain of this function?

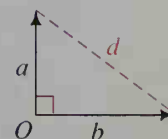


Ex. 3



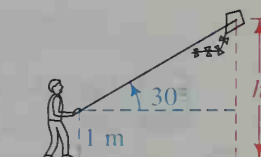
Ex. 4

- $P(x, y)$  is a point on the parabola  $y = x^2$ , as shown at the right above.
  - Express the distance from  $P$  to  $A(2, 0)$  as a function of  $x$  and  $y$ .
  - Express the distance as a function of  $x$  alone.
  - How can you find the minimum value of the function in part (b)?
- A runner starts north from point  $O$  at 6 m/s. At the same time, a second runner sprints east from  $O$  at 8 m/s. (See the figure at the right.) Find the distance  $d$  between the runners  $t$  seconds later.

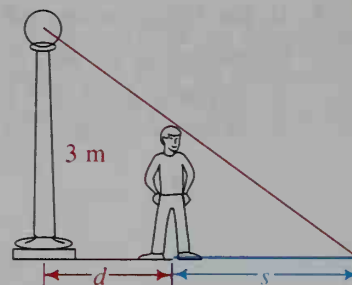


## WRITTEN EXERCISES

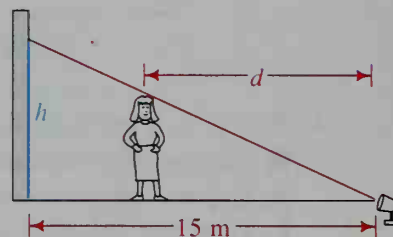
- A**
- Express the area  $A$  of a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle as a function of the length  $h$  of the hypotenuse.
  - Express the area  $A$  of an equilateral triangle as a function of the perimeter  $P$ .
  - A tourist walks  $n$  km at 4 km/h and then travels  $2n$  km at 36 km/h by bus. Express the total traveling time  $t$  (in hours) as a function of  $n$ .
  - A student holds a ball of string attached to a kite, as shown at the right. The string is held 1 m above the ground and rises at a  $30^\circ$  angle to the horizontal. If the student lets the string out at a rate of 2 m/s, express the kite's height  $h$  (in meters) as a function of the time  $t$  (in seconds) after the kite begins to fly.
  - A store owner bought  $n$  dozen toy boats at a cost of \$3.00 per dozen, and sold them at \$.75 apiece. Express the profit  $P$  (in dollars) as a function of  $n$ .
  - The cost of renting a large boat is 30 dollars per hour plus a usage fee roughly equivalent to  $x^3$  cents per hour when the boat is operated at a speed of  $x$  km/h. Express the cost  $C$  (in cents per kilometer) as a function of  $x$ .
  - The height of a cylinder is twice the diameter. Express the total surface area  $A$  as a function of the height  $h$ .
  - A pile of sand is in the shape of a cone with a diameter that is twice the height. Express the volume  $V$  of sand as a function of the height  $h$ .



9. A light 3 m above the ground causes a boy 1.8 m tall to cast a shadow  $s$  meters long measured along the ground, as shown at the right. Express  $s$  as a function of  $d$ , the boy's distance in meters from the light.
10. When a girl 1.75 m tall stands between a wall and a light on the ground 15 m away, she casts a shadow  $h$  meters high on the wall, as shown at the right. Express  $h$  as a function of  $d$ , the girl's distance in meters from the light.
11. A box with a square base has a surface area (including the top) of  $3 \text{ m}^2$ . Express the volume  $V$  of the box as a function of the width  $w$  of the base.
12. A box with a square base and no top has a volume of  $6 \text{ m}^3$ . Express the total surface area  $A$  of the box as a function of the width  $w$  of the base.



Ex. 9



Ex. 10

13. A stone is thrown into a lake, and  $t$  seconds after the splash the diameter of the circle of ripples is  $t$  meters.
- Express the circumference  $C$  of this circle as a function of  $t$ .
  - Express the area  $A$  of this circle as a function of  $t$ .
14. A balloon is inflated in such a way that its volume increases at a rate of  $20 \text{ cm}^3/\text{s}$ .
- If the volume of the balloon was  $100 \text{ cm}^3$  when the process of inflation began, what will the volume be after  $t$  seconds of inflation?
  - Assuming that the balloon is spherical while it is being inflated, express the radius  $r$  of the balloon as a function of  $t$ .



Part (b) of Exercises 15 and 16 requires the use of a computer or graphing calculator.

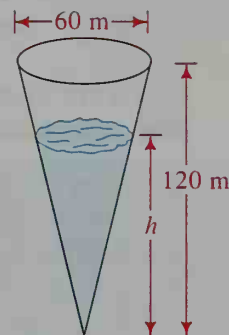
- B** 15. **Manufacturing** A box with a square base and no top has volume  $8 \text{ m}^3$ . The material for the base costs \$8 per square meter, and the material for the sides costs \$6 per square meter.
- Express the cost  $C$  of the materials used to make the box as a function of the width  $w$  of the base.
  - Use a computer or graphing calculator to find the minimum cost.
16. **Manufacturing** A cylindrical can has a volume of  $400\pi \text{ cm}^3$ . The material for the top and bottom costs 2¢ per square centimeter. The material for the vertical surface costs 1¢ per square centimeter.
- Express the cost  $C$  of the materials used to make the can as a function of the radius  $r$ .
  - Use a computer or graphing calculator to find the minimum cost.



17. At 2:00 P.M. bike A is 4 km north of point C and traveling south at 16 km/h. At the same time, bike B is 2 km east of C and traveling east at 12 km/h.
- a. Show that  $t$  hours after 2:00 P.M. the distance between the bikes is:

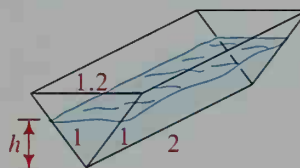
$$\sqrt{400t^2 - 80t + 20}$$

- b. At what time is the distance between the bikes the least?
- c. What is the distance between the bikes when they are closest?
18. A car leaves Oak Corners at 11:33 A.M. traveling south at 70 km/h. At the same time, another car is 65 km west of Oak Corners traveling east at 90 km/h.
- a. Express the distance  $d$  between the cars as a function of the time  $t$  after the first car left Oak Corners.
- b. Show that the cars are closest to each other at noon.
19. **Engineering** Water is flowing at a rate of  $5 \text{ m}^3/\text{s}$  into the conical tank shown at the right.
- a. Find the volume  $V$  of the water as a function of the water level  $h$ .
- b. Find  $h$  as a function of the time  $t$  during which water has been flowing into the tank.



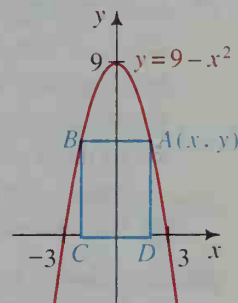
Ex. 19

20. **Engineering** A trough is 2 m long, and its ends are triangles with sides of length 1 m, 1 m, and 1.2 m as shown at the right.
- a. Find the volume  $V$  of the water in the trough as a function of the water level  $h$ .
- b. If water is pumped into the empty trough at the rate of 6 L/min, find the water level  $h$  as a function of the time  $t$  after the pumping begins. ( $1 \text{ m}^3 = 1000 \text{ L}$ )



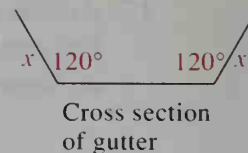
Ex. 20

21.  $P(x, y)$  is an arbitrary point on the line  $2x + y = 10$ .
- a. Express the distance  $d$  from the origin to  $P$  as a function of the  $x$ -coordinate of  $P$ .
- b. What are the domain and range of this function?
22.  $P(x, y)$  is an arbitrary point on the parabola  $y = x^2$ .
- a. Express the distance  $d$  from  $P$  to the point  $A(0, 1)$  as a function of the  $y$ -coordinate of  $P$ .
- b. What is the minimum distance  $d$ ?
23. As shown at the right, rectangle  $ABCD$  has vertices  $C$  and  $D$  on the  $x$ -axis and vertices  $A$  and  $B$  on the part of the parabola  $y = 9 - x^2$  that is above the  $x$ -axis.
- a. Express the perimeter  $P$  of the rectangle as a function of the  $x$ -coordinate of  $A$ .
- b. What is the domain of the perimeter function?
- c. For what value of  $x$  is the perimeter a maximum?



Ex. 23

24. **Manufacturing** A sheet of metal is 60 cm wide and 10 m long. It is bent along its width to form a gutter with a cross section that is an isosceles trapezoid with  $120^\circ$  angles, as shown at the right.

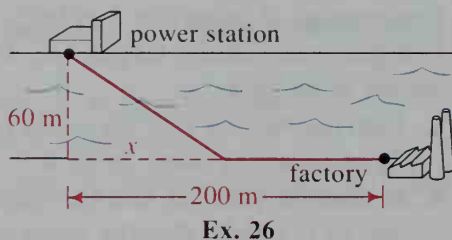
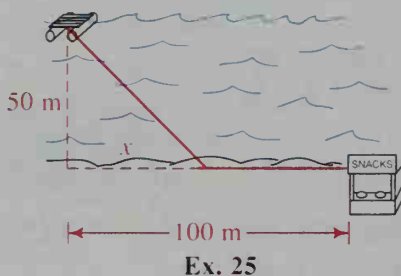


- Express the volume  $V$  of the gutter as a function of  $x$ , the length in centimeters of one of the equal sides.  
(Hint: Volume = area of trapezoid  $\times$  length of gutter)
- For what value of  $x$  is the volume of the gutter a maximum?



Part (b) of Exercises 25–29 requires the use of a computer or graphing calculator.

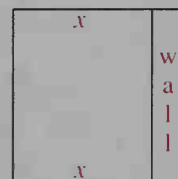
25. From a raft 50 m offshore, a lifeguard wants to swim to shore and run to a snack bar 100 m down the beach, as shown at the left below.
- If the lifeguard swims at 1 m/s and runs at 3 m/s, express the total swimming and running time  $t$  as a function of the distance  $x$  shown in the diagram.
  - Use a computer or graphing calculator to find the minimum time.



26. **Engineering** A power station and a factory are on opposite sides of a river 60 m wide, as shown at the right above. A cable must be run from the power station to the factory. It costs \$25 per meter to run the cable in the river and \$20 per meter on land.

- Express the total cost  $C$  as a function of  $x$ , the distance downstream from the power station to the point where the cable touches the land.
- Use a computer or graphing calculator to find the minimum cost.

27. **Landscaping** A rectangular area of  $60 \text{ m}^2$  has a wall as one of its sides, as shown. The sides perpendicular to the wall are made of fencing that costs \$6 per meter. The side parallel to the wall is made of decorative fencing that costs \$8 per meter.

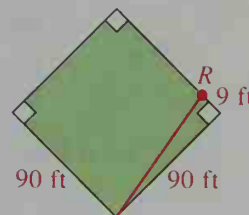


- Express the total cost  $C$  of the fencing as a function of the length  $x$  of a side perpendicular to the wall.
- Use a computer or graphing calculator to find the minimum cost.



28. A cylinder is inscribed in a sphere of radius 1.
- Express the volume  $V$  of the cylinder as a function of the base radius  $r$ .
  - Use a computer or graphing calculator to find the radius and height of the cylinder having maximum volume.

29. A cone circumscribes a sphere with radius 1.
- Express the volume  $V$  of the cone as a function of the base radius  $r$ .
  - Use a computer or graphing calculator to find the radius and height of the cone having minimum volume.
30. **Sports** A baseball diamond is a square with 90 ft sides. A runner ( $R$  in the diagram at the right) has taken a 9 ft lead from first base. At the moment the ball is pitched, the runner runs toward second base at 27 ft/s.
- Express the runner's straight-line distance  $d$  from home plate as a function of the time  $t$  after the ball is thrown.
  - What are the domain and range of this function?
31. **Sports** A baseball player hits a ball into the farthest corner of the outfield and tries for an inside-the-park home run; that is, the player tries to run the bases and make it to home plate safely. Suppose that the player runs at 30 ft/s and stays strictly on the base lines, as shown in Exercise 30.
- Express the player's straight-line distance  $d$  from home plate as a function of the time  $t$  after the ball was hit. (*Hint:* The function involves different rules for different intervals of time.)
  - Sketch the graph of the function in part (a).



## Chapter Summary

- A *function* consists of a set of real numbers, called the *domain* of the function, and a rule that assigns to each element in the domain exactly one real number. The set of real numbers assigned by the rule is called the *range* of the function. The *graph* of a function is the set of points corresponding to the ordered pairs that satisfy the functional rule.
- Functions can be *added*, *subtracted*, *multiplied*, or *divided*:
 
$$(f + g)(x) = f(x) + g(x) \quad (f - g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ provided } g(x) \neq 0$$
- The *composite* of  $f$  and  $g$ , denoted  $f \circ g$ , is defined by  $(f \circ g)(x) = f(g(x))$  such that  $x$  is in the domain of  $g$  and  $g(x)$  is in the domain of  $f$ .
- Simple changes in an equation of a graph can cause the graph to be reflected, translated, stretched, or shrunk. The chart on page 142 summarizes these changes.

5. Methods for testing an equation or its graph for symmetry in the  $x$ -axis, in the  $y$ -axis, in the origin, or in the line  $y = x$  are summarized on page 134.
6. Properties of functions can be used to help graph and analyze functions:
  - $f$  is an *even* function if  $f(-x) = f(x)$  for all  $x$  in the domain of  $f$ ;  
the graph of an even function has symmetry in the  $y$ -axis.
  - $f$  is an *odd* function if  $f(-x) = -f(x)$  for all  $x$  in the domain of  $f$ ;  
the graph of an odd function has symmetry in the origin.
7. A function  $y = f(x)$  is *periodic* if there is a positive number  $p$ , called a *period* of  $f$ , such that  $f(x + p) = f(x)$  for all  $x$  in the domain of  $f$ . The smallest period of  $f$  is called the *fundamental period*. If  $f$  has maximum value  $M$  and minimum value  $m$ , then the *amplitude* of  $f$  is defined to be  $\frac{M - m}{2}$ .
8. If  $f$  is periodic with period  $p$  and amplitude  $A$ , then  $y = cf(x)$  has period  $p$  and amplitude  $cA$ , and  $y = f(cx)$  has period  $\frac{p}{c}$  and amplitude  $A$ .
9. If  $f(g(x)) = x$  for all  $x$  in the domain of  $g$  and  $g(f(x)) = x$  for all  $x$  in the domain of  $f$ , then  $f$  and  $g$  are *inverses*. The inverse of  $f$  is denoted  $f^{-1}$  and its graph is the reflection of the graph of  $f$  in the line  $y = x$ .  $f^{-1}$  exists if  $f$  is *one-to-one*, that is, every horizontal line intersects the graph of  $f$  in at most one point.
10. In many situations, a problem is modeled by a function of more than one variable. By expressing one variable in terms of the other, a function of two variables can often be transformed into a function of one variable.

### Key vocabulary and ideas

function, domain, range (p. 119)	periodic function, period (p. 138)
relation (p. 121)	amplitude (p. 139)
vertical-line test (p. 121)	vertical stretch or shrink (p. 140)
sum and difference of $f$ and $g$ (p. 125)	horizontal stretch or shrink (p. 140)
product and quotient of $f$ and $g$ (p. 125)	translation (p. 141)
composite function (p. 126)	inverse functions (p. 146)
reflection (pp. 131–133)	one-to-one function (p. 148)
axis of symmetry (p. 133)	horizontal-line test (p. 148)
point of symmetry (p. 133)	function of two variables (p. 151)

## Chapter Test

1. Give the domain, range, and zeros of the function  $f(x) = \sqrt{9 - x^2}$ . 4-1
2. Graph  $g(x) = \begin{cases} x + 1 & \text{if } x < -1 \\ 1 - x^2 & \text{if } x \geq -1 \end{cases}$ . Find the range and zeros of  $g$ .
3. Let  $f(x) = x^2 + 2x$  and  $g(x) = x + 2$ . Find: 4-2
  - a.  $(f + g)(x)$
  - b.  $(f - g)(x)$
  - c.  $(f \cdot g)(x)$
  - d.  $\left(\frac{f}{g}\right)(x)$

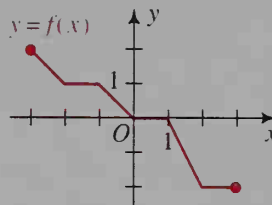
4. Using the functions  $f$  and  $g$  in Exercise 3, find:

- a.  $(f \circ g)(x)$                       b.  $(g \circ f)(x)$

5. Determine whether the graph of  $x^2 - xy = 4$  has symmetry in: (i) the  $x$ -axis, (ii) the  $y$ -axis, (iii) the line  $y = x$ , and (iv) the origin. 4-3

6. Given the graph of  $y = f(x)$  shown at the right, sketch the graph of each of the following.

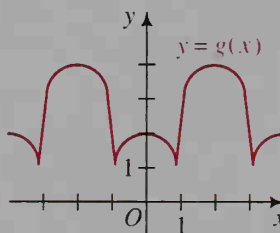
- a.  $y = 2f(x)$                       b.  $y = |f(x)|$   
c.  $y = -f(x)$                       d.  $y = f(x + 2)$



Ex. 6

7. The graph of a periodic function  $y = g(x)$  is shown at the right.

- a. What is the fundamental period of  $g$ ?  
b. What are the maximum and minimum values of  $g$ ?  
c. What is the amplitude of  $g$ ?



Ex. 7

8. **Writing** Describe and illustrate what happens when the graph of a periodic function with period  $p$  is horizontally translated  $p$  units. Then use the definition of a periodic function to explain why this happens.


9. a. Which one of the two functions  $f(x) = \sqrt{3 + x^2}$  and  $g(x) = 3 + x$  has an inverse? Find a rule for the inverse. 4-5

b. Explain why the other function does not have an inverse.

10. The area  $A$  of a triangle is a function of the base  $b$  and height  $h$ . 4-6

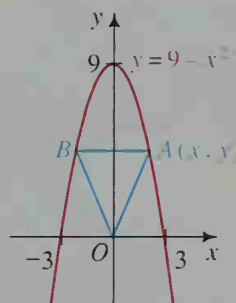
- a. Express  $A$  as a function of  $b$  and  $h$ .  
b. Find  $A(3, 4)$  and  $A(6, 5)$ .  
c. Draw the curve of constant area  $A(b, h) = 3$  in a  $bh$ -plane.

11. A cylindrical tank 4 ft in diameter fills with water at the rate of  $10 \text{ ft}^3/\text{s}$ . Express the depth of the water in the tank as a function of the time  $t$  in seconds. Assume the tank is empty at time  $t = 0$ . 4-7

 **Part (c) of Exercise 12 requires the use of a computer or graphing calculator.**

12. As shown at the right, triangle  $OAB$  is an isosceles triangle with vertex  $O$  at the origin and vertices  $A$  and  $B$  on the part of the parabola  $y = 9 - x^2$  that is above the  $x$ -axis.

- a. Express the area of the triangle as a function of the  $x$ -coordinate of  $A$ .  
b. What is the domain of the area function?  
c. Use a computer or graphing calculator to find the maximum area.





# 5

## *Exponents and Logarithms*



# Exponents

## 5-1 Growth and Decay: Integral Exponents

**Objective** To define and apply integral exponents.

Suppose that the cost of a hamburger has been increasing at the rate of 9% per year. Then, each year the cost is 1.09 times the cost in the previous year. Suppose that the cost now is \$4. Some projected future costs are given in the table below.

Time (years from now)	0	1	2	3	$t$
Cost (dollars)	4	$4(1.09)$	$4(1.09)^2$	$4(1.09)^3$	$4(1.09)^t$

$\xrightarrow{\times 1.09}$     $\xrightarrow{\times 1.09}$     $\xrightarrow{\times 1.09}$     $\xrightarrow{\times 1.09}$

The table suggests that the cost is a function of time  $t$ . Since the variable  $t$  occurs as an exponent, the cost is said to be an exponential function of time:

$$C(t) = 4(1.09)^t$$

When  $t > 0$  the function gives future costs, and when  $t < 0$  it gives costs in the past. Example 1 illustrates this.

**Example 1** Use the cost function,  $C(t) = 4(1.09)^t$ , to find the cost of a hamburger (a) 5 years from now and (b) 5 years ago.

**Solution** Values of  $C(t)$  are most easily found with a scientific calculator. (Use the exponent key.)

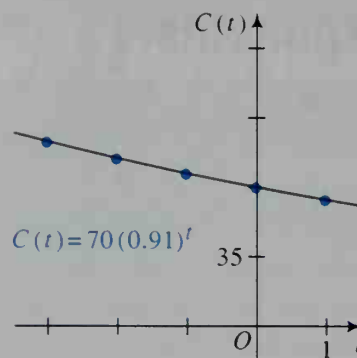
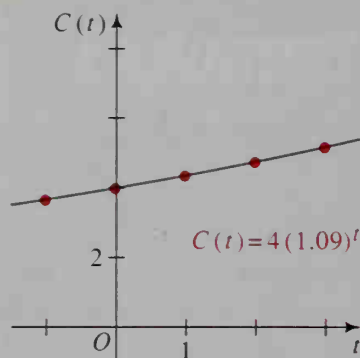
a.  $C(5) = 4(1.09)^5 \approx 6.15$ . The cost will be about \$6.15.

b.  $C(-5) = 4(1.09)^{-5} \approx 2.60$ . The cost was about \$2.60.

We will soon give exact meaning to expressions that involve negative exponents. However, first let us contrast the cost of a hamburger that has been increasing at 9% per year with the cost of a graphing calculator that has been *decreasing* at 9% per year. Each year the calculator cost is  $100\% - 9\%$ , or 91%, of what it was the year before.

Hamburger cost: \$4 now Cost increasing at 9% per year $C(t) = 4(1.09)^t$	Calculator cost: \$70 now Cost decreasing at 9% per year $C(t) = 70(0.91)^t$
---	--

◀ Ancient Egyptian astronomers used geometry to predict the future position of stars and planets. Current archeologists use mathematics to explore the Egyptian past. *Exponential functions*, for instance, are used in the carbon dating of ancient artifacts, such as this ceremonial boat dating from the reign of King Cheops, around 2400 B.C.



The graph at the left above shows exponential growth. The one at the right above shows exponential decay. Growth and decay can be modeled by:

$$A(t) = A_0(1 + r)^t.$$

where  $A_0$  is the initial amount, the amount at time  $t = 0$ , and  $r$  is the growth rate. If  $r > 0$ , then the initial amount grows exponentially. If  $-1 < r < 0$ , then the initial amount decays exponentially.

### Example 2

Suppose that a radioactive isotope decays so that the radioactivity present decreases by 15% per day. If 40 kg are present now, find the amount present (a) 6 days from now and (b) 6 days ago.

### Solution

$$A(t) = A_0(1 + r)^t = 40(1 - 0.15)^t = 40(0.85)^t$$

$$\text{a. } A(6) = 40(0.85)^6 \approx 15.1$$

$$\text{b. } A(-6) = 40(0.85)^{-6} \approx 106.1$$

There will be about 15.1 kg.

There was about 106.1 kg.

Although we have used a calculator to evaluate expressions involving negative exponents, we must still define them. To do this, we will first review the laws of exponents for positive integers.

## Laws of Exponents

*Same bases*

$$1. b^x \cdot b^y = b^{x+y}$$

$$2. \frac{b^x}{b^y} = b^{x-y} \quad (b \neq 0)$$

$$3. \text{ If } b \neq 0, 1, \text{ or } -1, \text{ then } b^x = b^y \text{ if and only if } x = y.$$

*Same exponents*

$$4. (ab)^x = a^x b^x$$

$$5. \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \quad (b \neq 0)$$

$$6. \text{ If } x \neq 0, a > 0, \text{ and } b > 0, \text{ then } a^x = b^x \text{ if and only if } a = b.$$

*Power of a power*

$$7. (b^x)^y = b^{xy}$$

If the laws on the preceding page are to make sense for the zero exponent and negative integral exponents, then such exponents must be defined as follows.

*Definition of  $b^0$ :* If law 1 is to hold for  $y = 0$ , then we have

$$b^x \cdot b^0 = b^{x+0} = b^x.$$

Since  $b^0$  behaves like the number 1, we define it to be 1:

$$b^0 = 1 \quad (b \neq 0)$$

*Definition of  $b^{-x}$ :* If law 1 is to hold for  $y = -x$  and  $b \neq 0$ , then we have

$$b^x \cdot b^{-x} = b^{x+(-x)} = b^0 = 1.$$

Since  $b^x$  and  $b^{-x}$  have a product of 1, they are reciprocals of each other. Therefore, we define:

$$b^{-x} = \frac{1}{b^x} \quad (x > 0 \text{ and } b \neq 0)$$

An expression is simplified when it contains neither negative exponents nor powers of powers. This is illustrated in the examples below.

**Example 3** Simplify  $\left(\frac{b^2}{a}\right)^{-2} \left(\frac{a^2}{b}\right)^{-3}$ , where  $a \neq 0$  and  $b \neq 0$ .

**Solution**  $\left(\frac{b^2}{a}\right)^{-2} \left(\frac{a^2}{b}\right)^{-3} = \frac{(b^2)^{-2}}{a^{-2}} \cdot \frac{(a^2)^{-3}}{b^{-3}} = \frac{b^{-4}a^{-6}}{a^{-2}b^{-3}} = \frac{1}{a^4b}$

**Example 4** Simplify  $(a^{-2} + b^{-2})^{-1}$ , where  $a \neq 0$  and  $b \neq 0$ .

**Solution**  $(a^{-2} + b^{-2})^{-1} = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{-1} = \left(\frac{b^2 + a^2}{a^2b^2}\right)^{-1} = \frac{a^2b^2}{b^2 + a^2}$

As Example 4 shows, you do not simplify a power of a sum or difference by distributing the exponent over the individual terms. Laws 4 and 5 apply only to a power of a product or quotient:

$$(a^{-2}b^{-2})^{-1} = a^2b^2 \quad \text{but} \quad (a^{-2} + b^{-2})^{-1} \neq a^2 + b^2$$

**Example 5** Simplify (a)  $\frac{x^5 + x^{-2}}{x^{-3}}$  and (b)  $\frac{x^5 \cdot x^{-2}}{x^{-3}}$ , where  $x \neq 0$ .

**Solution** a.  $\frac{x^5 + x^{-2}}{x^{-3}} = \frac{x^5}{x^{-3}} + \frac{x^{-2}}{x^{-3}} = x^8 + x$  b.  $\frac{x^5 \cdot x^{-2}}{x^{-3}} = \frac{x^3}{x^{-3}} = x^6$

We shall assume throughout the rest of this book that variables are restricted so that there are no denominators of zero. In this section only, we assume that variables appearing as exponents represent integers.



## CLASS EXERCISES

Copy and complete each table.

1. a.	If $A_0$ increases by	3%	15%	4.6%	120%	?
b.	then multiply $A_0$ by	?	?	?	?	1.105

2. a.	If $A_0$ decreases by	12%	7.5%	80%	?	100%
b.	then multiply $A_0$ by	?	?	?	0.68	?

3.	Item	Annual rate of increase	Cost now	Cost in $t$ years
a.	Bike	5%	\$200.00	?
b.	Jeans	8%	\$20.00	?
c.	Loaf of bread	?	\$1.25	$1.25(1.06)^t$

4.	Item	Annual rate of decrease	Value now	Value in $t$ years
a.	Car	20%	\$9800	?
b.	Boat	15%	\$2200	?
c.	Skates	?	\$100	$100(0.75)^t$



Simplify.

5. a.  $8^{-1}$

b.  $8^{-2}$

6. a.  $\left(\frac{2}{3}\right)^{-1}$

b.  $\left(\frac{2}{3}\right)^{-2}$

7. a.  $4 \cdot 3^{-2}$

b.  $(4 \cdot 3)^{-2}$

8. a.  $(2^{-1} \cdot 4^{-1})^{-1}$

b.  $(2^{-1} + 4^{-1})^{-1}$

9.  $\frac{12^3}{6^3}$

10.  $\frac{8^n \cdot 3^n}{4^n}$

11.  $\frac{(-2n)^2}{-2n^2}$

12.  $x^{-3}(x^5 + x^3)$

13.  $\frac{3a^3 \cdot 6a^6}{a^{-1}}$

14.  $\frac{3a^3 - 6a^6}{a^{-1}}$

15.  $\frac{5b^3 + 10b^6}{b^{-2}}$

16.  $\frac{5a^3 \cdot 10b^6}{b^{-2}}$



# WRITTEN EXERCISES

Simplify each expression.

- A**
1. a.  $(-4)^{-2}$       b.  $-4^{-2}$       2. a.  $(-3)^{-4}$       b.  $-3^{-4}$
  3. a.  $5 \cdot 2^{-3}$       b.  $(5 \cdot 2)^{-3}$       4. a.  $2 \div 4^{-3}$       b.  $(2 \div 4)^{-3}$
  5. a.  $(x^{-2})^{-1}$       b.  $((x^{-2})^{-1})^0$       6. a.  $(2x^{-3})^{-4}$       b.  $2x^{-3} \cdot x^{-4}$
  7. a.  $(3a^{-1})^{-1}$       b.  $(3 + a^{-1})^{-1}$       8. a.  $(5x^2 \cdot x^{-2})^2$       b.  $(5x^2 + x^{-2})^2$
  9. a.  $(2^{-2} + 2^{-3})^{-1}$       b.  $(2^{-2} \cdot 2^{-3})^{-1}$       10. a.  $(4^{-1} - 2^{-1})^2$       b.  $(4^{-1} \div 2^{-1})^2$
  11. a.  $(a^{-1} - b^{-1})^{-1}$       b.  $(a^{-1} \cdot b^{-1})^{-1}$       12. a.  $(2 + x^{-2})^{-2}$       b.  $(2 \cdot x^{-2})^{-2}$

Copy and complete the table. The cost of each item grows exponentially.

	Item	Annual rate of increase	Cost now	Cost in 10 years	Cost in 20 years
13.	Airplane ticket	15%	\$300	?	?
14.	Swim suit	8%	\$35	?	?
15.	Jar of mustard	7%	\$1	?	?
16.	College tuition	10%	\$12,000	?	?

Copy and complete the table. The value of each item decays exponentially.

	Item	Annual rate of decrease	Value now	Value in 3 years	Value in 6 years
17.	Farm tractor	25%	\$65,000	?	?
18.	Industrial equipment	10%	\$200,000	?	?
19.	Value of the dollar	6%	\$1	?	?
20.	Value of the dollar	8%	\$1	?	?

Simplify each expression.

21.  $(3a^{-2})^3 \cdot 3a^5$       22.  $(-4x^3)^2 \cdot 3x^{-2}$       23.  $(3n^2)^{-1} (3n^2)^7$
24.  $(2r^{-1})^4 (4r^2)^{-2}$       25.  $\frac{(2a^{-1})^2}{(2a^{-1})^{-2}}$       26.  $\frac{(-3n^{-3})^2}{-9n^{-4}}$
27.  $\left(\frac{a}{b^2}\right)^{-2} \left(\frac{a}{b^2}\right)^{-3}$       28.  $\frac{(-2r)^4}{(-2r)^{-2}}$       29.  $2x^{-3}(x^5 - 2x^3)$
30.  $xy^{-2}(xy^2 - 3y^3)$       31.  $\frac{6a^{-2} + 9a^2}{3a^{-2}}$       32.  $\frac{8n^4 - 4n^{-2}}{2n^{-2}}$

33. **Business** The value of a new car decreases 20% each year. Complete the table. The value  $V(t)$  of the car is in dollars and its age  $t$  is in years. Give each value to the nearest hundred. Using the values in your table, make a graph to show the relationship between  $V(t)$  and  $t$ .

$t$	0	1	2	3	4	5
$V(t)$	10,000	?	?	?	?	?



34. **Business** The value in dollars of a car  $t$  years from now is  $V(t) = 12,500(0.85)^t$ . (a) What is the annual rate of depreciation, the rate at which the car loses value? (b) In how many years will the value of the car be approximately half what it is now?
35. **Consumer Economics** If grocery prices increase 1% per month for a whole year, how much would groceries that cost \$100 at the beginning of the year cost at the end of the year?
36. **Consumer Economics** The cost of goods and services in an urban area increased 1.5% last month. At this rate, what will be the annual rate of increase?

Simplify by using powers of the same base.

37. a.  $\frac{3^5 \cdot 9^4}{27^4}$       b.  $\frac{125^{-3} \cdot 25}{5^{-8}}$       c.  $\sqrt{\frac{8^n \cdot 2^7}{4^{-n}}}$

38. a.  $\frac{4^9 \cdot 8^{-4}}{16^3}$       b.  $\frac{3^7 \cdot 9^5}{\sqrt{27^{12}}}$       c.  $\sqrt[3]{\frac{125^n \cdot 5^{4n}}{25^{-n}}}$

Write as a power of  $b$ .

39. a.  $\frac{(b^n)^3}{b^n \cdot b^n}$       b.  $\frac{(b^n)^2}{b^n \cdot b^{n+2}}$       c.  $\frac{b \cdot b^n}{(b^3)^n}$

40. a.  $\sqrt{\frac{b^{2n}}{b^{-2n}}}$       b.  $\frac{(b \cdot b^n)^2}{(b^2)^n}$       c.  $\sqrt{\frac{b^{1-n}}{b^{n-1}}}$

Simplify. (*Hint:* In Exercise 41(a), multiply the numerator and the denominator by  $2^3$ .)

**B** 41. a.  $\frac{2^{-1}}{2^{-2} + 2^{-3}}$       b.  $\frac{4^{-5}}{4^{-2} + 4^{-3}}$       42. a.  $\frac{3^{-2}}{3^{-3} + 3^{-2}}$       b.  $\frac{2^{-1} - 2^{-2}}{2^{-1} + 2^{-2}}$

43. a.  $\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$       b.  $\frac{1 - y^{-1}}{y - y^{-1}}$       44. a.  $\frac{x^{-1}}{x - x^{-1}}$       b.  $\frac{4 - x^{-4}}{2 - x^{-2}}$

45. **Discussion** According to legend Manhattan Island was purchased in 1626 for trinkets worth about \$24. If the \$24 had been invested instead at a rate of 6% interest per year, what would be the value of the money in 1996? Compare this with a recent total of \$34,000,000,000 in assessed values for Manhattan Island.

- C** 46. Solve  $2^x + 8 \cdot 2^{-x} = 9$ .  
 47. Solve  $2^x + 2^{-x} = \frac{5}{2}$ .  
 48. Solve  $2^{2x} - 3 \cdot 2^{x+1} + 8 = 0$ .  
 49. Solve  $3^{2x+1} - 10 \cdot 3^x + 3 = 0$ .



## 5-2 Growth and Decay: Rational Exponents

**Objective** To define and apply rational exponents.

In Section 5-1, we considered a 9% annual growth in the cost of a hamburger. We saw that if a hamburger costs \$4 now (time  $t = 0$ ), then its cost  $t$  years from now will be  $C(t) = 4(1.09)^t$ . To find the cost one-half year from now, we must evaluate an expression with a fractional exponent:

$$C\left(\frac{1}{2}\right) = 4(1.09)^{1/2}$$

Although a scientific calculator easily gives an approximate value of \$4.18, there remains the question of what a fractional exponent means.

The definition of a rational exponent given below is made in such a way that the laws for integral exponents on page 170 will continue to hold. However, the base  $b$  must now be a positive real number other than 1.

**Definition of  $b^{1/2}$ :** If exponent law 7 is to hold for rational exponents, then

$$(b^{1/2})^2 = b^{(1/2)(2)} = b^1.$$

Since we know  $(\sqrt{b})^2 = b$ , we define  $b^{1/2}$  to be  $\sqrt{b}$ .

**Definition of  $b^{3/2}$ :**  $b^{3/2} = (b^{1/2})^3 = (\sqrt{b})^3$  and  $b^{3/2} = (b^3)^{1/2} = \sqrt{b^3}$

Either  $(\sqrt{b})^3$  or  $\sqrt{b^3}$  can be used as a definition of  $b^{3/2}$ .

**Definition of  $b^{p/q}$ :** Using reasoning similar to that above, we make these definitions:

$$b^{1/q} = \sqrt[q]{b} \quad \text{and} \quad b^{p/q} = (\sqrt[q]{b})^p \quad \text{or} \quad \sqrt[q]{b^p}$$

**Example 1** Simplify: a.  $16^{1/4}$       b.  $16^{-1/4}$       c.  $8^{2/3}$       d.  $8^{-2/3}$

**Solution**      a.  $16^{1/4} = \sqrt[4]{16} = 2$       b.  $16^{-1/4} = (16^{1/4})^{-1} = 2^{-1} = \frac{1}{2}$   
c.  $8^{2/3} = (\sqrt[3]{8})^2 = 4$       d.  $8^{-2/3} = (8^{2/3})^{-1} = 4^{-1} = \frac{1}{4}$

You can use a calculator to evaluate expressions that contain rational powers. For example,  $5^{2/3} \approx 2.9240$ .

**Example 2** Suppose a car presently worth \$8200 depreciates 20% per year. About how much will it be worth 2 years and 3 months from now?

**Solution** Express 2 years and 3 months as 2.25 years. Then:

$$C(t) = 8200(1 - 0.20)^t = 8200(0.80)^t$$
$$C(2.25) = 8200(0.80)^{2.25} \approx 4963$$

It will be worth about \$4963.

Examples 3 and 4 show how the laws of exponents can be used to solve two kinds of equations containing exponents.

**Example 3** Solve (a)  $2^x = \frac{1}{8}$  and (b)  $9^{x+1} = \sqrt{27}$ .

**Solution** Express both sides of each equation as powers of the same base. Then apply law 3.

a. $2^x = \frac{1}{8}$ $2^x = 2^{-3}$ $x = -3$	b. $9^{x+1} = \sqrt{27}$ $(3^2)^{x+1} = \sqrt{3^3}$ $3^{2x+2} = 3^{3/2}$ $2x + 2 = \frac{3}{2}$ $x = -\frac{1}{4}$
--	--

**Example 4** Solve (a)  $4x^{3/2} = 32$  and (b)  $(x - 1)^{-1/4} - 2 = 0$ .

a. $4x^{3/2} = 32$ $x^{3/2} = 8$ $(x^{3/2})^{2/3} = 8^{2/3}$ $x = 4$	b. $(x - 1)^{-1/4} - 2 = 0$ $(x - 1)^{-1/4} = 2$ $((x - 1)^{-1/4})^{-4} = 2^{-4}$ $x - 1 = \frac{1}{16}$ $x = \frac{17}{16}$
---	--



**Example 5**

A house bought five years ago for \$100,000 was just sold for \$135,000. To the nearest tenth of a percent what was the annual growth rate?

**Solution**

$$A(t) = A_0(1 + r)^t$$

Since  $A_0 = 100,000$

and  $A(5) = 135,000$ ,

$$135,000 = 100,000(1 + r)^5.$$

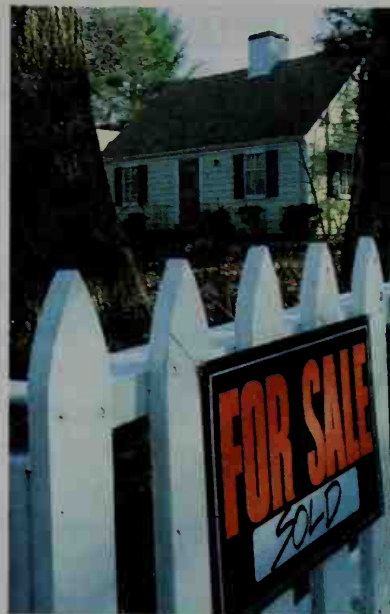
Therefore:

$$1.35 = (1 + r)^5$$

$$(1.35)^{1/5} = (1 + r)$$

$$r \approx 0.0619$$

To the nearest tenth of a percent, the growth rate was 6.2%.

**CLASS EXERCISES**

Simplify each expression.

1. a.  $4^{1/2}$

b.  $4^{-1/2}$

2. a.  $4^{3/2}$

b.  $4^{-3/2}$

3. a.  $-9^{1/2}$

b.  $-9^{-1/2}$

4. a.  $(3^{1/2} \cdot 5^{1/2})^2$

b.  $(3^{1/2} + 5^{1/2})^2$

5.  $\left(\frac{49}{25}\right)^{-1/2}$

6.  $\left(\frac{4}{9}\right)^{3/2}$

7.  $(8^{-1/6})^{-2}$

8.  $8^{3/2} \cdot 2^{3/2}$

9.  $(2x^{-1/3})^3$

10.  $\left(\frac{125}{x^6}\right)^{1/3}$

11.  $\frac{x^{1/3}}{2x^{-2/3}}$

12.  $2x^{3/2} \cdot 4x^{-1/2}$

13. **Consumer Economics** The cost in dollars of a new pair of running shoes  $t$  years from now is  $C(t) = 62(1.05)^t$ .

a. What is the cost now?

b. To find the cost in 2.5 years, use  $t = \underline{\quad ? \quad}$ .

c. To find the cost 9 months ago, use  $t = \underline{\quad ? \quad}$ .

Solve.

14.  $3^{2x} = 3^{12}$

15.  $9^x = 3^5$

16.  $x^{2/3} = 9$

17.  $x^{-1/2} = 4$

18. **Discussion** If the exponent is an integer  $n$ , then the laws of exponents are true for both positive and negative bases. When the definition of exponent is extended to rational numbers, the base must be restricted to positive numbers. Discuss why this restriction must be made.



## WRITTEN EXERCISES

Write each expression using a radical sign and no negative exponents.

- A** 1. a.  $x^{2/3}$       b.  $x^{3/2}$       c.  $5^{1/2} \cdot x^{-1/2}$       d.  $6^{1/3} \cdot x^{2/3}$   
 2. a.  $3y^{2/5}$       b.  $(3y)^{2/5}$       c.  $a^{4/7} \cdot b^{-4/7}$       d.  $a^{1/10} \cdot b^{-1/5}$

Write each expression using positive rational exponents.

3. a.  $\sqrt{x^5}$       b.  $\sqrt[3]{y^2}$       c.  $(\sqrt[6]{2a})^5$       d.  $\sqrt{x} \cdot \sqrt[3]{x} \cdot \sqrt[6]{x}$   
 4. a.  $\sqrt[3]{8x^7}$       b.  $(\sqrt[4]{16x})^3$       c.  $\sqrt[3]{27x^{-6}y^2}$       d.  $\sqrt[4]{x} \cdot \sqrt[3]{x} \div \sqrt[6]{x}$

Simplify.

5. a.  $\left(\frac{9}{25}\right)^{1/2}$       b.  $\left(\frac{9}{25}\right)^{-1/2}$       c.  $\left(\frac{9}{25}\right)^{5/2}$       d.  $\left(\frac{9}{25}\right)^{-1.5}$   
 6. a.  $\left(\frac{27}{8}\right)^{1/3}$       b.  $\left(\frac{27}{8}\right)^{2/3}$       c.  $\left(\frac{27}{8}\right)^{-2/3}$       d.  $\left(\frac{27}{8}\right)^0$   
 7.  $(16^{-3/5})^{5/4}$       8.  $(25^{-1/3})^{-3/2}$       9.  $(81^{1/2} - 9^{1/2})^2$       10.  $(3^{-2} + 4^{-2})^{-1/2}$   
 11.  $(8a^{-6})^{-2/3}$       12.  $(9n^{-5})^{-3/2}$       13.  $(4x^{-3})^{-1/2} \cdot 4x^{1/2}$       14.  $(4a^3)^{1/3} \div (4a^3)^{-2/3}$

15. **Consumer Economics** The cost of a certain brand of camera has been increasing at 8% per year. If a camera now costs \$150, find the cost:

a. 2 years and 6 months from now      b. 4 years and 3 months ago

16. **Business** The value of a computer depreciates at the rate of 25% per year. If a computer is now worth \$2400, find its approximate value:

a. 3 years and 6 months from now      b. 20 months ago

Simplify.

17.  $a^{1/2}(a^{3/2} - 2a^{1/2})$       18.  $2n^{1/3}(n^{2/3} + n^{-1/3})$       19.  $x^{-1/2}(x^{5/2} - 2x^{3/2})$   
 20.  $2n^{-2/3}(n^{8/3} - 3n^{5/3})$       21.  $\frac{x^{1/2} - 2x^{-1/2}}{x^{-1/2}}$       22.  $\frac{y^{-1/3} - 3y^{2/3}}{y^{-4/3}}$   
 23.  $\frac{2n^{1/3} - 4n^{-2/3}}{2n^{-2/3}}$       24.  $\frac{x^{-1/2}(2x^{1/2} - x^{-1/2})}{x^{-1}}$       25.  $\frac{2n^{1/3}(3n^{1/3} - 4n^{4/3})}{2n^{-1/3}}$   
 26.  $\frac{4ab^{-1/2} - 2ab^{1/2}}{(a^2b)^{-1/2}}$       27.  $\frac{(\sqrt[3]{4a})^2}{\sqrt[6]{4a}}$       28.  $\frac{(\sqrt{2x})^5}{(\sqrt{2x})^9}$

Solve.

29.  $8^x = 2^6$       30.  $9^{4x} = 81$       31.  $8^{x-1} = 2^{x+1}$   
 32.  $9^x = 3^{10}$       33.  $8^x = 2^7 \cdot 4^9$       34.  $27^{1-x} = \left(\frac{1}{9}\right)^{2-x}$   
 35. a.  $(8x)^{-3} = 64$       b.  $8x^{-3} = 64$       c.  $(8+x)^{-3} = 64$   
 36. a.  $(2x)^{-2} = 16$       b.  $2x^{-2} = 16$       c.  $4(x-2)^{-2} = 16$

- B** 37. **Finance** A house bought for \$50,000 in 1980 was sold for \$150,000 in 1990. To the nearest percent, what was the annual rate of appreciation (increase) in the value of the house?
38. **Consumer Economics** The price of firewood four years ago was \$140 per cord. Today, a cord of wood costs \$182. To the nearest percent, what has been the annual rate of increase in the cost?
39. **Economics** The *consumer price index* (CPI) is a measure of the average cost of goods and services. The United States government set the index at 100 for the period 1982-1984. In 1988, the index was 118.3. What was the average annual rate of increase (to the nearest tenth of a percent) from 1984 to 1988?
40. **Research** Look in an almanac to find the current consumer price index. Then determine the average annual rate of increase (to the nearest tenth of a percent) for this index since 1984. (See Exercise 39.)
41. **Education** Yearly expenses at a state university have increased from \$14,000 to \$18,500 in the last 4 years. What has been the average annual growth rate in expenses? If this growth rate continues, what will the expenses be 4 years from now?
42. **Research** Find the present cost of something that interests you and its cost several years ago. Find the average annual growth rate.



Solve.

43. a.  $(4x)^3 = 9^6$

b.  $3^{4x} = 9^6$

44. a.  $(4 - x)^{1/2} = 8$     b.  $\left(\frac{1}{2}\right)^{4-x} = 8$

45.  $\frac{2^{x^2}}{2^x} = 64$

46.  $\frac{5^{x^2}}{(5^x)^2} = 125$

47.  $\sqrt{\frac{9^{x+3}}{27^x}} = 81$

48.  $\sqrt[3]{\frac{8^{x+1}}{16^x}} = 32$

Factor. (In Exercise 49(a), factor out  $a^{1/2}b^{1/2}$ .)

49. a.  $a^{3/2}b^{1/2} - a^{1/2}b^{3/2}$

b.  $a^{1/2}b^{-1/2} - a^{3/2}b^{1/2}$

50. a.  $(x - 1)^{1/2} - x(x - 1)^{-1/2}$

b.  $(x + 1)^{3/2} - 4(x + 1)^{1/2}$

51. a.  $(x^2 + 1)^{3/2} - x^2(x^2 + 1)^{1/2}$

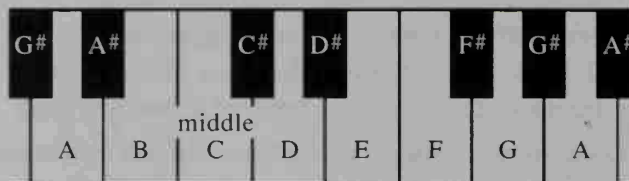
b.  $(x^2 + 2)^{1/2} - x^2(x^2 + 2)^{-1/2}$

52. a.  $(2x + 1)^{2/3} - 4(2x + 1)^{-1/3}$

b.  $(1 + x^2)^{-3/2} - (1 + x^2)^{-1/2}$

If you play the A below middle C on a piano, the piano string vibrates with a frequency of 220 vibrations per second, or 220 hertz (Hz). Frequencies for some notes above A are as follows:

Note	A <sup>#</sup> (A sharp)	B	Middle C	C <sup>#</sup>
Frequency	$220 \cdot 2^{1/12}$	$220 \cdot 2^{2/12}$	$220 \cdot 2^{3/12}$	$220 \cdot 2^{4/12}$



53. **Music** Give the frequencies of the notes D through A above middle C.
54. a. **Music** The frequency of G<sup>#</sup> below middle C is  $220 \cdot 2^{-1/12}$  Hz and the frequency of G<sup>#</sup> above middle C is  $220 \cdot 2^{11/12}$  Hz. Find the ratio of these frequencies. (Note: The interval between these two notes is called an “octave.”)
- b. If a song is played in the key of C, the interval from C to the G above is called a “fifth.” Show that the ratio of these two frequencies is approximately 3 to 2.

Solve for  $x$  by rewriting the equation in quadratic form.

55.  $x^{2/3} - 7x^{1/3} + 12 = 0$  (Hint:  $x^{2/3} = (x^{1/3})^2$ )      56.  $x^{4/3} - 6x^{2/3} + 8 = 0$
57.  $9^{2x} - 2 \cdot 9^x - 3 = 0$       58.  $4^{2x} - 10 \cdot 4^x + 16 = 0$

## 5-3 Exponential Functions

**Objective** To define and use exponential functions.

We have defined rational exponents, but not irrational ones. For example, how is  $3^{\sqrt{2}}$  defined? To find out, complete the following activity.

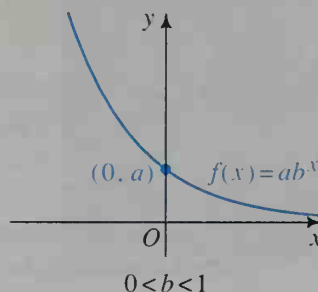
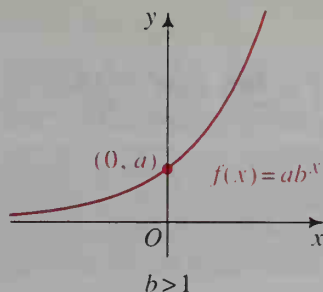
### Activity

The first row in the table shows a sequence of decimal approximations for  $\sqrt{2}$ . Copy the table. Complete it by using the power key on a calculator. (Write all the digits displayed.) What do you notice about  $3^x$ ?

$x$	1.4	1.41	1.414	1.4142	1.41421
$3^x$	?	?	?	?	?

As you discovered, the sequence  $3^{1.4}, 3^{1.41}, 3^{1.414}, \dots$  seems to approach a fixed number. It can be proved that if any sequence has  $\sqrt{2}$  as its limit, then the corresponding sequence of powers of 3 also has a limit, defined to be  $3^{\sqrt{2}}$ .

Any function of the form  $f(x) = ab^x$ , where  $a > 0$ ,  $b > 0$ , and  $b \neq 1$  is called an **exponential function with base  $b$** . Its domain is the set of all real numbers. The range is the set of positive real numbers. See the graphs below.



**Example 1** If  $f$  is an exponential function,  $f(0) = 3$ , and  $f(2) = 12$ , find  $f(-2)$ .

**Solution**

Since  $f$  is an exponential function,  $f(x) = ab^x$  for some constants  $a$  and  $b$ . We use the given values of  $f$  to find  $a$  and  $b$ .

Since  $f(0) = 3$ , then  $ab^0 = 3$ . So  $a = 3$ .

Since  $a = 3$  and  $f(2) = 12$ , then  $3b^2 = 12$ . So  $b = 2$ .

Therefore,  $f(x) = 3 \cdot 2^x$ . Thus,  $f(-2) = 3 \cdot 2^{-2} = \frac{3}{4}$ .

When exponential functions are used to describe exponential growth and decay, the variable  $t$  is often used to represent time. Although these functions can always be written as  $f(t) = ab^t$ , they are often written in other forms such as the two listed below.

(1)  $A(t) = A_0(1 + r)^t$       $A_0$  = amount at time  $t = 0$  and  $r$  = growth rate

(2)  $A(t) = A_0b^{t/k}$       $k$  = time needed to multiply  $A_0$  by  $b$

**Example 2** A bank advertises that if you open a savings account, you can double your money in 12 years. Express  $A(t)$ , the amount of money after  $t$  years, in each of the two forms listed above.

**Solution**

Since 12 years is the time needed to multiply  $A_0$  by 2, form (2) gives:

$$A(t) = A_0 \cdot 2^{t/12}$$

Notice that  $A(12) = A_0 \cdot 2^{12/12} = 2A_0$ .

To express  $A(t)$  in form (1), reason as follows:

$$\begin{aligned} A(t) &= A_0 \cdot 2^{t/12} = A_0 \cdot (2^{1/12})^t \\ &\approx A_0(1.059)^t = A_0(1 + 0.059)^t \end{aligned}$$

The so-called rule of 72 stated below provides an estimate of the time it takes for a quantity to double.

### The Rule of 72

If a quantity is growing at  $r\%$  per year (or month), then the **doubling time** is approximately  $(72 \div r)$  years (or months).

For example, if a quantity grows at 8% per month, then its doubling time will be about  $72 \div 8 = 9$  months. If a population grows exponentially at 2% per year, then it will double in about  $72 \div 2 = 36$  years.

**Example 3** A radioactive isotope has a **half-life** of 5 days. This means that half the substance decays in 5 days. At what rate does the substance decay each day?

**Solution** We use the form:  $A(t) = A_0 b^{t/k} = A_0 \left(\frac{1}{2}\right)^{t/5}$   
(Notice that  $A(5) = A_0 \cdot \frac{1}{2}$ , which agrees with the half-life being 5 days.)  
To find the daily decay rate, we rewrite  $A(t)$  as follows.

$$A(t) = A_0 \left( \left( \frac{1}{2} \right)^{1/5} \right)^t \approx A_0 (0.87)^t = A_0 (1 - 0.13)^t$$

Thus, the daily decay rate is approximately 13%.

## CLASS EXERCISES

1. The graph of  $y = ab^x$  has y-intercept 7. Find the value of  $a$ .
2. If  $h(x) = ab^x$ ,  $h(0) = 5$ , and  $h(1) = 15$ , find the values of  $a$  and  $b$ .
3. For what values of  $b$  is  $g(x) = b^x$  an increasing function? a decreasing function? (See the definitions of increasing and decreasing on page 138.)
4. State the domain and range of  $f(x) = 3 \cdot 4^x$ .
5. Give an integer estimate for  $5^\pi$ . Find  $5^\pi$  on a calculator. What is the difference between your estimate and the calculator value?
6. **Finance** According to the rule of 72, how long does it take to double an investment that grows at the rate of 9% per year? 6% per month?
7. **Finance** Suppose that today you invest some money that grows to the amount  $A(t) = 1000 \cdot 2^{t/10}$  in  $t$  years.
  - a. How much money did you invest?
  - b. How long does it take to double your money?
8. **Biology** Suppose that  $t$  hours from now the population of a bacteria colony is given by  $P(t) = 90(100)^{t/8}$ .
  - a. What is the population when  $t = 0$ ?
  - b. How long does it take for the population to be multiplied by 100?



9. **Biology** A colony of bacteria decays so that the population  $t$  days from now is given by  $A(t) = 1000\left(\frac{1}{2}\right)^{t/4}$ .

- What is the amount present when  $t = 0$ ?
- How much will be present in 4 days?
- What is the half-life?

10. **Discussion** Can the data below be described as exponential growth?

$x$	0	1	2	3	4
$y$	4	4.4	4.8	5.3	5.8



Justify your answer by using a graph, an equation, or a logical argument. Can you think of a situation that the data might describe?

## WRITTEN EXERCISES

Evaluate each expression with a calculator.

- A** 1.  $6^\pi$  and  $\pi^6$       2.  $3.6^{\sqrt{2}}$  and  $\sqrt{2}^{3.6}$

Find an exponential function having the given values.

- $f(0) = 3, f(1) = 15$
- $f(0) = 5, f(3) = 40$
- $f(0) = 64, f(2) = 4$
- $f(0) = 80, f(4) = 5$
- Physics** The half-life of a radioactive isotope is 4 days. If 3.2 kg are present now, how much will be present after:
  - 4 days?
  - 8 days?
  - 20 days?
  - $t$  days?
- Physics** The half-life of radium is about 1600 years. If 1 kg is present now, how much will be present after:
  - 3200 years?
  - 16,000 years?
  - 800 years?
  - $t$  years?
- Physics** The table shows the amount  $A(t)$  in grams of a radioactive element present after  $t$  days. Suppose that  $A(t)$  decays exponentially.

$t(\text{days})$	0	2	4	6	8	10
$A(t)$	320	226	160	115	80	57

- What is the half-life of the element?
- About how much will be present after 16 days?
- Find an equation for  $A(t)$ .

10. **Geography** The table shows the population  $P(t)$  (in thousands) for a small mythical nation at various times.

$t$ (year)	1825	1850	1875	1900	1925	1950	1975
$P(t)$	200	252	318	401	504	635	800

- Does it appear that this population is growing exponentially?
  - About how long does it take for the population to double?
  - Find an equation for  $P(t)$ . (*Hint:* The exponent contains  $t - 1825$ .)
11. **Business** The value of a car  $t$  years from now is given by  $V(t) = 4000(0.85)^t$ .
- What is the annual rate of depreciation?
  - In how many years will the value of the car be about half what it is now?
12. a. **Geography** Suppose the population of a nation grows at 3% per year. If the population was 30,000,000 people in 1990, what will be the population, to the nearest million, in the year 2000?
- According to the rule of 72, how long does it take for the population to double?
13. a. **Finance** If \$1000 is invested so that it grows at the rate of 10% per year, what will the investment be worth in 20 years?
- According to the rule of 72, in approximately how many years will the investment double in value?
14. **Biology** A bacteria colony triples every 4 days. The population is  $P_0$  bacteria. What will the population  $P(t)$  be  $t$  days later?
15. **Consumer Economics** If the price of sneakers increases 6% per year, about how long will it take for the price to double?
16. **Medicine** When a certain medicine enters the blood stream, it gradually dilutes, decreasing exponentially with a half-life of 3 days. The initial amount of the medicine in the blood stream is  $A_0$  milliliters. What will the amount be 30 days later?
17. **Medicine** An amount  $A_0$  of radioactive iodine has a half-life of 8.1 days. In terms of  $A_0$ , how much is present after 5 days? (Radioactive iodine is used to evaluate the health of the thyroid gland.)
18. a. Let  $f(x) = 2^x$ . Complete the table.

$x$	-2	-1	0	1	2
$f(x)$	?	?	?	?	?

- Graph the function by plotting points.

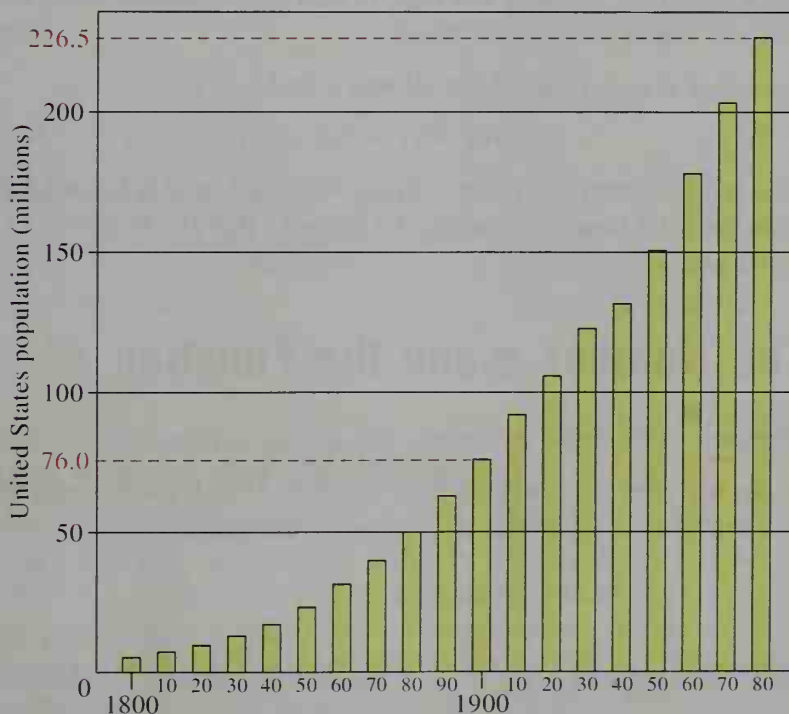


- c. Explain why  $f$  has an inverse.
- d. Graph  $f^{-1}$  on the same set of axes as  $f$ .
- e. State the domain and range of  $f$  and of  $f^{-1}$ .



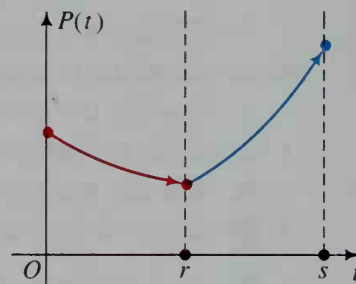
You may find it helpful to have a graphing calculator to complete Exercises 19–21. You need a graphing calculator for Exercise 22.

- B** 19. a. Graph the functions  $y = 2^x$  and  $y = \left(\frac{1}{2}\right)^x$  on a single set of axes.  
 b. How are the graphs related to each other?
20. a. Graph the functions  $y = 4^x$  and  $y = 4^{x/2}$  on a single set of axes.  
 b. How are the graphs related to each other?
21. a. Graph the functions  $y = 2^x$  and  $y = 2^{x-1}$  on a single set of axes.  
 b. How are these graphs related to each other?
22. a. Graph the functions  $y = x^2$  and  $y = 2^x$  on a single set of axes.  
 b. Use a graphing calculator to solve  $x^2 = 2^x$ .
23. a. **Geography** The bar graph below gives the population of the United States for each census from 1800 to 1980. The tops of the bars lie approximately on the curve  $y = ab^x$ , where  $x$  is the number of years since 1800. Find the values of  $a$  and  $b$  to the nearest thousandth.  
 b. Use part (a) to predict the population in the years 2000 and 2050.



24. a. If  $f(x) = 2^x$ , show that  $f(x+1) = 2f(x)$ .  
 b. Translating the graph of  $f$  1 unit (left or right?) is equivalent to (vertically or horizontally?) stretching the graph of  $f$  by ? units.

25. **Investigation** Go to a bank in your area.
- What kinds of savings accounts does the bank offer? List the characteristics of each type of account. Write down the interest rate, any conditions placed on opening the account, any penalties placed on withdrawals, and so forth.
  - Suppose that you have \$1000 to invest. Which of the plans available would you choose?
  - Make a table showing how the money in the account would grow.
  - Discuss your investigation with the class.
26. **Visual Thinking** In the diagram, the red and blue curves are exponential in behavior.
- Describe the behavior of  $P(t)$  on the interval  $0 \leq t \leq r$ .
  - Describe the behavior of  $P(t)$  on the interval  $r < t \leq s$ .
  - Describe a real-world situation for which the diagram could be a model.
27. If  $f(x) = 10^{2x+1}$  and  $x$  is a nonnegative integer, show that  $f(x+2) - f(x)$  is divisible by 99.
28. Over the past 60 years, the population of a country increased from 100 million people to 200 million people.
- Find an equation that gives the population  $t$  years from now.
  - Use a calculator and trial and error to find approximately when the population will reach 300 million people.



- C** 29. Suppose that  $f(x)$  is positive for all real  $x$  and

$$f(x+y) = f(x) \cdot f(y)$$

for all  $x$  and  $y$ . Prove that  $f(0) = 1$ ,  $f(2x) = [f(x)]^2$ , and  $f(3x) = [f(x)]^3$ . Give an equation for a function that has the property that  $f(x+y) = f(x) \cdot f(y)$  for all real  $x$  and  $y$ .

## 5-4 The Number $e$ and the Function $e^x$

**Objective** To define and apply the natural exponential function.

You have already seen many exponential functions. In advanced mathematics, the most important base is the irrational number  $e$ , defined as

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

This is read “the limit of  $\left(1 + \frac{1}{n}\right)^n$  as  $n$  approaches infinity.” Although you will not study limits until Chapter 13, you can get an idea of the value of  $e$  by completing the Activity on the next page.

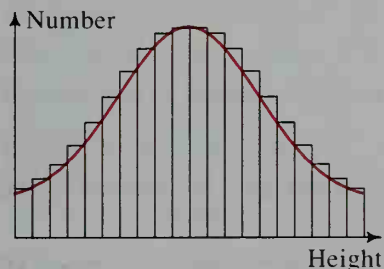
## Activity



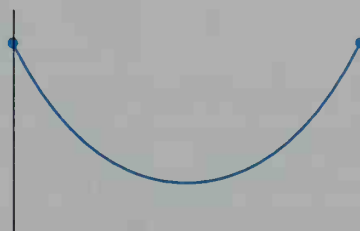
1. Copy and complete the table by using a calculator. What do you notice about the successive values of  $\left(1 + \frac{1}{n}\right)^n$ ?
2. Using a graphing calculator, graph  $y = \left(1 + \frac{1}{x}\right)^x$ . What happens to  $y$  when  $x$  becomes large?

$n$	$\left(1 + \frac{1}{n}\right)^n$
10	?
100	?
1000	?
10,000	?
100,000	?

You can see that as  $n$  increases,  $\left(1 + \frac{1}{n}\right)^n$  appears to get closer and closer to 2.718 . . . . The Swiss mathematician Leonhard Euler (1707–1783) proved this, and the limit is called  $e$  in his honor. Values of  $e^x$  can be obtained using a calculator or table (page 821). The function  $e^x$  is called the **natural exponential function**. In Exercise 13, you will be asked to draw its graph and the graph of its inverse. The number  $e$  is extremely important in advanced mathematics. It appears in unexpected places including statistics and physics.



Statistics: Distribution of heights of 18-year-olds



Physics: Hanging rope

The equations of these two graphs contain  $e^x$ .

## Compound Interest and the Number $e$

Suppose you invest  $P$  dollars (the *principal*) at 12% interest compounded semi-annually. Each half year, your money grows by 6%. At the end of the year, you have  $P(1.06)^2 = 1.1236P$  dollars. If the interest had been compounded quarterly (4 times per year), then it would grow by 3% each quarter. At the end of the year, you would have  $P(1.03)^4 = 1.1255P$  dollars. See the table at the top of the next page. The table also shows the results when the 12% annual interest rate is compounded monthly and daily. For simplicity, the investment is \$1.



Interest period	% growth each period	Growth factor during period	Amount
Annually	12%	$1 + \frac{0.12}{1}$	$1.12^1 = 1.12$
Semiannually	6%	$1 + \frac{0.12}{2}$	$1.06^2 = 1.1236$
Quarterly	3%	$1 + \frac{0.12}{4}$	$1.03^4 \approx 1.1255$
Monthly	1%	$1 + \frac{0.12}{12}$	$1.01^{12} \approx 1.1268$
Daily (360 days)	$\frac{12}{360}\%$	$1 + \frac{0.12}{360}$	$\left(1 + \frac{0.12}{360}\right)^{360} \approx 1.1275$
$k$ times per year	$\frac{12}{k}\%$	$1 + \frac{0.12}{k}$	$\left(1 + \frac{0.12}{k}\right)^k$

The numbers in the right-hand column of the table suggest that the amount approaches a fixed value. To find this number, note that

$$\left(1 + \frac{0.12}{k}\right)^k = \left(1 + \frac{\frac{1}{k}}{\frac{0.12}{1}}\right)^k = \left[\left(1 + \frac{1}{\frac{k}{0.12}}\right)^{\frac{k}{0.12}}\right]^{0.12} = \left[\left(1 + \frac{1}{n}\right)^n\right]^{0.12}$$

where  $n = \frac{k}{0.12}$ . Since the limit of  $\left(1 + \frac{1}{n}\right)^n$  as  $n$  increases is  $e$ , the limit of the expression above is  $e^{0.12}$ , approximately 1.1275. Since \$1 grows to \$1.1275 in one year, a bank that compounds daily will advertise that its 12% annual interest is equivalent to a 12.75% *effective annual yield*.

In general, if you invest  $P$  dollars at an annual rate  $r$  (expressed as a decimal) compounded continuously, then  $t$  years later your investment will be worth  $Pe^{rt}$  dollars. (See Written Exercises 17 and 18.)

The same principle applies to any quantity, such as population, where compounding takes place “all the time.” If  $P_0$  is the initial amount, then the amount at any future time  $t$  is  $P(t) = P_0e^{rt}$ .

## CLASS EXERCISES

Use a calculator to evaluate the following.

- $e^2$
- $e^{3.2}$
- $e^{-4}$
- $e^{\sqrt{2}}$
- $e^1$
- Which is larger,  $e^\pi$  or  $\pi^e$ ? Try to estimate before using your calculator.
- Reading** From your reading, how is  $e$  defined?
  - Estimate the value of  $\left(1 + \frac{1}{1,000,000}\right)^{1,000,000}$ .

8. **Finance** If money is invested at 8% compounded semiannually, then each year the investment is multiplied by  $1.04^2$ . What is the investment multiplied by if interest is compounded:
- a. quarterly?                      b. 12 times a year?                      c. continuously?
9. **Finance** A bank advertises that its 5% annual interest rate compounded daily is equivalent to a 5.13% effective annual yield. What does this mean?

## WRITTEN EXERCISES

- A** 1. a. Evaluate  $\left(1 + \frac{1}{n}\right)^n$  when  $n = 5000$ , and  $n = 5,000,000$ .  
 b. Compare your answers in part (a) with an approximation for  $e$ .
2. a. Evaluate  $\left(1 - \frac{1}{n}\right)^n$  for  $n = 100$ ,  $n = 10,000$ , and  $n = 1,000,000$ .  
 b. Compare your answers in part (a) with an approximation for  $e^{-1}$ .  
 c. What appears to be  $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n$ ?
3. Which is larger,  $e^{\sqrt{2}}$  or  $\sqrt{2}^e$ ? Try to estimate before using your calculator.
4. Evaluate:      a.  $e^{0.08}$                       b.  $e^{-0.08}$                       c.  $e^{4/3}$
5. **Finance** Suppose you invest \$1.00 at 6% annual interest. Calculate the amount that you would have after one year if the interest is compounded (a) quarterly, (b) monthly, (c) continuously.
6. **Finance** Repeat Exercise 5 if the annual rate is 8%.
7. **Finance** One hundred dollars deposited in a bank that compounds interest quarterly yields \$107.50 over 1 year. Find the effective annual yield.
8. **Finance** After a year during which interest is compounded quarterly, an investment of \$800 is worth \$851. What is the effective annual yield?
9. With which plan would an investor earn more, Plan A or B?  
 Plan A: A 6% annual rate compounded annually over a 10-year period  
 Plan B: A 5.5% annual rate compounded quarterly over a 10-year period
10. With which plan would an investor earn more, Plan A or B?  
 Plan A: An 8% annual rate compounded quarterly for 5 years  
 Plan B: A 7.5% annual rate compounded daily for 5 years
11. **Finance** Suppose that \$1000 is invested at 7% interest compounded continuously. How much money would be in the bank after 5 years?
12. **Biology** A population of ladybugs rapidly multiplies so that the population  $t$  days from now is given by  $A(t) = 3000e^{0.017t}$ .  
 a. How many ladybugs are present now?  
 b. How many will there be after a week?





You might find it helpful to have a graphing calculator to complete Exercises 13–16.

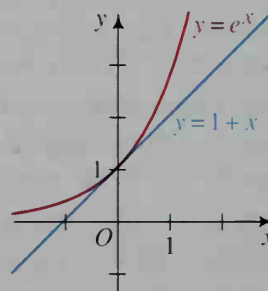
13. a. Use a calculator to sketch the graph of  $f(x) = e^x$ .  
b. Explain why this function has an inverse.  
c. Without using a calculator, graph the inverse.
14. On a single set of axes, graph  $y = e^x$  and  $y = e^{-x}$ . How are the graphs related?
15. Graph  $y = \frac{e^x + e^{-x}}{2}$ . (See the hanging rope on page 187.)
16. Sketch the graph of  $y = e^{-x^2}$ . (See the “bell-shaped curve” on page 187.)



17. a. **Finance** Suppose that  $P$  dollars is invested at an annual interest rate  $r$  ( $r$  a decimal) with interest compounded  $k$  times a year. Explain why the value of the investment at the end of the year is  $P\left(1 + \frac{r}{k}\right)^k$ .  
b. Show that the expression in part (a) is approximately  $Pe^r$  if  $k$  is large.
18. a. **Finance** Suppose that \$10,000 is invested at an annual rate of 9% and that interest is compounded every second for 365 days. Find the value of this investment at the end of one year by using the expression in part (a) of Exercise 17. (First find  $k$ , the number of seconds in one year.)  
b. Compare your answer in part (a) with the value of  $10,000e^{0.09}$ .



19. It can be proved that  $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$ .  
Approximate  $e$  by using the first five terms. (Note:  $n!$ , read “ $n$  factorial,” denotes  $n(n-1)(n-2) \cdots 2 \cdot 1$ . For example,  $3! = 3 \cdot 2 \cdot 1 = 6$ .)
20. The points  $A(0, f(0))$  and  $B(h, f(h))$  are on the graph of  $f(x) = e^x$ . Find the slope of  $\overline{AB}$  if  $h$  is (a) 1, (b) 0.1, and (c) 0.01.
21. It can be proved that the line  $y = 1 + x$  is tangent to the graph of  $y = e^x$  at  $(0, 1)$ . Thus,  $e^x \approx 1 + x$  when  $|x|$  is small. Show that  $(1 + x)^{1/x}$  is approximately  $e$ .
22. Show that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n = \sqrt{e}$  by using the definition of  $e$ .
23. Determine the value of  $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$ .  
(See Exercise 22.)



## ////COMPUTER EXERCISE

Write a program that prints a table giving the value of a \$1000 investment compounded quarterly at 4.5%, 5%, 5.5%, 6%, and 6.5% annual interest rates after each of ten successive years.

# Logarithms

## 5-5 Logarithmic Functions

**Objective** To define and apply logarithms.

Now we will explore *logarithms*, numbers used to measure the severity of earthquakes, the loudness of sounds, and the brightness of stars.

We define the *common logarithm* of an integral power of 10 to be its exponent, as shown in the chart below.

		$\times 10$	$\times 10$	$\times 10$	$\times 10$		
		$\curvearrowright$	$\curvearrowright$	$\curvearrowright$	$\curvearrowright$		
Number	$10^0$	$10^1$	$10^2$	$10^3$	$10^4$	$\cdots$	$10^k$
Common logarithm	0	1	2	3	4	$\cdots$	$k$
		$\curvearrowleft$	$\curvearrowleft$	$\curvearrowleft$	$\curvearrowleft$		
		+1	+1	+1	+1		

We denote the common logarithm of  $x$  by  $\log x$ . For example,

$$\log 10^3 = 3, \log 10^4 = 4, \text{ and } \log 10^k = k.$$

In general, the **common logarithm** of any positive real number  $x$  is defined to be the exponent you get when you write  $x$  as a power of 10.

$$\log x = a \text{ if and only if } 10^a = x.$$

For example:

$$\log 6.3 \approx 0.8 \text{ because } 6.3 \approx 10^{0.8}$$

You can verify this by evaluating  $10^{0.8}$  on a calculator.

You can also use a scientific calculator to find common logarithms directly. On most calculators, you enter the number whose common logarithm you wish to find and then press the “log” key.

The table above illustrates that common logarithms increase arithmetically when numbers increase exponentially. This fact enables us to make a linear scale to measure quantities that increase exponentially.

Common logarithms are useful in applications involving the perception of sound. Every sound has an intensity level due to the power of the sound wave. In the table on the next page,  $I_0$  represents the intensity of a sound barely audible. The intensity level  $I$  of any other sound is measured in terms of  $I_0$ . (See column 2 of the table on the next page.) The human ear perceives a sound as soft or loud. The unit for measuring the loudness of a sound is the decibel (dB) and it is related to the intensity of a sound by:

$$\text{decibel level of } I = 10 \log \frac{I}{I_0}$$

(See column 3 of the table below.) For example, a loud stereo set has an intensity level of  $10^8 I_0$  and is perceived to have a decibel level of:

$$\begin{aligned} 10 \log \frac{10^8 I_0}{I_0} &= 10 \log 10^8 \\ &= 10 \times 8 = 80 \end{aligned}$$

Sound	$I$	Decibel level
Barely audible	$I_0$	0
Whisper	$10I_0$	10
Leaves in a breeze	$10^2 I_0$	20
Soft recorded music	$10^4 I_0$	40
Two-person conversation	$10^6 I_0$	60
Loud stereo set	$10^8 I_0$	80
Subway train	$10^{10} I_0$	100
Jet at takeoff	$10^{12} I_0$	120
Pain in eardrum	$10^{13} I_0$	130

Many people think that when the intensity of a sound is doubled, the decibel level is doubled also. The following example shows that this is not so.

**Example 1** Two loud stereos are playing the same music simultaneously at 80 dB each. What is the decibel level of the combined sound? By how many decibels is the decibel level of the two stereos greater than the decibel level of one stereo?

**Solution** Since one stereo at 80 dB has an intensity  $10^8 I_0$ , two stereos will have an intensity that is twice that amount.

$$2(10^8 I_0)$$

With a calculator, we find that the decibel level corresponding to the two stereos is

$$10 \log \left( \frac{2 \times 10^8 I_0}{I_0} \right) \approx 83.$$

Since the decibel level of one stereo is 80 dB, there is only about a 3 dB increase in the decibel level when the two stereos are played at the same time.

The decibel scale is an example of a *logarithmic scale*. Such a scale is also used to measure acidity and brightness. (See Exercises 45 and 46.)



Common logarithms have base 10. Logarithms to other bases are sometimes used. The *logarithm to base  $b$*  of a positive number  $x$ , denoted by  $\log_b x$ , is defined to be the exponent  $a$  that you get when you write  $x$  as a power of  $b$ . (Note:  $b > 0$ ,  $b \neq 1$ )

$$\log_b x = a \text{ if and only if } x = b^a.$$

### Example 2

$$\begin{array}{lll} \log_5 25 = 2 & \text{because} & 5^2 = 25. \\ \log_5 125 = 3 & \text{because} & 5^3 = 125. \\ \log_2 \frac{1}{8} = -3 & \text{because} & 2^{-3} = \frac{1}{8}. \end{array}$$

The base  $b$  logarithmic function, whose graph is shown in blue below, is the inverse of the base  $b$  exponential function, whose graph is shown in red. Notice the domain and range of each function.

Base  $b$  exponential function:

$$f(x) = b^x$$

Domain: All reals

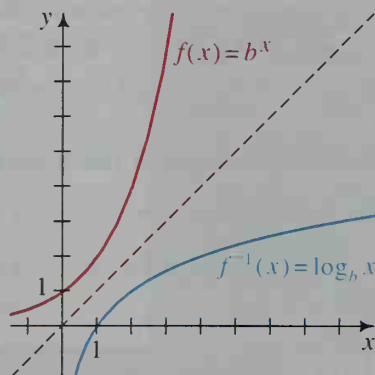
Range: Positive reals

Base  $b$  logarithmic function:

$$f^{-1}(x) = \log_b x$$

Domain: Positive reals

Range: All reals



The most important logarithmic function in advanced mathematics and statistics has the number  $e$  as its base. This function is called the *natural logarithm function*. The natural logarithm of  $x$  is usually denoted  $\ln x$  although sometimes it is written  $\log_e x$ . The value of  $\ln x$  can be found by using a scientific calculator.

$$\ln x = k \text{ if and only if } e^k = x.$$

For example,

$$\ln 5 \approx 1.6 \text{ because } e^{1.6} \approx 5.$$

### Example 3

Find the value of  $x$  to the nearest hundredth.

a.  $10^x = 75$       b.  $e^x = 75$

### Solution

- By definition,  $x$  is the common logarithm of 75.  
Thus,  $x = \log 75 \approx 1.88$ .
- By definition,  $x$  is the natural logarithm of 75.  
Thus,  $x = \ln 75 \approx 4.32$ .

## CLASS EXERCISES

- Use a calculator to approximate each logarithm to four decimal places.
  - $\log 30$
  - $\log 587$
  - $\log 6800$
  - $\log 0.3$
  - $\log 0.0003$
- Try to find  $\log(-7)$  on a calculator. What happens? Why?
- Suppose you are told that  $\log 8 \approx 0.9031$ . Find each of the following without using a calculator. Then check your answers with a calculator.
  - $\log 80$
  - $\log 800$
  - $\log 80,000$
  - $\log 0.8$
  - $\log 0.08$
- The statement  $\log_2 16 = 4$  means that  $16 = \underline{\quad ? \quad}$ .
  - The statement  $\log 31 \approx 1.49$  means that  $31 \approx \underline{\quad ? \quad}$ .
  - The statement  $\log 61 \approx 1.79$  means that  $61 \approx \underline{\quad ? \quad}$ .
- Find each logarithm without using a calculator.
  - $\log_7 49$
  - $\log_2 16$
  - $\log_2 \frac{1}{8}$
  - $\log_5 \frac{1}{5}$
  - $\log_5 \sqrt{5}$
- Use a calculator to approximate each logarithm to four decimal places.
  - $\ln 2$
  - $\ln 3$
  - $\ln 2.7$
  - $\ln 2.8$
  - $\ln e$
- Find the value of  $x$  to the nearest hundredth: (a)  $10^x = 50$  (b)  $e^x = 50$

Solve each equation. (Do not use a calculator.)

- $\log_5 x = 2$
  - $\log_6 x = 2$
  - $\log x = 2$
  - $\ln x = 2$
- $\log_x 121 = 2$
  - $\log_x 64 = 3$
  - $\log_x \left(\frac{1}{2}\right) = -1$
  - $\log_x \sqrt{6} = \frac{1}{2}$

## WRITTEN EXERCISES

Write each equation in exponential form.

- A**
- $\log_4 16 = 2$
  - $\log_4 64 = 3$
  - $\log_6 \left(\frac{1}{36}\right) = -2$
  - $\log_4 8 = 1.5$
  - $\log 1000 = 3$
  - $\log 40 \approx 1.6$
  - $\ln 8 \approx 2.1$
  - $\ln 0.2 \approx -1.6$
  - What does it mean to say that  $x$  is the common logarithm of  $N$ ?
    - Solve (1)  $10^x = 7$  and (2)  $10^x = 0.562$  for  $x$  to the nearest hundredth.
    - What does it mean to say that  $x$  is the natural logarithm of  $N$ ?
    - Solve (1)  $e^x = 12$  and (2)  $e^x = 0.06$  for  $x$  to the nearest hundredth.
  - Find the value of  $x$  to the nearest hundredth: (a)  $10^x = 170$  (b)  $e^x = 500$


Find each logarithm. (Do not use a calculator.)

- $\log 100$
  - $\log 10,000$
  - $\log 0.01$
  - $\log 0.0001$
- $\log_2 4$
  - $\log_2 32$
  - $\log_2 64$
  - $\log_2 2^{10}$
- $\log_3 9$
  - $\log_3 27$
  - $\log_3 243$
  - $\log_3 3^8$
- $\log_5 0.2$
  - $\log_5 \frac{1}{125}$
  - $\log_5 \sqrt[3]{5}$
  - $\log_5 1$

15. a.  $\log_4 64$       b.  $\log_4 \frac{1}{64}$       c.  $\log_4 \sqrt[4]{4}$       d.  $\log_4 1$   
 16. a.  $\log_6 36$       b.  $\log_{36} 6$       c.  $\log_6 6\sqrt{6}$       d.  $\log_6 \sqrt[3]{\frac{1}{6}}$   
 17. a.  $\ln e$       b.  $\ln e^2$       c.  $\ln \frac{1}{e}$       d.  $\ln \sqrt{e}$   
 18. a.  $\log 10^8$       b.  $\log_2 2^8$       c.  $\log_5 5^8$       d.  $\ln e^8$   
 19. Given  $\log 4.17 \approx 0.6201$ , find:    a.  $\log 417$     b.  $\log 0.417$     c.  $\log 0.0417$   
 20. Given  $\log 6.92 \approx 0.8401$ , find:    a.  $\log 692$     b.  $\log 0.692$     c.  $\log 0.00692$   
 21. Given  $\ln 10 \approx 2.3026$ , find:    a.  $\ln 0.1$     b.  $\ln 0.01$     c.  $\ln 100$   
 22. Given  $\ln 5 \approx 1.6094$ , find:    a.  $\ln 0.2$     b.  $\ln 25$     c.  $\ln 0.04$

- B** 23. **Physics** Find the decibel level for each sound with the given intensity  $I$ .  
 a. Average car at 70 km/h,  $I = 10^{6.8}I_0$       b. Whisper,  $I = 10^{1.5}I_0$   
 24. **Physics** Find the decibel level for each sound with the given intensity  $I$ .  
 a. Softly played flute,  $I = 10^{4.1}I_0$       b. Vacuum cleaner,  $I = 10^{7.5}I_0$   
 25. a. **Physics** Find the decibel level of two stereos, playing the same music simultaneously at 62 dB.  
 b. Find the decibel level if three stereos play instead of two.  
 26. **Physics** The decibel level of one car accelerating from rest to 50 km/h is 80 dB. Find the decibel level of four similar cars accelerating at once.  
 27. Graph  $f(x) = 2^x$  and  $f^{-1}(x) = \log_2 x$  on a single set of axes. Give the domain and range of each function.  
 28. a. Find  $2^{\log_2 8}$ ,  $5^{\log_5 25}$ , and  $3^{\log_3 x}$ .  
 b. If  $f(x) = 3^x$ , what is  $f^{-1}(x)$ ? What is  $(f \circ f^{-1})(x)$ ?



 Part (b) of Exercises 29 and 30 requires the use of a computer or a graphing calculator. You may also use a computer or a graphing calculator to confirm your answers to part (b) of Exercises 31 and 32.

29. a. Consider the functions  $f(x) = e^x$  and  $g(x) = \ln x$ . Find rules for  $(f \circ g)(x)$  and  $(g \circ f)(x)$  and give the domain of each composite function.  
 b. To confirm the domains in part (a), graph the composite functions using a computer or a graphing calculator. Enter the equations of the composite functions in their *unsimplified* form. (See Example 2 on page 127.)  
 30. Repeat Exercise 29 for the functions  $f(x) = e^{-x}$  and  $g(x) = -\ln x$ .  
 31. a. On separate sets of axes, graph  $y = \log |x|$  and  $y = |\log x|$ .  
 b. Give the domain and range of each function.  
 32. Repeat Exercise 31 for the functions  $y = -\ln x$  and  $y = \ln(-x)$ .  
 33. Give the domain, range, and zeros of  $y = \log x + 3$  and  $y = \log(x + 3)$ .  
 34. Give the domain, range, and zeros of  $y = \log_2(x - 2)$  and  $y = \log_2 x - 2$ .

Solve for  $x$  without using a calculator. You may leave answers in terms of  $e$  if necessary.

- |                                |                        |                       |
|--------------------------------|------------------------|-----------------------|
| 35. a. $\log x = 3$            | b. $\log  x  = 3$      | c. $\log  x - 1  = 3$ |
| 36. a. $\log_6 x = 2$          | b. $\ln x = 2$         | c. $\ln  x  = 2$      |
| 37. a. $\log_4 x = 1.5$        | b. $\ln x = 1.5$       | c. $\ln x = 0$        |
| 38. a. $\log (x^2 - 1) = 2$    | b. $\ln (x^2 - 1) = 2$ | c. $\ln  x  = 1$      |
| 39. a. $\log_5 (\log_3 x) = 0$ | b. $\log (\log x) = 1$ | c. $\ln (\ln x) = 1$  |
| 40. a. $\log_6 (\log_2 x) = 1$ | b. $\ln (x - 2) = 1$   | c. $(\log x)^2 = 4$   |

Solve for  $x$  using a calculator. Give answers to the nearest hundredth.

- |                       |                    |                    |
|-----------------------|--------------------|--------------------|
| 41. a. $\log x = 0.7$ | b. $\log x = 3.7$  | c. $\log x = -0.3$ |
| 42. a. $\log x = 1.4$ | b. $\log x = 0.4$  | c. $\log x = -0.6$ |
| 43. a. $\ln x = 4.2$  | b. $\ln x = -1.5$  | c. $e^x = 5$       |
| 44. a. $\ln x = 1.73$ | b. $\ln x = -0.52$ | c. $e^x = 16$      |

45. **Chemistry** The pH of a solution is a measure of how acidic or alkaline the solution is. The pH of a solution is given by:

$$\text{pH} = -\log_{10} (\text{hydrogen ion concentration})$$

Pure water, which has a pH of 7, is considered neutral. A solution with pH less than 7 is acidic; a solution with pH greater than 7 is alkaline. Find the pH of the following solutions and classify them as acidic, neutral, or alkaline.

Solution	Hydrogen ion concentration (moles per liter)
Human gastric juices	$10^{-2}$
Acid rain	$3 \times 10^{-5}$
Pure water	$10^{-7}$
Good soil for vegetables	$5 \times 10^{-7}$
Sea water	$10^{-8}$

46. **Astronomy** The observed brightness of stars is classified by magnitude. Two stars can be compared by giving their magnitude difference  $d$  or their brightness ratio  $r$ . The numbers  $d$  and  $r$  are related by the equation  $d = 2.5 \log r$ . Comparing a first magnitude star with a sixth magnitude star, we have that  $d = 6 - 1 = 5$ . Find the value of  $r$ . What can you say about the relative brightness of the two stars?



47. a. Compare  $\log_4 16$  and  $\log_{16} 4$ .  
 b. Compare  $\log_9 27$  and  $\log_{27} 9$ .  
 c. State and prove a generalization based on parts (a) and (b).
48. a. Show that  $\log_2 4 + \log_2 8 = \log_2 32$  by finding the three logarithms.  
 b. Verify that  $\log_9 3 + \log_9 27 = \log_9 81$ .  
 c. State and prove a generalization based on parts (a) and (b).
49. a. If  $\log y = 1.5x - 2$ , show that  $y \approx 0.01(31.6)^x$ .  
 b. If  $\log y = 0.5x + 1$ , express  $y$  in terms of  $x$ .
50. a. If  $\ln y = 4x + 2$ , show that  $y \approx 7.4(54.6)^x$ .  
 b. If  $\ln y = 1 - 0.1x$ , express  $y$  in terms of  $x$ .
- C** 51. In advanced mathematics, it can be proved that the number of prime numbers less than a positive integer  $n$  is approximately  $\frac{n}{\ln n}$ .
- a. About how many primes are less than (1) 1000 and (2) 1,000,000?  
 b. There are four prime numbers less than 10. We say that the density of the primes in the interval from 1 to 10 is  $\frac{4}{10}$ . Use your answers to part (a) to find the approximate density of primes in the intervals from 1 to 1000 and from 1 to 1,000,000.  
 c. What happens to the density of primes less than  $n$  as  $n$  increases?
52. Prove that  $\log 2$  is irrational. (Hint: Assume that  $\log 2 = \frac{p}{q}$ , where  $p$  and  $q$  are integers and  $\frac{p}{q}$  is in lowest terms.)

## 5-6 Laws of Logarithms

**Objective** To prove and apply laws of logarithms.

Since the logarithmic function  $y = \log_b x$  is the inverse of the exponential function  $y = b^x$ , it is not surprising that the laws of logarithms are very closely related to the laws of exponents on page 170.

### Laws of Logarithms

If  $M$  and  $N$  are positive real numbers and  $b$  is a positive number other than 1, then:

- $\log_b MN = \log_b M + \log_b N$
- $\log_b \frac{M}{N} = \log_b M - \log_b N$
- $\log_b M = \log_b N$  if and only if  $M = N$
- $\log_b M^k = k \log_b M$ , for any real number  $k$



To prove law 1, let  $\log_b M = x$  and  $\log_b N = y$ . Then  $M = b^x$  and  $N = b^y$ .

$$MN = b^x \cdot b^y = b^{x+y}$$

Therefore:

$$\log_b MN = x + y$$

$$\log_b MN = \log_b M + \log_b N$$

Laws 2 and 4 are proved in a similar fashion. Law 3 is a restatement, in terms of logarithms, of the third law of exponents on page 170.

If you know the logarithms of  $M$  and  $N$ , then you can use the laws of logarithms to find the logarithm of a more complicated expression in  $M$  and  $N$ .

**Example 1** Express  $\log_b MN^2$  in terms of  $\log_b M$  and  $\log_b N$ .

**Solution**

$$\begin{aligned}\log_b MN^2 &= \log_b M + \log_b N^2 && \text{(law 1)} \\ &= \log_b M + 2 \log_b N && \text{(law 4)}\end{aligned}$$

**Example 2** Express  $\log_b \sqrt{\frac{M^3}{N}}$  in terms of  $\log_b M$  and  $\log_b N$ .

**Solution**

$$\begin{aligned}\log_b \sqrt{\frac{M^3}{N}} &= \log_b \left( \frac{M^3}{N} \right)^{1/2} = \frac{1}{2} \log_b \left( \frac{M^3}{N} \right) && \text{(law 4)} \\ &= \frac{1}{2} (\log_b M^3 - \log_b N) && \text{(law 2)} \\ &= \frac{1}{2} (3 \log_b M - \log_b N) && \text{(law 4)}\end{aligned}$$

In Examples 1 and 2, the logarithm of an expression was written in terms of separate logarithms. In Examples 3 and 4, separate logarithms are combined into a single logarithm.

**Example 3** Simplify  $\log 45 - 2 \log 3$ .

**Solution**

$$\begin{aligned}\log 45 - 2 \log 3 &= \log 45 - \log 3^2 && \text{(law 4)} \\ &= \log \frac{45}{3^2} && \text{(law 2)} \\ &= \log 5\end{aligned}$$

**Example 4** Express  $y$  in terms of  $x$  if  $\ln y = \frac{1}{3} \ln x + \ln 4$ .

**Solution**

$$\begin{aligned}\ln y &= \frac{1}{3} \ln x + \ln 4 = \ln x^{1/3} + \ln 4 && \text{(law 4)} \\ \ln y &= \ln 4x^{1/3} && \text{(law 1)} \\ y &= 4x^{1/3} && \text{(law 3)}\end{aligned}$$

Our final example shows how properties of logarithms can be used to solve certain equations.

**Example 5** Solve  $\log_2 x + \log_2 (x - 2) = 3$ .

**Solution**

$$\begin{aligned}\log_2 x + \log_2 (x - 2) &= 3 \\ \log_2 x(x - 2) &= 3 \\ x(x - 2) &= 8 \quad \leftarrow 2^3 = 8 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \\ x &= 4 \text{ or } -2\end{aligned}$$

Since  $\log x$  is not defined for negative  $x$ ,  $-2$  is *not* a solution. The only solution is  $x = 4$ .

## CLASS EXERCISES

Express the common logarithm of each of the following in terms of  $\log M$  and  $\log N$ .

1.  $M^2N$
2.  $\frac{M^2}{N}$
3.  $\sqrt{\frac{M}{N}}$
4.  $\sqrt[3]{MN}$
5.  $M\sqrt{N}$
6.  $\frac{M^2}{N^3}$

Use the laws of logarithms to express each of the following as a single logarithm.

7.  $\log_5 2 + \log_5 3$
8.  $\log_3 5 + \log_3 4$
9.  $\log 12 - \log 3$
10.  $\log 3 + \log 6 - \log 2$
11.  $\ln 4 + 2 \ln 3$
12.  $\frac{1}{2} \ln 25 - \ln 2$
13.  $\log M + 2 \log N$
14.  $2 \log P - \log Q$
15.  $\log_b M + \log_b N + \log_b P$
16.  $\log_b M + \log_b N - 3 \log_b P$
17.  $\frac{1}{2} \ln a - \frac{1}{2} \ln b$
18.  $\ln c + \frac{1}{3} \ln d$

In Exercises 19 and 20, give an example to show that in general each statement is false.

19.  $\log_b (M + N) = \log_b M + \log_b N$
20.  $\log_b \left( \frac{M}{N} \right) = \frac{\log_b M}{\log_b N}$

21. **Discussion** What is wrong with the following argument?

Since  $\frac{1}{8} < \frac{1}{4}$ ,  $\log \frac{1}{8} < \log \frac{1}{4}$ . Therefore,  $\log \left( \frac{1}{2} \right)^3 < \log \left( \frac{1}{2} \right)^2$ . This means that  $3 \log \left( \frac{1}{2} \right) < 2 \log \left( \frac{1}{2} \right)$ . Thus,  $3 < 2$ .

## WRITTEN EXERCISES

In Exercises 1–6, write each expression in terms of  $\log M$  and  $\log N$ .

- A** 1.  $\log (MN)^2$     2.  $\log \frac{M}{N^2}$     3.  $\log \sqrt[3]{\frac{M}{N}}$     4.  $\log M\sqrt[4]{N}$     5.  $\log M^2\sqrt{N}$     6.  $\log \frac{1}{M}$

Write each expression as a rational number or as a single logarithm.

- |   |   |
|---|---|
| 7. $\log 2 + \log 3 + \log 4$                       | 8. $\log 8 + \log 5 - \log 4$                     |
| 9. $\frac{1}{2}\log_6 9 + \log_6 5$                 | 10. $\log_2 48 - \frac{1}{3}\log_2 27$            |
| 11. $2 \ln 6 - \ln 3$                               | 12. $\frac{1}{2}\ln 5 + 3 \ln 2$                  |
| 13. $\log M - 3 \log N$                             | 14. $4 \log M + \frac{1}{2}\log N$                |
| 15. $\log A + 2 \log B - 3 \log C$                  | 16. $\frac{1}{2}(\log_b M + \log_b N - \log_b P)$ |
| 17. $\frac{1}{3}(2 \log_b M - \log_b N - \log_b P)$ | 18. $5(\log_b A + \log_b B) - 2 \log_b C$         |
| 19. $\log \pi + 2 \log r$                           | 20. $\log 4 - \log 3 + \log \pi + 3 \log r$       |
| 21. $\ln 2 + \ln 6 - \frac{1}{2}\ln 9$              | 22. $\ln 10 - \ln 5 - \frac{1}{3}\ln 8$           |

Simplify each expression.

- |                        |                        |                        |                       |
|------------------------|------------------------|------------------------|-----------------------|
| 23. a. $\ln e^2$       | b. $\ln e^3$           | c. $\ln \frac{1}{e}$   | d. $\ln \sqrt{e}$     |
| 24. a. $\ln e^4$       | b. $\ln \frac{1}{e^3}$ | c. $\ln \sqrt[3]{e}$   | d. $\ln 1$            |
| 25. a. $\ln e^x$       | b. $e^{\ln x}$         | c. $e^{2 \ln x}$       | d. $e^{-\ln x}$       |
| 26. a. $\ln e^{3x}$    | b. $e^{3 \ln x}$       | c. $e^{\ln \sqrt{x}}$  | d. $e^{(-1/2) \ln x}$ |
| 27. a. $10^{\log 6}$   | b. $10^{2 \log 6}$     | c. $10^{3 + \log 4}$   | d. $e^{3 + \ln 4}$    |
| 28. a. $10^{3 \log 5}$ | b. $e^{3 \ln 5}$       | c. $10^{1 + 2 \log x}$ | d. $e^{1 + 2 \ln x}$  |

Express  $y$  in terms of  $x$ .

- |  |   |
|--|---|
| 29. a. $\log y = 2 \log x$   | b. $\log y = 3 \log x + \log 5$         |
| 30. a. $\ln y - \ln x = 2 \ln 7$   | b. $\ln y = 2 \ln x - \ln 4$            |
| 31. a. $\log y = -\log x$  | b. $\log y = 2 \log x + \log 2$         |
| 32. a. $\log y + \frac{1}{2}\log x = \log 3$<br><i><math>\log 3 - \frac{1}{2}\log x</math></i> | b. $\ln y = \frac{1}{3}(\ln 4 + \ln x)$ |
| 33. a. $\log y = 1.2x - 1$   | b. $\ln y = 1.2x - 1$                   |
| 34. a. $\log y = 3 - 0.5x$   | b. $\ln y = 3 - 0.5x$                   |



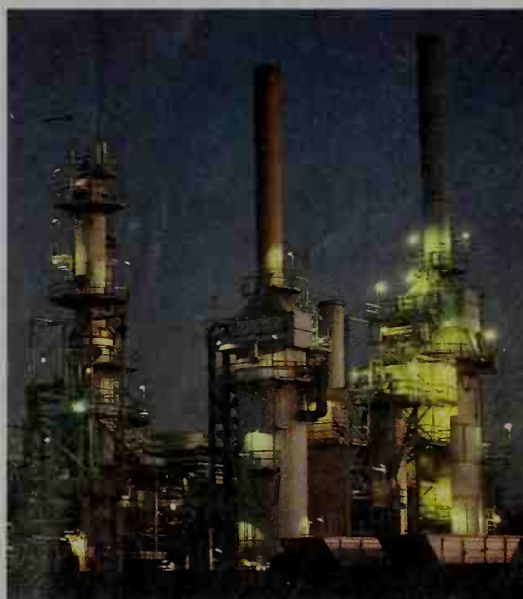
You may find it helpful to have a graphing calculator to complete Exercises 35 and 36.

- B** 35. a. Predict how the graph of  $y = \log_b \left( \frac{1}{x} \right)$  is related to the graph of  $y = \log_b x$ .
- b. Check your prediction by graphing  $y = \log x$  and  $y = \log \left( \frac{1}{x} \right)$  on a single set of axes.
36. a. Predict how the graph of  $y = \log_b x^2$  is related to the graph of  $y = \log_b x$ .
- b. Check your prediction by graphing  $y = \log x$  and  $y = \log x^2$  on a single set of axes.
37. a. If  $f(x) = \log_2 x$ , show that  $f(2x) = f(x) + 1$ .
- b. Part (a) shows that horizontally shrinking the graph of  $f$  by a factor of 2 is equivalent to ?.
38. a. If  $f(x) = \log_3 x$ , show that  $f\left(\frac{x}{3}\right) = f(x) - 1$ .
- b. Make a statement similar to that in part (b) of Exercise 37.
39. Suppose that  $\log y = ax + b$ , where  $a$  and  $b$  are real numbers and  $a$  is nonzero. Express  $y$  in terms of  $x$ . What kind of expression defines  $\log y$  in terms of  $x$ ? What kind of expression defines  $y$  in terms of  $x$ ?
40. **Conservation** The expiration time  $T$  of a natural resource is the time remaining until it is all used. If one assumes that the current growth rate of consumption remains constant, then the expiration time in years is given by

$$T = \frac{1}{r} \ln \left( \frac{rR}{C} + 1 \right),$$

where  $C$  = current consumption,  $r$  = current growth rate of consumption, and  $R$  = resource size. Suppose that the world's consumption of oil is growing at the rate of 7% per year ( $r = 0.07$ ) and current consumption is approximately  $17 \times 10^9$  barrels per year. Find the expiration time for the following estimates of  $R$ .

- a.  $R \approx 1691 \times 10^9$  barrels  
(estimate of remaining crude oil)
- b.  $R \approx 1881 \times 10^9$  barrels  
(estimate of remaining crude plus shale oil)



41. If  $\log_8 3 = r$  and  $\log_8 5 = s$ , express each logarithm in terms of  $r$  and  $s$ .
- a.  $\log_8 75$       b.  $\log_8 225$       c.  $\log_8 0.12$       d.  $\log_8 \frac{3}{64}$
42. If  $\log_9 5 = x$  and  $\log_9 4 = y$ , express each logarithm in terms of  $x$  and  $y$ .
- a.  $\log_9 100$       b.  $\log_9 36$       c.  $\log_9 \left(6\frac{1}{4}\right)$       d.  $\log_9 3.2$

In Exercises 43–46, solve the given equation.

43. a.  $\log_2 (x + 2) + \log_2 5 = 4$       b.  $\log_4 (2x + 1) - \log_4 (x - 2) = 1$
44. a.  $\log_6 (x + 1) + \log_6 x = 1$       b.  $\log_3 x + \log_3 (x - 2) = 1$
45. a.  $\log_4 (x - 4) + \log_4 x = \log_4 5$       b.  $\log_2 (x^2 + 8) = \log_2 x + \log_2 6$
46. a.  $\ln (x^2) = 16$       b.  $\ln \frac{1}{x} = -5$
47. a. For what values of  $M$  is  $\log_2 M < 0$ ?
- b. Use your answer to part (a) to solve the inequality  $\log_2 \left(\frac{x-1}{2}\right) < 0$ .
- c. For what values of  $M$  is  $\log_3 M > 2$ ?
- d. Use your answer to part (c) to solve the inequality  $\log_3 (x^2 + 5) > 2$ .
- e. Use the fact that  $\log_b M > \log_b N$  only if  $M > N$  to solve the inequality  $\log_6 5x > 2 \log_6 x$ .

In Exercises 48–50, solve the given inequality.

48. a.  $\ln (x - 4) + \ln 3 \leq 0$       b.  $\log (5 - x) - \log 7 > 0$
49. a.  $2 \log x < \log (2x - 1)$       b.  $\ln (x + 1) - \ln 2 > 3$
50. a.  $\log_2 (x + 5) + \log_2 (x - 2) \geq 3$       b.  $\log_4 (x - 1) + \log_4 (x + 1) < \log_4 6$

The *Richter scale* is a system for rating the severity of an earthquake. The severity can be measured either by the amplitude of the seismic wave or by the energy released by the earthquake. A one-point increase in the Richter scale number corresponds to a ten-fold increase in the *amplitude* of the seismic wave and to a thirty-one-fold increase in the *energy* released. Exercises 51 and 52 give some details.

51. **Geology** Richter scale numbers for some earthquakes that have occurred in this century are given in the table at the right. To find the ratio of the seismic wave amplitudes for the quakes in Japan and Alaska, we write:

$$\frac{10^{8.9}}{10^{8.4}} = 10^{0.5} \approx 3.16$$

- a. Find the ratio of wave amplitudes for the 1906 and 1989 California earthquakes.
- b. Find the ratio of wave amplitudes for the earthquakes in (1) Iran (1968) and Yugoslavia, and (2) Iran (1990) and Yugoslavia.

Year	Place	Richter scale
1906	California	8.3
1933	Japan	8.9
1963	Yugoslavia	6.0
1964	Alaska	8.4
1968	Iran	7.4
1989	California	7.1
1990	Iran	7.7



- C** 52. **Geology** The Richter scale was proposed in 1935 by Charles Richter. It was re-fined in 1979. The Richter magnitude,  $R$ , of an earthquake is given by,

$$R = 0.67 \log (0.37E) + 1.46,$$

where  $E$  is the energy in  $\text{kW} \cdot \text{h}$  released by the earthquake.

- Show that  $E = 2.7 \cdot 10^{(R - 1.46)/0.67}$ .
  - Show that if  $R$  increases by 1 unit,  $E$  increases by a factor of about 31.
53. If  $9^{(9^9)}$  is multiplied out and is typed on a strip of paper, 3 digits per centimeter, about how many kilometers long would the paper be? (*Hint*: The common logarithm of a number can tell you how many digits the number has.)
54. Suppose that all you know about a function  $f$  is that  $f(ab) = f(a) + f(b)$  for all positive numbers  $a$  and  $b$ .
- Find  $f(1)$ .
  - Prove that  $f(a^2) = 2f(a)$  and  $f(a^3) = 3f(a)$ . What generalization does this suggest?
  - Prove that  $f(\sqrt{a}) = \frac{1}{2}f(a)$  and  $f(\sqrt[3]{a}) = \frac{1}{3}f(a)$ . What generalization does this suggest?
  - Prove that  $f\left(\frac{1}{b}\right) = -f(b)$ .
  - Prove that  $f\left(\frac{a}{b}\right) = f(a) - f(b)$ .
  - Try to find a function  $f$  that satisfies the original equation.
  - If  $f(10) = 1$ , find the values of  $x$  for which  $f(x) = 2$  and  $f(x) = 3$ .



The San Andreas fault

## 5-7 Exponential Equations; Changing Bases

### Objective

To solve exponential equations and to change logarithms from one base to another.

An **exponential equation** is an equation that contains a variable in the exponent. Here are exponential equations you can solve from Section 5-2.

$$2^{t-3} = 8 \qquad 9^{2t} = 3\sqrt[3]{3}$$

These exponential equations are special because both sides of each equation can easily be expressed as powers of the same number. Usually exponential equations cannot be solved this way. In this section, you will see how to use logarithms to solve exponential equations involving a variable such as time,  $t$ .

**Example 1**

In 1990, there were about 5.4 billion people in the world. If the population has been growing at 1.95% per year, estimate the year when the population will be 8 billion people.

**Solution**

$$A_0(1+r)^t = A(t)$$

$$5.4(1 + 0.0195)^t = 8$$

$$1.0195^t = 1.4815$$

$$\log(1.0195)^t = \log 1.4815$$

$$t \log(1.0195) = \log 1.4815$$

$$\text{Thus: } t = \frac{\log 1.4815}{\log 1.0195} \approx 20.35$$



The population will reach 8 billion people around the year 2010.

**Example 2**

Suppose you invest  $P$  dollars at an annual rate of 6% compounded daily. How long does it take (a) to increase your investment by 50%? (b) to double your money?

**Solution**

- a. Since the interest is compounded daily, we can reliably use the formula for continuous compounding,  $P(t) = Pe^{rt}$ . We need to find the value of  $t$  for which  $P(t) = P + 50\%P = 1.5P$ .

$$P(t) = Pe^{rt} = 1.5P$$

$$Pe^{0.06t} = 1.5P$$

$$e^{0.06t} = 1.5$$

The most convenient logarithm to use is the natural logarithm.

$$\ln e^{0.06t} = \ln 1.5$$

$$0.06t \cdot \ln e = \ln 1.5$$

$$(0.06t)1 \approx 0.4055$$

$$\text{Thus: } t \approx 6.76$$

It will take about 6.76 years, or 6 years and 9 months.

- b. Use the technique shown in part (a).

$$Pe^{0.06t} = 2P$$

$$e^{0.06t} = 2$$

$$0.06t = \ln 2$$

$$\text{Thus: } t = \frac{\ln 2}{0.06} \approx \frac{0.693}{0.06} = 11.55$$

It will take about 11.55 years, or 11 years and 6 months.

(Note that the rule of 72 gives 12 years as an estimate.)

The **change-of-base formula** enables us to find the logarithm of a number in one base if logarithms in another base are known. The formula is stated below, and you are asked to derive it in Exercise 21.

$$\log_b c = \frac{\log_a c}{\log_a b}$$

For example,  $\log_5 8 = \frac{\log 8}{\log 5} \approx \frac{0.903}{0.700} = 1.2900$ .

## CLASS EXERCISES

Solve each equation. Leave answers in radical form when possible.

1.  $x^3 = 81$

2.  $3^x = 81$

3.  $4^x = 81$

4.  $x^4 = 8$

Use a calculator to find the value of  $x$  to the nearest hundredth.

5.  $10^x = 3$

6.  $10^x = 8.1$

7.  $10^x = 256$

8.  $100^x = 302$



You may find it helpful to have a graphing calculator to complete Exercise 9.

9. **Discussion** Explain how the change-of-base formula enables you to use a graphing calculator or computer to graph  $y = \log_2 x$ .

## WRITTEN EXERCISES

Use a calculator to find the value of  $x$  to the nearest hundredth.

**A**

1.  $3^x = 12$

2.  $2^x = 100$

3.  $(1.06)^x = 3$

4.  $(0.98)^x = 0.5$

5.  $e^x = 18$

6.  $e^{-x} = 0.01$

7.  $\sqrt{e^x} = 50$

8.  $(e^x)^3 = 200$

For each pair of equations, solve one of them by using powers of the same number. To the nearest hundredth, solve the other by using logarithms.

9. a.  $4^x = 16\sqrt{2}$

b.  $4^x = 20$

10. a.  $9^x = \frac{3}{3^x}$

b.  $9^x = 4$

11. a.  $25^x = \sqrt[5]{5^x}$

b.  $25^x = 2$

12. a.  $8^x = \sqrt[3]{\frac{2}{4^x}}$

b.  $8^x = \sqrt[3]{5}$

13. **Geography** The population of Kenya reached 25,000,000 people in 1990. When will it reach 50,000,000 people? Assume an annual rate of increase of 4.1%.
14. **Finance** An investment is made at 7% annual interest compounded daily. How long does it take to triple the investment?

15. **Finance** A \$10,000 certificate of deposit at a certain bank will double in value in 9 years.
- Give a formula for the accumulated amount  $t$  years after the investment is made.
  - How long does it take for the money to triple in value?
- B** 16. **Finance** According to the rule of 72, an investment at  $r\%$  interest compounded continuously will double in approximately  $\frac{72}{r}$  years. Show that a more accurate doubling time is  $\frac{69.3}{r}$  years. (Note that the usual formula,  $P(t) = Pe^{rt}$ , where  $r$  is a decimal, must be rewritten as  $P(t) = Pe^{0.01rt}$  where  $r$  is a percent.)
17. **Finance** Prove that the time needed to triple an investment at  $r\%$  interest compounded continuously is approximately  $\frac{110}{r}$ .
18. **Finance** Tell how long it takes for \$100 to become \$1000 if it is invested at 8% interest compounded:
- annually
  - quarterly
  - daily
19. **Physics** A radioactive isotope has a half-life of 9.6 h.
- If there is 1 kg of the isotope now, how much will there be in 24 h?
  - How long does it take for the isotope to decay to 1 g?
20.
  - Find  $\log_6 88$  by using the change-of-base formula.
  - Find  $\log_6 88$  by solving  $6^x = 88$ .
21. Derive the change-of-base formula.
22. Find the value of  $x$  to the nearest tenth.
- $x^5 = 98$
  - $5^x = 98$



You may find it helpful to have a graphing calculator to complete Exercises 23 and 24.

23. **Discussion** Suppose that you wished to solve  $x = 2^x$ .
- What happens when you try to solve  $x = 2^x$  by taking the logarithm of each side of the equation?
  - Give another approach to solve  $x = 2^x$ . How many solutions exist?
  - Discuss the approach you would take to solve  $x + 1 = 2^x$  and to solve  $x + 2 = 2^x$ . How many solutions are there in each case?
  - Based on your answers to parts (a)–(c), how many solutions are there to  $x + c = 2^x$ , where  $c$  is a whole number?
24. Given:  $x^x = \pi$ . Find the value of  $x$  to the nearest hundredth.

**Solve. Express  $x$  as a logarithm if necessary.**

- $2^{2x} - 2^x - 6 = 0$  (Hint:  $2^{2x} = (2^x)^2$ )
- $3^{2x} - 5 \cdot 3^x + 4 = 0$
- $e^{2x} - 5e^x + 6 = 0$
- $e^{2x} - e^x - 6 = 0$
- $3^{2x+1} - 7 \cdot 3^x + 2 = 0$
- $e^x + e^{-x} = 4$

31. a. If  $b^m > b^n$  and  $b > 1$ , what can you say about  $m$  and  $n$ ?  
 b. If  $b^m > b^n$  and  $0 < b < 1$ , what can you say about  $m$  and  $n$ ?
32. Solve: a.  $8^x > 8^{7-x}$     b.  $0.6^{5x} > 0.6^{x/2}$     c.  $e^{3x} < e^x$     d.  $\left(\frac{1}{2}\right)^x > 2^{6-x}$

33. **Archaeology** All living organisms contain a small amount of carbon 14, denoted  $C^{14}$ , a radioactive isotope. When an organism dies, the amount of  $C^{14}$  present decays exponentially. By measuring the radioactivity  $N(t)$  of, say, an ancient skeleton of an animal and by comparing that radioactivity with the radioactivity  $N_0$  of living animals, archaeologists can tell approximately when the animal died.

- a. Given that the half-life of  $C^{14}$  is about 5700 years, write an equation relating  $N(t)$ ,  $N_0$ , and the time  $t$  since the animal's death.

- b. Suppose it is found that, for a certain animal,  

$$N(t) = \frac{1}{10}N_0.$$
 To the nearest 100 years, how long ago did the animal die?



34. **Archaeology** An archaeologist unearths a piece of wood that may have come from the Hanging Gardens of Babylon, about 600 B.C. The amount of radioactive  $C^{14}$  in the wood is  $N(t) = 0.8N_0$ . Is it possible that the wood could be from the Hanging Gardens? (The half-life of  $C^{14}$  is about 5700 years.)

35. Prove:  $\log_b c = \frac{1}{\log_c b}$

36. Prove:  $(\log_a b)(\log_b c) = \log_a c$

Evaluate each expression. (Use the results of Exercises 35 and 36.)

37.  $\log_3 2 \cdot \log_2 27$     38.  $\log_{25} 8 \cdot \log_8 5$     39.  $\frac{1}{\log_2 6} + \frac{1}{\log_3 6}$     40.  $\frac{1}{\log_4 6} + \frac{1}{\log_9 6}$



For Exercises 41 and 42, use a computer or a graphing calculator to solve each inequality. Give answers to the nearest hundredth.

41. On a single set of axes, graph  $y = \log_2 (x - 1)$  and  $y = \log_3 x$ . (See Class Exercise 9 on page 205.) Use your graph to solve  $\log_2 (x - 1) > \log_3 x$ .
42. Solve each inequality using the method suggested in Exercise 41.  
 a.  $e^x < \ln (x + 5)$     b.  $2^x \leq \log_5 x$     c.  $\log 20x > 2^{-x}$     d.  $\log x \geq \log_4 x^2$

43. **Oceanography** After passing through a material  $t$  centimeters thick, the intensity  $I(t)$  of a light beam is given by  $I(t) = (4^{-ct})I_0$ , where  $I_0$  is the initial intensity and  $c$  is a constant called the absorption factor. Ocean water absorbs light with an absorption factor of  $c = 0.0101$ . At what depth will a beam of light be reduced to 50% of its initial intensity? 2% of its initial intensity?



44. Prove:  $a^{\log b} = b^{\log a}$

45. Prove:  $\frac{1}{\log_a ab} + \frac{1}{\log_b ab} = 1$



## Chapter Summary

1. An *exponential function* has the form  $f(x) = ab^x$ , where  $a > 0$ ,  $b > 0$ , and  $b \neq 1$ . Such functions are commonly used for calculating such things as the value of an investment deposited at a certain rate of compound interest, how much of a substance remains after a period of radioactive decay, or the size of a population growing at a certain rate.
2. The *laws of exponents* are given on page 170. They are used to define the zero exponent and negative integral exponents:

$$b^0 = 1$$

$$b^{-x} = \frac{1}{b^x}$$

Integral exponents are used to define rational exponents:

$$b^{p/q} = (\sqrt[q]{b})^p = \sqrt[q]{b^p},$$

where  $b > 0$ ,  $p$  and  $q$  are integers, and  $q \neq 0$ .

3. The *rule of 72* provides an approximation of the doubling time for exponential growth. If a quantity is growing at  $r\%$  per year, then

$$\text{doubling time} \approx 72 \div r.$$

4. The number  $e$  is defined as

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

The number  $e$  is the base for the *natural exponential function* and its inverse, the *natural logarithm function*.

5. The *logarithm* of  $x$  to the base  $b$  ( $b > 0$ ,  $b \neq 1$ ) is the exponent  $a$  such that  $x = b^a$ . Thus,

$$\log_b x = a \text{ if and only if } x = b^a.$$

Common logarithms, logarithms to base 10, are usually denoted  $\log x$ , while natural logarithms, those to base  $e$ , are usually denoted  $\ln x$ .

6. Since logarithms are exponents, the laws of logarithms, found on page 197, are closely related to the laws of exponents stated on page 170.
7. *Exponential equations* can be solved by writing both sides of the equation in terms of a common base (Section 5-2) or taking logarithms of both sides (Section 5-7).
8. The *change-of-base formula* enables you to write logarithms in any given base in terms of logarithms in any other base.

$$\log_b c = \frac{\log_a c}{\log_a b}$$

### Key vocabulary and ideas

exponential function (p. 169, p. 181)  
exponential growth and decay (p. 170)  
zero exponent (p. 171)  
negative exponent (p. 171)  
rational exponent (p. 175)  
the rule of 72 (p. 182)  
the number  $e$  (p. 186)

natural exponential function (p. 187)  
common logarithm (p. 191)  
logarithm to base  $b$  (p. 193)  
logarithmic function (p. 193)  
natural logarithm function (p. 193)  
exponential equation (p. 203)  
change-of-base formula (p. 205)

## Chapter Test

1. Give a general expression for the value of a piece of property  $t$  years from now if its current value is \$150,000 and property values are increasing at the rate of 9% per year. 5-1
2. Evaluate the following.  
a.  $\frac{2^5 \cdot 2^{-4}}{2^{-2}}$       b.  $(5^{-2} + 5^0)^{-1}$       c.  $\sqrt{\frac{4^6}{2^{-4}}}$       d.  $\frac{3^{-3} + 9^{-2}}{3^{-3}}$
3. Solve each equation: a.  $2^{6-x} = 4^{2+x}$       b.  $3\sqrt{27} = 9^{2x}$  5-2
4. A gallon of milk cost \$1.99 two years ago. Now it costs \$2.19. To the nearest percent, what has been the annual rate of increase in the cost?
5. Graph  $y = 2^x$  and  $y = 2^{-x}$  on a single set of axes. How are the graphs related? 5-3
6. **Writing** Explain how  $P(t) = P_0(1 + r)^t$  can be used to model exponential growth and decay. Give an example of each.
7. Suppose that \$1500 is invested at an interest rate of 8.5%. How much is the investment worth after 18 months if interest is compounded (a) quarterly? (b) continuously? 5-4
8. Given that  $\log 25 \approx 1.3979$ , find the value of:  
a.  $\log 2.5$       b.  $\log 2500$       c.  $\log 0.04$  5-5
9. Find the exact value of: a.  $\log_2 8$       b.  $\log_8 2$       c.  $2^{\log_4 64}$
10. Express  $y$  as a function of  $x$ . 5-6  
a.  $\log_2 y = 2 \log_2 (2x)$       b.  $\log_2 y = 2 + \log_2 x$
11. Express each of the following in terms of  $\log_b M$  and  $\log_b N$ .  
a.  $\log_b \sqrt[3]{\frac{M^2}{N}}$       b.  $\log_b M^2 N^3$
12. Solve  $\log_5 x + \log_5 (x - 4) = 1$ .
13. a. Between what two consecutive integers must  $\log_5 21$  lie? 5-7  
b. Use the change-of-base formula to express  $\log_5 21$  in terms of common logarithms and then evaluate it to the nearest thousandth.
14. Solve  $5^x = 8$  for  $x$  to the nearest hundredth.

## Newton's Law of Cooling

The best way to get a real feeling for mathematics is to see it in action. You've already seen functions that describe various processes. Did you ever wonder how you might develop a function to describe a process yourself?

Since this might be your first attempt, let's consider a relatively simple occurrence, such as cooling. Intuitively, you know that a cup of hot water set in a cool room will cool to room temperature. There must be some relationship between the temperature of the water over time and the temperature of the room.

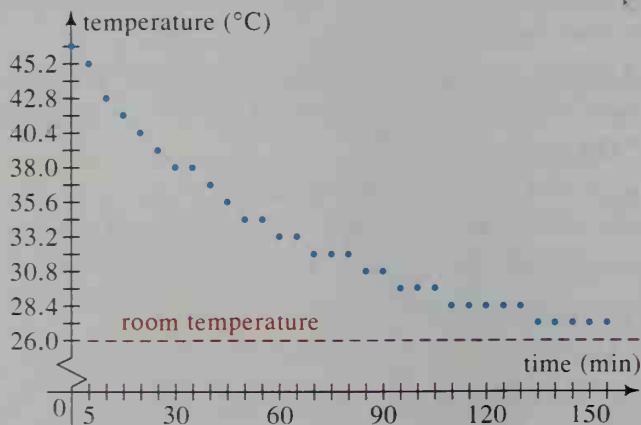
In this project, you will see how to gather data for a cooling experiment, model the data, report your findings, and explore the experiment further.

### Materials:

- |                            |                                   |
|----------------------------|-----------------------------------|
| a standard lab thermometer | a graphing calculator or software |
| a cup of hot water         | a refrigerator                    |

### Gather the Data.

- Record the temperature,  $T_r$ , of the room.
- Next, fill a cup with some hot tap water. Place the thermometer in the cup. (Make certain that the water is not too hot for the thermometer scale.)
- Record the initial water temperature. Every five minutes thereafter, record the elapsed time  $t$  and the water temperature  $T(t)$  until the temperature stops changing.
- Using a software package or a programmable graphing calculator, graph the ordered pairs  $(t, T(t))$ . Typically, your plot should resemble the graph below.



The graph probably confirms your expectations. The water temperature decreases, and the water cools more rapidly at first and then more slowly.

### Model the Data.

In this part of the project, you will look for a function that reasonably represents the table of ordered pairs and the graph obtained. Which of the types of functions that you have studied so far might be a good model?

- Let's consider the exponential function  $T(t) = Ce^{-kt}$ , where  $C$  and  $k$  are positive constants. Why would there be a negative sign in front of  $k$ ? (For a hint, see Exercise 14 in Section 5-4.)

As  $t$  increases, what does  $Ce^{-kt}$  approach? (Use your calculator.) Now think about this. After a long period of time, what would you expect the temperature of the water to approach? If the temperature of the water approaches room temperature, then our exponential should approach  $T_r$ , or 26.0, instead of 0. Let's adjust our formula for  $T(t)$  accordingly and write:

$$T(t) = Ce^{-kt} + T_r$$

Notice that our function  $T(t)$  is the exponential  $Ce^{-kt}$  translated  $T_r$  units vertically (Section 4-4).

- Next, find the value of  $T(0)$ . What does this tell us about the value of  $C$ ? More specifically, then:

$$T(t) = (T(0) - T_r)e^{-kt} + T_r$$

- How can we find a value for  $k$ ? If the function fits all of the data, then any one of the ordered pairs should satisfy the function. For instance, we can use the ordered pair (30, 38.0) from the given data to find that  $k \approx 0.02$ . Then:

$$T(t) = 20.4e^{-0.02t} + 26.0$$

Try it! Find a value of  $k$  and the corresponding function by using one of the other ordered pairs.

- How well do you think the function above will fit the ordered pairs? To find out how well, try graphing the data points and the function on a single set of axes. Because of the variation in measurements and conditions, chances are high that there will be places where the data points do not lie exactly on the graph of the function. The value of  $k$ , after all, is just an estimate. A more accurate method of fitting an exponential function to a set of data involves all of the data points. You will see the method in Chapter 18.

## Report Your Results.

A good written report should be logically organized and clearly written. It should enable someone to reproduce the experiment successfully. Be sure to state conditions, such as room temperature and initial water temperature, a listing of your ordered pairs of data, and your graph. Specify your calculations for finding the values of  $C$  and  $k$ , and your resulting function for modeling the data. In your report, analyze how well your function seems to fit your data. Explain why you do or do not think that cooling can be described by an exponential function.

## Extend the Project.

Let's go one step further. What do you think would happen if instead of letting the cup of hot water cool to room temperature, you placed it in the refrigerator? To get an accurate comparison, heat the water to the same initial temperature and record your data over the same time intervals. Graph both your new and old ordered pairs on a single set of axes. How has the colder environment affected the cooling process?

# 6

## *Analytic Geometry*

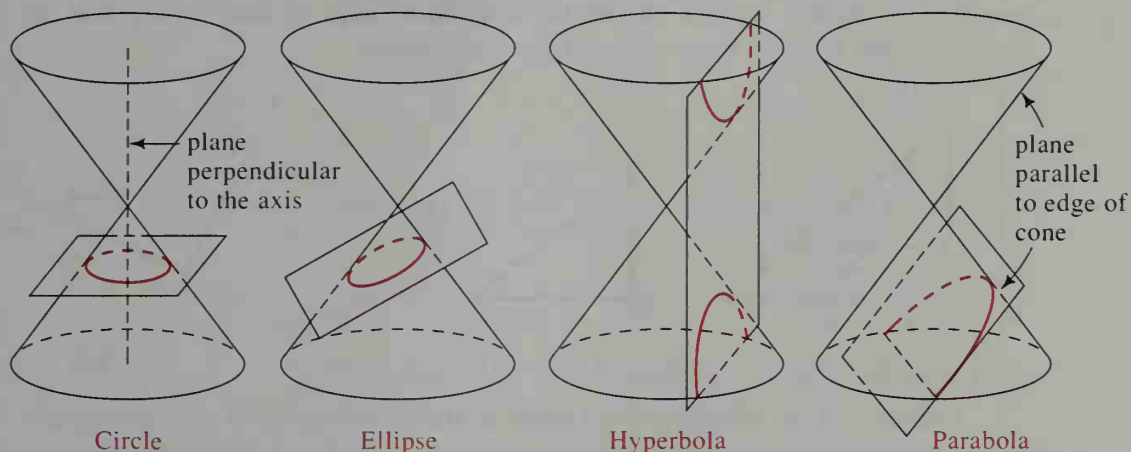




# Introduction

**Analytic geometry** is the study of geometric problems by means of analytic (or algebraic) methods. You saw some analytic geometry when you studied the distance formula, midpoint formula, and equations of lines in Sections 1-1 to 1-3. You saw more analytic geometry in Section 1-7 when you studied quadratic functions and their graphs, parabolas.

Imagine that the double cones shown below are extended indefinitely up and down. When these cones are sliced by a plane tilted at various angles, the resulting cross sections are called **conic sections**. As shown below, a circle, an ellipse, a hyperbola, and a parabola are conic sections.



It is also possible to slice the double cone to obtain a single point, a line, or a pair of lines. Do you see how? These extreme cases are called *degenerate conic sections*.

Using analytic geometry, we can find equations for these curves. Each equation is a special case of the **general second-degree equation** in two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

For example,

$$\begin{aligned}x^2 + y^2 &= 9 \\x^2 + 4y^2 &= 9 \\x^2 - y^2 &= 9 \\\frac{1}{4}x^2 - y &= 0\end{aligned}$$

are equations of a *circle*, an *ellipse*, a *hyperbola*, and a *parabola*, respectively.

Before we study the conic sections, we will show how to use analytic geometry to prove theorems.

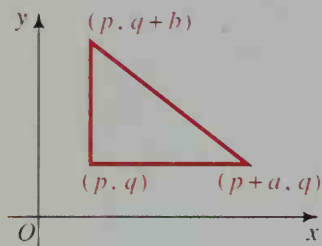
◀ In this Cassegrain reflecting telescope at Arizona's Kitt Peak Observatory, a large parabolic mirror reflects light to its focus, which is also one of the foci of a smaller hyperbolic mirror. The mirror reflects the light to the eyepiece, located at the second hyperbolic focus.

## 6-1 Coordinate Proofs

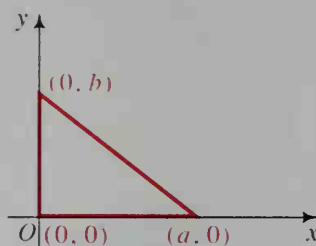
**Objective** To prove theorems from geometry by using coordinates.

In this section, you will see how to use analytic methods to prove theorems from geometry.

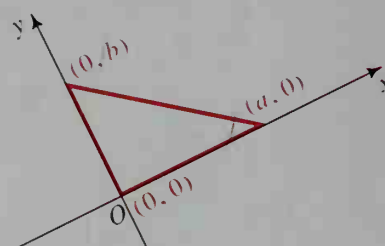
If a theorem is about a right triangle, there are several ways that we can place the coordinate axes on the triangle and then assign coordinates to the vertices of the triangle. For example, figures (1) and (2) below illustrate a right triangle with legs  $a$  and  $b$  units long. Most people prefer to work with figure (2). Because more of the coordinates are zero, the work is easier. Figure (3) shows that even if the right triangle is “tilted,” we can choose the coordinate axes in such a way that the vertices of the triangle have several zero coordinates.



(1)

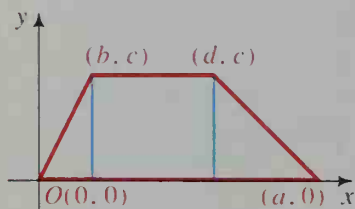


(2)

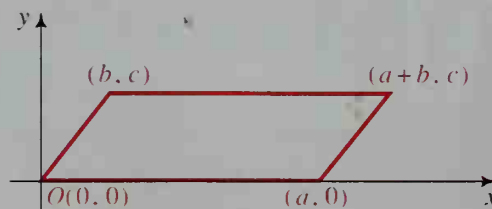


(3)

Similarly, if we wish to prove a theorem about a trapezoid or a parallelogram, we can always choose the axes in such a way that one of the vertices of the figure is at the origin and one of its parallel sides lies on the  $x$ -axis.



(4)



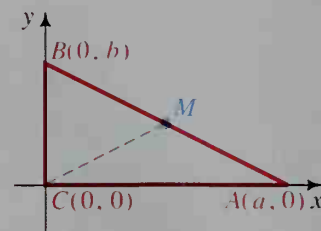
(5)

**Example 1** Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

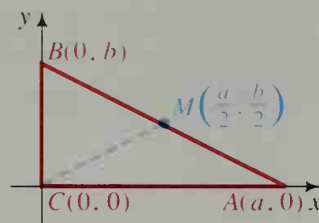
**Solution** **Step 1** First, we make a coordinate diagram of the triangle and note what we are given and what we must prove.

*Given:*  $\angle C$  is a right angle.  
 $M$  is the midpoint of  $\overline{AB}$ .

*Prove:*  $MC = MA$   
(We already know that  $MB = MA$ .)



**Step 2** Next, we use what is given to add information to the diagram or to express algebraically any given fact not shown in the original diagram. In this example, we use the given fact that  $M$  is the midpoint of  $\overline{AB}$  to find the coordinates of  $M$ .



**Step 3** Finally, we reword what we are trying to prove in algebraic terms. To prove  $MC = MA$ :

$$\begin{aligned} MC &= \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} \\ &= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} \end{aligned}$$

$$\begin{aligned} MA &= \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} \\ &= \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} \end{aligned}$$

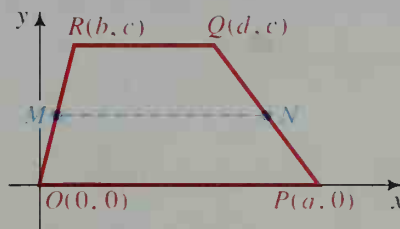
Therefore,  $MC = MA$ . Since  $MA = MB$ , we have that  $MA = MB = MC$ .

## Example 2

Prove that the median of a trapezoid is parallel to the bases and has length equal to the average of the lengths of the bases.

## Solution

**Step 1** We show a diagram, and the “Given” and “Prove.” Place the  $x$ -axis along the longer base of the trapezoid, with the origin at the endpoint of the longer base. Since the bases of a trapezoid are parallel and base  $\overline{OP}$  has been chosen to be horizontal, base  $\overline{RQ}$  is also horizontal. Thus  $R$  and  $Q$  have the same  $y$ -coordinate.

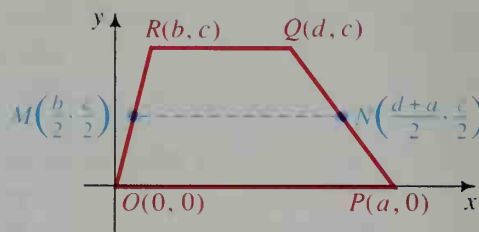


**Given:** Figure  $OPQR$  is a trapezoid. Points  $M$  and  $N$  are midpoints of  $\overline{OR}$  and  $\overline{PQ}$ , respectively.

**Prove:** (1)  $\overline{MN} \parallel \overline{OP}$  and (2)  $MN = \frac{OP + RQ}{2}$ .

(Solution continues on the next page.)

**Step 2** Next, we use what is given to add information to the diagram. In this example, we use the fact that  $M$  and  $N$  are midpoints to find their coordinates.



**Step 3** We reword what we are to prove in algebraic terms.

(1) To prove  $\overline{MN} \parallel \overline{OP}$ , we must show that  $\overline{MN}$  and  $\overline{OP}$  have the same slope. A quick check shows that both slopes are zero, so this part of the proof is done.

(2) Lastly, we use algebra to show that  $MN = \frac{1}{2}(OP + RQ)$ .

$$MN = \frac{d+a}{2} - \frac{b}{2}$$

$$\frac{1}{2}(OP + RQ) = \frac{1}{2}(a + (d - b)) = \frac{d+a}{2} - \frac{b}{2}$$

$$\text{Therefore, } MN = \frac{1}{2}(OP + RQ).$$

### Example 3

Prove that the altitudes of a triangle meet in one point, that is, they are concurrent.

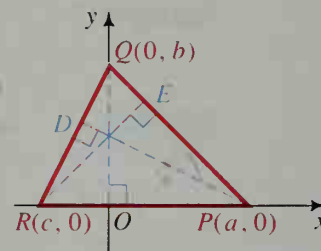
### Solution

**Step 1** We show a diagram and the “Given” and “Prove.”

Given:  $\triangle PQR$  with altitudes  $\overline{PD}$ ,  $\overline{QE}$ , and  $\overline{RE}$ .

Prove: Lines  $\overline{PD}$ ,  $\overline{QE}$ , and  $\overline{RE}$  have a point in common.

Notice that the axes are placed in such a way that one of the altitudes lies on the y-axis.



**Step 2** We use the given information to express algebraically the fact that  $\overline{PD}$ ,  $\overline{QE}$ , and  $\overline{RE}$  are altitudes.

a. To find the slope of line  $\overline{PD}$ , we note that the slope of line  $\overline{QR}$  is  $\frac{b-0}{0-c} = -\frac{b}{c}$ , so that the slope of line  $\overline{PD}$  is  $\frac{c}{b}$ .

Since line  $\overline{PD}$  contains the point  $(a, 0)$ , its equation is

$$\frac{y-0}{x-a} = \frac{c}{b}, \text{ or } cx - by = ca.$$

b. Likewise, an equation of line  $\overline{RE}$  is

$$\frac{y-0}{x-c} = \frac{a}{b}, \text{ or } ax - by = ca.$$

c. An equation of the vertical line  $\overline{QE}$  is  $x = 0$ .

**Step 3** We reword what we are to prove in algebraic terms. To prove that lines  $PD$ ,  $QO$ , and  $RE$  have a point in common, we must show that their equations have a common solution. Using subtraction to solve

$$cx - by = ca$$

$$ax - by = ca,$$

we get

$$cx - ax = 0.$$

Therefore,  $x(c - a) = 0$ . Since  $c \neq a$ ,  $x = 0$ . Substituting 0 for  $x$  in the equation  $cx - by = ca$ , we get  $y = -\frac{ca}{b}$ . Thus, the lines

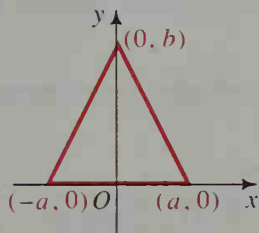
$PD$  and  $RE$  intersect at  $\left(0, -\frac{ca}{b}\right)$ , a point on the  $y$ -axis, that is, on altitude  $QO$ , so we are done. (The point of concurrency of the altitudes is called the *orthocenter* of the triangle.)

### Summary of Methods Commonly Used in Coordinate Proofs

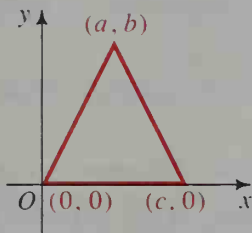
1. To prove line segments equal, use the distance formula to show that they have the same length.
2. To prove nonvertical lines parallel, show that they have the same slope.
3. To prove lines perpendicular, show that the product of their slopes is  $-1$ .
4. To prove that two line segments bisect each other, use the midpoint formula to show that each segment has the same midpoint.
5. To show that lines are concurrent, show that their equations have a common solution.

### CLASS EXERCISES

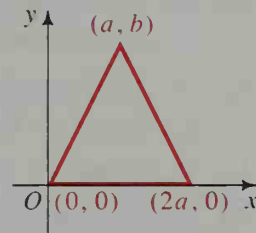
1. Study the coordinates of the vertices in the following diagrams and tell which figures represent isosceles triangles.



(a)



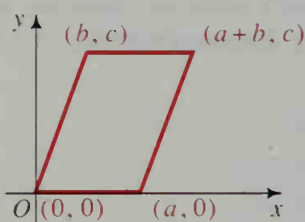
(b)



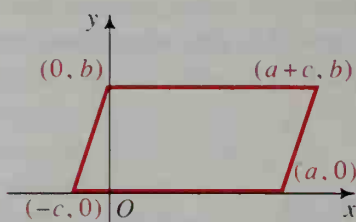
(c)



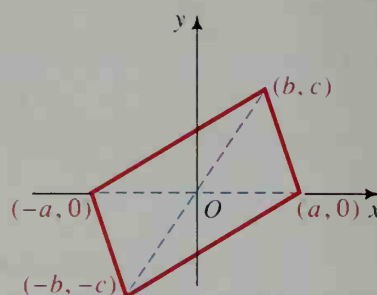
2. Which of the following diagrams represent parallelograms?



(a)



(b)



(c)

## WRITTEN EXERCISES

In these exercises, do not use any theorems from geometry other than the Pythagorean Theorem. You may use the formulas and theorems in Sections 1-1 and 1-2, and results proved in earlier exercises of this section.

- A**
1. Use figure (a) or (c) of Class Exercise 1 to prove that the medians to the legs of an isosceles triangle are equal in length.
  2. Prove that if a triangle has two congruent medians, then it is isosceles.
  3. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has length half that of the third side.
  4. Use either figure (a) or (b) of Class Exercise 2 to prove that the diagonals of a parallelogram bisect each other.
  5. Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. How does this result show that any parallelogram can be represented by figure (c) of Class Exercise 2?
  6. Prove that the lengths of the diagonals of a rectangle are equal.
  7. Prove that the lengths of the diagonals of an isosceles trapezoid are equal.
  8. Prove that if the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.
  9. Prove that the line segments joining the midpoints of successive sides of any quadrilateral form a parallelogram.
  10. Prove that the line segments joining the midpoints of successive sides of any rectangle form a rhombus.
- B**
11. Prove that if the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.
  12. Prove that the diagonals of a rhombus are perpendicular.
  13. Suppose that  $Q$  is any point in the plane of rectangle  $RSTU$ . Prove that  $(QR)^2 + (QT)^2 = (QS)^2 + (QU)^2$ .

14. Use the diagram at the right to prove that  $P$  is on the perpendicular bisector of  $\overline{AB}$  if  $PA = PB$ .

15. a. Use a figure similar to the one in Example 3 to prove that the medians of  $\triangle PQR$  meet in a point  $G$ . (This point is called the *centroid* of the triangle.)

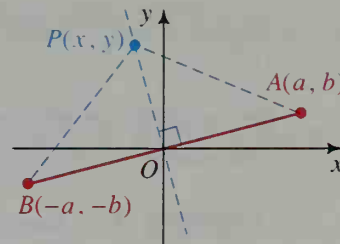
b. Show that the  $x$ -coordinate of the centroid is the average of the  $x$ -coordinates of  $P$ ,  $Q$ , and  $R$ , and that the  $y$ -coordinate of the centroid is the average of the  $y$ -coordinates of  $P$ ,  $Q$ , and  $R$ .

16. Using Exercise 15, prove that the centroid  $G$  divides each median in a 2:1 ratio.

17. a. Use a figure similar to the one in Example 3 to prove that the perpendicular bisectors of the sides of  $\triangle PQR$  meet in a point  $C$ . This point is called the *circumcenter* of the triangle.

b. Prove that  $C$  is equidistant from  $P$ ,  $Q$ , and  $R$ .

**C** 18. Let  $G$  be the intersection point of the medians of a triangle, let  $C$  be the intersection of the perpendicular bisectors of the sides, and let  $H$  be the intersection of the altitudes. Prove that  $G$ ,  $C$ , and  $H$  are collinear and that  $GH = 2GC$ . (*Hint*: Use the results of Example 3 and Exercises 15 and 17.)



Ex. 14

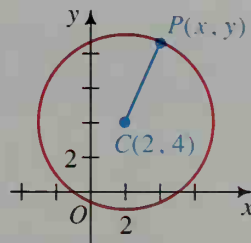
## 6-2 Equations of Circles

### Objective

To find equations of circles and to find the coordinates of any points where circles and lines meet.

The set of all points  $P(x, y)$  in the plane that are 5 units from the point  $C(2, 4)$  is a *circle*. To find an equation of this circle, we use the distance formula:

$$\begin{aligned} PC &= 5 \\ \sqrt{(x - 2)^2 + (y - 4)^2} &= 5 \\ (x - 2)^2 + (y - 4)^2 &= 25 \end{aligned}$$



In general, if  $P(x, y)$  is on the circle with center  $C(h, k)$  and radius  $r$ , then:

$$\begin{aligned} PC &= r \\ \sqrt{(x - h)^2 + (y - k)^2} &= r \\ (x - h)^2 + (y - k)^2 &= r^2 \end{aligned}$$

The steps leading to the last equation can be reversed to show that any point  $P(x, y)$  satisfying the equation is on the circle with center  $(h, k)$  and radius  $r$ .

If the center of the circle is the origin, the equation becomes:

$$x^2 + y^2 = r^2$$

For brevity, we often refer to a circle with equation  $x^2 + y^2 = r^2$ , for example, as “the circle  $x^2 + y^2 = r^2$ .”

**Example 1** Find the center and radius of each circle.

- a.  $(x - 3)^2 + (y + 7)^2 = 19$
- b.  $x^2 + y^2 - 6x + 4y - 12 = 0$

**Solution**

- a. Since  $(x - 3)^2 + (y + 7)^2 = (x - 3)^2 + (y - (-7))^2$ , the center of the circle is  $(3, -7)$ . The radius is  $\sqrt{19}$ .
- b. We rewrite the equation in *center-radius form* by completing the squares in  $x$  and  $y$ .

$$\text{Original equation: } x^2 + y^2 - 6x + 4y - 12 = 0$$

$$(x^2 - 6x \quad) + (y^2 + 4y \quad) = 12$$

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4$$

$$\text{Center-radius form: } (x - 3)^2 + (y + 2)^2 = 25$$

Thus, the center of the circle is  $(3, -2)$  and the radius is 5.



In Example 2, you will see how to use a graphing calculator (or computer) to graph an equation that represents a circle.

**Example 2** Use a graphing calculator to graph the equation in Example 1(b).

**Solution**

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

$$(x - 3)^2 + (y + 2)^2 = 25 \quad \leftarrow \text{center-radius form}$$

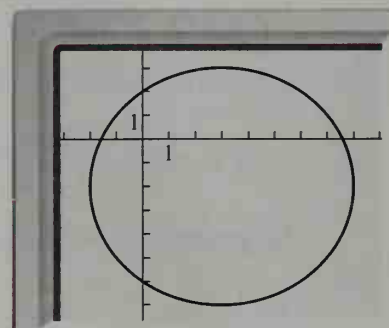
Solve for  $y$  in terms of  $x$ .

$$(y + 2)^2 = 25 - (x - 3)^2$$

$$y_1 = -2 + \sqrt{25 - (x - 3)^2} \quad (1)$$

$$y_2 = -2 - \sqrt{25 - (x - 3)^2} \quad (2)$$

The figure at the right was obtained by graphing each of the *two* equations above on the same set of axes. Equation (1) represents the top half of the circle; equation (2) represents the bottom half.



Notice that the graph does not appear to be a perfect circle. You should consult your calculator manual to determine how to adjust the display to make the graph appear as a circle.

**Example 3**

Find the coordinates of the points where the line  $y = 2x - 2$  and the circle  $x^2 + y^2 = 25$  intersect.

**Solution**

**Step 1** To find the coordinates of their intersection points,  $A$  and  $B$ , we solve these two equations simultaneously:

$$y = 2x - 2 \quad (1)$$

$$x^2 + y^2 = 25 \quad (2)$$

**Step 2** Substituting for  $y$  in equation (2), we get:

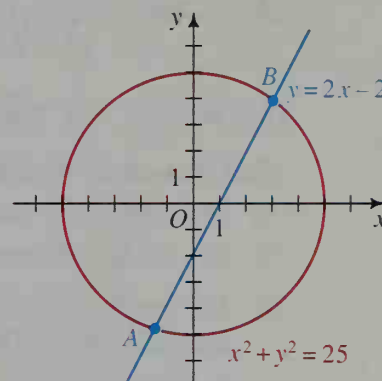
$$x^2 + (2x - 2)^2 = 25$$

$$x^2 + 4x^2 - 8x + 4 = 25$$

$$5x^2 - 8x - 21 = 0$$

$$(5x + 7)(x - 3) = 0$$

Therefore,  $x = -\frac{7}{5}$  or  $x = 3$ .



**Step 3** Substituting these values for  $x$  in equation (1), we get:

$$y = 2\left(-\frac{7}{5}\right) - 2 = -\frac{24}{5} \quad \text{and} \quad y = 2(3) - 2 = 4$$

Thus,  $A = \left(-\frac{7}{5}, -\frac{24}{5}\right)$  and  $B = (3, 4)$ . Check this result by substituting the coordinates of  $A$  and  $B$  in equations (1) and (2).

The solution to Example 3 involves an algebraic method, which is summarized below. You can also find the solution by simultaneously graphing the line and the circle with a graphing calculator or computer software.

### To Find the Intersection of a Line and a Circle Algebraically:

1. Solve the linear equation for  $y$  in terms of  $x$  (or  $x$  in terms of  $y$ ).
2. Substitute this expression for  $y$  (or  $x$ ) in the equation of the circle. Then solve the resulting quadratic equation.
3. Substitute each real  $x$ -solution from Step 2 in the *linear* equation to get the corresponding value of  $y$  (or vice versa). Each point  $(x, y)$  is an intersection point.
4. You can check your result by substituting the coordinates of the intersection points in the two original equations.

When a line and a circle do not intersect, the quadratic equation in Step 2 will have a negative discriminant and the equation will have only imaginary roots. If the discriminant is zero, then there is only one real root. This indicates that the line intersects the circle in a single point and is tangent to the circle.

## CLASS EXERCISES

Find the center and radius of each circle whose equation is given.

1.  $x^2 + y^2 = 16$
2.  $(x - 2)^2 + (y - 7)^2 = 36$
3.  $(x - 4)^2 + (y + 7)^2 = 7$
4.  $x^2 + y^2 + 12y = 0$
5.  $x^2 - 2x + y^2 - 6y = 9$
6.  $4x^2 + 4y^2 = 36$

In Exercises 7–9, find an equation of the circle described.

7. The circle with center (7, 3) and radius 6.
8. The circle with center (-5, 4) and radius  $\sqrt{2}$ .
9. The circle with center (0, 0) that passes through (-5, 12).
10. The graph of  $x^2 + y^2 = 25$  consists of all points in the plane that are 5 units from the origin. Describe the graphs of:
  - a.  $x^2 + y^2 < 25$
  - b.  $x^2 + y^2 > 25$
11. **Discussion** Suppose that you wish to find where the line  $3y + x = 6$  intersects the circle  $x^2 + y^2 = 10$ . Describe what you would do.
12. **Discussion** To find the intersection of the line  $y = x + 8$  and the circle  $x^2 + y^2 = 16$ , Janice solved the equations simultaneously and found that  $x = -4 \pm 2i\sqrt{2}$ . What do the imaginary roots tell her?
13. **Discussion** Describe how to use a graphing calculator to graph  $x^2 + y^2 = 9$  so that the circle does not appear distorted or flattened.

## WRITTEN EXERCISES

In Exercises 1–12, write an equation of the circle described.

- A**
1.  $C(4, 3)$ ,  $r = 2$
  2.  $C(5, -6)$ ,  $r = 7$
  3.  $C(-4, -9)$ ,  $r = 3$
  4.  $C(a, b)$ ,  $r = f$
  5.  $C(6, 0)$ ,  $r = \sqrt{15}$
  6.  $C(-4, 2)$ ,  $r = \sqrt{7}$
  7. The center is (2, 3); the circle passes through (5, 6).
  8. The points (8, 0) and (0, 6) are endpoints of a diameter.
  9. The center is (5, -4) and the circle is tangent to the  $x$ -axis.
  10. The center is (-3, 1) and the circle is tangent to the line  $x = 4$ .
  11. The circle is tangent to the  $x$ -axis at (4, 0) and has  $y$ -intercepts -2 and -8.
  12. The circle contains (-2, 16) and has  $x$ -intercepts -2 and -32.

Write each equation in center-radius form. Give the center and radius.

13.  $x^2 + y^2 - 2x - 8y + 16 = 0$
14.  $x^2 + y^2 - 4x + 6y + 4 = 0$
15.  $x^2 + y^2 - 12y + 25 = 0$
16.  $x^2 + y^2 + 14x = 0$
17.  $2x^2 + 2y^2 - 10x - 18y = 1$
18.  $2x^2 + 2y^2 - 5x + y = 0$



19. Show that the line  $y = 2x + 8$  contains the center of the circle  $x^2 + y^2 + 6x - 4y + 8 = 0$ .
20. Determine whether the line  $3x + 2y = 6$  contains the center of the circle  $x^2 + y^2 + 4x - 12y + 24 = 0$ .



In Exercises 21–23, use a graphing calculator or computer software to graph each equation.

21.  $x^2 + y^2 = 50$       22.  $(x - 3)^2 + y^2 = 36$       23.  $x^2 + y^2 - 6y = 40$

In Exercises 24–26, on a single set of axes, sketch the graph of each semicircle whose equation is given.

24. a.  $y = \sqrt{9 - x^2}$       b.  $y = -\sqrt{9 - x^2}$   
 25. a.  $x = \sqrt{9 - y^2}$       b.  $x = -\sqrt{9 - y^2}$   
 26. a.  $y = \sqrt{16 - (x - 5)^2}$       b.  $x = 5 - \sqrt{16 - y^2}$

In Exercises 27–34, graph the equations. Solve the equations simultaneously to find the coordinates of any intersection points of their graphs. If the graphs are tangent or fail to intersect, say so.

27.  $x + y = 23$ ,  $x^2 + y^2 = 289$       28.  $9y - 8x = 10$ ,  $x^2 + y^2 = 100$   
 29.  $2x - y = 7$ ,  $x^2 + y^2 = 7$       30.  $x + 2y = 10$ ,  $x^2 + y^2 = 20$   
 31.  $5x + 2y = -1$ ,  $x^2 + y^2 = 169$       32.  $y = 5$ ,  $x^2 + y^2 - 4x - 6y = -9$   
 33.  $y = \sqrt{3}x$ ,  $x^2 + (y - 4)^2 = 16$       34.  $x - y = 3$ ,  $x^2 + y^2 - 10x + 4y = -13$

35. **Writing** Write a description of the graphs of  $x^2 + y^2 < 1$  and  $x^2 + y^2 > 1$ .
36. Sketch the graph of  $(x - 3)^2 + (y - 4)^2 \leq 25$ .



37. The line  $x - 2y = 15$  intersects the circle  $x^2 + y^2 = 50$  in points  $A$  and  $B$ . Show that the line joining the center of the circle to the midpoint of  $\overline{AB}$  is perpendicular to  $\overline{AB}$ .
38. Find the length of a tangent line segment from  $(10, 5)$  to the circle  $x^2 + y^2 = 25$ .
39.  $P(2, 3)$  is on the circle with center  $O(0, 0)$ .  
 a. Write an equation of the circle.  
 b. Write an equation for the tangent  $l$  to the circle at  $P$ . (Hint:  $l$  is perpendicular to  $\overline{OP}$ .)
40. Show that  $P(4, 2)$  is on the circle with equation  $(x - 3)^2 + (y - 4)^2 = 5$ . Find an equation of the tangent to the circle at  $P$ . (Hint: See the hint for Exercise 39.)
41. A circle with center  $C(2, 4)$  has radius 13.  
 a. Verify that  $A(14, 9)$  and  $B(7, 16)$  are points on this circle.  
 b. If  $M$  is the midpoint of  $\overline{AB}$ , show that  $\overline{CM} \perp \overline{AB}$ .



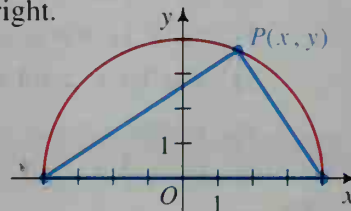
42. A circle with center  $C(-4, 0)$  has radius 15.
- Verify that  $A(8, 9)$  and  $B(-13, 12)$  are points on this circle.
  - Write an equation of the perpendicular bisector of  $\overline{AB}$  and show that the coordinates of point  $C$  satisfy the equation.
  - What theorem from geometry does this exercise illustrate?
43. A diameter of a circle has endpoints  $A(13, 0)$  and  $B(-13, 0)$ .
- Show that  $P(-5, 12)$  is a point on this circle.
  - Show that  $\overline{PA}$  and  $\overline{PB}$  are perpendicular.
44. a. Find the coordinates of  $A$  and of  $B$  if  $\overline{AB}$  is a horizontal diameter of the circle  $x^2 + y^2 - 34x = 0$ .
- b. Show that  $P(2, 8)$  is a point on this circle and that  $\overline{PA} \perp \overline{PB}$ .
45. Given  $O(0, 0)$  and  $N(12, 0)$ , find an equation in terms of  $x$  and  $y$  for all points  $P(x, y)$  such that  $\overline{PO} \perp \overline{PN}$ . Simplify this equation and show that  $P$  is on a circle. What are the center and radius of the circle?
46. **Discussion** Given  $A(6, 8)$  and  $B(-6, -8)$ , write an equation in terms of  $x$  and  $y$  for all points  $P(x, y)$  such that  $\overline{PA} \perp \overline{PB}$ . Simplify the equation and interpret your answer.



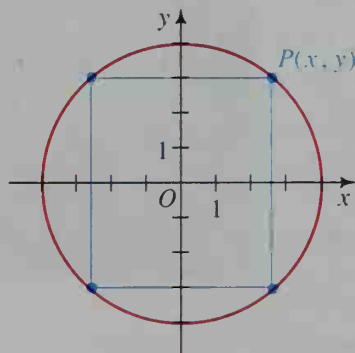
You may find it helpful to have a graphing calculator to complete Exercises 47–49.

47. A triangle is inscribed in a semicircle as shown at the right.

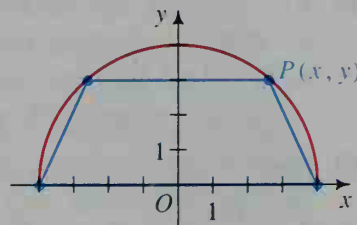
- Find an equation for the semicircle.
- Write a function  $A(x)$  for the area of the triangle.
- What is the domain of the function from part (b)?
- Graph  $y = A(x)$ .
- Use the graph from part (d) to find the value of  $x$  that maximizes  $A(x)$ .
- What is the maximum value of  $A(x)$ ?



48. a. A rectangle is inscribed in a circle of radius 4 as shown at the left below. Write a function  $A(x)$  for the area of the rectangle.
- b. Graph  $y = A(x)$ . Use the graph from part (a) to find the value of  $x$  that maximizes  $A(x)$ . What is the maximum value of  $A(x)$ ?



Ex. 48



Ex. 49

49. a. An isosceles trapezoid is inscribed in a semicircle of radius 4 as shown at the bottom of page 224. Write a function  $A(x)$  for the area of the trapezoid.  
 b. Find the value of  $x$  that maximizes  $A(x)$ .

Find an equation of the circle that contains the given points.

50.  $A(0, 0)$ ,  $B(2, 0)$ , and  $C(2, 2)$       51.  $P(0, 0)$ ,  $Q(6, 0)$ , and  $R(0, 8)$   
 52.  $L(8, 2)$ ,  $M(1, 9)$ , and  $N(1, 1)$       53.  $D(7, 5)$ ,  $E(1, -7)$ , and  $F(9, -1)$

In Exercises 54–57, describe the set of points satisfying each equation.

54.  $x^2 + y^2 + 2x + 2y + 2 = 0$       55.  $x^2 + y^2 - 6x + 8y + 26 = 0$   
**C** 56.  $(x^2 + y^2 - 1)(x^2 + y^2 - 4) = 0$       57.  $x^3y + xy^3 - xy = 0$   
 58. a. A point  $(x, y)$  lies inside the circle  $x^2 + y^2 = 2$  and above the line  $y = 1$ . Give two inequalities that must be satisfied.  
 b. Sketch the region in which the point lies and find the area of the region.  
 59. a. Sketch the set of points that satisfies  $|x| \geq 1$  and  $x^2 + y^2 \leq 4$ .  
 b. Find the area of the region.  
 60. Suppose that point  $P(a, b)$  is any point on the circle with center  $O(0, 0)$  and radius  $r$ . Suppose that line  $l$  is perpendicular to  $\overline{OP}$  at  $P$ . Prove that  $l$  is tangent to the circle as follows:  
 a. Show that the equation of  $l$  can be written  $ax + by = r^2$ .  
 b. Solve the equations of  $l$  and the circle simultaneously. Show that there is only one solution.

## 6-3 Ellipses

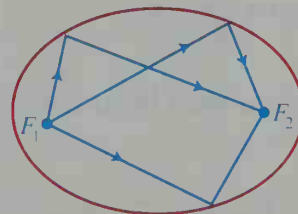
### Objective

To find equations of ellipses and to graph them.

If you stand a few feet away from a wall and shine a flashlight against it, you can make a lighted area in the form of an oval, or *ellipse*. The ellipse appears to be an elongated circle. Try it.

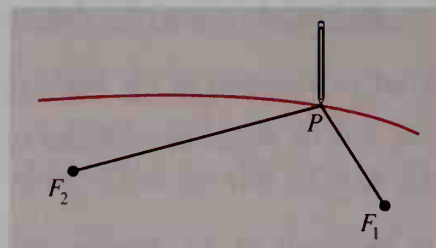
Ellipses are found in many applications. Light or sound waves from one *focus*  $F_1$  are reflected from the ellipse to the other focus  $F_2$ . This reflection property is used to make a whispering chamber, where a person whispering at  $F_1$  can be heard at  $F_2$ . The United States Capitol Building has such a chamber.

To sketch an ellipse, complete the Activity on the next page.



## Activity

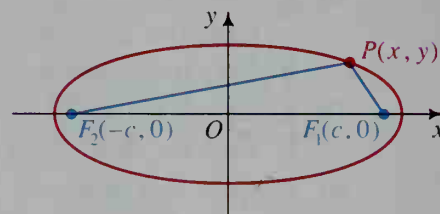
Fasten the ends of a string to a piece of cardboard with thumbtacks. Make sure the string has some slack. Keeping the string taut, draw a curve on the cardboard as shown. Describe the curve traced by the pencil point  $P$ . Repeat the experiment by moving the tacks farther apart or closer together.



If  $F_1(c, 0)$  and  $F_2(-c, 0)$  are two fixed points in the plane and  $a$  is a constant,  $0 < c < a$ , then the set of all points  $P$  in the plane such that

$$PF_1 + PF_2 = 2a$$

is an **ellipse**. This is the *geometric definition* of an ellipse. Points  $F_1$  and  $F_2$  are called the **foci** of the ellipse. (*Foci* is the plural of *focus*.) In the Activity above, the thumbtacks are the foci and the string has length  $2a$ .



To find an equation for the ellipse with foci  $F_1(c, 0)$  and  $F_2(-c, 0)$ , we let  $P(x, y)$  be any point on the ellipse, then express  $PF_1$  and  $PF_2$  in terms of  $x$ ,  $y$ , and  $c$ . Since  $PF_1 + PF_2 = 2a$ , we have:

$$\begin{aligned}\sqrt{(x - c)^2 + y^2} + \sqrt{(x - (-c))^2 + y^2} &= 2a \\ \sqrt{(x + c)^2 + y^2} &= 2a - \sqrt{(x - c)^2 + y^2}\end{aligned}$$

$$\begin{aligned}\text{Squaring: } (x + c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + [(x - c)^2 + y^2] \\ x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + x^2 - 2cx + c^2 + y^2\end{aligned}$$

$$\text{Simplifying: } 4cx = 4a^2 - 4a\sqrt{(x - c)^2 + y^2}$$

Dividing by 4 and rearranging terms:

$$cx - a^2 = -a\sqrt{(x - c)^2 + y^2}$$

$$\begin{aligned}\text{Squaring again: } c^2x^2 - 2ca^2x + a^4 &= a^2[(x - c)^2 + y^2] \\ &= a^2x^2 - 2ca^2x + a^2c^2 + a^2y^2\end{aligned}$$

$$\text{Simplifying: } c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$\begin{aligned}\text{Rearranging terms: } a^4 - a^2c^2 &= a^2x^2 - c^2x^2 + a^2y^2 \\ a^2(a^2 - c^2) &= (a^2 - c^2)x^2 + a^2y^2\end{aligned}$$

Since  $a$  must be greater than  $c$ , substitute  $b^2 = a^2 - c^2$ :

$$a^2b^2 = b^2x^2 + a^2y^2$$

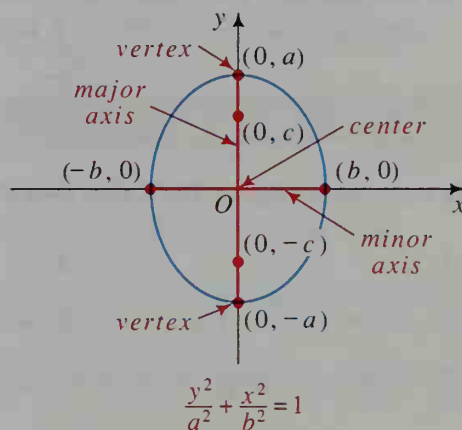
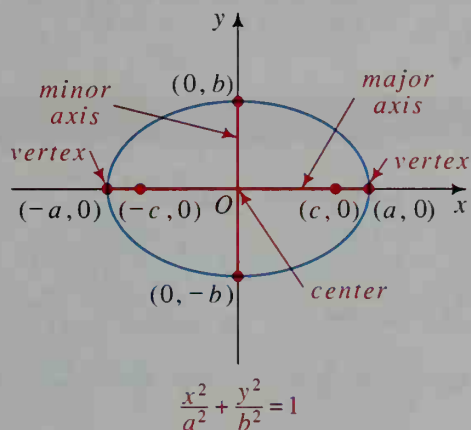
$$\text{Dividing by } a^2b^2, \text{ we have: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2 - c^2$$

The equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is an *algebraic definition* of an ellipse with center at the origin and foci on the  $x$ -axis. Sometimes it is convenient to consider an ellipse with foci  $(0, c)$  and  $(0, -c)$ . In this case, we can derive an equation using similar reasoning and obtain:

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \text{ where } b^2 = a^2 - c^2$$

The steps in the derivation of these equations can be reversed to show that any point satisfying the equations must also satisfy the condition  $PF_1 + PF_2 = 2a$ . Thus the algebraic and geometric definitions are equivalent.

All ellipses have two axes of symmetry. In the figures below, the axes of symmetry are the  $x$ - and  $y$ -axes. The portions of the axes of symmetry that lie on or within the ellipse are called the **major axis** and **minor axis** of the ellipse. The endpoints of the major axis are called the **vertices** of the ellipse. The midpoint of the major axis is the **center** of the ellipse.



### Example

Find an equation of the ellipse with center at the origin, one vertex at  $(0, 5)$ , and one focus at  $(0, 2)$ . Sketch the ellipse, and label the vertices and the endpoints of the minor axis.

### Solution

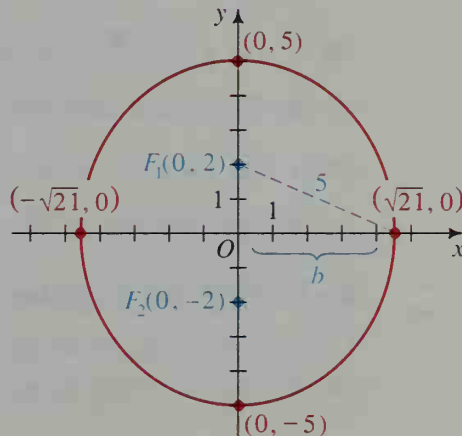
Since one focus is  $(0, 2)$ , we have that  $c = 2$ . One vertex is at  $(0, 5)$ , so  $a = 5$  and:

$$b^2 = a^2 - c^2 = 5^2 - 2^2 = 21$$

We substitute to write an equation of the ellipse:

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

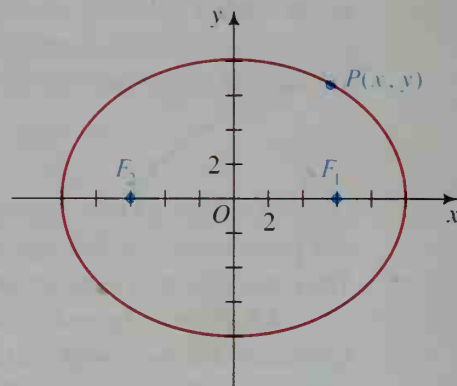
$$\frac{y^2}{25} + \frac{x^2}{21} = 1$$





## CLASS EXERCISES

- Study the ellipse shown and state:
  - whether its major axis is horizontal or vertical.
  - its vertices and foci.
  - the constant value of  $PF_1 + PF_2$ .  
(Hint: Suppose  $P$  is at a vertex.)
  - an equation of the ellipse.
- An ellipse has equation  $\frac{x^2}{25} + \frac{y^2}{169} = 1$ .
  - Is its major axis horizontal or vertical?
  - Find the coordinates of its vertices and foci.
- Describe the graphs of  $\frac{x^2}{9} + y^2 < 1$  and  $\frac{x^2}{9} + y^2 > 1$ .
- Discussion** If  $F_1$  and  $F_2$  are fixed points in space, describe the set of points  $P$  such that (a)  $PF_1 + PF_2 = 8$  and (b)  $PF_1 + PF_2 < 8$ .



## WRITTEN EXERCISES

Sketch each ellipse. Find the coordinates of its vertices and foci.

- A**
- $\frac{x^2}{36} + \frac{y^2}{16} = 1$
  - $\frac{x^2}{4} + \frac{y^2}{9} = 1$
  - $\frac{x^2}{16} + \frac{y^2}{25} = 1$
  - $4x^2 + 25y^2 = 100$
  - $9x^2 + 25y^2 = 225$
  - $6.25x^2 + 4y^2 = 25$
  - Sketch the graph of each inequality: (a)  $\frac{x^2}{25} + \frac{y^2}{9} \leq 1$  (b)  $\frac{x^2}{4} + \frac{y^2}{16} \geq 1$



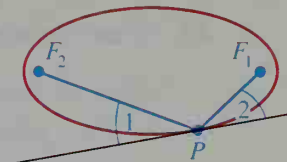
Find the domain and range of each function. Then graph the function. You may find a graphing calculator helpful.

- $y = 3\sqrt{1-x^2}$
- $y = -\frac{1}{3}\sqrt{36-x^2}$
- On a single set of axes, graph  $x^2 + y^2 = 1$  and  $\left(\frac{x}{3}\right)^2 + y^2 = 1$ .
  - Give the area of the circle and guess the area of the ellipse.
- On a single set of axes, graph  $x^2 + y^2 = 1$  and  $x^2 + \left(\frac{y}{2}\right)^2 = 1$ .
  - Give the area of the circle and guess the area of the ellipse.

Each ellipse has its center at the origin. Find an equation of the ellipse.

- Vertex  $(7, 0)$ ; minor axis 2 units long
- Vertex  $(0, -9)$ ; minor axis 6 units long
- Vertex  $(0, -13)$ ; focus  $(0, -5)$
- Vertex  $(17, 0)$ ; focus  $(8, 0)$

16. a. **Investigation** Use the method described in the Activity to draw a large ellipse filling most of a full page of paper. Label the thumbtack holes  $F_2$  and  $F_1$ . Then pick any point  $P$  on the ellipse. Draw  $\overline{F_2P}$ ,  $\overline{F_1P}$ , and the tangent to the ellipse at  $P$ . Use a protractor to show that the angle between  $\overline{F_2P}$  and the tangent is congruent to the angle between  $\overline{F_1P}$  and the tangent. Repeat the experiment for another point  $P$ .
- b. What property of the ellipse does part (a) illustrate?
17. **Investigation** Refer to the Activity on page 226. What happens to an ellipse as its foci  $F_1$  and  $F_2$  move toward each other? if  $F_1$  and  $F_2$  coincide?



**Sketch the graphs of the given equations on a single set of axes. Then determine algebraically where the graphs intersect.**

18.  $9x^2 + 2y^2 = 18$   
 $3x + y = -3$
19.  $x^2 + 4y^2 = 400$   
 $x - 2y = 28$
20.  $2x^2 + y^2 = 9$   
 $y - 4x = 9$
21.  $x^2 + 4y^2 = 16$   
 $|x| = 2$
22. a. What happens to the graph of an equation of an ellipse when  $x$  is replaced by  $x - 5$ ?
- b. Graph  $\frac{x^2}{4} + y^2 = 1$  and  $\frac{(x - 5)^2}{4} + y^2 = 1$  on a single set of axes.
23. a. What happens to the graph of an equation of an ellipse when  $y$  is replaced by  $y + 6$ ?
- b. Graph  $\frac{x^2}{4} + \frac{y^2}{9} = 1$  and  $\frac{x^2}{4} + \frac{(y + 6)^2}{9} = 1$  on a single set of axes.

**Sketch each ellipse. Label the center and vertices.**

- B** 24.  $\frac{(x - 3)^2}{4} + \frac{(y - 6)^2}{25} = 1$
25.  $\frac{(x + 5)^2}{25} + \frac{(y - 4)^2}{16} = 1$
26.  $(x + 7)^2 + \frac{(y - 5)^2}{9} = 1$
27.  $4(x + 2)^2 + (y - 5)^2 = 4$

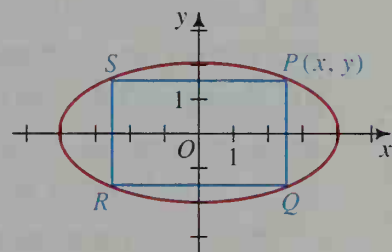


You may find it helpful to have a graphing calculator to complete Exercise 28(b).

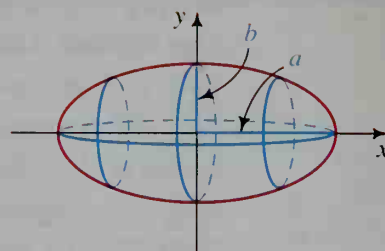
28. In the figure at the right, rectangle  $PQRS$  is inscribed in the ellipse  $\frac{x^2}{16} + \frac{y^2}{4} = 1$ .

- a. Show that the area of the rectangle is  $A(x) = 4x\sqrt{4 - \frac{x^2}{4}}$ .

- b. Approximate the maximum area to the nearest tenth of a square unit.



29. When the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is rotated about either of its axes, an *ellipsoid* is formed.
- The volume of the ellipsoid shown is  $V = \frac{4}{3}\pi b^2 a$ . Interpret this volume if the original ellipse is a circle.
  - Sketch the ellipsoid formed by rotating the given ellipse about its minor axis and guess its volume.



Sketch each ellipse. Find the coordinates of its vertices and foci.

- $\frac{(x-5)^2}{25} + \frac{(y+3)^2}{9} = 1$
- $\frac{(x+6)^2}{12} + \frac{(y-4)^2}{16} = 1$
- $9(x-3)^2 + 4(y+5)^2 = 36$
- $(x+1)^2 + 4(y+3)^2 = 9$
- $x^2 + 25y^2 - 6x - 100y + 84 = 0$   
(Hint: Complete the squares in  $x$  and in  $y$ . Begin by rewriting the equation in this form:  $(x^2 - 6x + \quad) + 25(y^2 - 4y + \quad) = -84$ .)
- $9x^2 + y^2 + 18x - 6y + 9 = 0$
- $9x^2 + 16y^2 - 18x - 64y - 71 = 0$



You may find a graphing calculator or graphing software helpful to complete Exercise 37.

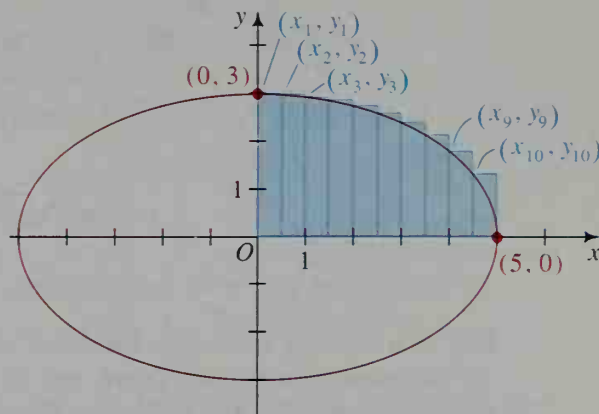
- A graphing calculator or computer software can be used to sketch the graph of an equation that represents a function.
  - Solve  $\frac{x^2}{36} + \frac{y^2}{16} = 1$  for  $y$ . The result involves two equations.
  - Graph the ellipse in part (a). Does the result agree with the graph in Exercise 1?

In Exercises 38–41, find an equation of the ellipse described.

- Center is  $(3, 7)$ ; one focus is  $(6, 7)$ ; one vertex is  $(8, 7)$ .
- Center is  $(4, -1)$ ; one vertex is  $(4, -5)$ ; one focus is  $(4, -3.5)$ .
- Vertices are  $(5, 9)$  and  $(5, 1)$ ; one focus is  $(5, 7)$ .
- Center is  $(5, 6)$ ; the ellipse is tangent to both axes.
- Suppose that  $F_1 = (3, 0)$ ,  $F_2 = (-3, 0)$ , and  $P = (x, y)$ . Using the distance formula, write an equation that expresses the fact that  $PF_1 + PF_2 = 10$ .
  - Simplify your equation to one of the form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

## //////COMPUTER EXERCISES

1. a. To approximate the area of the ellipse  $\frac{x^2}{25} + \frac{y^2}{9} = 1$ , shown at the right, you can add the areas of the ten rectangles in the first quadrant and multiply the sum by 4. The area of the leftmost rectangle is  $\frac{1}{2} \cdot 3$ ; the area of the next is  $\frac{1}{2}y_2$ ; and the area of the next is  $\frac{1}{2}y_3$ . How do you find  $y_2$  and  $y_3$ ?



- b. Write a program to print the approximate area of the ellipse.
2. Modify the program to print a more accurate approximation of the area.

## 6-4 Hyperbolas

### Objective

To find equations of hyperbolas and to graph them.

The equation of a *hyperbola* with center at the origin and horizontal and vertical axes of symmetry is very similar to the equation of an ellipse. Moreover, the terminology for ellipses and hyperbolas is very similar.

Suppose that  $F_1(c, 0)$  and  $F_2(-c, 0)$  are fixed points in the plane, and that  $a$  is a constant,  $a < c$ . The *geometric definition* of a **hyperbola** states that a hyperbola is the set of all points  $P(x, y)$  in the plane such that

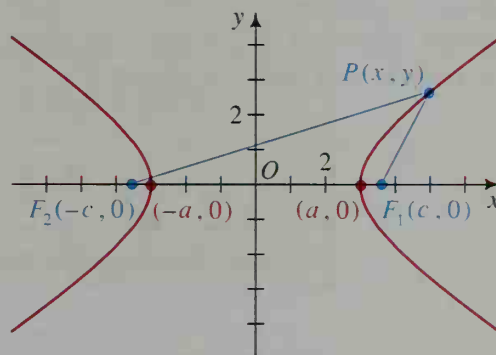
$$|PF_1 - PF_2| = 2a.$$

To derive an equation of a hyperbola with foci  $F_1(c, 0)$  and  $F_2(-c, 0)$ , we begin as follows:

$$|PF_1 - PF_2| = 2a$$

$$\sqrt{(x - c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = \pm 2a$$

The rest of the derivation is similar to that for the ellipse on page 226, except that we let  $b^2 = c^2 - a^2$  rather than  $b^2 = a^2 - c^2$ , since  $c > a$ . The equation that results is  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .



## Asymptotes

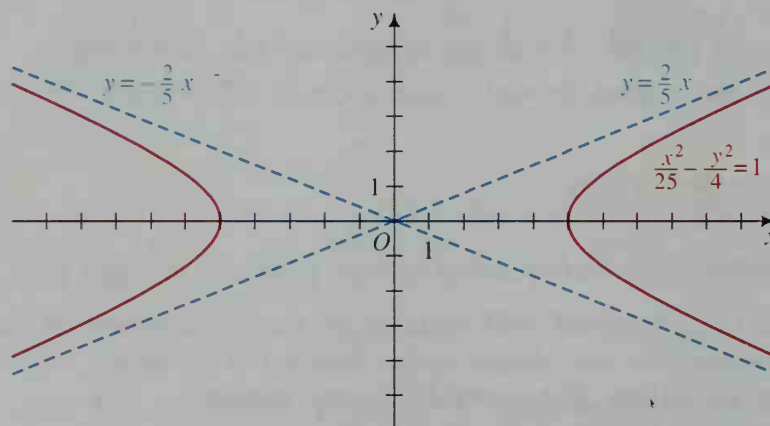
When we solve  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  for  $y$ , we obtain:

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$$

If  $|x|$  is very large,  $\sqrt{x^2 - a^2}$  is approximately the same as  $\sqrt{x^2} = |x|$ . Therefore, when  $|x|$  is very large,

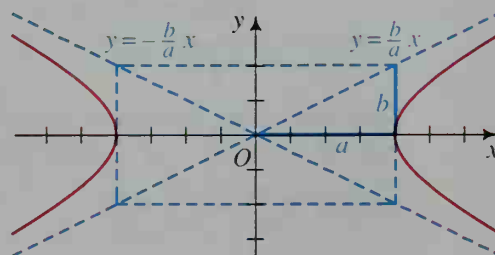
$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2} \approx \pm \frac{b}{a} x.$$

The lines  $y = \pm \frac{b}{a} x$  are called the *asymptotes* of the hyperbola. An **asymptote** of a curve is a line that the curve approaches more and more closely, that is, the distance between the curve and its asymptote becomes less and less as  $|x|$  becomes large. See the following figure and table.



$x$	10	20	100	1000
$y$ (on asymptote) $y = \frac{2}{5}x$	4	8	40	400
$y$ (on hyperbola) $y = \frac{2}{5}\sqrt{x^2 - 25}$	3.46	7.75	39.95	399.995
vertical distance (difference in $y$ 's)	0.54	0.25	0.05	0.005

The asymptotes for  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  contain the diagonals of a rectangle with dimensions  $2a$  and  $2b$ . The asymptotes and the rectangle are shown in blue. Drawing the rectangle and its diagonals is a good first step in graphing the equation.



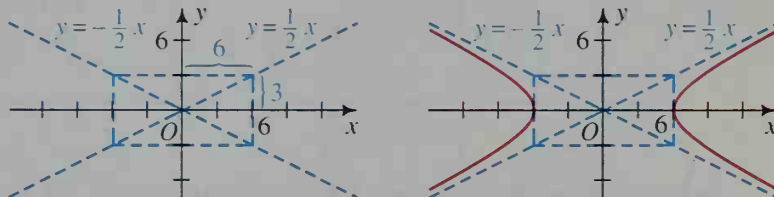


**Example 1**

Graph the hyperbola  $\frac{x^2}{36} - \frac{y^2}{9} = 1$ . Find its foci.

**Solution**

1. Since the hyperbola has  $x$ -intercepts  $\pm 6$ , the hyperbola contains  $(6, 0)$  and  $(-6, 0)$ . Draw the rectangle centered at the origin having length  $2 \cdot 6 = 12$  and width  $2 \cdot 3 = 6$ . Sketch the diagonals of the rectangle.



2. Sketch the curves through the vertices,  $(\pm 6, 0)$ , extending them towards the asymptotes as shown at the right above.
3. To find the foci, note that  $c^2 = a^2 + b^2 = 36 + 9 = 45$ . Thus,  $c = \pm\sqrt{45} = \pm 3\sqrt{5}$ . So the foci are  $(\pm 3\sqrt{5}, 0)$ .

The graph of

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

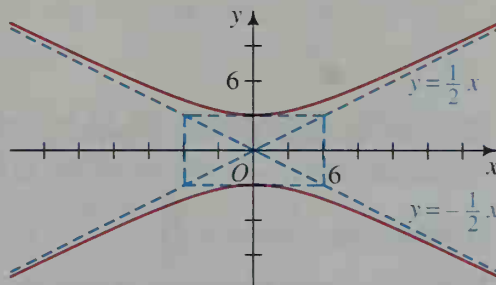
is also a hyperbola, but with a vertical major axis. Its asymptotes have equations  $y = \pm \frac{a}{b}x$ . Contrast the hyperbola in Example 2 with the one in Example 1.

**Example 2**

Graph the hyperbola  $\frac{y^2}{9} - \frac{x^2}{36} = 1$ . Find its foci.

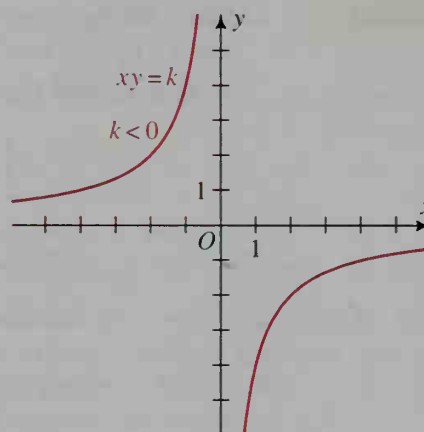
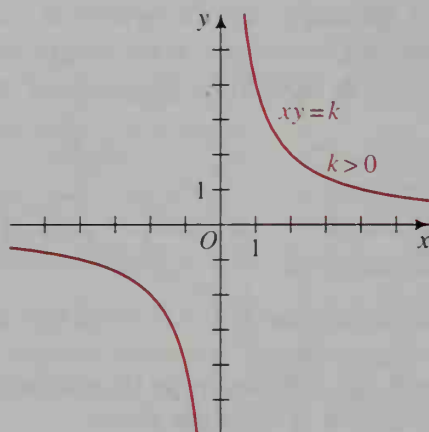
**Solution**

1. Since the hyperbola has  $y$ -intercepts  $\pm 3$ , the hyperbola contains  $(0, -3)$  and  $(0, 3)$ . Draw the rectangle centered at the origin having length  $2 \cdot 6 = 12$  and width  $2 \cdot 3 = 6$ . Sketch the diagonals of the rectangle.



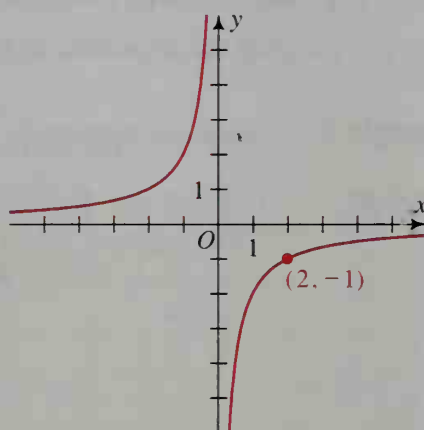
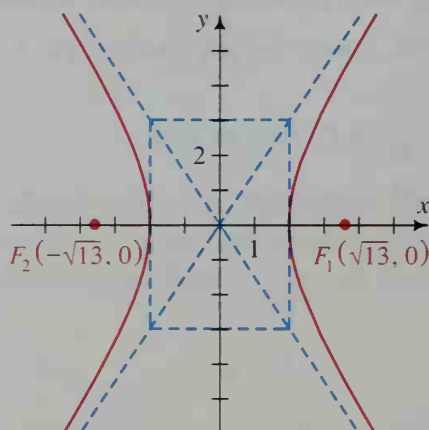
2. Sketch the curves through the vertices,  $(0, \pm 3)$ , extending them towards the asymptotes as shown above.
3. To find the foci, note that  $c^2 = a^2 + b^2 = 9 + 36 = 45$  and  $c = \pm 3\sqrt{5}$ . So the foci are  $(0, \pm 3\sqrt{5})$ .

The diagrams below show hyperbolas with simple equations, that is, equations of the form  $xy = k$  where  $k$  is a nonzero constant. The coordinate axes are the asymptotes for the hyperbolas.



## CLASS EXERCISES

- Give the vertices and foci for the hyperbola at the left below.
  - Is the major axis horizontal or vertical?
  - Find an equation for the hyperbola and equations of its asymptotes.



- Give an equation for the hyperbola at the right above.
  - Rotate the hyperbola  $90^\circ$  about the origin. What is its new equation?
- A hyperbola has equation  $\frac{y^2}{25} - \frac{x^2}{1} = 1$ .
  - Is its major axis horizontal or vertical? Explain.
  - What are its vertices and foci?
  - What are the equations of its asymptotes?
  - If the hyperbola were translated 6 units to the right and 5 units down, what would be its new equation?
- Describe the graphs of  $x^2 - y^2 < 4$  and  $x^2 - y^2 > 4$ .

## WRITTEN EXERCISES

In Exercises 1–8, sketch each hyperbola and its asymptotes. Give equations for the asymptotes.

**A**

1.  $\frac{x^2}{9} - \frac{y^2}{4} = 1$

3.  $x^2 - y^2 = 1$

5.  $4y^2 - x^2 = 4$

7.  $xy = 4$

2.  $\frac{y^2}{16} - \frac{x^2}{4} = 1$

4.  $y^2 - x^2 = 1$

6.  $25x^2 - 16y^2 = 400$

8.  $xy = -4$



In Exercises 9 and 10, give the domain and the range of each function. Then graph the function. You may use a graphing calculator if you wish.

9.  $y = \sqrt{x^2 - 4}$

10.  $y = -\sqrt{9 + x^2}$

In Exercises 11 and 12, sketch each hyperbola and its asymptotes. Give equations for the asymptotes.

11. a.  $xy = 12$

b.  $xy = -12$

c.  $(x + 3)(y + 4) = 12$

12. a.  $xy = 8$

b.  $xy = -8$

c.  $(x - 2)(y + 4) = -8$

13. On a single set of axes, graph the hyperbolas with equations  $x^2 - y^2 = k$ , where  $k = 4, 1, -1$ , and  $-4$ .

14. On a single set of axes, graph the hyperbolas with equations  $xy = k$ , where  $k = 12, 8, 4$ , and  $1$ .

In Exercises 15–18, sketch the graphs of the given equations on a single set of axes. Then determine algebraically the coordinates of any points where the graphs intersect.

15.  $x^2 - y^2 = 16$   
 $x + y = 5$

16.  $y^2 - x^2 = 94$   
 $x - y = 3$

17.  $x + 2y = 6$   
 $xy = -20$

18.  $2x + y = -14$   
 $xy = 24$

In Exercises 19–22, find an equation of the hyperbola, with center at the origin, that satisfies the given conditions.

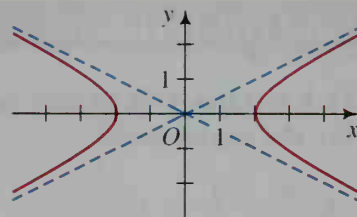
19. A vertex at  $(6, 0)$  and a focus at  $(10, 0)$

20. A vertex at  $(0, -12)$  and a focus at  $(0, -13)$

21. A vertex at  $(0, -2)$  and an asymptote with equation  $y = -x$

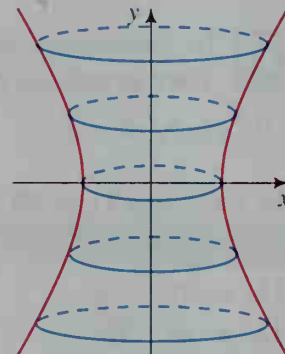
22. A vertex at  $(8, 0)$  and an asymptote with equation  $y = \frac{1}{2}x$

23. a. The diagram shows the graph of the hyperbola  $\frac{x^2}{4} - y^2 = 1$ . Show that this equation can be rewritten as  $y = \pm \frac{1}{2}\sqrt{x^2 - 4}$ .



- b. Find the vertical distance between the hyperbola and its asymptote  $y = \frac{1}{2}x$  when  $x = 5$ ,  $x = 10$ , and  $x = 100$ .
24. Graph  $y^2 - x^2 = 9$  and its asymptotes. Find a value of  $x$  for which the vertical distance between the hyperbola and an asymptote is less than 0.001.

- B** 25. **Visual Thinking** When the hyperbola with equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is rotated about its vertical axis of symmetry, a *hyperboloid of one sheet* is formed. (See the figure at the right.) When the hyperbola is rotated about its horizontal axis of symmetry, a *hyperboloid of two sheets* is formed. Sketch this hyperboloid.



26. a. **Visual Thinking** The region enclosed by the hyperbola  $y^2 - x^2 = 16$  and the lines  $x = \pm 3$  is rotated about the  $x$ -axis. Sketch the solid formed.
- b. Think of the solid as inscribed in a cylinder. Find the volume of the cylinder by using  $V = \pi r^2 h$ .

27. **Investigation** This exercise illustrates a reflection property of a hyperbola: The lines drawn from a point on a hyperbola to the foci form equal angles with the tangent at that point.



- a. Carefully trace the hyperbola and the foci of the hyperbola at the right.
- b. On one of the branches, choose a point  $P$  different from the one shown. Carefully sketch a line tangent to the hyperbola at  $P$ . Draw lines  $PF_1$  and  $PF_2$ .
- c. Measure  $\angle 1$  and  $\angle 2$ . What is the relationship between these angles?
28. a. Suppose that  $F_1 = (5, 0)$ ,  $F_2 = (-5, 0)$ , and  $P = (x, y)$ . Write an equation that expresses the fact that  $|PF_1 - PF_2| = 8$ .
- b. Simplify your equation to the form  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .
29. a. Suppose that  $F_1 = (0, 10)$ ,  $F_2 = (0, -10)$ , and  $P = (x, y)$ . Write an equation that expresses the fact that  $|PF_1 - PF_2| = 12$ .
- b. Simplify your equation to the form  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ .

On a single set of axes, sketch the graphs of the given equations.

30.  $x^2 - y^2 = 1$

$(x - 4)^2 - (y - 3)^2 = 1$

31.  $y^2 - 4x^2 = 4$

$(y + 5)^2 - 4x^2 = 4$

32. Suppose the hyperbola in Example 1 is translated 8 units right and 6 units up. Give the equation of the new hyperbola and the equations of its asymptotes.

33. Suppose the hyperbola in Example 1 is rotated  $90^\circ$  about the origin. Give the equation of the new hyperbola and the equations of its asymptotes.

Sketch the graph of each inequality.

34. a.  $xy \geq 8$

b.  $xy \geq 1$

c.  $xy \geq 0$

35. a.  $xy \leq -4$

b.  $xy \leq -1$

c.  $xy \leq 0$

36.  $x^2 - 4y^2 < 4$

37.  $y^2 - x^2 > 1$

38.  $(x - 5)^2 - (y - 4)^2 \geq 1$

Sketch the hyperbola whose equation is given. Find the coordinates of the vertices and the foci, and equations for the asymptotes.

39.  $\frac{(x - 6)^2}{36} - \frac{(y - 8)^2}{64} = 1$

40.  $\frac{(y + 5)^2}{16} - \frac{x^2}{9} = 1$

41.  $y^2 - x^2 - 2y + 4x - 4 = 0$

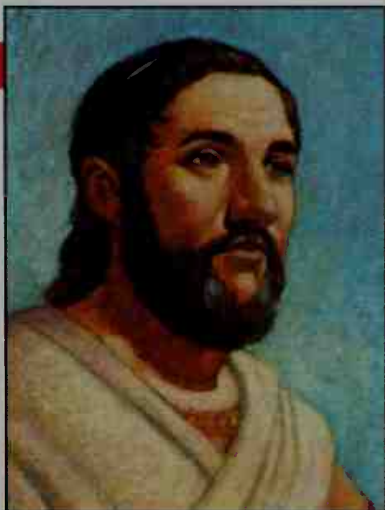
42.  $x^2 - 4y^2 - 2x + 16y - 19 = 0$

Find an equation of the hyperbola described.

43. Center is  $(5, 0)$ ; one vertex is  $(9, 0)$ ; one focus is  $(10, 0)$ .

44. Vertices are  $(4, 0)$  and  $(4, 8)$ ; asymptotes have slopes  $\pm 1$ .

**C** 45. Find the values of  $a$  and  $b$  that make the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  congruent to the hyperbola  $xy = 1$ .



*Apollonius (262–200 B.C.)*

Apollonius of Perga was an astronomer and geometer noted for his eight-volume work entitled *On Conic Sections*. In it, he derived all the conic sections from a plane intersecting a right double cone. He was the first to use the terms *ellipse*, *parabola*, and *hyperbola* in reference to conic sections.

The problem of constructing a circle tangent to three given circles in the plane bears his name. It is called the problem of Apollonius.



## 6-5 Parabolas

**Objective** To find equations of parabolas and to graph them.

When we discussed quadratic functions in Section 1-7, we stated that their graphs are parabolas. In this section, we will state the geometric and algebraic definitions of a parabola and examine some of the properties of a parabola.

A **parabola** is the set of all points  $P$  in the plane that are equidistant from a fixed point  $F$ , the **focus**, and a fixed line  $d$  (not containing  $F$ ), the **directrix**. That is, if  $F$  and  $d$  are fixed, then the set of all points  $P$  such that  $PF = PN$  is a parabola.

The line through the focus  $F$  and perpendicular to the directrix  $d$  is the line of symmetry for the parabola. It intersects the parabola at its **vertex**.

If the line  $y = -p$ ,  $p \neq 0$ , is the directrix of a parabola with focus  $F(0, p)$ , then by the geometric definition above:

$$PF = PN$$

$$\sqrt{(x-0)^2 + (y-p)^2} = |y+p|$$

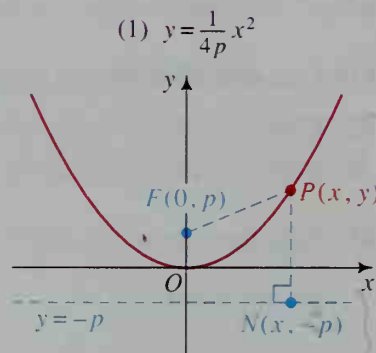
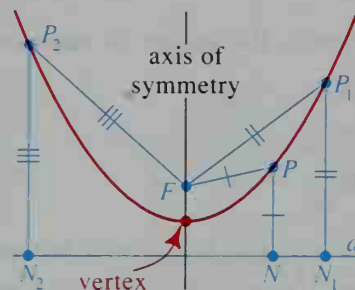
Squaring and simplifying we get:

$$x^2 + (y-p)^2 = (y+p)^2$$

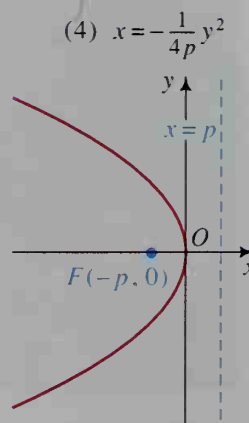
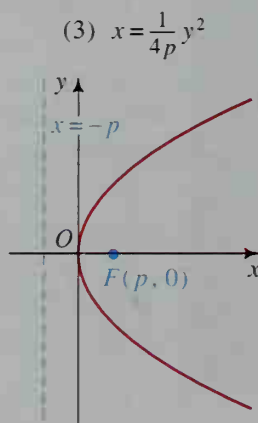
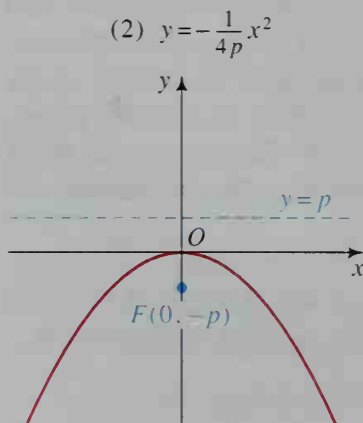
$$x^2 + y^2 - 2yp + p^2 = y^2 + 2yp + p^2$$

$$x^2 = 4py$$

$$(1) y = \frac{1}{4p}x^2$$



The figures below show other parabolas with vertex  $(0, 0)$ . The derivation of each equation is analogous to that shown above.



**Example 1** Find the focus and directrix of each parabola whose equation is given.

a.  $y = 2x^2$

b.  $x = \frac{1}{20}y^2$

**Solution**

a. Comparing  $y = 2x^2$  with  $y = \frac{1}{4p}x^2$ , we see that  $p = \frac{1}{8}$ . Thus, the focus is  $F(0, \frac{1}{8})$  and the directrix is  $y = -\frac{1}{8}$ .

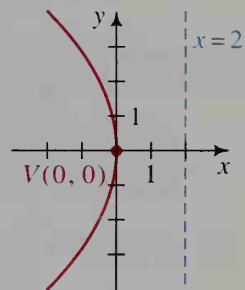
b. Comparing  $x = \frac{1}{20}y^2$  with  $x = \frac{1}{4p}y^2$ , we see that  $p = 5$ . Thus, the focus is  $F(5, 0)$  and the directrix is  $x = -5$ .

**Example 2**

Find an equation of the parabola with vertex  $(0, 0)$  and directrix  $x = 2$ .

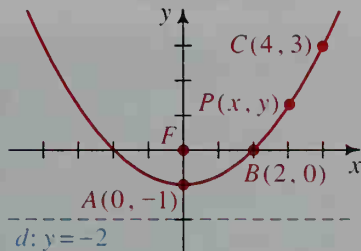
**Solution**

A sketch shows that the parabola has an equation of type (4),  $x = -\frac{1}{4p}y^2$ . Since  $p$  is the distance from the vertex to the directrix,  $p = 2$ . Thus,  $x = -\frac{1}{8}y^2$ .

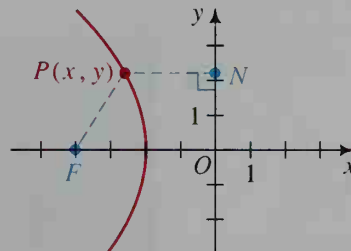


## CLASS EXERCISES

- Points  $A$ ,  $B$ ,  $C$ , and  $P$  are on the parabola  $y = \frac{1}{4}x^2 - 1$ , whose focus is  $F(0, 0)$  and whose directrix  $d$  is the line  $y = -2$ . (See the figure at the left below.)
  - Find the distances from  $A$  to  $F$  and  $A$  to  $d$ .
  - Find the distances from  $B$  to  $F$  and  $B$  to  $d$ .
  - Find the distances from  $C$  to  $F$  and  $C$  to  $d$ .
  - Find the distances from  $P$  to  $F$  and  $P$  to  $d$ .



Ex. 1



Ex. 2

- The point  $P(x, y)$  is on the parabola whose focus  $F$  is  $(-4, 0)$  and whose directrix is the  $y$ -axis. (See the figure at the right above.)
  - Express (1) the distance  $PF$  and (2) the distance  $PN$  in terms of  $x$  and  $y$ .
  - Explain how to derive an equation of the parabola.

3. **Writing** In a few sentences, describe how to tell whether a parabola opens up, down, left, or right when you are given an equation for the parabola.

Give the focus and the directrix of each parabola. Tell whether the parabola opens up, down, left, or right.

4.  $y = \frac{1}{4}x^2$       5.  $y = -\frac{1}{4}x^2$       6.  $x = y^2$       7.  $x = -y^2$

8. **Discussion** You know that a parabola can open up, down, left, or right. Is it possible for a parabola to open in some other direction? Explain.
9. **Discussion** Explain how to use the graph of the equation  $y = x^2$  to graph the equation  $y - k = (x - h)^2$ .

## WRITTEN EXERCISES

For each parabola give the coordinates of its vertex and focus and the equation of its directrix.

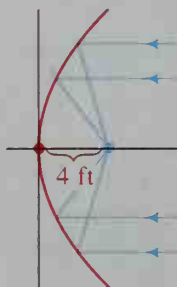
- A** 1. a.  $y = \frac{1}{8}x^2$       b.  $x = \frac{1}{8}y^2$       2. a.  $y = -\frac{1}{12}x^2$       b.  $x = -\frac{1}{12}y^2$
3. a.  $y = -2x^2$       b.  $x = -2y^2$       4. a.  $y = x^2 + 1$       b.  $x = y^2 + 1$

In Exercises 5–8, for each parabola give the coordinates of its vertex and focus and the equation of its directrix. (Use translations.)

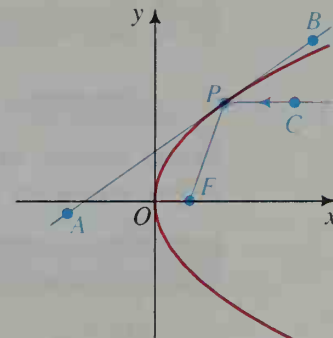
5.  $y - 1 = \frac{1}{4}(x - 2)^2$       6.  $y + 3 = \frac{1}{8}(x - 5)^2$

7.  $x - 4 = (y - 7)^2$       8.  $x + 2 = -2(y - 3)^2$

9. **Telecommunications** A satellite dish used to receive television signals has a parabolic cross-section as shown at the left below. Incoming signals that are parallel to the axis of the parabola are reflected to its focus. Thus, the incoming signal is magnified. Find an equation of the parabola.




10. **Investigation** On a sheet of graph paper, carefully draw a *large* graph of the equation  $x = \frac{1}{16}y^2$ . Draw a tangent line to the parabola at any point  $P$ . Mark two points  $A$  and  $B$  on the tangent line. Join  $P$  to the focus  $F$  (which you must first determine). Draw a ray through  $P$  parallel to the axis of the parabola. Label a point  $C$  on the ray. With a protractor, measure  $\angle CPB$  and  $\angle FPA$ . What do you observe? Repeat the experiment for another point  $P$  on the parabola.



Find an equation of each parabola. Sketch its graph.

- B** 11. Focus,  $(-1, 0)$ ; directrix,  $x = 1$       12. Focus,  $(0, 0.25)$ ; directrix,  $y = -0.25$   
 13. Vertex,  $(0, 0)$ ; focus,  $(0, -0.25)$       14. Vertex,  $(0, 0)$ ; focus,  $(5, 0)$   
 15. Vertex,  $(0, 0)$ ; directrix,  $x = 4$       16. Vertex,  $(0, 0)$ ; directrix,  $y = -2$   
 17. Focus,  $(0, 2)$ ; directrix,  $y = 0$       18. Focus,  $(3, 0)$ ; directrix,  $x = -3$

 Sketch the graph of each “half-parabola.” You may use a graphing calculator if you wish.

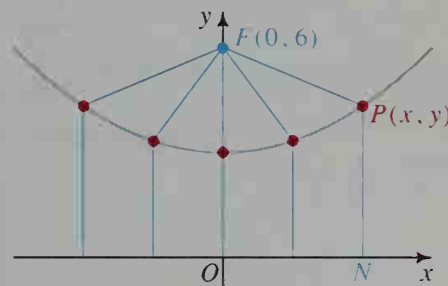
19. a.  $y = \sqrt{x}$       b.  $y = -\sqrt{x}$       c.  $y = -\sqrt{x-3}$       d.  $y = \sqrt{x+2} - 1$   
 20. a.  $x = \sqrt{y}$       b.  $x = -\sqrt{y}$       c.  $x = \sqrt{y-4}$       d.  $x = -\sqrt{y+3} + 1$

21. The diagram at the right shows many points  $P(x, y)$  that are equidistant from  $F(0, 6)$  and the  $x$ -axis.

a. Write an equation that states that  $PF = PN$ .


b. Rewrite the equation in the form

$$y - k = \frac{1}{4p} (x - h)^2.$$



The coordinates of  $F$  and an equation of line  $d$  are given. Write and simplify an equation that specifies the set of points  $P(x, y)$  that are equidistant from  $F$  and  $d$ .

22.  $F(0, 2)$       23.  $F(0, -3)$       24.  $F(-1, 3)$       25.  $F(-7, -5)$   
 $d: y = -2$        $d: x = -2$        $d: x = 0$        $d: y = -7$

 Find the vertex, focus, and directrix of each parabola. Graph the equation. Check your graph with a graphing calculator.

26.  $y = 2x^2 - 8x + 3$       27.  $y = 5 - 6x - 3x^2$   
 28.  $6x - x^2 = 8y + 1$       29.  $4y = x^2 - 8x + 12$   
 30.  $x = y^2 - 2y - 5$       31.  $y^2 - 6y + 16x + 25 = 0$

## 6-6 Systems of Second-Degree Equations

### Objective

To solve systems of second-degree equations.

In this section, we will investigate three methods of solving a system of second-degree equations in two variables.

### Methods for Solving Systems of Second-Degree Equations

**Method 1** Algebraic approaches, such as substitution or elimination

**Method 2** A graphing calculator or a computer graphing utility

**Method 3** A combination of algebraic and graphing methods



In the first solution to Example 1, you can see how to use technology to solve a system of second-degree equations.

### Example 1

Solve the system:  $xy = 6$  and  $x^2 + 4y^2 = 64$

### Solution 1

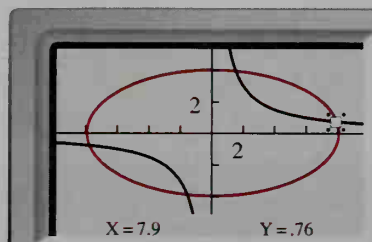
To solve the system graphically, you must first solve each equation for  $y$ .

$$y = \frac{6}{x} \quad \text{hyperbola}$$

$$y = \sqrt{16 - \frac{x^2}{4}} \quad \text{upper half of ellipse}$$

$$y = -\sqrt{16 - \frac{x^2}{4}} \quad \text{lower half of ellipse}$$

The figure shows a calculator or computer display of the graphs.



As you can see, there are four solutions. The figure suggests that the solutions in the third quadrant are reflections in the origin of the two solutions in the first quadrant. (You can verify this: If  $(a, b)$  satisfies each equation, show that  $(-a, -b)$  satisfies each equation.) Using a calculator or computer, you can find that approximate solutions are  $(1.5, 3.9)$ ,  $(7.9, 0.76)$ ,  $(-1.5, -3.9)$ , and  $(-7.9, -0.76)$ .

Alternatively, the system in Example 1 can be solved algebraically by using substitution.



**Solution 2**

**Step 1** Solve one equation for  $x$  or  $y$ . Solve  $xy = 6$  for  $x$ :  $x = \frac{6}{y}$

**Step 2** Substitute in the other equation and solve. Substitute  $\frac{6}{y}$  for  $x$  in the second equation.

$$\begin{aligned}\left(\frac{6}{y}\right)^2 + 4y^2 &= \frac{36}{y^2} + 4y^2 = 64 \\ 4y^4 - 64y^2 + 36 &= 0 \\ y^4 - 16y^2 + 9 &= 0\end{aligned}$$

Using the quadratic formula, we get:

$$\begin{aligned}y^2 &= \frac{16 \pm \sqrt{16^2 - 4 \cdot 9}}{2} = 8 \pm \sqrt{55} \approx 15.4 \text{ or } 0.584 \\ y &\approx \pm 3.9 \text{ or } \pm 0.76\end{aligned}$$

**Step 3** Find the corresponding values of  $x$ . When  $y = 3.9$ , for example,  $x = 6 \div 3.9 \approx 1.5$ . The four approximate solutions are  $(1.5, 3.9)$ ,  $(7.9, 0.76)$ ,  $(-1.5, -3.9)$ , and  $(-7.9, -0.76)$ .

**Example 2**

The figure shows two circles with equations  $x^2 + y^2 = 20$  and  $(x - 5)^2 + (y - 5)^2 = 10$ . Find the coordinates of their points of intersection, points  $A$  and  $B$ .

**Solution**

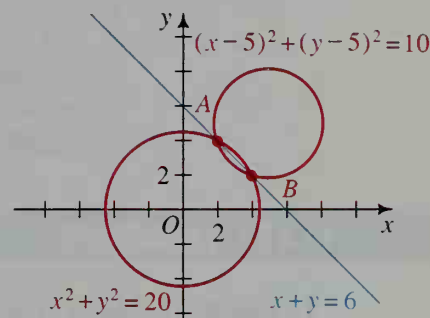
Write both equations in expanded form and then subtract.

$$\begin{array}{rcl}x^2 + y^2 & = & 20 \quad (1) \\ x^2 + y^2 - 10x - 10y & = & -40 \quad (2) \\ \hline 10x + 10y & = & 60 \\ x + y & = & 6 \quad (3)\end{array}$$

Equation (3) is the line containing  $A$  and  $B$ . From (3),  $y = 6 - x$ . Then substituting in (1) we get:

$$\begin{aligned}x^2 + (6 - x)^2 &= 20 \\ x^2 - 6x + 8 &= 0 \\ (x - 2)(x - 4) &= 0 \\ x &= 2 \text{ or } 4\end{aligned}$$

When  $x = 2$ ,  $y = 6 - x = 4$ . When  $x = 4$ ,  $y = 6 - x = 2$ . Thus, the intersection points are  $A(2, 4)$  and  $B(4, 2)$ .



**Example 3** Solve  $2x^2 + y^2 = 4$  and  $x^2 - 2y^2 = 12$  simultaneously.

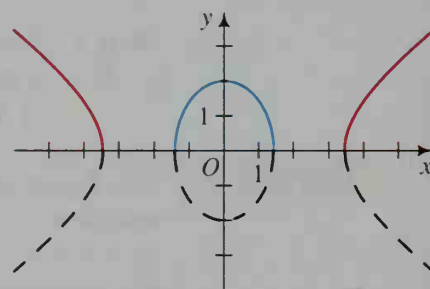
**Solution 1** Multiply the first equation by 2. Then add.

$$\begin{array}{r} 4x^2 + 2y^2 = 8 \\ x^2 - 2y^2 = 12 \\ \hline 5x^2 = 20 \end{array}$$

Therefore,  $x = \pm 2$ . Substitute  $x = \pm 2$  in either equation to obtain  $y = \pm 2i$ . Since there is no solution  $(x, y)$  where  $x$  and  $y$  are real numbers,  $2x^2 + y^2 = 4$  and  $x^2 - 2y^2 = 12$  have no common solution.

Alternatively, you can solve the system by graphing the upper half of the ellipse and the hyperbola and then using symmetry.

**Solution 2** Solving  $2x^2 + y^2 = 4$  and  $x^2 - 2y^2 = 12$  for  $y$ , we get  $y = \sqrt{4 - 2x^2}$  for the upper half of the ellipse and  $y = \sqrt{\frac{x^2}{2} - 6}$  for the upper half of the hyperbola. By symmetry, we can see that there are no solutions.



## CLASS EXERCISES

- Visual Thinking** Two quadratic equations in two variables may have 4, 3, 2, 1, or 0 real solutions. Illustrate each of these cases with a circle and a parabola.
- Compare the graphs of  $x^2 + y^2 = 9$  and  $(x + y)^2 = 9$ .
- Compare the graphs of  $x^2 - y^2 = 4$  and  $(x - y)^2 = 4$ .

## WRITTEN EXERCISES

Sketch the graphs of each pair of circles to determine the number of points of intersection. If the circles are tangent or fail to intersect, say so. Then solve the system.

- A**
- $x^2 + y^2 = 16$ ,  $(x - 4)^2 + y^2 = 16$
  - $x^2 + y^2 = 4$ ,  $x^2 + (y - 6)^2 = 25$
  - $x^2 + y^2 = 20$ ,  $(x - 2)^2 + (y + 1)^2 = 13$
  - $x^2 + y^2 - 4y = 0$ ,  $x^2 + y^2 - 2x = 4$
  - $x^2 + y^2 = 5$ ,  $x^2 + y^2 - 12x + 6y = -25$
  - $x^2 + y^2 = 4$ ,  $x^2 + y^2 - 6x - 6y = -14$

7. **Visual Thinking** Consider the ellipses  $4x^2 + 5y^2 = 81$  and  $5x^2 + 4y^2 = 81$ .
- In how many points do they intersect? Give a convincing argument to justify your answer. Illustrate your answer graphically.
  - Add the given equations. What is the graph of the resulting equation?
  - Why do the intersection points of the ellipses lie on the graph from part (b)?
8. Two ellipses have equations  $5x^2 + 3y^2 = 64$  and  $3x^2 + 5y^2 = 64$ . Show algebraically that their points of intersection are on the circle  $x^2 + y^2 = 16$ .



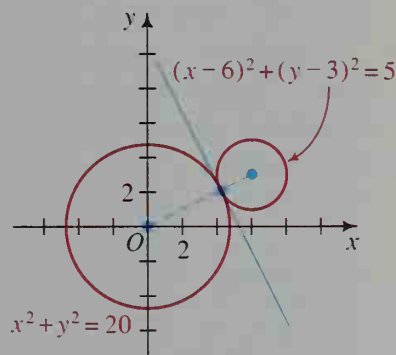
Solve each system algebraically or with a graphing calculator or computer software. (Note: For many of these systems, an algebraic approach may be more efficient.)

- |  |  |   |
|--|--|---|
| 9. $x^2 + 4y^2 = 16$<br>$x^2 + y^2 = 4$    | 10. $4x^2 + y^2 = 16$<br>$x^2 - y^2 = -4$  | 11. $x^2 + 9y^2 = 36$<br>$x^2 - 2y = 4$   |
| 12. $4x^2 + 4y^2 = 25$<br>$2x + y^2 = 1$   | 13. $x^2 + y^2 = 25$<br>$xy = -12$         | 14. $y^2 - x^2 = 64$<br>$xy = 24$         |
| 15. $x^2 - y^2 = 1$<br>$y = -1 - x^2$      | 16. $9x^2 + 4y^2 = 36$<br>$x^2 + y^2 = 16$ | 17. $xy = 4$<br>$y = -4x^2$               |
| 18. $9x^2 - 16y^2 = 144$<br>$x + y^2 = -4$ | 19. $(x + 3)^2 + y^2 = 1$<br>$x = -y^2$    | 20. $x^2 + 6y^2 = 9$<br>$5x^2 + y^2 = 16$ |

**B**

21.  $x^2 + y^2 = 9$   
 $x^2 + y^2 + 8x + 7 = 0$
22.  $4x^2 + y^2 - 4y - 32 = 0$   
 $x^2 - y - 7 = 0$

23. a. The circles  $x^2 + y^2 = 20$  and  $(x - 6)^2 + (y - 3)^2 = 5$  are tangent. Find the coordinates of the point of tangency and then find an equation of the common internal tangent shown. (Hint: How is this line related to the line containing the centers?)



- b. If you subtract the equations of the two circles from each other, you get a linear equation. Find this equation. Compare it with the equation found in part (a).
24. a. Find the radii and the distance between the centers of the circles with equations  $x^2 + y^2 = 225$  and  $(x - 6)^2 + (y - 8)^2 = 25$ .
- b. Use your answers to part (a) to conclude that the circles must be internally tangent. Make a sketch.
- c. Find an equation of the common tangent of the two circles.

Graph the solution set of each inequality or system of inequalities.

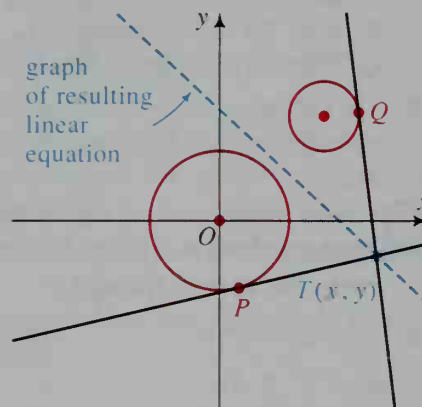
- |  |                                  |
|--|----------------------------------|
| 25. $x^2 - 9y^2 \leq 0$                  | 26. $y^2 - 2xy \geq 0$           |
| 27. $x^2 + 4y^2 - 10x + 24y + 61 \leq 0$ | 28. $x^2 - y^2 + 2x - 2y \geq 0$ |

$$29. \begin{aligned} 9x^2 - y^2 &\geq 0 \\ x^2 + y^2 &\leq 10 \end{aligned}$$

- C** 31. **Carpentry** A board 90 cm long just fits inside the bottom of a box 80 cm long and 60 cm wide. How wide is the board?

32. a. Given equations of two non-intersecting circles, show that the graph of the difference of the equations consists of those points  $T(x, y)$  for which tangents  $\overline{TP}$  and  $\overline{TQ}$  are congruent.  
b. Does the result in part (a) hold if the circles intersect?

$$30. \begin{aligned} x^2 + 4y^2 &\leq 16 \\ x^2 - 2xy &\geq 0 \end{aligned}$$



## COMMUNICATION: Reading

René Descartes was the first mathematician to use letters from the end of the alphabet to represent variables and letters from the beginning of the alphabet to represent constants, much as we do today. In the equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , the letters  $A, B, C, D, E$ , and  $F$  represent constants, and the letters  $x$  and  $y$  represent variables.

The *American Heritage Dictionary* entry for the letter  $c$  is given below. Pick a letter of the alphabet. Write as many mathematical or scientific uses of that letter as you can without using a dictionary. Use a dictionary to expand your list.

**c, C** (*sē*) *n.*, *pl. c's* or *rare cs. C's* or **Cs**. 1. The third letter of the modern English alphabet. See **alphabet**. 2. Any of the speech sounds represented by this letter.  
**c, C, c., C. Note:** As an abbreviation or symbol,  $c$  may be a small or a capital letter, with or without a period. Established forms or those generally preferred precede the definition. When no form is given, all four forms are in general use in that sense. 1. **c** *Physics*. candle. 2. **C** *Electricity*. capacitance. 3. **c., C.** capacity. 4. **c., C.** cape. 5. **c** carat. 6. **C** The symbol for the element carbon. 7. **c., C.** carton. 8. **c., C.** case. 9. **c., C.** *Baseball*. catcher. 10. **C.** Catholic. 11. **C** Celsius. 12. **C.** Celtic. 13. **c., C.** cent. 14. **c** centi-. 15. **C** centigrade. 16. **c., C.** centime. 17. **c., C.** century. 18. **C.** chancellor. 19. **c., C.** chapter. 20. **C** *Physics*. charge conjugation. 21. **C.** chief. 22. **c., C.** church. 23. **c., C.** circa. 24. **C.** city. 25. **c.** cloudy. 26. **C.** companion. 27. **c., C.** congius. 28. **C.** Congress. 29. **C.** Conservative. 30. **c, C** *Mathematics*. constant. 31. **c., C.** consul. 32. **c., C.** copy. 33. **c., C.** copyright. 34. **c., C.** corps. 35. **C** coulomb. 36. **C.** court. 37. **c** cubic. 38. **c.** cup. 39. **C** The Roman numeral for 100 (Latin *centum*). 40. The third in a series. 41. **C** The third highest in quality or rank; a *mark of C on a term paper*. 42. **C** *Music*. a. The first tone in the scale of C major, or the third tone in the relative minor scale. b. The key or a scale in which C is the tonic. c. A written or printed note representing this tone. d. A string, key, or pipe tuned to the pitch of this tone.

## 6-7 A New Look at Conic Sections

### Objective

To define conic sections in terms of eccentricity and to classify the graph of a second-degree equation by examining the coefficients in the equation.

Although the various conic sections look quite different from one another, they do have some common properties.

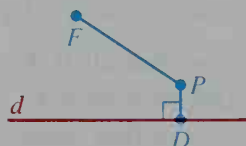
1. The conic sections result from slicing a double cone with a plane. (See page 213.)
2. The conic sections can all be obtained from the single definition given below.
3. The conic sections all have second-degree equations. (See the discussion and the theorem on the next page.)

### Common Definition of Conic Sections

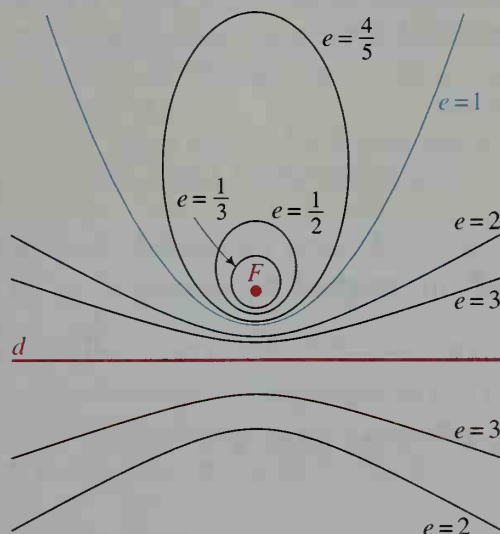
Let  $F$  (the **focus**) be a fixed point not on a fixed line  $d$  (the **directrix**). Let  $P$  be a point in the plane of  $d$  and  $F$ , and let  $PD$  be the perpendicular distance from  $P$  to  $d$ . Consider the set of points for which the ratio  $PF:PD$  is the constant  $e$ . The number  $e$  is the **eccentricity** of the conic section.

This set of points is:

- (1) an ellipse if  $0 < e < 1$ ;
- (2) a parabola if  $e = 1$ ;
- (3) a hyperbola if  $e > 1$ .



The diagram below suggests that once line  $d$  and point  $F$  are specified, the parabola (shown in blue) with directrix  $d$  and focus  $F$  separates the set of ellipses and the set of hyperbolas having this same directrix and focus. Notice that as the value of  $e$  approaches 0, the ellipse becomes more circular. The eccentricity of a circle is 0.





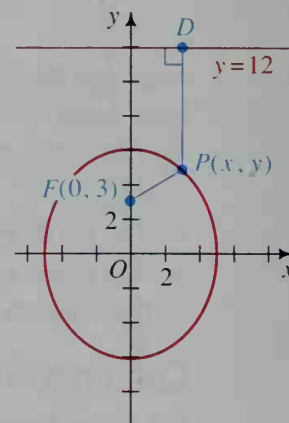
**Example 1**

Let the focus  $F$  be  $(0, 3)$  and let the directrix  $d$  have equation  $y = 12$ . Find the equation of the set of points  $P$  for which  $\frac{PF}{PD} = \frac{1}{2}$ , and identify the graph.

**Solution**

$$\begin{aligned}
 2(PF) &= PD \\
 2\sqrt{(x-0)^2 + (y-3)^2} &= |12-y| \\
 4[x^2 + (y-3)^2] &= (12-y)^2 \\
 4x^2 + 4y^2 - 24y + 36 &= 144 - 24y + y^2 \\
 4x^2 + 3y^2 &= 108 \\
 \frac{x^2}{27} + \frac{y^2}{36} &= 1
 \end{aligned}$$

The graph is an ellipse with vertices  $(0, \pm 6)$  and foci  $(0, \pm 3)$ .

**Second-Degree Equations in Two Variables**

A conic section can be considered the graph of a *second-degree equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where  $A$ ,  $B$ , and  $C$  are not all 0. For example, if  $A = 4$ ,  $B = 0$ ,  $C = 3$ ,  $D = 0$ ,  $E = 0$ , and  $F = -108$ , then we have  $4x^2 + 3y^2 - 108 = 0$ . In the example above, we saw that the graph of this equation is an ellipse.

So far we have considered primarily those second-degree equations in which  $B = 0$ . The one exception has been the hyperbola of the form  $xy = k$ . The theorem below enables us to classify a conic section by looking at its equation.

**Classifying Second-Degree Equations**

Consider the equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ . If  $A$ ,  $B$ , and  $C$  are not all 0 and if the graph is not degenerate, then:

- (1) the graph is a circle or an ellipse if  $B^2 - 4AC < 0$ ; (In a circle,  $B = 0$  and  $A = C$ .)
- (2) the graph is a parabola if  $B^2 - 4AC = 0$ ;
- (3) the graph is a hyperbola if  $B^2 - 4AC > 0$ .

**Example 2**

Identify the graph of the equation  $x^2 - 2xy + 3y^2 - 1 = 0$ .

**Solution**

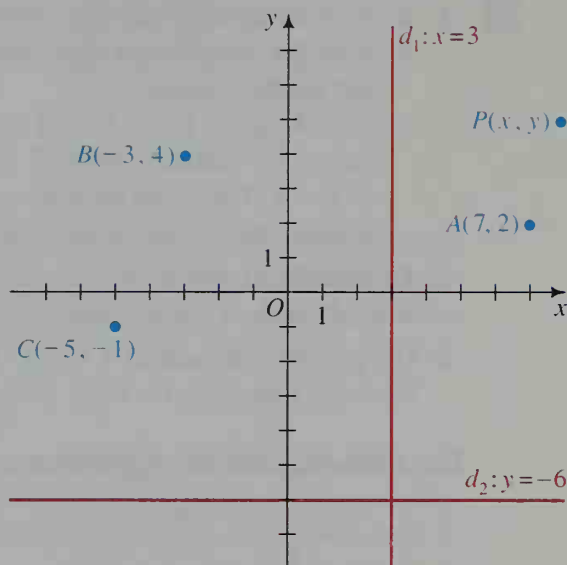
$$\begin{aligned}
 B^2 - 4AC &= (-2)^2 - 4(1)(3) \\
 &= -8 < 0
 \end{aligned}$$

Since  $B \neq 0$ , the graph is an ellipse.

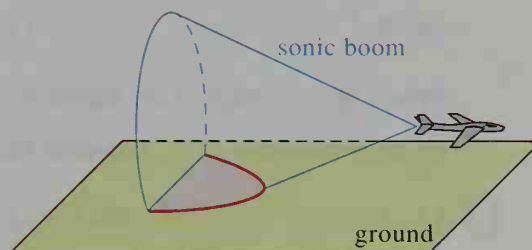
## CLASS EXERCISES

Use the diagram for Exercises 1 and 2.

- Find the distances from  $A$ ,  $B$ ,  $C$ , and  $P$  to:
  - line  $d_1$  with equation  $x = 3$ .
  - line  $d_2$  with equation  $y = -6$ .
- State an equation that expresses the fact that:
  - $PO = 2PA$
  - $PO = \frac{1}{2}PB$
  - the distance between  $P$  and  $C$  is equal to the distance between  $P$  and  $d_2$ .
  - the distance between  $P$  and  $B$  is two times the distance between  $P$  and  $d_2$ .
  - the distance between  $P$  and  $A$  is  $\frac{1}{2}$  the distance between  $P$  and  $d_2$ .



- Aviation** The shock wave of a supersonic jet flying parallel to the ground is a cone. Points on the intersection of the cone and the ground receive a sonic boom at the same time. What conic section is the intersection?



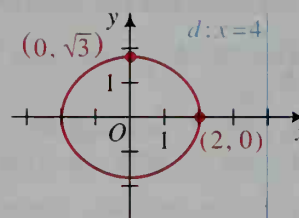
- Given that the following equations are not degenerate conics, tell whether the graph of each is a circle, an ellipse, a parabola, or a hyperbola.
  - $x^2 - 3xy + 2y^2 + 2x - y + 6 = 0$
  - $3x^2 - 4xy + 2y^2 - 3y = 0$
  - $x^2 - 6xy + 9y^2 + x - y - 1 = 0$
  - $144x^2 + 144y^2 - 216x + 96y - 47 = 0$

Describe the graph of each degenerate conic. If there is no graph, say so.

- $x^2 + 9y^2 = 0$
- $x^2 - 9y^2 = 0$
- $x^2 + 9y^2 + 1 = 0$
- Discussion** Describe the graph of  $Ax^2 + Cy^2 = 0$  if  $A$  and  $C$  are not both 0.
- Investigation** Shine a flashlight against a wall so that the edge of the lighted area is (a) a circle, (b) a parabola, and (c) a hyperbola. Relate the results of your investigation to the diagrams of the conic sections on page 213.

## WRITTEN EXERCISES

- A**
1. a. Find an equation for the ellipse.  
 b. Find the coordinates of  $F$ , the focus on the positive  $x$ -axis.  
 c. The distance from  $P(x, y)$  to  $F$  is 0.5 times the distance from  $P$  to the line  $d: x = 4$ . Write and simplify an equation to show that  $P$  lies on the ellipse shown.



2. Given  $F(0, 8)$  and  $d: y = \frac{25}{2}$ . The distance from  $P(x, y)$  to  $F$  is  $\frac{4}{5}$  the distance from  $P$  to  $d$ . Write an equation to show that  $P$  lies on an ellipse.
3. Given  $F(-6, 0)$  and  $d: x = -\frac{3}{2}$ . The distance from  $P(x, y)$  to  $F$  is twice the distance from  $P$  to  $d$ . Write an equation to show that  $P$  lies on a hyperbola.

The following are not degenerate conics. Identify the graph of each.

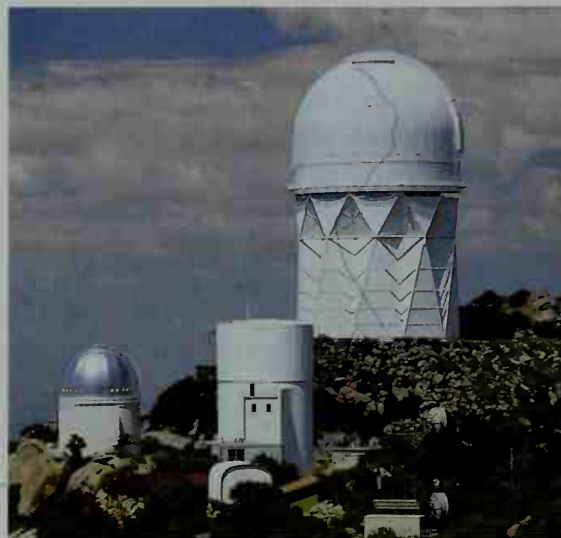
- |   |   |
|---|---|
| 4. $x^2 - 2xy - y^2 = 4$                          | 5. $x^2 - xy + 2y^2 = 2$                        |
| 6. $y = x - \frac{1}{x}$                          | 7. $x^2 - 3xy + y^2 = 5$                        |
| 8. $x^2 + xy + y^2 - 4x\sqrt{2} - 4y\sqrt{2} = 0$ | 9. $x^2 + xy + y^2 - x\sqrt{2} - y\sqrt{2} = 0$ |

Describe the graph of each equation. If there is no graph, say so.

- |                      |                       |                     |
|----------------------|-----------------------|---------------------|
| 10. $4x^2 - y^2 = 0$ | 11. $4x^2 + y^2 = -1$ | 12. $y^2 - 3yx = 0$ |
|----------------------|-----------------------|---------------------|

When an equation of an ellipse is given in the form shown on page 226, it can be shown that  $e = \frac{c}{a}$ . Use this fact in Exercises 13 and 14.

- B**
13. **Astronomy** Earth's orbit is an ellipse with the sun almost at one focus. During January, Earth is closest to the sun, a distance of  $9.14 \times 10^7$  mi. During July, Earth is farthest from the sun, a distance of  $9.44 \times 10^7$  mi. Find the eccentricity of Earth's orbit.
  14. **Astronomy** Halley's comet has an elliptical orbit with the sun at a focus. When it is closest to the sun, it passes within  $8.8 \times 10^7$  km of the sun. Its greatest distance from the sun is about  $5.282 \times 10^9$  km. Sketch the orbit and give its eccentricity.



Solve.

15.  $x^2 + y^2 = 9$

$(x + y)^2 = 9$

17.  $x^2 + y^2 = 40$

$x^2 + 2xy - 3y^2 = 0$

**C** 19.  $x^2 + xy + y^2 = 3$

$x^2 - y^2 = 3$

16.  $2x^2 + 3xy - 2y^2 = 0$

$x^2 + 2y^2 = 6$

18.  $3x^2 - 4xy - 4y^2 = 0$

$x - 2y^2 + 4 = 0$

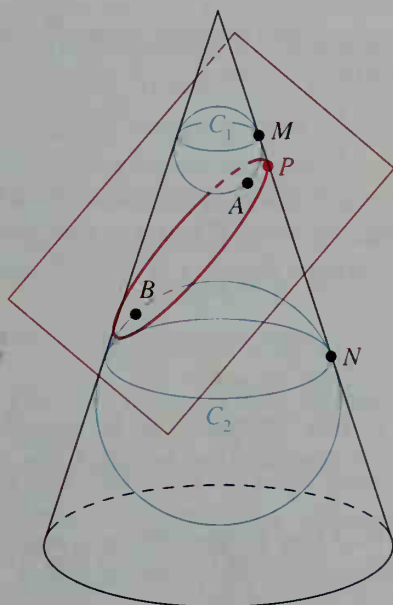
20.  $x^2 + 3y^2 = 3$

$3x^2 - xy = 6$

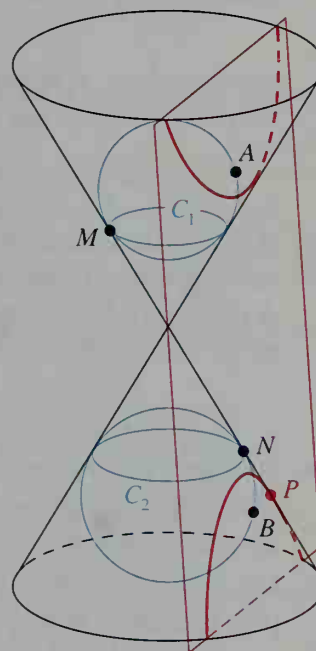
21. The diagram at the left below shows an ellipse formed by a plane cutting through a cone. It also shows two spheres tangent to the plane of the ellipse at  $A$  and  $B$ . They are also tangent to the cone, touching the cone in the parallel circles  $C_1$  and  $C_2$ . Let  $P$  be any point on the ellipse. (For simplicity,  $P$  is shown where it is.)

a. Explain why  $PA = PM$  and  $PB = PN$ .

b. Prove  $PA + PB$  is a constant, thus proving that  $A$  and  $B$  are the foci of an ellipse. (The spheres are called **Dandelin spheres** after the Belgian mathematician Germinal Pierre Dandelin (1794–1847) who discovered them in 1822.)



Ex. 21



Ex. 22

22. The diagram at the right above shows a hyperbola formed by a plane cutting a double cone. Two spheres are tangent to the plane of the hyperbola at  $A$  and  $B$ . They are also tangent to the cone, touching the cone in the parallel circles  $C_1$  and  $C_2$ . Let  $P$  be any point on the hyperbola. Show that  $PA - PB$  is a constant, thus proving that  $A$  and  $B$  are the foci of a hyperbola.

## Chapter Summary

1. To prove a theorem from geometry by using coordinates, introduce a coordinate system in which the figure involved has as many zero coordinates as possible. Choose one or more of the methods listed on page 217.

2. A circle with radius  $r$  and center  $C(h, k)$  has an equation of the form

$$(x - h)^2 + (y - k)^2 = r^2.$$

To find the intersection of a line and circle, follow the procedure on page 221.

3. If  $F_1$  and  $F_2$  are two fixed points (the *foci*) in the plane and  $a$  is a positive real number, then the set of all points  $P(x, y)$  in the plane such that

$$PF_1 + PF_2 = 2a$$

is an *ellipse*. An ellipse centered at the origin and having horizontal or vertical major axis has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

4. If  $F_1$  and  $F_2$  are two fixed points (the *foci*) in the plane and  $a$  is a positive real number, then the set of all points  $P(x, y)$  in the plane such that

$$|PF_1 - PF_2| = 2a$$

is a *hyperbola*. A hyperbola centered at the origin and opening either horizontally or vertically has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

5. If  $F$  is a fixed point (the *focus*) in the plane and  $d$  (the *directrix*) is a fixed line not containing  $F$ , then the set of points equidistant from  $F$  and  $d$  is a *parabola*. A parabola whose vertex is at the origin and whose directrix is either horizontal or vertical has an equation of the form

$$y = \pm \frac{1}{4p}x^2 \text{ or } x = \pm \frac{1}{4p}y^2.$$

6. A system of two second-degree equations in two variables can be solved by algebraic methods, graphical methods, or a combination of algebraic and graphical methods.
7. Let  $F$  (the *focus*) be a fixed point not on a fixed line  $d$  (the *directrix*) and let  $e$  (the *eccentricity*) be positive. Let  $P$  be any point in the plane such that

$$\frac{PF}{PD} = e, \text{ where } PD \text{ is the distance from } P \text{ to } d.$$

If  $0 < e < 1$ , the set of points is an *ellipse*.

If  $e = 1$ , the set of points is a *parabola*.

If  $e > 1$ , the set of points is a *hyperbola*.

Conic sections are graphs of second-degree equations. A conic section can be identified by examining the coefficients of its equation.



### Key vocabulary and ideas

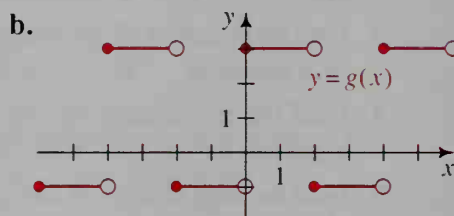
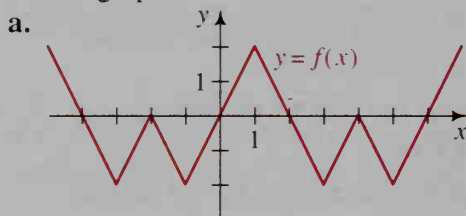
conic sections (p. 213, p. 247)  
second-degree equation (p. 213, p. 248)  
circle, center, radius (p. 219)  
ellipse, foci of an ellipse (p. 226)  
major axis, minor axis (p. 227)

vertices, center of an ellipse (p. 227)  
hyperbola (p. 231)  
asymptotes of a hyperbola (p. 232)  
parabola, focus, directrix (p. 238)  
eccentricity (p. 247)

## Chapter Test

1. Prove that the diagonals of a square are perpendicular and congruent. **6-1**
2. Find the coordinates of any points where the line  $x - y = 2$  and the circle  $x^2 + y^2 = 4$  intersect. **6-2**
3. Find an equation of the circle having  $(2, 5)$  and  $(-2, -1)$  as endpoints of a diameter.
4. Find the coordinates of the vertices and the foci of the ellipse with equation  $9x^2 + 5y^2 = 45$ . Sketch the ellipse. **6-3**
5. Find the coordinates of any points where the line  $2x + 3y = 6$  and the ellipse  $4x^2 + 9y^2 = 36$  intersect.
6. Find an equation for the ellipse that has  $(0, -4)$  and  $(0, 4)$  as vertices and  $(-3, 0)$  and  $(3, 0)$  as endpoints of its minor axis.
7. Find an equation for the hyperbola that satisfies the given conditions. **6-4**
  - a. Center at  $(0, 0)$ , a vertex at  $(0, -3)$ , and a focus at  $(0, -\sqrt{13})$
  - b. A vertex at  $(2, 0)$  and asymptotes with equations  $y = \pm 2x$
8. Graph the hyperbola  $xy = -9$  and give equations for its asymptotes.
9. Find the coordinates of the vertex and focus, and the equation of the directrix, of the parabola  $y = \frac{1}{6}x^2$ . **6-5**
10. Find an equation for the parabola whose directrix is  $x = -3$  and whose focus is  $(3, 0)$ .
11. On a single set of axes, graph  $x^2 + y^2 = 25$  and  $x^2 + 10y^2 = 169$ . Solve these equations simultaneously. **6-6**
12. Given that the following equations are not degenerate conics, tell whether the graph is a circle, an ellipse, a hyperbola, or a parabola. **6-7**
  - a.  $4x^2 + 4y^2 - 8x + 24y - 15 = 0$
  - b.  $-x^2 + 2xy - 3y^2 + 12 = 0$
  - c.  $2x^2 - 12xy + 18y^2 + 2x - 6y + 24 = 0$
13. Find and simplify an equation that expresses that  $P(x, y)$  is equidistant from the point  $F(1, 3)$  and the line  $y = -4$ .
14. **Writing** In a few sentences, describe the possibilities for the graph of the equation  $Ax^2 + Bxy + Cy^2 = 0$ , where  $AC = 0$ .

- Sketch the graph of the function  $f(x) = x^2 - 2x$ . Then use the graph to find the range and zeros of  $f$ .
- If  $f(x) = x - 1$  and  $g(x) = 1 - x$ , find:
  - $(f + g)(x)$
  - $(f - g)(x)$
  - $(f \cdot g)(x)$
  - $\left(\frac{f}{g}\right)(x)$
  - $(f \circ g)(x)$
  - $(g \circ f)(x)$
- Tell whether the graph of each equation has symmetry in (i) the  $x$ -axis, (ii) the  $y$ -axis, (iii) the line  $y = x$ , and (iv) the origin.
  - $xy^2 + x^2y = 4$
  - $y = x^4 + 2x^2 + 1$
  - $y^2 = |x| + 1$
- Give the fundamental period and amplitude of each periodic function whose graph is shown.



- Using the graph of  $y = f(x)$  in part (a) of Exercise 4, sketch the graph of  $y = f(x + 1)$ .
  - Using the graph of  $y = g(x)$  in part (b) of Exercise 4, sketch the graph of  $y = g(-x) - 1$ .
- Tell whether each function  $f$  has an inverse. If  $f^{-1}$  exists, find a rule for  $f^{-1}(x)$  and show that  $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ .
  - $f(x) = 8 - 2x$
  - $f(x) = x^3 - x^2$
  - $f(x) = \frac{1}{x - 2}$
- Tickets to a show cost \$6.00 for adults and \$4.00 for children. Let  $x$  and  $y$  represent the numbers of tickets sold to adults and children, respectively.
  - Find a rule for the total sales function  $S(x, y)$ .
  - Find  $S(40, 25)$  and  $S(32, 48)$ .
  - Sketch the graphs of the constant sales curves  $S(x, y) = 300$  and  $S(x, y) = 1200$  in an  $xy$ -plane.
- Simplify.
  - $(4^{-1} + 2^{-1})^2$
  - $(8y^4)(2y^{-3})^{-2}$
  - $\frac{(3x^2)^{-1}}{6x^{-3}}$
  - $\left(\frac{81}{64}\right)^{-1/2}$
  - $(27^{-2})^{-1/3}$
  - $a^{3/4}(a^{-3/4} - a^{1/4})$
- Solve.
  - $3^{4x+1} = 81$
  - $\frac{1}{5}x^{2/3} = 20$
  - $(x - 1)^{-2} = 25$
- Find an exponential function  $f$  such that  $f(0) = \frac{1}{2}$  and  $f(4) = \frac{9}{2}$ .

11. The table shows the number  $N(t)$  of bacteria in a colony at various times  $t$  (in hours).

$t$	0	1	2	3	4	5
$N(t)$	150	189	238	300	378	476

- What is the doubling time for this colony of bacteria?
  - Find an equation for  $N(t)$ .
  - How many bacteria will there be after one full day?
12. Suppose you invest \$100 at 7% annual interest. Calculate the amount that you would have after one year if interest is compounded:
- quarterly
  - monthly
  - continuously
13. Which plan yields a greater return on an investment?
- Plan A: 8.1% annual rate compounded quarterly
- Plan B: 8% annual rate compounded continuously
14. Simplify.
- $\log \frac{1}{100}$
  - $\log_2 16$
  - $\log_{125} 5$
  - $\ln \frac{1}{e^5}$
  - $e^{2 \ln 3}$
  - $10^{1 + \log 5}$
15. Write each expression in terms of  $\log M$  and  $\log N$ .
- $\log M^2 N^3$
  - $\log \sqrt{\frac{M^3}{N}}$
  - $\log 100M\sqrt{N}$
16. Express  $y$  in terms of  $x$ .
- $\log y = \log x + 2$
  - $\ln y = \ln 2 - 3 \ln x$
  - $\log y = 2.5x + 1$
17. To the nearest hundredth, solve each equation.
- $(e^x)^2 = 64$
  - $2^{x+1} = 50$
  - $(1.04)^x = 2$
18. Prove that the line segments joining the midpoints of successive sides of any rhombus form a rectangle.
19. Find the center and radius of the circle  $x^2 + y^2 - 4x + 14y + 28 = 0$ .
20. Sketch the ellipse  $\frac{x^2}{9} + \frac{y^2}{16} = 1$ . Find the coordinates of its center, vertices, and foci.
21. Find an equation of the hyperbola centered at the origin that has a focus at  $(\sqrt{5}, 0)$  and a vertex at  $(2, 0)$ .
22. Sketch the parabola  $x + 2 = -\frac{1}{4}(y - 1)^2$ , and give its vertex, focus, and directrix.
23. Solve the system:
- $$\begin{aligned} x^2 + y^2 &= 25 \\ 2x^2 - 3y &= 6 \end{aligned}$$
24. Given that  $x^2 - 2xy + 4y^2 = 4$  is not a degenerate conic, identify the graph of the equation.

# 7

## *Trigonometric Functions*





# Angles, Arcs, and Sectors

## 7-1 Measurement of Angles

### Objective

To find the measure of an angle in either degrees or radians and to find coterminal angles.

The word *trigonometry* comes from two Greek words, *trigonon* and *metron*, meaning “triangle measurement.” The earliest use of trigonometry may have been for surveying land in ancient Egypt after the Nile River’s annual flooding washed away property boundaries. In Chapter 9 we will discuss this use of trigonometry in greater detail. In Chapter 8 we will discuss more modern applications of trigonometry, such as the analysis of radio waves. The foundation for these applications is laid in this chapter, where we discuss the definitions and properties of the trigonometric functions.

In trigonometry, an angle often represents a rotation about a point. Thus, the angle  $\theta$  shown is the result of rotating its *initial ray* to its *terminal ray*.

A common unit for measuring very large angles is the **revolution**, a complete circular motion. For example, when a car with wheels of radius 14 in. is driven at 35 mi/h, the wheels turn at an approximate rate of 420 revolutions per minute (abbreviated rpm).

A common unit for measuring smaller angles is the **degree**, of which there are 360 in one revolution. For example, when a door is opened, the doorknob is usually turned  $\frac{1}{4}$  revolution, or 90 degrees.

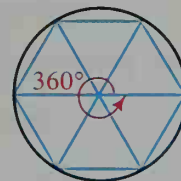
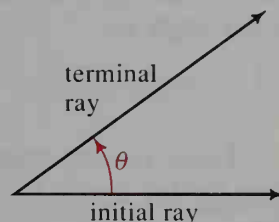
The convention of having 360 degrees in 1 revolution can be traced to the fact that the Babylonian numeration system was based on the number 60. One theory suggests that Babylonian mathematicians subdivided the angles of an equilateral triangle into 60 equal parts (eventually called degrees). Since six equilateral triangles can be arranged within a circle, 1 revolution contained  $6 \times 60 = 360$  degrees.

Angles can be measured more precisely by dividing 1 degree into 60 **minutes**, and by dividing 1 minute into 60 **seconds**. For example, an angle of 25 degrees, 20 minutes, and 6 seconds is written  $25^{\circ}20'6''$ .

Angles can also be measured in decimal degrees. To convert between decimal degrees and degrees, minutes, and seconds, you can reason as follows:

$$12.3^{\circ} = 12^{\circ} + 0.3(60)' = 12^{\circ}18'$$

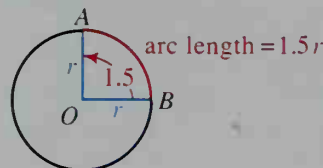
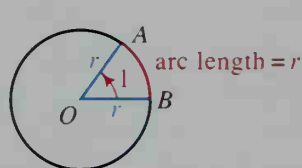
$$25^{\circ}20'6'' = 25^{\circ} + \left(\frac{20}{60}\right)^{\circ} + \left(\frac{6}{3600}\right)^{\circ} = 25.335^{\circ}$$



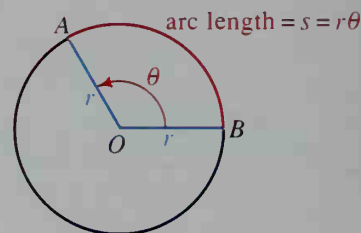
◀ Swiftly falling water propels this water wheel. Can you see how the radius of the wheel and the speed of the water determine the speed of revolution of the shaft of the water wheel?



Relatively recently in mathematical history, another unit of angle measurement, the *radian*, has come into widespread use. When an arc of a circle has the same length as the radius of the circle, as shown at the left below, the measure of the central angle,  $\angle AOB$ , is by definition 1 radian. Likewise, a central angle has a measure of 1.5 radians when the length of the intercepted arc is 1.5 times the radius, as shown at the right below.



In general, the **radian measure** of the central angle,  $\angle AOB$ , is the number of *radius* units in the length of arc  $AB$ . This accounts for the name radian. In the diagram at the right, the measure  $\theta$  (Greek theta) of the central angle is:



$$\theta = \frac{s}{r}$$

Let us use this equation to see how many radians correspond to 1 revolution. Since the arc length of 1 revolution is the circumference of the circle,  $2\pi r$ , we have

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi.$$

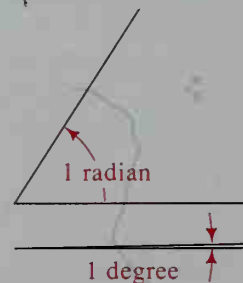
Thus, 1 revolution measured in radians is  $2\pi$  and measured in degrees is 360. We therefore have  $2\pi$  radians = 360 degrees, or

$$\pi \text{ radians} = 180 \text{ degrees.}$$

This gives us the following two conversion formulas:

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.2958 \text{ degrees}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians} \approx 0.0174533 \text{ radians}$$



### Example 1

- Convert  $196^\circ$  to radians (to the nearest hundredth).
- Convert 1.35 radians to decimal degrees (to the nearest tenth) and to degrees and minutes (to the nearest ten minutes).

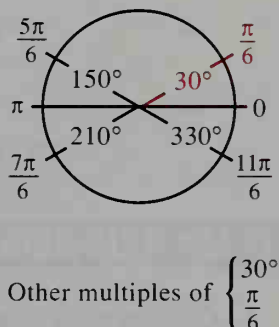
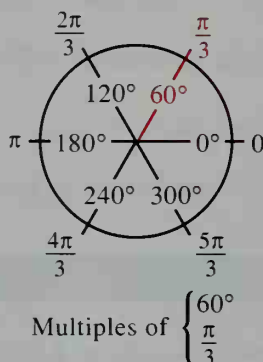
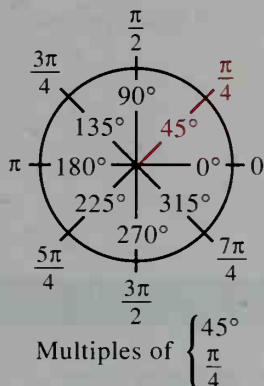
### Solution

Use a calculator and the conversion formulas above. Note that some calculators have the conversion formulas already built in; consult the instruction manual for your calculator.

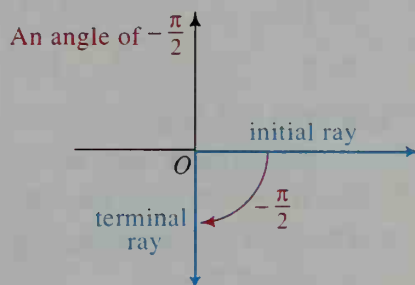
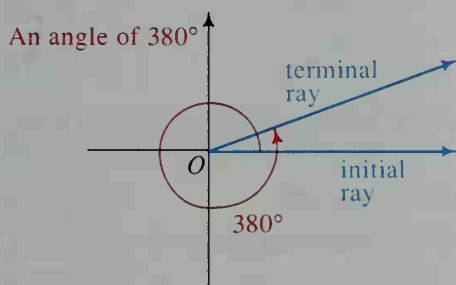
$$\text{a. } 196^\circ = 196 \times \frac{\pi}{180} \approx 3.42 \text{ radians}$$

$$\text{b. } 1.35 \text{ radians} = 1.35 \times \frac{180}{\pi} \approx 77.3^\circ \approx 77^\circ 20'$$

Angle measures that can be expressed evenly in degrees cannot be expressed evenly in radians, and vice versa. That is why angles measured in radians are frequently given as fractional multiples of  $\pi$ . Angles whose measures are multiples of  $\frac{\pi}{4}$ ,  $\frac{\pi}{3}$ , and  $\frac{\pi}{6}$  appear often in trigonometry. The diagrams below will help you keep the degree conversions for these special angles in mind. Note that a degree measure, such as  $45^\circ$ , is usually written with the degree symbol ( $^\circ$ ), while a radian measure, such as  $\frac{\pi}{4}$ , is usually written without any symbol.



When an angle is shown in a coordinate plane, it usually appears in **standard position**, with its vertex at the origin and its initial ray along the positive  $x$ -axis. Moreover, we consider a counterclockwise rotation to be positive and a clockwise rotation to be negative. The diagrams below give examples of positive and negative angles. (In this book we often do not distinguish between an angle and its measure. Thus, by “positive and negative angles” we mean angles with positive and negative measures.)



If the terminal ray of an angle in standard position lies in the first quadrant, as shown at the left above, the angle is said to be a first-quadrant angle. Second-, third-, and fourth-quadrant angles are similarly defined. If the terminal ray of an angle in standard position lies along an axis, as shown at the right above, the angle is called a **quadrantal angle**. The measure of a quadrantal angle is always a multiple of  $90^\circ$ , or  $\frac{\pi}{2}$ .

Two angles in standard position are called **coterminal** angles if they have the same terminal ray. For any given angle there are infinitely many coterminal angles.

**Example 2** Find two angles, one positive and one negative, that are coterminal with the angle  $\frac{\pi}{4}$ . Sketch all three angles.

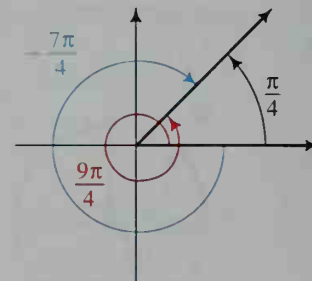
**Solution** A positive angle coterminal with  $\frac{\pi}{4}$  is:

$$\frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$$

A negative angle coterminal with  $\frac{\pi}{4}$  is:

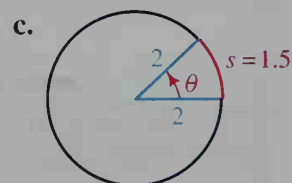
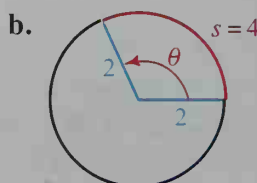
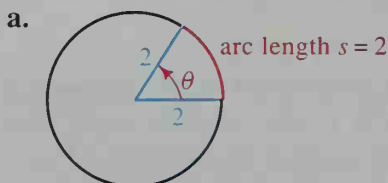
$$\frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}$$

The three angles are shown at the right.



## CLASS EXERCISES

- Convert each degree measure to radians. Leave answers in terms of  $\pi$ .
  - $180^\circ$
  - $90^\circ$
  - $315^\circ$
  - $60^\circ$
  - $120^\circ$
  - $240^\circ$
  - $30^\circ$
  - $1^\circ$
- Convert each radian measure to degrees.
  - $2\pi$
  - $\pi$
  - $\frac{\pi}{2}$
  - $\frac{\pi}{4}$
  - $\frac{3\pi}{4}$
  - $\frac{5\pi}{3}$
  - $\frac{11\pi}{6}$
  - $\frac{5\pi}{6}$
- Find two angles, one positive and one negative, that are coterminal with each given angle.
  - $10^\circ$
  - $100^\circ$
  - $-5^\circ$
  - $400^\circ$
  - $\pi$
  - $\frac{\pi}{2}$
  - $-\frac{\pi}{3}$
  - $4\pi$
- The equation  $\theta = (60 + 360n)^\circ$ , where  $n$  is an integer, represents all angles  $\theta$  coterminal with an angle of  $\underline{\hspace{1cm}}^\circ$ .
  - What would be the equivalent equation in radians?
- Give the radian measure of  $\theta$  in each of the following diagrams.



- Find the degree measure of an angle formed by each rotation described.
  - $1\frac{2}{3}$  revolutions counterclockwise
  - $2\frac{3}{4}$  revolutions clockwise

## WRITTEN EXERCISES

Convert each degree measure to radians. Leave answers in terms of  $\pi$ .

**A**

- |                    |                 |                 |                 |
|--------------------|-----------------|-----------------|-----------------|
| 1. a. $315^\circ$  | b. $225^\circ$  | c. $15^\circ$   | d. $-45^\circ$  |
| 2. a. $-90^\circ$  | b. $135^\circ$  | c. $-180^\circ$ | d. $-225^\circ$ |
| 3. a. $-120^\circ$ | b. $-240^\circ$ | c. $300^\circ$  | d. $360^\circ$  |
| 4. a. $210^\circ$  | b. $-135^\circ$ | c. $-210^\circ$ | d. $-315^\circ$ |

Convert each radian measure to degrees.

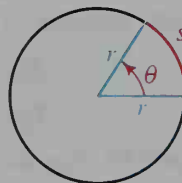
- |                         |                      |                      |                      |
|-------------------------|----------------------|----------------------|----------------------|
| 5. a. $-\frac{\pi}{2}$  | b. $\frac{4\pi}{3}$  | c. $-\frac{3\pi}{4}$ | d. $-\frac{\pi}{6}$  |
| 6. a. $-\frac{5\pi}{6}$ | b. $-2\pi$           | c. $\frac{5\pi}{4}$  | d. $-\frac{\pi}{3}$  |
| 7. a. $\pi$             | b. $-\frac{3\pi}{2}$ | c. $\frac{2\pi}{3}$  | d. $\frac{7\pi}{6}$  |
| 8. a. $-\frac{\pi}{4}$  | b. $\frac{7\pi}{4}$  | c. $4\pi$            | d. $\frac{11\pi}{6}$ |

9. Give the radian measure of  $\theta$  if:

- a.  $r = 5$  and  $s = 6$   
 b.  $r = 8$  and  $s = 6$

10. Give the radian measure of  $\theta$  if:

- a.  $r = 4$  and  $s = 5$   
 b.  $r = 6$  and  $s = 15$



Exs. 9, 10

Convert each degree measure to radians. Give answers to the nearest hundredth of a radian.

- |                    |                |                    |                  |
|--------------------|----------------|--------------------|------------------|
| 11. a. $95^\circ$  | b. $110^\circ$ | c. $95^\circ 10'$  | d. $119.2^\circ$ |
| 12. a. $212^\circ$ | b. $365^\circ$ | c. $200^\circ 40'$ | d. $240.8^\circ$ |

Convert each radian measure to degrees. Give answers to the nearest ten minutes or tenth of a degree.

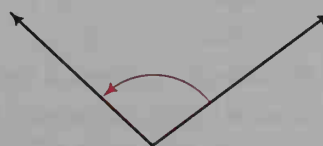
- |            |        |         |         |
|------------|--------|---------|---------|
| 13. a. 1.6 | b. 1.7 | c. 1.21 | d. 1.32 |
| 14. a. 2.2 | b. 3.7 | c. 2.82 | d. 3.41 |

**Visual Thinking** Estimate (by sight) the size in radians of each angle shown below. Then measure each angle with a protractor and convert from degrees to radians to find its actual size.

15.



16.



Find two angles, one positive and one negative, that are coterminal with each given angle.

17. a.  $500^\circ$                       b.  $-60^\circ$                       c.  $\frac{\pi}{4}$                       d.  $-\frac{2\pi}{3}$
18. a.  $1000^\circ$                       b.  $-100^\circ$                       c.  $\frac{4\pi}{3}$                       d.  $-\frac{\pi}{6}$
- B** 19. a.  $28.5^\circ$                       b.  $116.3^\circ$                       c.  $-60.4^\circ$                       d.  $-315.3^\circ$
20. a.  $38.4^\circ$                       b.  $127.6^\circ$                       c.  $-50.8^\circ$                       d.  $-320.7^\circ$
21. a.  $360^\circ 30'$                       b.  $-90^\circ 40'$                       c.  $3^\circ 21'$                       d.  $115^\circ 15'$
22. a.  $180^\circ 20'$                       b.  $-270^\circ 30'$                       c.  $11^\circ 44'$                       d.  $172^\circ 11'$
23. Give an expression in terms of the integer  $n$  for the measure of all angles that are coterminal with an angle of  $29.7^\circ$ .
24. Give an expression in terms of the integer  $n$  for the measure of all angles that are coterminal with an angle of  $-116^\circ 10'$ .

Each of Exercises 25–30 gives the speed of a revolving gear. Find (a) the number of degrees per minute through which each gear turns and (b) the number of radians per minute. Give answers to the nearest hundredth.

25. 35 rpm                      26. 27 rpm                      27. 2.5 rpm
28. 6.5 rpm                      29. 14.6 rpm                      30. 19.8 rpm

31. **Reading** On page 257, you were told that when a car with wheels of radius 14 in. is driven at 35 mi/h, the wheels turn at an approximate rate of 420 rpm. Show how to obtain this rate of turn.



32. **Recreation** Suppose you can ride a bicycle a distance of 5 mi in 15 min. If you ride at a constant speed and if the bicycle's wheels have diameter 27 in., find the wheels' approximate rate of turn (in rpm).
33. **Research** Consult an encyclopedia or an atlas to see how points on a world map are located by using *latitude* and *longitude* coordinates given in degrees, minutes, and seconds.
- If you travel south from a given point on Earth, about how many miles do you have to go to traverse an angle of  $1^\circ$ ?
  - Explain why your answer to part (a) might be different if you travel west instead of south.
34. **Research** Consult a book of astronomy or a star atlas to see how stars on a celestial map are located by using angles of *right ascension* and *declination*. Describe how each of these angles is measured, and give examples.



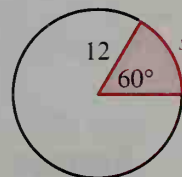
## 7-2 Sectors of Circles

### Objective

To find the arc length and area of a sector of a circle and to solve problems involving apparent size.

A **sector** of a circle, shaded in red at the right below, is the region bounded by a central angle and the intercepted arc. Your geometrical intuition should tell you that the length  $s$  of the arc is some fraction of the circumference of the circle and that the area  $K$  of the sector is the same fraction of the area of the circle.

For example, suppose the central angle of a sector is  $60^\circ$  and the radius is 12. Then the arc length of the sector is  $\frac{60}{360} = \frac{1}{6}$  of the whole circumference, or  $\frac{1}{6}(2\pi r) = \frac{1}{6}(2\pi \cdot 12) = 4\pi$ . Similarly, the area of the sector is  $\frac{1}{6}$  of the area of the whole circle, or  $\frac{1}{6}\pi r^2 = \frac{1}{6}\pi \cdot 12^2 = 24\pi$ .



In general, we have the following formulas for the arc length  $s$  and area  $K$  of a sector with central angle  $\theta$ .

If  $\theta$  is in degrees, then: (1)  $s = \frac{\theta}{360} \cdot 2\pi r$  (2)  $K = \frac{\theta}{360} \cdot \pi r^2$

If  $\theta$  is in radians, then: (1a)  $s = r\theta$  (2a)  $K = \frac{1}{2}r^2\theta$

Notice that formulas (1a) and (2a) are more straightforward than formulas (1) and (2). In fact, one reason for using radian measure is that many formulas in calculus are expressed more simply in radians than in degrees.

By combining formulas (1a) and (2a), we can obtain a third area formula:

$$K = \frac{1}{2}r^2\theta = \frac{1}{2}r(r\theta) = \frac{1}{2}rs$$

$$(2b) K = \frac{1}{2}rs$$

### Example 1

A sector of a circle has arc length 6 cm and area  $75 \text{ cm}^2$ . Find its radius and the measure of its central angle.

### Solution

Using formula (2b), we have:  $75 = \frac{1}{2}r(6)$

$$r = 25$$

Then, using formula (1a), we have:  $6 = 25\theta$

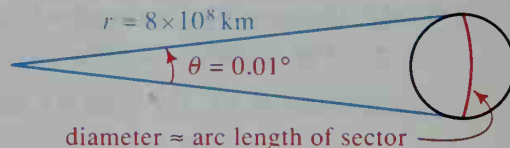
$$\theta = \frac{6}{25} = 0.24$$

Thus, the radius is 25 cm and the central angle is 0.24 radians  $\approx 14^\circ$ .

## Apparent Size

When there is nothing in our field of vision against which to judge the size of an object, we perceive the object to be smaller when it is farther away. For example, the sun is much larger than the moon, but we perceive the sun to be about the same size as the moon because the sun is so much farther from Earth. Thus, how big an object looks depends not only on its size but also on the angle that it subtends at our eyes. The measure of this angle is called the object's **apparent size**.

**Example 2** Jupiter has an apparent size of  $0.01^\circ$  when it is  $8 \times 10^8$  km from Earth. Find the approximate diameter of Jupiter.



### Solution

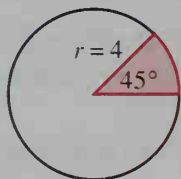
As the exaggerated diagram above indicates, the diameter of Jupiter is approximately the same as the arc length of a sector with central angle  $0.01^\circ$  and radius  $8 \times 10^8$  km. Using formula (1), we have:

$$\text{diameter} \approx s \approx \frac{0.01}{360} (2\pi)(8 \times 10^8) \approx 140,000 \text{ km}$$

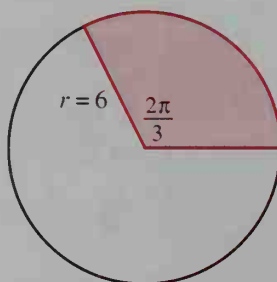
## CLASS EXERCISES

Find the arc length and area of each sector.

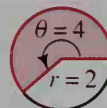
1.



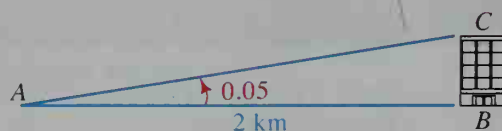
2.



3.



4. a. The apparent size of a tall building 2 km away is 0.05 radians. What is the building's approximate height?  
 b. Explain why you can apply an arc length formula to  $\triangle ABC$  and get a good approximation of  $BC$ .

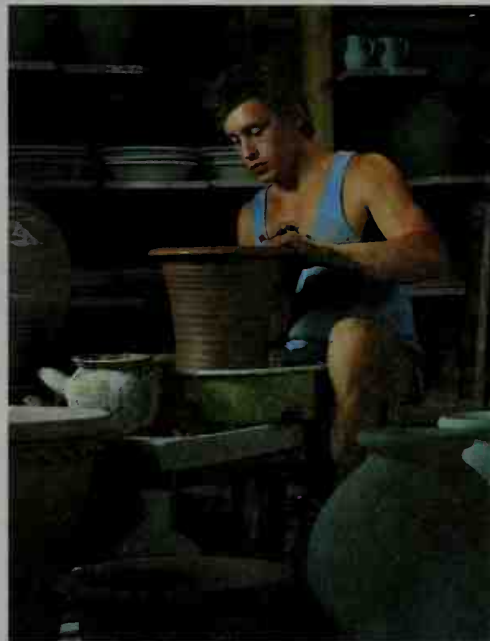


## WRITTEN EXERCISES

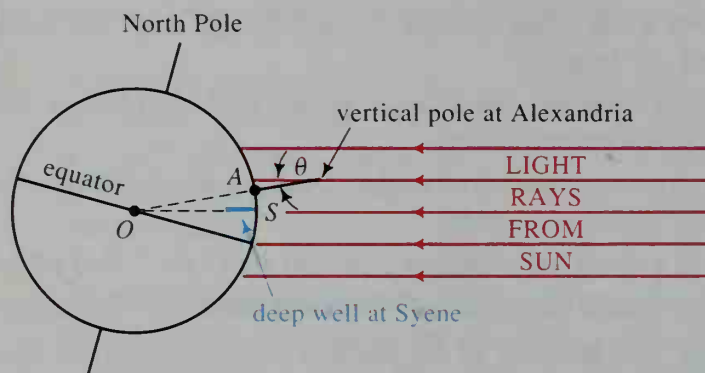
**A**

1. A sector of a circle has radius 6 cm and central angle 0.5 radians. Find its arc length and area.
2. A sector of a circle has radius 5 cm and central angle 3 radians. Find its arc length and area.

3. A sector of a circle has arc length 11 cm and central angle 2.2 radians. Find its radius and area.
  4. A sector of a circle has arc length 2 cm and central angle 0.4 radians. Find its radius and area.
  5. A sector of a circle has area  $25 \text{ cm}^2$  and central angle 0.5 radians. Find its radius and arc length.
  6. A sector of a circle has area  $90 \text{ cm}^2$  and central angle 0.2 radians. Find its radius and arc length.
  7. A sector of a circle has central angle  $30^\circ$  and arc length 3.5 cm. Find its area to the nearest square centimeter.
  8. A sector of a circle has central angle  $24^\circ$  and arc length 8.4 cm. Find its area to the nearest square centimeter.
  9. A sector of a circle has perimeter 7 cm and area  $3 \text{ cm}^2$ . Find all possible radii.
  10. A sector of a circle has perimeter 12 cm and area  $8 \text{ cm}^2$ . Find all possible radii.
  11. **Astronomy** The diameter of the moon is about 3500 km. Its apparent size is about 0.0087 radians. About how far is it from Earth?
  12. **Astronomy** At its closest approach, Mars is about  $5.6 \times 10^7$  km from Earth and its apparent size is about 0.00012 radians. What is the approximate diameter of Mars?
  13. **Physics** A compact disc player uses a laser to read music from a disc. The player varies the rotational speed of the disc depending on the position of the laser. When the laser is at the outer edge of the disc, the player spins the disc at the slowest speed, 200 rpm.
    - a. At the slowest speed, through how many degrees does the disc turn in a minute? Through how many radians does it turn in a minute?
    - b. If the diameter of the disc is 11.9 cm, find the approximate distance that a point on the outer edge travels at the slowest speed in 1 min.
    - c. Use part (b) to give the speed in cm/s.
  14. **Physics** To make a clay vase, an artist uses a potter's wheel that has a diameter of 13 in. and spins at 120 rpm. Find the approximate distance traveled in 1 min by a point on the outer edge of the wheel.
- B** 15. **Astronomy** The moon and the sun have approximately the same apparent size for viewers on Earth. The distances from Earth to the moon and to the sun are about  $4 \times 10^5$  km and  $1.5 \times 10^8$  km, respectively. The diameter of the moon is about 3500 km. What is the approximate diameter of the sun?



16. **Astronomy** This exercise will show how the Greek mathematician and astronomer Eratosthenes (about 276 B.C.–194 B.C.) determined the circumference of Earth. It was reported to him that at noon on the first day of summer the sun was directly overhead in the city of Syene because there was no shadow in a deep well. Eratosthenes observed at this same time in the city of Alexandria that the sun's rays made an angle  $\theta = 7.2^\circ$  with a vertical pole.

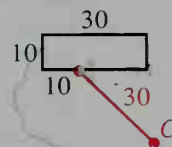


- How did Eratosthenes conclude that the measure of  $\angle AOS = \theta = 7.2^\circ$ ?
- If Alexandria was known to be 5000 stadia due north of Syene, show how Eratosthenes could conclude that the circumference of Earth was about 250,000 stadia. (1 stadium  $\approx 0.168$  km)
- Given that

$$\text{percent difference} = \frac{\text{old value} - \text{modern value}}{\text{modern value}} \times 100,$$

what is the percent difference between Eratosthenes' value for the circumference of Earth and the modern value of 40,067 km?

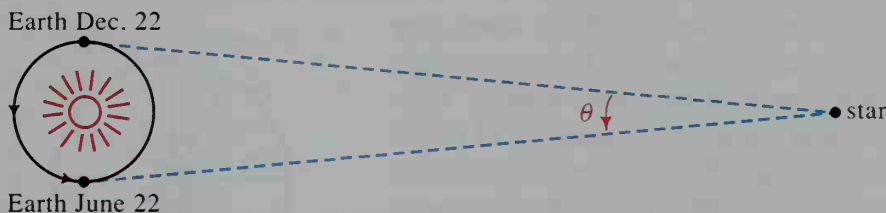
17. **Farming** A cow at  $C$  is tethered to a post alongside a barn 10 m wide and 30 m long. If the post is 10 m from a corner of the barn and if the rope is 30 m long, find the cow's total grazing area to the nearest square meter.



- Optics** What is the apparent size of an object 1 cm long held 80 cm from your eyes?
- Optics** You are traveling in a car toward a certain mountain at a speed of 80 km/h. The apparent size of the mountain is  $0.5^\circ$ . Fifteen minutes later the same mountain has an apparent size of  $1^\circ$ . About how tall is the mountain?
- Optics** A ship is approaching a lighthouse known to be 20 m high. The apparent size of the lighthouse is 0.005 radians. Ten minutes later the lighthouse has an apparent size of 0.010 radians. What is the approximate speed of the ship (in km/h)?

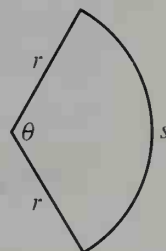


21. **Astronomy** Some stars are so far away that their positions appear fixed as Earth orbits the sun. Other stars, however, appear over time to shift their positions relative to the background of “fixed” stars. Suppose that the star shown below appears to shift through an arc of  $\theta = 0^\circ 0' 1.5''$  when viewed on the first day of winter and the first day of summer. If the distance from Earth to the sun is about  $1.5 \times 10^8$  km, find the approximate distance from Earth to the star.

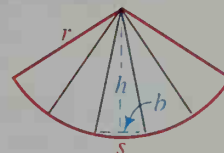


22. **Astronomy** Give the distance found in Exercise 21 in light years. (A *light year* is the distance light travels in one year. Use the fact that light travels  $3.00 \times 10^8$  m/s.)
23. **Writing** A point traveling along a circle has both a *linear* speed, defined as the length of arc traversed per unit of time, and an *angular* speed, defined as the measure of angle moved through per unit of time. Write a paragraph in which you compare the linear and angular speeds of two points on a rotating line (for example, the tip of a clock's hand versus a point on the hand closer to the center of rotation). Then discuss the implication this has for ice skaters who form a rotating line by interlocking their arms and skating in a circle.
24. The sector shown at the right has perimeter 20 cm.

- Show that  $\theta = \frac{20}{r} - 2$  and that the area of the sector is  $K = 10r - r^2$ .
- What value of  $r$  gives the maximum possible area of a sector of perimeter 20 cm? (Hint:  $K$  is a quadratic function of  $r$ .)
- What is the measure of the central angle of the sector of maximum area?



25. The purpose of this exercise is to derive the formula for the area of a circle by first deriving the formula for the area of a sector. Consider the sector with radius  $r$  and arc length  $s$  shown in the diagram at the right. Inscribed in the sector are  $n$  congruent isosceles triangles, each with height  $h$  and base  $b$ .



- Show that the total area of the inscribed triangles is  $\frac{1}{2}nbh$ .
- As  $n$  increases,  $h$  gets closer and closer to  $\underline{\hspace{1cm}}$ , and  $nb$  gets closer and closer to  $\underline{\hspace{1cm}}$ .
- Use parts (a) and (b) to derive the formula  $K = \frac{1}{2}rs$ .
- Derive the formula for the area of a circle from  $K = \frac{1}{2}rs$ .



# The Trigonometric Functions

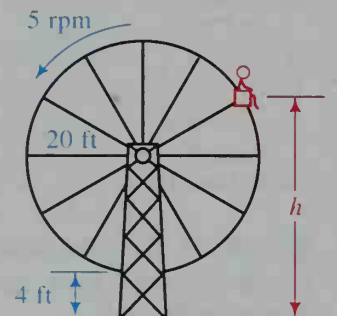
## 7-3 The Sine and Cosine Functions

### Objective

To use the definitions of sine and cosine to find values of these functions and to solve simple trigonometric equations.

If you have ever ridden on a Ferris wheel, you may have wondered how to find your height above the ground at any given moment. Suppose a Ferris wheel has a radius of 20 ft and revolves at 5 rpm. If the bottom of the Ferris wheel sits 4 ft off the ground, then  $t$  seconds after the ride begins, a rider's height  $h$  above the ground is given in feet by:

$$h = 24 + 20 \sin(30t - 90)^\circ$$



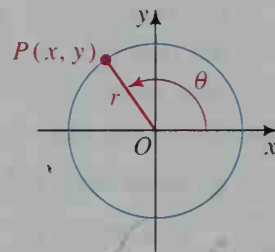
The “sin” that appears in the equation above is an abbreviation of the *sine* function, one of the two trigonometric functions that we will discuss in this section.

Suppose  $P(x, y)$  is a point on the circle  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ , as shown at the right. We define the **sine** of  $\theta$ , denoted  $\sin \theta$ , by:

$$\sin \theta = \frac{y}{r}$$

and we define the **cosine** of  $\theta$ , denoted  $\cos \theta$ , by:

$$\cos \theta = \frac{x}{r}$$



### Example 1

If the terminal ray of an angle  $\theta$  in standard position passes through  $(-3, 2)$ , find  $\sin \theta$  and  $\cos \theta$ .

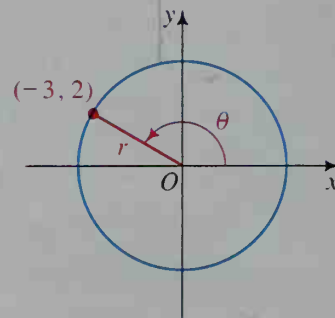
### Solution

Make a sketch as shown. To find the radius  $r$  of the circle, use the equation  $x^2 + y^2 = r^2$  with  $x = -3$  and  $y = 2$ :

$$\begin{aligned} (-3)^2 + 2^2 &= 13 = r^2 \\ \sqrt{13} &= r \end{aligned}$$

$$\text{Thus: } \sin \theta = \frac{y}{r} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$\text{and } \cos \theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$



**Example 2** If  $\theta$  is a fourth-quadrant angle and  $\sin \theta = -\frac{5}{13}$ , find  $\cos \theta$ .

**Solution** Make a sketch of a circle with radius 13 as shown. Since  $\sin \theta = \frac{y}{r} = -\frac{5}{13}$  and  $r$  is always positive,  $y = -5$ . To find  $x$ , use the circle's equation,  $x^2 + y^2 = r^2$ :

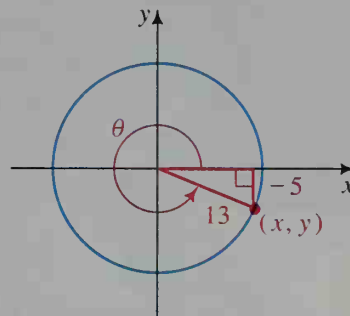
$$x^2 + (-5)^2 = 13^2$$

$$x^2 + 25 = 169$$

$$x^2 = 144$$

$$x = \pm 12$$

Since  $\theta$  is a fourth-quadrant angle,  $x = 12$ . Thus,  $\cos \theta = \frac{x}{r} = \frac{12}{13}$ .



Although the definitions of  $\sin \theta$  and  $\cos \theta$  involve the radius  $r$  of a circle, the values of  $\sin \theta$  and  $\cos \theta$  depend only on  $\theta$ , as the following activity shows.

### Activity

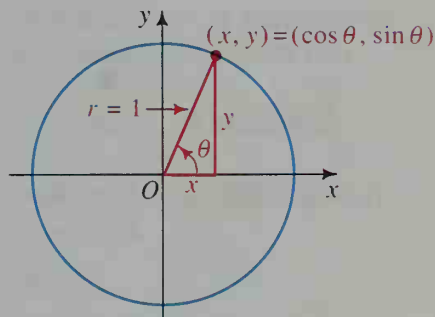
You will need graph paper, a ruler, a compass, a protractor, and a calculator.

- Using graph paper, draw an  $xy$ -plane and an acute angle  $\theta$  in standard position.
- Draw three concentric circles centered at the origin, and mark the points  $P_1$ ,  $P_2$ , and  $P_3$  where the circles intersect the terminal ray of  $\theta$ .
- Carefully measure the radii  $r_1$ ,  $r_2$ , and  $r_3$  of the three circles as well as the vertical distances  $y_1$ ,  $y_2$ , and  $y_3$  between  $P_1$ ,  $P_2$ , and  $P_3$  and the  $x$ -axis.
- Use a calculator to compute  $\frac{y_1}{r_1}$ ,  $\frac{y_2}{r_2}$ , and  $\frac{y_3}{r_3}$  to the nearest hundredth. Each ratio is an approximation of the sine of  $\theta$ . What do you observe about the ratios?
- Use your knowledge of geometry to support your observation from part (d).

The circle  $x^2 + y^2 = 1$  has radius 1 and is therefore called the **unit circle**. This circle is the easiest one with which to work because, as the diagram shows,  $\sin \theta$  and  $\cos \theta$  are simply the  $y$ - and  $x$ -coordinates of the point where the terminal ray of  $\theta$  intersects the circle.

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y$$

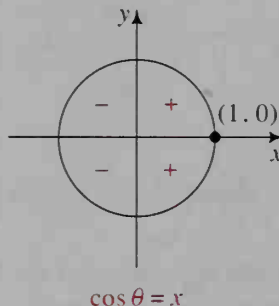
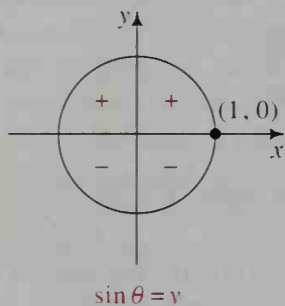
$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x$$



When a circle is used to define the trigonometric functions, they are sometimes called *circular functions*. (See Exercise 44 for another way to use the unit circle to define the trigonometric, or circular, functions.)

From the definitions and diagram at the bottom of the preceding page, we can see that the domain of both the sine and cosine functions is the set of all real numbers, since  $\sin \theta$  and  $\cos \theta$  are defined for any angle  $\theta$ . Also, the range of both functions is the set of all real numbers between  $-1$  and  $1$  inclusive, since  $\sin \theta$  and  $\cos \theta$  are the coordinates of points on the unit circle.

The diagrams below indicate where the sine and cosine functions have positive and negative values. For example, if  $\theta$  is a second-quadrant angle,  $\sin \theta$  is positive and  $\cos \theta$  is negative.

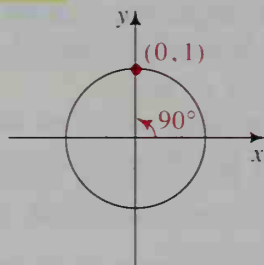


### Example 3

Find: a.  $\sin 90^\circ$       b.  $\sin 450^\circ$       c.  $\cos(-\pi)$

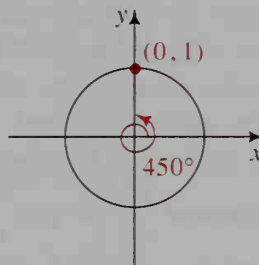
### Solution

a.



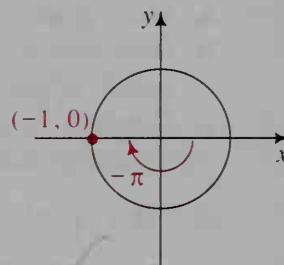
$$\sin 90^\circ = y\text{-coordinate} = 1$$

b.



$$\sin 450^\circ = y\text{-coordinate} = 1$$

c.



$$\cos(-\pi) = x\text{-coordinate} = -1$$

As figures (a) and (b) in Example 3 show,  $\theta = 90^\circ$  and  $\theta = 450^\circ$  are two solutions of the trigonometric equation  $\sin \theta = 1$ . The following example shows that there are infinitely many solutions of this equation.

### Example 4

Solve  $\sin \theta = 1$  for  $\theta$  in degrees.

### Solution

You already know that  $\theta = 90^\circ$  is one solution of the equation  $\sin \theta = 1$ . Since any angle coterminal with  $90^\circ$  also has 1 as its sine value,

$$\theta = 90^\circ, 90^\circ \pm 360^\circ, 90^\circ \pm 2 \cdot 360^\circ, 90^\circ \pm 3 \cdot 360^\circ, \dots$$

are all solutions of the equation. They can be written more conveniently as  $\theta = 90^\circ + n \cdot 360^\circ$ , where  $n$  is an integer. (In radians, the solutions would

be written as  $\theta = \frac{\pi}{2} + n \cdot 2\pi$  or  $\theta = \frac{\pi}{2} + 2n\pi$ .)

From Example 4 and the definitions of  $\sin \theta$  and  $\cos \theta$ , you can see that the sine and cosine functions repeat their values every  $360^\circ$  or  $2\pi$  radians. Formally this means that for all  $\theta$ :

$$\sin(\theta + 360^\circ) = \sin \theta$$

$$\cos(\theta + 360^\circ) = \cos \theta$$

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\cos(\theta + 2\pi) = \cos \theta$$

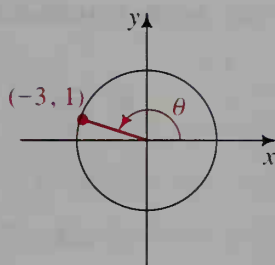
We summarize these facts by saying that the sine and cosine functions are *periodic* and that they have a *fundamental period* of  $360^\circ$ , or  $2\pi$  radians. It is the periodic nature of these functions that makes them useful in describing many repetitive phenomena such as tides, sound waves, and the orbital paths of satellites.



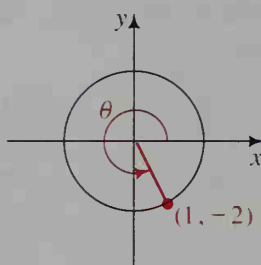
## CLASS EXERCISES

Find  $\sin \theta$  and  $\cos \theta$ .

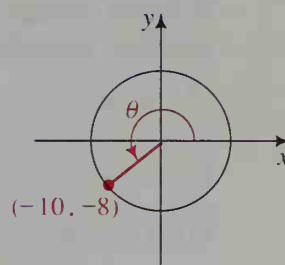
1.



2.



3.



4. State whether each expression is positive or negative.

a.  $\sin 165^\circ$

b.  $\sin 265^\circ$

c.  $\cos 210^\circ$

d.  $\cos 310^\circ$

e.  $\sin \frac{5\pi}{6}$

f.  $\cos \frac{5\pi}{6}$

g.  $\sin \frac{4\pi}{3}$

h.  $\cos \frac{5\pi}{3}$

i.  $\sin 2$

j.  $\cos 2$

k.  $\sin 4$

l.  $\cos 4$

5. Does  $\cos \theta$  increase or decrease as:

a.  $\theta$  increases from  $0^\circ$  to  $90^\circ$ ?

b.  $\theta$  increases from  $90^\circ$  to  $180^\circ$ ?

c.  $\theta$  increases from  $180^\circ$  to  $270^\circ$ ?

d.  $\theta$  increases from  $270^\circ$  to  $360^\circ$ ?

6. Answer Exercise 5 for  $\sin \theta$ .

7. Use the unit circle to justify the fact that for all  $\theta$ :

$$(\cos \theta)^2 + (\sin \theta)^2 = 1$$

8. There are infinitely many values of  $\theta$  for which  $\cos \theta = 0$ . Name several.

9. a. Explain the meaning of  $\theta = 45^\circ + n \cdot 360^\circ$ , where  $n$  is an integer.

b. What is the equivalent statement if  $\theta$  is expressed in radians?

## WRITTEN EXERCISES

Find the value of each expression without using a calculator or table.

- A**
- |                          |                                       |                          |  |
|--------------------------|---------------------------------------|--------------------------|--|
| 1. a. $\sin 180^\circ$   | b. $\cos 180^\circ$                   | c. $\sin 270^\circ$      | d. $\cos 270^\circ$                    |
| 2. a. $\sin (-90^\circ)$ | b. $\cos (-90^\circ)$                 | c. $\sin 360^\circ$      | d. $\cos 360^\circ$                    |
| 3. a. $\sin (-\pi)$      | b. $\cos \pi$                         | c. $\sin \frac{3\pi}{2}$ | d. $\cos \frac{\pi}{2}$                |
| 4. a. $\cos 2\pi$        | b. $\sin \left(-\frac{\pi}{2}\right)$ | c. $\sin 3\pi$           | d. $\cos \left(-\frac{3\pi}{2}\right)$ |

Name each quadrant described.

- |   |   |
|---|---|
| 5. a. $\sin \theta > 0$ and $\cos \theta < 0$ | b. $\sin \theta < 0$ and $\cos \theta < 0$              |
| 6. a. $\sin \theta < 0$ and $\cos \theta > 0$ | b. $\sin \theta > 0$ and $\sin (90^\circ + \theta) > 0$ |

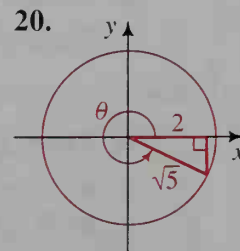
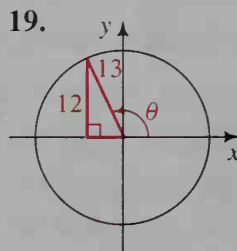
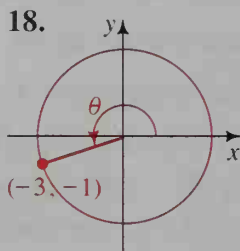
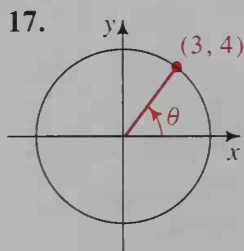
Without using a calculator or table, solve each equation for *all*  $\theta$  in radians.

- |                         |                       |                      |                       |
|-------------------------|-----------------------|----------------------|-----------------------|
| 7. a. $\sin \theta = 1$ | b. $\cos \theta = -1$ | c. $\sin \theta = 0$ | d. $\sin \theta = 2$  |
| 8. a. $\cos \theta = 1$ | b. $\sin \theta = -1$ | c. $\cos \theta = 0$ | d. $\cos \theta = -3$ |

Without using a calculator or table, state whether each expression is positive, negative, or zero.

- |   |                                       |                                       |                                       |
|---|---------------------------------------|---------------------------------------|---------------------------------------|
| 9. a. $\sin 4\pi$                         | b. $\cos \frac{7\pi}{6}$              | c. $\sin \left(-\frac{\pi}{4}\right)$ | d. $\cos \frac{3\pi}{4}$              |
| 10. a. $\cos 3\pi$                        | b. $\sin \frac{2\pi}{3}$              | c. $\sin \frac{11\pi}{6}$             | d. $\cos \left(-\frac{\pi}{2}\right)$ |
| 11. a. $\sin 60^\circ$                    | b. $\cos (-120^\circ)$                | c. $\cos 300^\circ$                   | d. $\sin (-210^\circ)$                |
| 12. a. $\cos 45^\circ$                    | b. $\sin 135^\circ$                   | c. $\cos (-225^\circ)$                | d. $\sin (-315^\circ)$                |
| 13. a. $\sin \frac{7\pi}{4}$              | b. $\sin \left(-\frac{\pi}{6}\right)$ | c. $\cos \frac{3\pi}{2}$              | d. $\cos \frac{\pi}{3}$               |
| 14. a. $\cos \left(-\frac{\pi}{3}\right)$ | b. $\sin \frac{\pi}{6}$               | c. $\sin \frac{5\pi}{4}$              | d. $\cos \frac{7\pi}{4}$              |
| 15. a. $\cos 89^\circ$                    | b. $\cos 91^\circ$                    | c. $\sin 720^\circ$                   | d. $\sin (-270^\circ)$                |
| 16. a. $\sin 1^\circ$                     | b. $\sin (-1^\circ)$                  | c. $\cos 90^\circ$                    | d. $\cos 540^\circ$                   |

Find  $\sin \theta$  and  $\cos \theta$ .



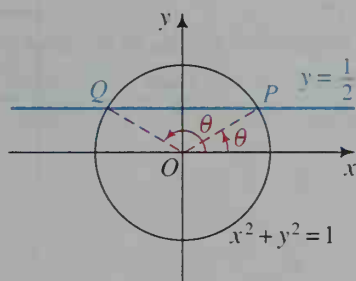


Complete the table. (A sketch like the one in Example 2 may be helpful.)

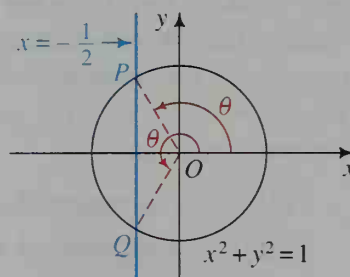
21. 22. 23. 24. 25. 26. 27. 28.

Quadrant	I	II	III	IV	II	III	IV	II
$\sin \theta$	$\frac{3}{5}$	$\frac{5}{13}$	?	?	$\frac{1}{5}$	$-\frac{3}{7}$	?	$\frac{1}{9}$
$\cos \theta$	?	?	$-\frac{24}{25}$	$\frac{15}{17}$	?	?	$\frac{3}{4}$	?

- B** 29. a. What are the coordinates of points  $P$  and  $Q$  where the line  $y = \frac{1}{2}$  intersects the unit circle? (Refer to the diagram at the left below.)  
 b. Explain how part (a) shows that if  $\sin \theta = \frac{1}{2}$ , then  $\cos \theta = \pm \frac{\sqrt{3}}{2}$ .

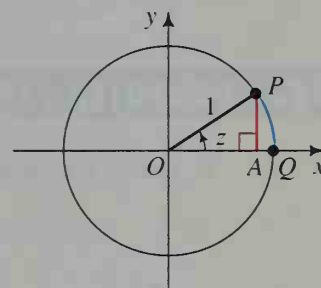


Ex. 29



Ex. 30

30. a. What are the coordinates of points  $P$  and  $Q$  where the line  $x = -\frac{1}{2}$  intersects the unit circle? (Refer to the diagram at the right above.)  
 b. Explain how part (a) shows that if  $\cos \theta = -\frac{1}{2}$ , then  $\sin \theta = \pm \frac{\sqrt{3}}{2}$ .
31. **Investigation** In the diagram of the unit circle at the right,  $z$  is measured in radians.
- Show that the length of arc  $PQ$  is  $z$ .
  - Show that the length of  $\overline{PA}$  is  $\sin z$ .
  - What do parts (a) and (b) imply about the relationship between  $\sin z$  and  $z$  for a small angle  $z$ ? Confirm this relationship by using a calculator to compare  $\sin z$  and  $z$  when  $z$  is a very small number of radians.



32. **Investigation** Refer to the diagram for Exercise 31.
- Show that the length of  $\overline{OA}$  is  $\cos z$ .
  - Use the results of part (c) of Exercise 31 and part (a) of this exercise to find an algebraic expression involving  $z$  (measure of the angle in radians) that approximates  $\cos z$  for a small angle  $z$ .
  - Use a calculator to check the accuracy of the approximation in part (b) when  $z$  is a very small number of radians.

Without using a calculator or table, complete each statement with one of the symbols  $<$ ,  $>$ , or  $=$ .

33.  $\sin 40^\circ$  ?  $\sin 30^\circ$

34.  $\cos 40^\circ$  ?  $\cos 30^\circ$

35.  $\sin 172^\circ$  ?  $\sin 8^\circ$

36.  $\sin 310^\circ$  ?  $\sin 230^\circ$

37.  $\sin 130^\circ$  ?  $\sin 50^\circ$

38.  $\cos 50^\circ$  ?  $\cos (-50^\circ)$

39.  $\cos 214^\circ$  ?  $\cos 213^\circ$

40.  $\sin 169^\circ$  ?  $\sin 168^\circ$

41. List in order of increasing size:  $\sin 1$ ,  $\sin 2$ ,  $\sin 3$ ,  $\sin 4$

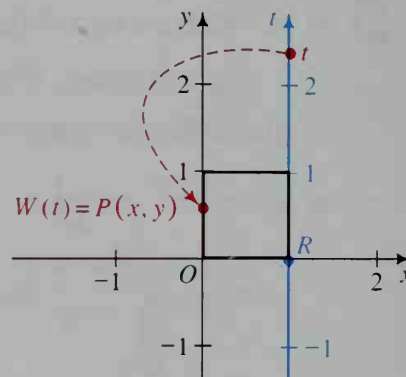
42. List in order of increasing size:  $\cos 1$ ,  $\cos 2$ ,  $\cos 3$ ,  $\cos 4$

- C** 43. Consider a special type of function called a *wrapping function*. This function, denoted by  $W$ , wraps a vertical number line whose origin is at  $R(1, 0)$  around a unit square, as shown at the right. With each real number  $t$  on the vertical number line,  $W$  associates a point  $P(x, y)$  on the square. For example,  $W(1) = (1, 1)$  and  $W(-1) = (0, 0)$ . From  $W$  we can define two simpler functions:

$$c(t) = x\text{-coordinate of } P,$$

and  $s(t) = y\text{-coordinate of } P.$

- Find  $W(2)$ ,  $W(3)$ ,  $W(4)$ , and  $W(5)$ .
  - Explain why  $W$  is a periodic function and give its fundamental period.
  - Explain how the periodicity of  $W$  guarantees the periodicity of  $c$  and  $s$ .
  - Sketch the graphs of  $u = c(t)$  and  $u = s(t)$  in separate  $tu$ -planes.
44. **Writing** Suppose the unit square in Exercise 43 is replaced with the unit circle. Write a paragraph in which you describe how the wrapping function can now be used to define the circular functions sine and cosine.



## //// COMPUTER EXERCISES

1. Use a computer to obtain the approximate value (to five decimal places) of

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

when  $x = 1$ ,  $x = 2$ , and  $x = \frac{\pi}{2} \approx 1.5708$ . Compare the results with the values of  $\text{SIN}(1)$ ,  $\text{SIN}(2)$ , and  $\text{SIN}(1.5708)$  given directly by the computer.

2. Use a computer to obtain the approximate value (to five decimal places) of

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

when  $x = 1$ ,  $x = 2$ , and  $x = \pi \approx 3.1416$ . Compare the results with the values of  $\text{COS}(1)$ ,  $\text{COS}(2)$ , and  $\text{COS}(3.1416)$  given directly by the computer.

## 7-4 Evaluating and Graphing Sine and Cosine

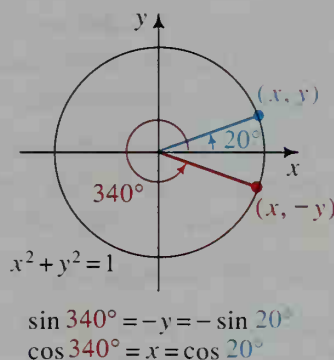
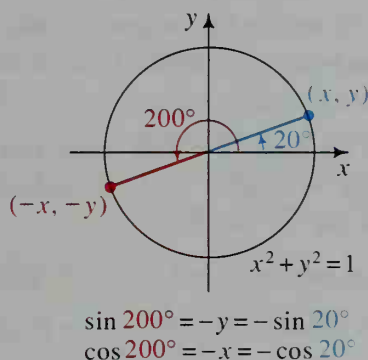
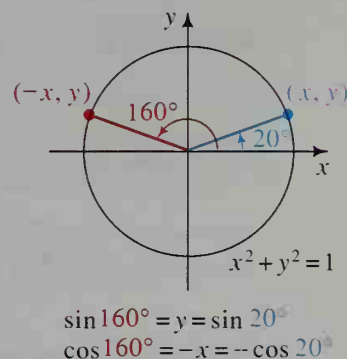
### Objective

To use reference angles, calculators or tables, and special angles to find values of the sine and cosine functions and to sketch the graphs of these functions.

### Reference Angles

Let  $\alpha$  (Greek alpha) be an acute angle in standard position. Suppose, for example, that  $\alpha = 20^\circ$ . Notice that the terminal ray of  $\alpha = 20^\circ$  and the terminal ray of  $180^\circ - \alpha = 160^\circ$  are symmetric in the  $y$ -axis. If the sine and cosine of  $\alpha = 20^\circ$  are known, then the sine and cosine of  $160^\circ$  can be deduced, as shown in the diagram at the right.

The angle  $\alpha = 20^\circ$  is called the *reference angle* for the  $160^\circ$  angle. It is also the reference angle for the  $200^\circ$  and  $340^\circ$  angles shown below.



In general, the acute angle  $\alpha$  is the **reference angle** for the angles  $180^\circ - \alpha$ ,  $180^\circ + \alpha$ , and  $360^\circ - \alpha$  as well as all coterminal angles. In other words, the reference angle for any angle  $\theta$  is the acute positive angle  $\alpha$  formed by the terminal ray of  $\theta$  and the  $x$ -axis.

### Example 1

Express  $\sin 695^\circ$  in terms of a reference angle.

### Solution

An angle between  $0^\circ$  and  $360^\circ$  that is coterminal with a  $695^\circ$  angle is:

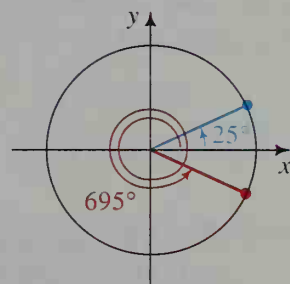
$$695^\circ - 360^\circ = 335^\circ$$

The reference angle for  $335^\circ$  is:

$$360^\circ - 335^\circ = 25^\circ$$

(See the diagram at the right.) Since  $695^\circ$  is a fourth-quadrant angle,  $\sin 695^\circ < 0$ . Thus:

$$\sin 695^\circ = -\sin 25^\circ$$



## Using Calculators or Tables

The easiest way to find the sine or cosine of most angles is to use a calculator that has the sine and cosine functions. Always be sure to check whether the calculator is in degree or radian mode.

If you do not have access to a calculator, there are tables at the back of the book that evaluate  $\sin \theta$  and  $\cos \theta$  for first-quadrant values of  $\theta$ . Instructions for using trigonometric tables are on page 800.

**Example 2** Find the value of each expression to four decimal places.  
a.  $\sin 122^\circ$     b.  $\cos 237^\circ$     c.  $\cos 5$     d.  $\sin (-2)$

**Solution** Note that in parts (a) and (b) the angles are given in degrees, while in parts (c) and (d) the angles are given in radians. Use your calculator and compare your results with those below.

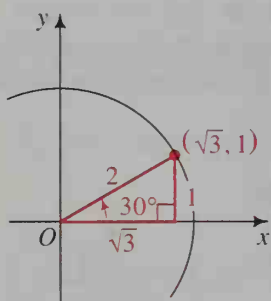
$$\begin{array}{ll} \text{a. } \sin 122^\circ \approx 0.8480 & \text{b. } \cos 237^\circ \approx -0.5446 \\ \text{c. } \cos 5 \approx 0.2837 & \text{d. } \sin (-2) \approx -0.9093 \end{array}$$

## Finding Sines and Cosines of Special Angles

Because angles that are multiples of  $30^\circ$  and  $45^\circ$  occur often in mathematics, it can be useful to know their sine and cosine values without resorting to a calculator or table. To do this, you need the following facts.

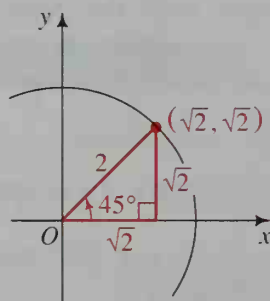
1. In a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle, the sides are in the ratio  $1:\sqrt{3}:2$ . (Note that in this ratio, 1 corresponds to the side opposite the  $30^\circ$  angle,  $\sqrt{3}$  to the side opposite the  $60^\circ$  angle, and 2 to the side opposite the  $90^\circ$  angle.)
2. In a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle, the sides are in the ratio  $1:1:\sqrt{2}$ , or  $\sqrt{2}:\sqrt{2}:2$ .

These facts are used in the diagrams below to obtain the values of  $\sin \theta$  and  $\cos \theta$  for  $\theta = 30^\circ$ ,  $\theta = 45^\circ$ , and  $\theta = 60^\circ$ .



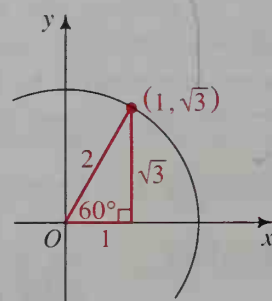
$$\sin 30^\circ = \frac{y}{r} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{x}{r} = \frac{\sqrt{3}}{2}$$



$$\sin 45^\circ = \frac{y}{r} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{x}{r} = \frac{\sqrt{2}}{2}$$



$$\sin 60^\circ = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{x}{r} = \frac{1}{2}$$

The information obtained from the diagrams above is summarized in the table at the top of the next page.

Although the table at the right only gives the sine and cosine values of special angles from  $0^\circ$  to  $90^\circ$ , reference angles can be used to find other multiples of  $30^\circ$  and  $45^\circ$ . For example:

$$\sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$$

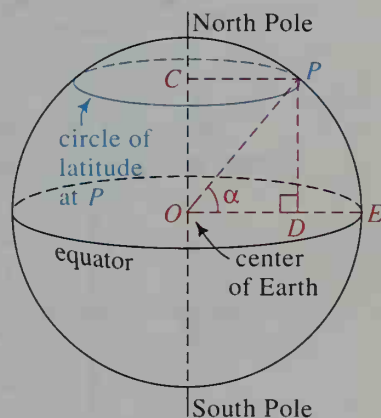
$$\cos 315^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$$

$\theta$ (degrees)	$\theta$ (radians)	$\sin \theta$	$\cos \theta$
$0^\circ$	0	0	1
$30^\circ$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45^\circ$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60^\circ$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$90^\circ$	$\frac{\pi}{2}$	1	0

As the table suggests,  $\sin \theta$  and  $\cos \theta$  are both one-to-one functions for first-quadrant  $\theta$ , a fact that we use in the next example to solve a geography problem.

### Example 3

The *latitude* of a point on Earth is the degree measure of the shortest arc from that point to the equator. For example, the latitude of point  $P$  in the diagram equals the degree measure of arc  $PE$ . At what latitude is the circumference of the circle of latitude at  $P$  half the distance around the equator?



### Solution

Let  $R$  be the radius of Earth, and let  $r$  be the radius of the circle of latitude at  $P$ . Then:

$$\text{circumference of circle of latitude at } P = \frac{1}{2}(\text{circumference of Earth})$$

$$2\pi r = \frac{1}{2}(2\pi R)$$

$$r = \frac{1}{2}R$$

As shown in the diagram, point  $D$  is the intersection of the perpendicular from  $P$  to  $\overline{OE}$ . Since planes containing circles of latitude are perpendicular to the north-south axis of Earth, quadrilateral  $OCPD$  is a rectangle, so that  $OD = CP = r$  and

$$\cos \alpha = \frac{OD}{OP} = \frac{r}{R} = \frac{\frac{1}{2}R}{R} = \frac{1}{2}.$$

Since  $\alpha$  is an acute angle, its measure must be  $60^\circ$  (see the table above). Thus, the latitude of point  $P$  is  $60^\circ\text{N}$  (north of the equator).



## Graphs of Sine and Cosine

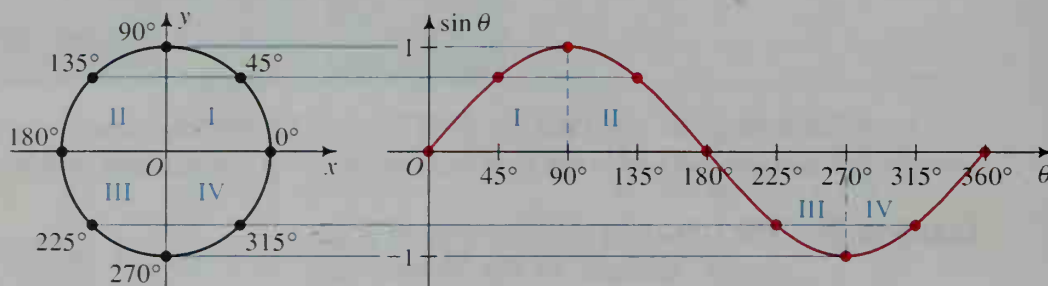
To graph the sine function, imagine a particle on the unit circle that starts at  $(1, 0)$  and rotates counterclockwise around the origin. Every position  $(x, y)$  of the particle corresponds to an angle  $\theta$ , where  $\sin \theta = y$  by definition. As the particle rotates through the four quadrants, we get the four pieces of the sine graph shown below.

I From  $0^\circ$  to  $90^\circ$ , the  $y$ -coordinate increases from 0 to 1.

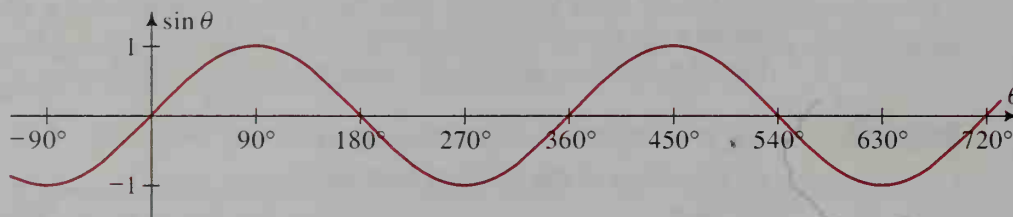
II From  $90^\circ$  to  $180^\circ$ , the  $y$ -coordinate decreases from 1 to 0.

III From  $180^\circ$  to  $270^\circ$ , the  $y$ -coordinate decreases from 0 to  $-1$ .

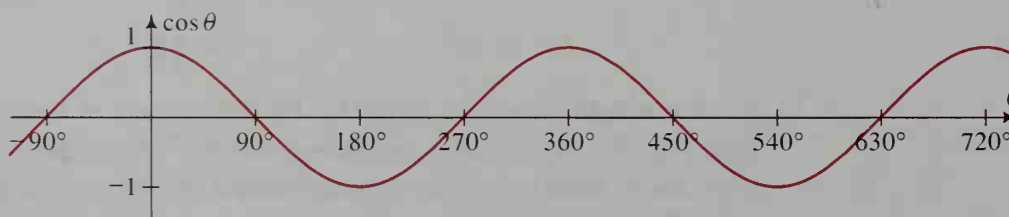
IV From  $270^\circ$  to  $360^\circ$ , the  $y$ -coordinate increases from  $-1$  to 0.



Since the sine function is periodic with a fundamental period of  $360^\circ$ , the graph above can be extended left and right as shown below.



To graph the cosine function, we analyze the  $x$ -coordinate of the rotating particle in a similar manner. The cosine graph is shown below.



### CLASS EXERCISES

- Find the reference angle for  $\theta$ .
  - $\theta = 170^\circ$
  - $\theta = 310^\circ$
  - $\theta = 205.1^\circ$
  - $\theta = 3$
- Name another angle between  $0^\circ$  and  $360^\circ$  with the same sine as  $70^\circ$ .

3. Name another angle between  $0^\circ$  and  $360^\circ$  with the same cosine as  $40^\circ$ .
4. Give each of the following in terms of the sine of a reference angle.
  - a.  $\sin 170^\circ$
  - b.  $\sin 330^\circ$
  - c.  $\sin (-15^\circ)$
  - d.  $\sin 400^\circ$
5. Give each of the following in terms of the cosine of a reference angle.
  - a.  $\cos 160^\circ$
  - b.  $\cos 182^\circ$
  - c.  $\cos (-100^\circ)$
  - d.  $\cos 365^\circ$

Use a calculator or table to find the value of each expression to four decimal places.

6. a.  $\sin 188^\circ$
- b.  $\sin (-110^\circ)$
- c.  $\cos 350^\circ$
- d.  $\cos (-230^\circ)$
7. a.  $\cos 10.2^\circ$
- b.  $\sin 28.6^\circ$
- c.  $\sin (-54.7^\circ)$
- d.  $\cos (-32.1^\circ)$
8. a.  $\sin 3$
- b.  $\cos 4$
- c.  $\sin (-1)$
- d.  $\cos (-2)$
9. a.  $\cos 2.5$
- b.  $\cos (-0.73)$
- c.  $\sin (-3.4)$
- d.  $\sin 0.39$

Study the sine and cosine values of  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ . Then give the exact value of each expression in simplest radical form.

10. a.  $\sin 45^\circ$
- b.  $\sin 135^\circ$
- c.  $\sin 225^\circ$
- d.  $\sin 315^\circ$
11. a.  $\cos 60^\circ$
- b.  $\cos 120^\circ$
- c.  $\cos 240^\circ$
- d.  $\cos 300^\circ$
12. a.  $\sin 30^\circ$
- b.  $\sin (-30^\circ)$
- c.  $\cos 30^\circ$
- d.  $\cos (-30^\circ)$
13. a.  $\sin 330^\circ$
- b.  $\cos 330^\circ$
- c.  $\sin \frac{7\pi}{6}$
- d.  $\cos \frac{7\pi}{6}$
14. a.  $\cos \frac{\pi}{4}$
- b.  $\sin \left(-\frac{\pi}{3}\right)$
- c.  $\cos \frac{5\pi}{6}$
- d.  $\sin (-300^\circ)$
15. For the graphs of the sine and cosine functions shown on the preceding page, the  $\theta$ -axis is labeled in degrees. Redraw each graph, labeling the  $\theta$ -axis in radians this time. (Your labels should be in terms of  $\pi$ .)
16. **Visual Thinking** Explain how translating the cosine graph can be used to justify the fact that for all  $\theta$ :

$$\cos (\theta - 90^\circ) = \sin \theta$$

17. a. What symmetry does the graph of the sine function have?
- b. What symmetry does the graph of the cosine function have?
18. Use Exercise 16 and part (b) of Exercise 17 to justify the fact that for all  $\theta$ :

$$\cos (90^\circ - \theta) = \sin \theta$$

## WRITTEN EXERCISES

Express each of the following in terms of a reference angle.

- A**
1. a.  $\sin 128^\circ$
  - b.  $\cos 128^\circ$
  - c.  $\sin (-37^\circ)$
  - d.  $\cos 500^\circ$
  2. a.  $\sin 310^\circ$
  - b.  $\cos 310^\circ$
  - c.  $\cos (-53^\circ)$
  - d.  $\sin 1000^\circ$
  3. a.  $\cos 224.5^\circ$
  - b.  $\cos 658^\circ$
  - c.  $\sin 145.7^\circ$
  - d.  $\sin (-201^\circ)$
  4. a.  $\cos 107.9^\circ$
  - b.  $\sin 271.3^\circ$
  - c.  $\sin 834^\circ$
  - d.  $\cos (-132^\circ)$

Use a calculator or table to find the value of each expression to four decimal places.

- |                        |                       |                       |                         |
|------------------------|-----------------------|-----------------------|-------------------------|
| 5. a. $\sin 28^\circ$  | b. $\sin 46^\circ$    | c. $\cos 65.3^\circ$  | d. $\sin 0.49$          |
| 6. a. $\sin 81^\circ$  | b. $\cos 0.3^\circ$   | c. $\cos 1.17$        | d. $\sin 1.11$          |
| 7. a. $\sin 160^\circ$ | b. $\cos 205.3^\circ$ | c. $\cos 302.1^\circ$ | d. $\sin 207^\circ 18'$ |
| 8. a. $\cos 238^\circ$ | b. $\sin 403^\circ$   | c. $\sin 285.7^\circ$ | d. $\cos 132^\circ 6'$  |
| 9. a. $\sin 1.2$       | b. $\sin 2.2$         | c. $\cos 3.5$         | d. $\cos 5$             |
| 10. a. $\sin 3.54$     | b. $\cos 6$           | c. $\sin (-2)$        | d. $\cos 3$             |

Study the sine and cosine values of  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$ . Then give the exact value of each expression in simplest radical form.

- |   |                                       |  |  |
|---|---------------------------------------|--|--|
| 11. a. $\cos 45^\circ$                    | b. $\cos 135^\circ$                   | c. $\sin 210^\circ$                    | d. $\cos 150^\circ$                    |
| 12. a. $\sin (-45^\circ)$                 | b. $\cos 315^\circ$                   | c. $\cos 225^\circ$                    | d. $\sin (-225^\circ)$                 |
| 13. a. $\sin 150^\circ$                   | b. $\cos (-240^\circ)$                | c. $\sin (-135^\circ)$                 | d. $\cos (-30^\circ)$                  |
| 14. a. $\cos 210^\circ$                   | b. $\cos 90^\circ$                    | c. $\sin (-120^\circ)$                 | d. $\sin (-315^\circ)$                 |
| 15. a. $\cos \frac{\pi}{6}$               | b. $\sin \frac{\pi}{3}$               | c. $\cos \frac{2\pi}{3}$               | d. $\sin \frac{3\pi}{4}$               |
| 16. a. $\cos \frac{\pi}{4}$               | b. $\sin \left(-\frac{\pi}{4}\right)$ | c. $\sin \frac{5\pi}{3}$               | d. $\cos \left(-\frac{7\pi}{6}\right)$ |
| 17. a. $\cos 2\pi$                        | b. $\sin \frac{11\pi}{6}$             | c. $\cos \left(-\frac{5\pi}{6}\right)$ | d. $\cos \frac{3\pi}{4}$               |
| 18. a. $\cos \left(-\frac{\pi}{3}\right)$ | b. $\sin \pi$                         | c. $\sin \frac{5\pi}{4}$               | d. $\sin \left(-\frac{\pi}{6}\right)$  |

19. The "natural" way to graph  $y = \sin x$  or  $y = \cos x$  is to measure  $x$  in radians and use the same real-number scale on both axes. Use this method to sketch the graph of  $y = \sin x$ . (Note that  $\pi$  is a little more than 3 units on the  $x$ -axis.)
20. Sketch the graph of  $y = \cos x$  using the method described in Exercise 19.



For part (b) of Exercises 21–23, you will need to use a computer or a graphing calculator. Be sure to use radian measure for  $x$ .

21. a. On a single set of axes, graph  $y = \sin x$  and  $y = 0.5x$ . How many solutions does the equation  $\sin x = 0.5x$  have?  
b. Find each solution of the equation to the nearest hundredth.
22. a. On a single set of axes, graph  $y = \cos x$  and  $y = 0.1x$ . How many solutions does the equation  $\cos x = 0.1x$  have?  
b. Find the smallest positive solution of the equation to the nearest tenth.
23. a. On a single set of axes, graph  $y = |\cos x|$  and  $y = x^2$ . How many solutions does the equation  $|\cos x| = x^2$  have?  
b. Find each solution of the equation to the nearest hundredth.
24. Sketch the graph of  $y = |\sin x|$ . Then give the domain, range, and fundamental period of the functions  $y = \sin x$  and  $y = |\sin x|$ .

25. **Visual Thinking** Imagine a particle starting at  $(1, 0)$  and making one counter-clockwise revolution on the unit circle. Let  $\theta$  be the angle in standard position that corresponds to the particle's position.
- At how many points along the path of the particle are the  $x$ - and  $y$ -coordinates equal?
  - What values of  $\theta$  correspond to the points in part (a)?
  - On a single set of axes, sketch the graphs of the sine and cosine functions for  $0^\circ \leq \theta \leq 360^\circ$ . Use the graphs to show where  $\sin \theta = \cos \theta$  in order to verify your answers in part (b).

For Exercises 26–33, use 3963 mi for the radius of Earth.

- B** 26. **Geography** The latitude of Durham, North Carolina, is  $36^\circ\text{N}$ . About how far from Durham is the North Pole?
27. **Geography** The latitude of Lima, Peru, is  $12^\circ\text{S}$ . About how far from Lima is the South Pole?
28. **Geography** Beijing, China, is due north of Perth, Australia. The latitude of Beijing is  $39^\circ55'\text{N}$  and the latitude of Perth is  $31^\circ58'\text{S}$ . About how far apart are the two cities?
29. **Geography** Memphis, Tennessee, is due north of New Orleans, Louisiana. The latitude of Memphis is  $35^\circ6'\text{N}$  and the latitude of New Orleans is  $30^\circ\text{N}$ . About how far apart are the two cities?



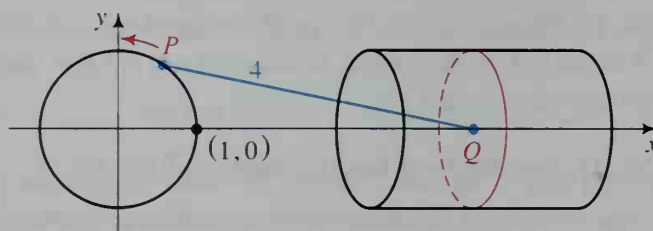
30. **Physics** Earth's rotational speed at the equator is found by dividing the circumference of the equator by 24 hours:

$$24,902 \text{ mi} \div 24 \text{ h} \approx 1038 \text{ mi/h}$$

What is Earth's rotational speed at (a) Bangor, Maine (latitude  $45^\circ\text{N}$ ) and (b) Esquina, Argentina (latitude  $30^\circ\text{S}$ )?

31. **Physics** What is Earth's rotational speed at (a) Anchorage, Alaska (latitude  $61^\circ\text{N}$ ) and (b) Rio de Janeiro, Brazil (latitude  $23^\circ\text{S}$ )?
32. a. **Physics** Show that at latitude  $L$ , Earth's rotational speed in miles per hour is approximately equal to  $1038 \cos L$ .  
 b. **Physics** Find Earth's rotational speed at the North Pole.
33. **Physics** Rome, Italy, and Boston, Massachusetts, have approximately the same latitude ( $42^\circ\text{N}$ ). A plane flying from Rome due west to Boston is able to "stay with the sun," leaving Rome with the sun overhead and landing in Boston with the sun overhead. How fast is the plane flying? (*Hint*: See part (a) of Exercise 32.)

- C** 34. **Mechanics** A piston rod  $\overline{PQ}$ , 4 units long, is connected to the rim of a wheel at point  $P$ , and to a piston at point  $Q$ . As  $P$  moves counterclockwise around the wheel at 1 radian per second,  $Q$  slides left and right in the piston. What are the coordinates of  $P$  and  $Q$  in terms of time  $t$  in seconds? Assume that  $P$  is at  $(1, 0)$  when  $t = 0$ .



## 7-5 The Other Trigonometric Functions

**Objective** To find values of the tangent, cotangent, secant, and cosecant functions and to sketch the functions' graphs.

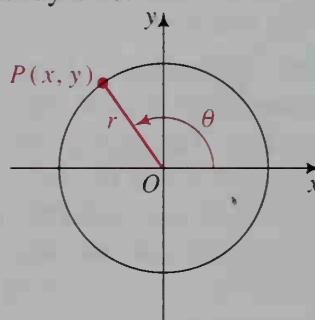
We can define four other trigonometric functions of an angle  $\theta$  in terms of the  $x$ - and  $y$ -coordinates of a point on the terminal ray of  $\theta$ .

tangent of  $\theta$ :  $\tan \theta = \frac{y}{x}, x \neq 0$

cotangent of  $\theta$ :  $\cot \theta = \frac{x}{y}, y \neq 0$

secant of  $\theta$ :  $\sec \theta = \frac{r}{x}, x \neq 0$

cosecant of  $\theta$ :  $\csc \theta = \frac{r}{y}, y \neq 0$



Since  $\cos \theta = \frac{x}{r}$  and  $\sin \theta = \frac{y}{r}$ , we can write these four new functions in terms of  $\cos \theta$  and  $\sin \theta$ :

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}$$

Notice that  $\sec \theta$  and  $\cos \theta$  are reciprocals, as are  $\csc \theta$  and  $\sin \theta$ . This is also true of  $\cot \theta$  and  $\tan \theta$ :

$$\cot \theta = \frac{1}{\tan \theta}$$

The signs of these functions in the various quadrants are summarized in the table at the right.

	I	II	III	IV
$\sin \theta$ and $\csc \theta$	+	+	-	-
$\cos \theta$ and $\sec \theta$	+	-	-	+
$\tan \theta$ and $\cot \theta$	+	-	+	-



**Example 1**

Find the value of each expression to four significant digits.

- a.  $\tan 203^\circ$       b.  $\cot 165^\circ$       c.  $\csc (-1)$       d.  $\sec 11$

**Solution**

Use a calculator to check each answer given below. If your calculator does not have the cotangent, secant, and cosecant as built-in functions, you must use the reciprocal relationships noted at the bottom of the preceding page. Be sure your calculator is in degree mode for parts (a) and (b), and in radian mode for parts (c) and (d).

a.  $\tan 203^\circ \approx 0.4245$

b.  $\cot 165^\circ = \frac{1}{\tan 165^\circ} \approx -3.732$

c.  $\csc (-1) = \frac{1}{\sin (-1)} \approx -1.188$

d.  $\sec 11 = \frac{1}{\cos 11} \approx 226.0$

**Example 2**

If  $\theta$  is a second-quadrant angle and  $\tan \theta = -\frac{3}{4}$ , find the values of the other five trigonometric functions.

**Solution**

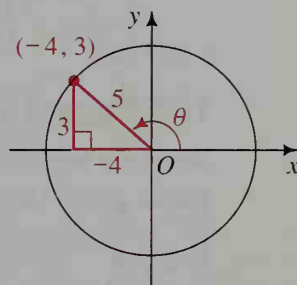
Since  $\theta$  is a second-quadrant angle and

$\tan \theta = \frac{y}{x} = -\frac{3}{4}$ , we can draw a diagram as shown at the right. Then:

$$\cos \theta = \frac{x}{r} = -\frac{4}{5} \quad \sec \theta = \frac{1}{\cos \theta} = -\frac{5}{4}$$

$$\sin \theta = \frac{y}{r} = \frac{3}{5} \quad \csc \theta = \frac{1}{\sin \theta} = \frac{5}{3}$$

$$\cot \theta = \frac{1}{\tan \theta} = -\frac{4}{3}$$

**The Tangent Graph**

Imagine on the unit circle a particle  $P$  that starts at  $(1, 0)$  and rotates counterclockwise around the origin. Every position  $(x, y)$  of the particle corresponds to an angle

$\theta$ , where  $\tan \theta = \frac{y}{x}$  by definition.

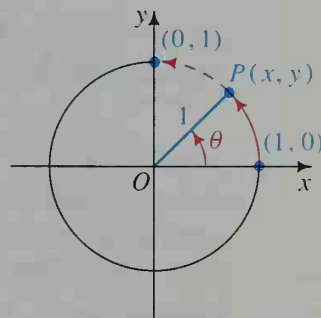
First consider what happens as the particle moves through the first quadrant. When  $P$  is at  $(1, 0)$ ,  $\theta = 0$  and  $\tan \theta = \frac{0}{1} = 0$ . As  $P$  moves toward  $(0, 1)$ ,  $y$  increases and  $x$

decreases, so that  $\tan \theta = \frac{y}{x}$  gets larger. When  $P$  reaches

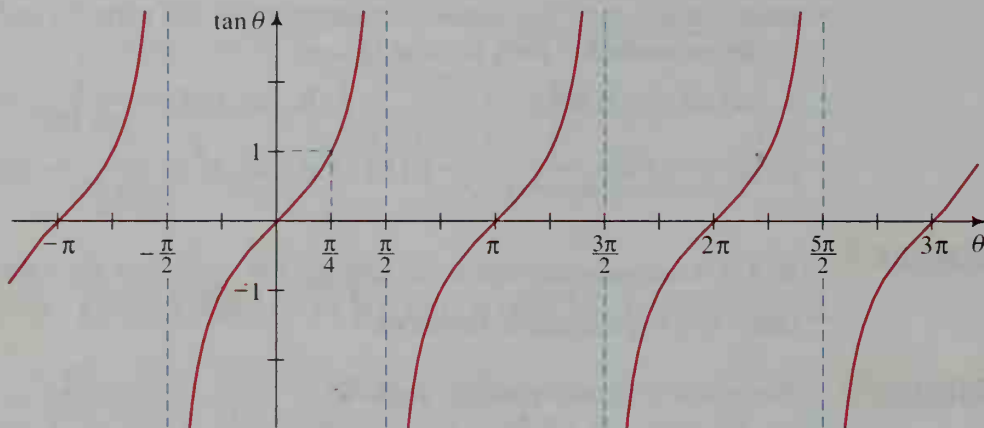
$(0, 1)$ ,  $\theta = \frac{\pi}{2}$ . Although the definition of tangent would

suggest that  $\tan \frac{\pi}{2}$  is  $\frac{1}{0}$ , this expression is undefined, so

we say that  $\tan \frac{\pi}{2}$  is also undefined.



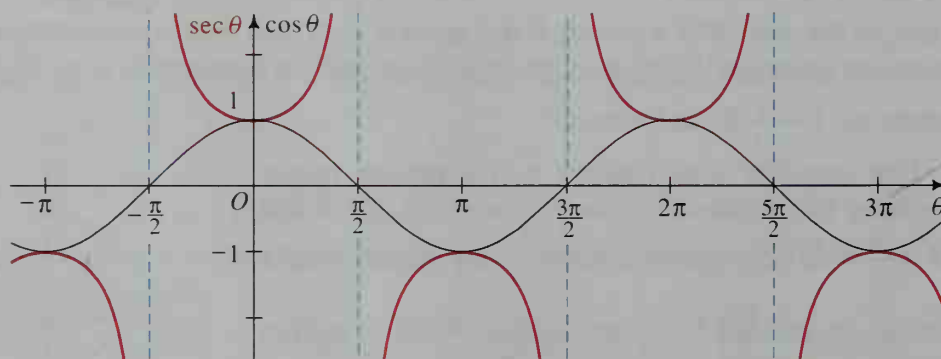
As the particle continues to move around the unit circle, we can analyze the other values of  $\tan \theta$  and obtain the tangent graph shown below. Notice that the graph has vertical asymptotes at odd multiples of  $\frac{\pi}{2}$ . Also notice that the tangent function is periodic with fundamental period  $\pi$  (or  $180^\circ$ ). The graph of the cotangent function is similar and is left for you to sketch (see Written Exercise 9).



## The Secant Graph

Since the secant function is the reciprocal of the cosine function, we can obtain the secant graph (shown in red below) using the cosine graph (black) and these facts:

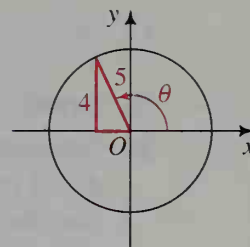
1.  $\sec \theta = 1$  when  $\cos \theta = 1$ : at  $\theta = 0, \pm 2\pi, \pm 4\pi, \dots$
2.  $\sec \theta = -1$  when  $\cos \theta = -1$ : at  $\theta = \pm \pi, \pm 3\pi, \pm 5\pi, \dots$
3.  $\sec \theta$  is undefined when  $\cos \theta = 0$ : at  $\theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$
4.  $|\sec \theta|$  gets larger as  $|\cos \theta|$  gets smaller.



Notice that the graph of the secant function has vertical asymptotes at odd multiples of  $\frac{\pi}{2}$ . Also notice that the secant function, like the cosine function, is periodic with fundamental period  $2\pi$  (or  $360^\circ$ ). The cosecant graph is like the secant graph and can be found by a similar analysis (see Written Exercise 10).

## CLASS EXERCISES

- Give the values of  $\theta$  (in degrees) for which:  
 a.  $\cot \theta = 0$     b.  $\csc \theta = 1$     c.  $\cot \theta = -1$     d.  $\csc \theta$  is undefined
- If  $\sec 15^\circ \approx 1.035$ , give the approximate value of:  
 a.  $\sec (-15^\circ)$     b.  $\sec 165^\circ$     c.  $\sec 195^\circ$     d.  $\sec 345^\circ$
- Find the value of each expression to four significant digits.  
 a.  $\tan 2$     b.  $\cot 185^\circ$     c.  $\csc 3$     d.  $\sec (-22^\circ)$
- In what quadrant is  $\theta$  if  $\csc \theta < 0$  and  $\tan \theta > 0$ ?
- The diagram shows an angle  $\theta$  with  $\sin \theta = \frac{4}{5}$ . Find:  
 a.  $\cos \theta$     b.  $\tan \theta$     c.  $\cot \theta$     d.  $\sec \theta$     e.  $\csc \theta$



## WRITTEN EXERCISES

Find the value of each expression to four significant digits.

- A** 1. a.  $\tan 100^\circ$     b.  $\cot 276^\circ$     c.  $\csc 5$     d.  $\sec 2.14$   
 2. a.  $\sec (-11^\circ)$     b.  $\csc 233^\circ$     c.  $\tan 3$     d.  $\cot 7.28$

Express each of the following in terms of a reference angle.

- a.  $\sin 195^\circ$     b.  $\sec 280^\circ$     c.  $\tan (-140^\circ)$     d.  $\sec 2$
- a.  $\cot 285^\circ$     b.  $\sec (-105^\circ)$     c.  $\csc 600^\circ$     d.  $\tan 3$
- a.  $\tan 820^\circ$     b.  $\sec 290^\circ$     c.  $\cot 185^\circ$     d.  $\csc 4$
- a.  $\tan 160^\circ$     b.  $\csc 115^\circ$     c.  $\sec 235^\circ$     d.  $\cot 5$

- Give the values of  $x$  (in radians) for which  $\csc x$  is:  
 a. undefined    b. 0    c. 1    d. -1
- Give the values of  $x$  (in radians) for which  $\cot x$  is:  
 a. undefined    b. 0    c. 1    d. -1
- Sketch a graph of  $\cot \theta$  versus  $\theta$  for  $-\pi \leq \theta \leq 3\pi$ . Be sure to show where any vertical asymptotes occur.



You may find it helpful to have a computer or a graphing calculator to complete Exercise 10.

- Writing** On a single set of axes, graph  $y = \sin x$  and  $y = \csc x$ , showing at least two full periods of each function. Using the reciprocal relationship between sine and cosecant, explain the features ( $x$ -intercepts, vertical asymptotes, periodicity, and so on) of the cosecant graph.
- On a single set of axes, sketch the graphs of  $y = \tan x$  and  $y = 2x$ . How many solutions does the equation  $\tan x = 2x$  have?
- On a single set of axes, sketch the graphs of  $y = \sin x$  and  $y = \sec x$ . How many solutions does the equation  $\sin x = \sec x$  have?

In Exercises 13–18, find the values of the other five trigonometric functions.

13.  $\sin x = \frac{5}{13}, \frac{\pi}{2} < x < \pi$

14.  $\cos x = \frac{24}{25}, -\frac{\pi}{2} < x < 0$

15.  $\tan x = \frac{3}{4}, \pi < x < 2\pi$

16.  $\cot x = -\frac{12}{5}, 0 < x < \pi$

17.  $\sec x = -3, 0 < x < \pi$

18.  $\csc x = -5, \frac{\pi}{2} < x < \frac{3\pi}{2}$

19. Explain why the cotangent graph has vertical asymptotes at multiples of  $\pi$ .

20. **Reading** Prepare a summary table for the six trigonometric functions introduced in this section and the preceding one. The table should give the definition, the domain, the range, the fundamental period, and a sketch of the graph of each function.

**B** 21. a. Verify that  $1 + \tan^2 \frac{\pi}{3} = \sec^2 \frac{\pi}{3}$ . [Note:  $\tan^2 \frac{\pi}{3}$  means  $(\tan \frac{\pi}{3})^2$ .]

b. Can you find any other values of  $x$  for which  $1 + \tan^2 x = \sec^2 x$ ?

22. a. Evaluate  $1 + \cot^2 x$  and  $\csc^2 x$  for  $x = \frac{\pi}{2}$ ,  $x = \frac{3\pi}{4}$ , and  $x = \frac{7\pi}{6}$ .

b. Make a conjecture about the relationship between  $1 + \cot^2 x$  and  $\csc^2 x$ . Prove your conjecture using Exercise 7 on page 271 and the definitions of cotangent and cosecant.

Find the exact value of each expression or state that the value is undefined.

23. a.  $\cos 120^\circ$

b.  $\sec 120^\circ$

c.  $\sin 120^\circ$

d.  $\tan 120^\circ$

24. a.  $\sin 225^\circ$

b.  $\csc 225^\circ$

c.  $\tan 225^\circ$

d.  $\sec 225^\circ$

25. a.  $\csc 90^\circ$

b.  $\sec 180^\circ$

c.  $\tan 240^\circ$

d.  $\cot 0^\circ$

26. a.  $\csc 150^\circ$

b.  $\csc 0^\circ$

c.  $\tan 315^\circ$

d.  $\sec 315^\circ$

27. a.  $\csc \pi$

b.  $\tan \frac{2\pi}{3}$

c.  $\cot \frac{\pi}{2}$

d.  $\sec \frac{5\pi}{6}$

28. a.  $\tan \frac{\pi}{2}$

b.  $\cot \frac{7\pi}{4}$

c.  $\sec (-3\pi)$

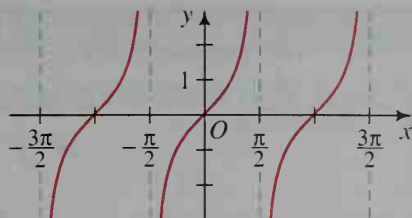
d.  $\csc \frac{7\pi}{6}$

## 7-6 The Inverse Trigonometric Functions

### Objective

To find values of the inverse trigonometric functions.

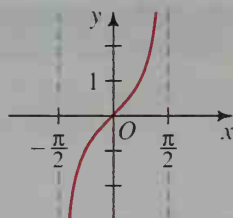
From the graph of  $f(x) = \tan x$  shown on the left at the top of the next page, we can see that the tangent function is not one-to-one and thus has no inverse. However, if we restrict  $x$  to the interval  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ , the restricted function, which we denote  $F(x) = \text{Tan } x$ , is one-to-one. Its inverse is denoted  $\text{Tan}^{-1} x$  and is read “the inverse tangent of  $x$ .” Notice that  $\text{Tan}^{-1} x = y$  means that  $\tan y = x$  and  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ .



$f(x) = \tan x$  has no inverse.

Domain:  $\{x \mid x \neq \frac{\pi}{2} + n\pi\}$

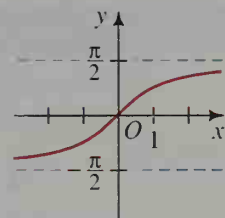
Range: Real numbers



$F(x) = \text{Tan } x$  has an inverse.

Domain:  $\{x \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$

Range: Real numbers



$F^{-1}(x) = \text{Tan}^{-1} x$

Domain: Real numbers

Range:  $\{y \mid -\frac{\pi}{2} < y < \frac{\pi}{2}\}$

**Example 1** Find  $\text{Tan}^{-1} 2$  with a calculator.

**Solution**

On most calculators, the inverse tangent function is symbolized by  $\tan^{-1}$ , INV tan, or ARC tan. Moreover, the calculator will give you answers in degrees or radians.

With the calculator in degree mode,  $\text{Tan}^{-1} 2 \approx 63.4^\circ$ .

With the calculator in radian mode,  $\text{Tan}^{-1} 2 \approx 1.11$ .

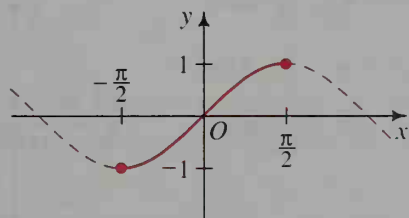
**Example 2** Find  $\text{Tan}^{-1}(-1)$  without a calculator.

**Solution**

$\text{Tan}^{-1}(-1) = x$  means that  $\tan x = -1$  and  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ .

Thus,  $\text{Tan}^{-1}(-1) = -\frac{\pi}{4}$ .

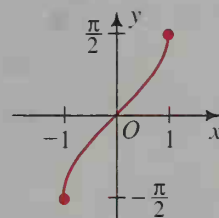
By considering only the solid portion of the graph of  $g(x) = \sin x$  shown at the left below, we obtain a new function  $G(x) = \text{Sin } x$  with domain  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . This function has an inverse,  $G^{-1}(x) = \text{Sin}^{-1} x$ , whose graph is shown at the right below. Note that  $\text{Sin}^{-1} x = y$  means that  $\sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .



$G(x) = \text{Sin } x$  has an inverse.

Domain:  $\{x \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$

Range:  $\{y \mid -1 \leq y \leq 1\}$



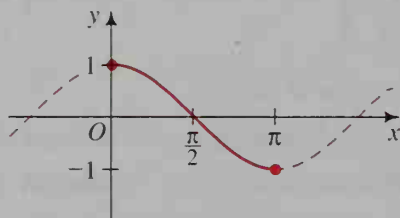
$G^{-1}(x) = \text{Sin}^{-1} x$

Domain:  $\{x \mid -1 \leq x \leq 1\}$

Range:  $\{y \mid -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$



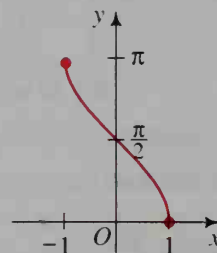
Similarly, by considering only the solid portion of the graph of  $h(x) = \cos x$  at the left below, we obtain a new function  $H(x) = \text{Cos } x$  with domain  $0 \leq x \leq \pi$ . This function has an inverse,  $H^{-1}(x) = \text{Cos}^{-1} x$ , whose graph is shown at the right below. Note that  $\text{Cos}^{-1} x = y$  means that  $\cos y = x$  and  $0 \leq y \leq \pi$ .



$H(x) = \text{Cos } x$  has an inverse.

Domain:  $\{x \mid 0 \leq x \leq \pi\}$

Range:  $\{y \mid -1 \leq y \leq 1\}$



$H^{-1}(x) = \text{Cos}^{-1} x$

Domain:  $\{x \mid -1 \leq x \leq 1\}$

Range:  $\{y \mid 0 \leq y \leq \pi\}$

### Example 3

- Find  $\text{Sin}^{-1}(-0.8)$  with a calculator.
- Find  $\text{Cos}^{-1}(-0.5)$  without a calculator.

### Solution

- With the calculator in radian mode,  $\text{Sin}^{-1}(-0.8) \approx -0.93$ .
- $\text{Cos}^{-1}(-0.5) = x$  means that  $\cos x = -0.5$  and  $0 \leq x \leq \pi$ .

Thus,  $\text{Cos}^{-1}(-0.5) = \frac{2\pi}{3}$ .

### Example 4

- Find  $\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right)$  with a calculator.
- Find  $\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right)$  without a calculator.

### Solution

- With the calculator in either degree or radian mode:

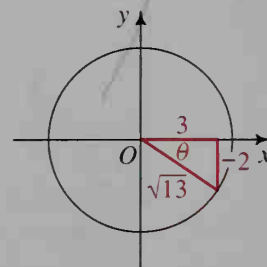
$$\cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right) \approx 0.83$$

- Let  $\theta = \text{Tan}^{-1}\left(-\frac{2}{3}\right)$ , so that  $\theta$  is a fourth-

quadrant angle such that  $\tan \theta = -\frac{2}{3}$ . After sketching a right triangle as in the diagram at the right, we see:

$$\begin{aligned} \cos\left(\text{Tan}^{-1}\left(-\frac{2}{3}\right)\right) &= \cos \theta \\ &= \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \end{aligned}$$

(Use a calculator to compare this result with part (a).)



## CLASS EXERCISES

Find the value of each expression to the nearest tenth of a degree.

1.  $\tan^{-1} 1.2$
2.  $\sin^{-1} (-0.3)$
3.  $\cos^{-1} (-0.425)$

Find the value of each expression to the nearest hundredth of a radian.

4.  $\tan^{-1} (-2.9)$
5.  $\sin^{-1} 0.75$
6.  $\cos^{-1} 0.058$

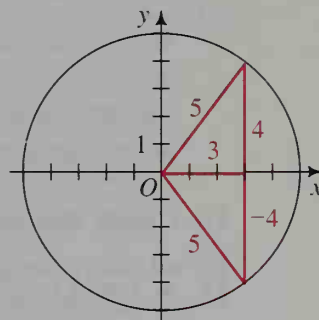
7. What happens when you try to evaluate  $\sin^{-1} 1.7$  with a calculator? Explain why this happens.

8. Use the diagram at the right to find the value of each expression.

a.  $\cos \left( \sin^{-1} \frac{4}{5} \right)$

b.  $\tan \left( \sin^{-1} \frac{4}{5} \right)$

c.  $\cot \left( \sin^{-1} \frac{4}{5} \right)$



## WRITTEN EXERCISES

Use a calculator or table to find the value of each expression to the nearest tenth of a degree.

- A** 1. a.  $\sin^{-1} 0.9$       b.  $\sin^{-1} (-0.9)$       c.  $\cos^{-1} \frac{3}{4}$       d.  $\cos^{-1} \left( -\frac{3}{4} \right)$
2. a.  $\tan^{-1} \frac{7}{3}$       b.  $\tan^{-1} \left( -\frac{7}{3} \right)$       c.  $\cos^{-1} 0.4$       d.  $\cos^{-1} (-0.4)$

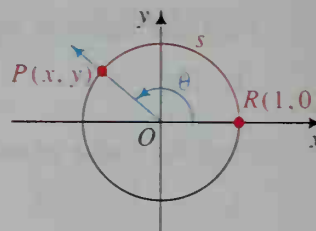
Use a calculator or table to find the value of each expression to the nearest hundredth of a radian.

3. a.  $\tan^{-1} 0.23$       b.  $\tan^{-1} (-0.23)$       c.  $\cos^{-1} 0.345$       d.  $\cos^{-1} (-0.345)$
4. a.  $\sin^{-1} \frac{3}{8}$       b.  $\sin^{-1} \left( -\frac{3}{8} \right)$       c.  $\cos^{-1} \frac{5}{6}$       d.  $\cos^{-1} \left( -\frac{5}{6} \right)$

Without using a calculator or table, find the value of each expression in radians. Many answers can be given in terms of  $\pi$ .

5. a.  $\sin^{-1} 0$       b.  $\cos^{-1} 0$       c.  $\tan^{-1} 1$       d.  $\tan^{-1} (-1)$
6. a.  $\sin^{-1} 1$       b.  $\sin^{-1} (-1)$       c.  $\cos^{-1} 1$       d.  $\cos^{-1} (-1)$
7. a.  $\sin^{-1} \frac{1}{2}$       b.  $\sin^{-1} \left( -\frac{1}{2} \right)$       c.  $\cos^{-1} \frac{1}{2}$       d.  $\cos^{-1} \left( -\frac{1}{2} \right)$
8. a.  $\sin^{-1} \frac{\sqrt{2}}{2}$       b.  $\sin^{-1} \left( -\frac{\sqrt{2}}{2} \right)$       c.  $\cos^{-1} \frac{\sqrt{2}}{2}$       d.  $\cos^{-1} \left( -\frac{\sqrt{2}}{2} \right)$

9. What is wrong with the expression  $\cos^{-1} 3$ ? What happens when you try to evaluate it with a calculator?
10. **Writing** Each of the trigonometric functions can be defined in terms of arc lengths instead of angle measures. In the diagram, angle  $\theta$  (in standard position) intercepts arc  $PR$  on the unit circle. If arc  $PR$  has length  $s$ , then we can define the cosine of  $s$  as:



$$\cos s = \cos \theta = x\text{-coordinate of } P$$

Write a similar definition for the inverse cosine function and explain why the inverse cosine function is sometimes denoted  $\text{Arccos}$  (read “arc cosine”). (By similar reasoning, the inverse sine function is sometimes denoted  $\text{Arcsin}$ , the inverse tangent function is sometimes denoted  $\text{Arctan}$ , and so on.)

Find both the approximate value and the exact value of each expression.

- |   |  |
|---|--|
| 11. a. $\tan\left(\cos^{-1} \frac{12}{13}\right)$ | b. $\tan\left(\cos^{-1}\left(-\frac{12}{13}\right)\right)$ |
| 12. a. $\sin\left(\cos^{-1} \frac{1}{5}\right)$   | b. $\sin\left(\cos^{-1}\left(-\frac{1}{5}\right)\right)$   |
| 13. a. $\csc(\tan^{-1} 1.05)$                     | b. $\sec(\sin^{-1}(-0.5))$                                 |
| 14. a. $\cos(\tan^{-1}(-0.2))$                    | b. $\sin(\tan^{-1}(-0.2))$                                 |



You may find it helpful to have a computer or a graphing calculator to complete Exercises 15–18. Be sure to use radian measure for  $x$ .

- B** 15. a. Graph  $y = \sin(\sin^{-1} x)$ . Give the domain, range, and a simplified rule for the function  $y = \sin(\sin^{-1} x)$ .  
 b. Graph  $y = \sin^{-1}(\sin x)$ .  
 c. Does your graph from part (b) contradict the property of inverse functions which states that  $f^{-1}(f(x)) = x$  for all  $x$  in the domain of  $f$ ? Explain.
16. a. Graph  $y = \cos^{-1}(\cos x)$ . Give the coordinates of the high and low points of the graph.  
 b. For what values of  $x$  is  $\cos^{-1}(\cos x) = x$ ?  
 c. Explain why  $\cos^{-1}(\cos x) = x$  is not true for all values of  $x$ .
17. **Writing** In a paragraph, compare the graphs of  $y = \tan(\tan^{-1} x)$  and  $y = \tan^{-1}(\tan x)$ .
18. Graph  $f(x) = \cos^{-1} x + \cos^{-1}(-x)$ . Give the domain and range of  $f$ .

Determine whether each equation is true for all  $x$  for which both sides of the equation are defined. If it is not true, give a counterexample.

- |  |  |
|--|--|
| 19. a. $\tan(\tan^{-1} x) = x$                     | b. $\tan^{-1}(\tan x) = x$                               |
| 20. a. $\sin^{-1}(-x) = -\sin^{-1} x$              | b. $\cos^{-1}(-x) = -\cos^{-1} x$                        |
| 21. a. $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$ | b. $\tan^{-1} \frac{1}{x} = \frac{\pi}{2} - \tan^{-1} x$ |

22. Let  $y = \cot x$  be a function identical to the cotangent function except that its domain is  $0 < x < \pi$ . State the domain and range of  $y = \cot^{-1} x$  and sketch its graph.
23. Let  $y = \sec x$  be a function identical to the secant function except that its domain is  $0 \leq x \leq \pi, x \neq \frac{\pi}{2}$ . State the domain and range of  $y = \sec^{-1} x$  and sketch its graph.
24. Let  $y = \csc x$  be a function identical to the cosecant function except that its domain is  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, x \neq 0$ . State the domain and range of  $y = \csc^{-1} x$  and sketch its graph.
25. a. Compare the values of  $\sec^{-1} 2$  and  $\cos^{-1} \frac{1}{2}$ .  
b. Define  $\sec^{-1} x$  in terms of the inverse cosine function.
26. Define  $\csc^{-1} x$  in terms of the inverse sine function.

In Exercises 27 and 28, the given expression always has the same value for all  $x$  between  $-1$  and  $1$ , inclusive. Find this value and explain why the expression has a constant value.

27.  $\cos^{-1} x + \cos^{-1} (-x)$

28.  $\sin^{-1} x + \sin^{-1} (-x)$

29. Using calculus, one can prove that

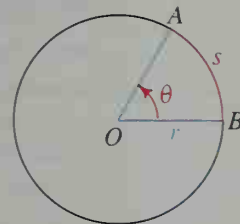
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \text{ for } |x| \leq 1.$$

Use this relationship to show that  $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$ .

- C** 30. If  $\cos^{-1} x + \cos^{-1} y = \frac{\pi}{2}$ , show that  $x^2 + y^2 = 1$ . (*Hint:* Use Exercise 18 on page 279 and Exercise 7 on page 271.)

## Chapter Summary

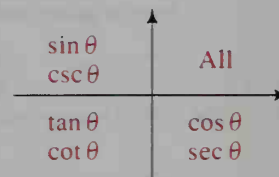
- In trigonometry, an angle is often formed by rotating an *initial ray* to a *terminal ray*. When an angle's vertex is at the origin and its initial ray is along the positive  $x$ -axis, the angle is in *standard position*. Two angles in standard position are *coterminal angles* if they have the same terminal ray.
- Angles can be measured in *revolutions*, *degrees*, or *radians*. Conversion formulas between degrees and radians are given on page 258.
- The radian measure of  $\angle AOB$ , shown at the right, is given by  $\theta = \frac{s}{r}$ . Thus, the length of arc  $AB$  is  $s = r\theta$ , and the area  $K$  of sector  $AOB$  is  $K = \frac{1}{2}r^2\theta$  ( $\theta$  in radians) or  $K = \frac{1}{2}rs$ .



4. If  $P(x, y)$  is a point on the unit circle  $x^2 + y^2 = 1$ , and if  $\theta$  is an angle in standard position with terminal ray  $OP$ , then the six *trigonometric functions* are defined as shown in the table below. Graphs of the sine, cosine, tangent, and secant functions are shown on pages 278 and 284.

Function	$\sin \theta = y$	$\cos \theta = x$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$	$\sec \theta = \frac{1}{\cos \theta}$	$\csc \theta = \frac{1}{\sin \theta}$
Domain	all $\theta$	all $\theta$	$\theta \neq \frac{\pi}{2} + n\pi$	$\theta \neq n\pi$	$\theta \neq \frac{\pi}{2} + n\pi$	$\theta \neq n\pi$
Range	$ \sin \theta  \leq 1$	$ \cos \theta  \leq 1$	all reals	all reals	$ \sec \theta  \geq 1$	$ \csc \theta  \geq 1$
Period	$2\pi$	$2\pi$	$\pi$	$\pi$	$2\pi$	$2\pi$

5. Values of the sine and cosine functions for the angles  $30^\circ$ ,  $45^\circ$ , and  $60^\circ$  are given in the table on page 277. To find the corresponding values of the other four trigonometric functions, use the definitions shown in the table above.
6. The acute angle  $\alpha$  is the *reference angle* of  $\theta = 180^\circ \pm \alpha$ ,  $\theta = 360^\circ \pm \alpha$ , and all coterminal angles. To find a trigonometric function of  $\theta$ , first evaluate the function at  $\alpha$ , the positive acute angle that  $\theta$  makes with the  $x$ -axis. Use a plus or a minus sign depending on the quadrant involved. The diagram shows which functions are positive in each quadrant.
7. a. By restricting the domains of the trigonometric functions, we can define new one-to-one functions that have inverses. The domains and ranges of these *inverse trigonometric functions* are as follows:



Function	$y = \sin^{-1} x$	$y = \cos^{-1} x$	$y = \tan^{-1} x$
Domain	$-1 \leq x \leq 1$	$-1 \leq x \leq 1$	all real $x$
Range	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	$0 \leq y \leq \pi$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

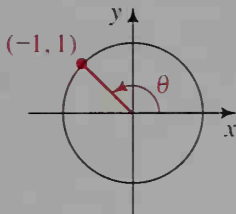
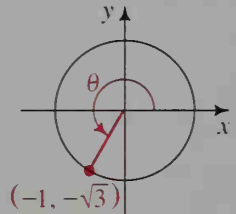
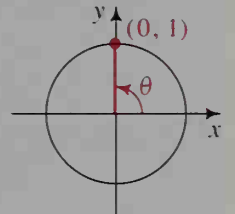
- b. The equation  $y = \sin^{-1} x$  is read “ $y$  is the inverse sine of  $x$ ” and means that  $y$  is the value for which  $\sin y = x$  where  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .
- c. Graphs of three inverse trigonometric functions are on pages 287 and 288.

### Key vocabulary and ideas

angle, initial ray, terminal ray (p. 257)    sine and cosine functions (p. 268)  
degree, minute, second (p. 257)    unit circle, circular function (p. 269)  
radian (p. 258)    reference angle (p. 275)  
standard position (p. 259)    tangent and cotangent functions (p. 282)  
coterminal angles (p. 260)    secant and cosecant functions (p. 282)  
length of arc, area of a sector (p. 263)     $\tan^{-1}$ ,  $\sin^{-1}$ ,  $\cos^{-1}$  (pp. 286–288)



## Chapter Test

1. a. Convert  $270^\circ$  to radians. Give the answer in terms of  $\pi$ . 7-1  
 b. Convert  $192^\circ$  to radians. Give the answer to the nearest hundredth.
2. a. Convert  $\frac{5\pi}{3}$  radians to degrees.  
 b. Convert 2.5 radians to degrees. Give the answer to the nearest ten minutes or tenth of a degree.
3. Find two angles, one positive and one negative, that are coterminal with each given angle.  
 a.  $-200^\circ$       b.  $313.2^\circ$       c.  $\frac{5\pi}{6}$       d.  $142^\circ 10'$
4. A sector of a circle has radius 5 cm and central angle  $48^\circ$ . Find its approximate arc length and area. 7-2
5. The moon is about  $4 \times 10^5$  km from Earth, and its apparent size is about 0.0087 radians. What is the moon's approximate diameter?
6. Find  $\sin \theta$  and  $\cos \theta$ . 7-3
  - a. 
  - b. 
  - c. 
7. Complete each statement with one of the symbols  $<$ ,  $>$ , or  $=$ .  
 a.  $\sin 60^\circ$  ?  $\sin 65^\circ$       b.  $\cos 60^\circ$  ?  $\cos 65^\circ$   
 c.  $\sin 20^\circ$  ?  $\sin 160^\circ$       d.  $\cos 184^\circ$  ?  $\cos 185^\circ$
8. Give the exact value of each expression in simplest radical form. 7-4  
 a.  $\sin \frac{5\pi}{4}$       b.  $\cos (-90^\circ)$       c.  $\sin 150^\circ$       d.  $\cos \frac{11\pi}{6}$
9. Give the exact value of each expression or state that the value is undefined. 7-5  
 a.  $\tan 135^\circ$       b.  $\cot \frac{2\pi}{3}$       c.  $\sec (-60^\circ)$       d.  $\csc (-\pi)$
10. If  $\tan x = -\frac{1}{3}$  where  $\frac{\pi}{2} < x < \pi$ , find the values of the other five trigonometric functions.
11. Give the exact value of each expression. 7-6  
 a.  $\tan^{-1} \sqrt{3}$       b.  $\cot \left( \sin^{-1} \frac{1}{2} \right)$       c.  $\sec \left( \cos^{-1} \left( -\frac{3}{5} \right) \right)$
12. **Writing** To obtain the inverse sine function, we restrict the domain of  $f(x) = \sin x$  to  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ . Write a short explanation in which you discuss why this restriction is necessary.

# 8

## *Trigonometric Equations and Applications*

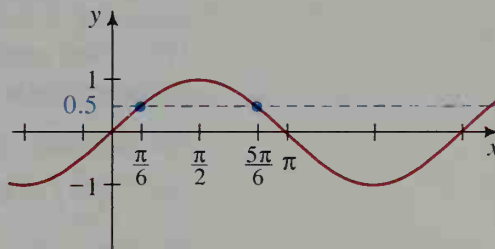


# Equations and Applications of Sine Waves

## 8-1 Simple Trigonometric Equations

**Objective** To solve simple trigonometric equations and to apply them.

The sine graph at the right illustrates that there are many solutions to the trigonometric equation  $\sin x = 0.5$ . We know that  $x = \frac{\pi}{6}$  and  $x = \frac{5\pi}{6}$  are two *particular* solutions. Since the period of  $\sin x$  is  $2\pi$ , we can add integral multiples of  $2\pi$  to get the other solutions.



$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi,$$

where  $n$  is any integer, are the *general solutions* of  $\sin x = 0.5$ .

When solving an equation such as  $\sin x = 0.6$ , we can use a calculator or a table of values.

**Example 1** Find the values of  $x$  between  $0$  and  $2\pi$  for which  $\sin x = 0.6$ .

### Solution

**Step 1** Set the calculator in radian mode and use the inverse sine key. Thus,

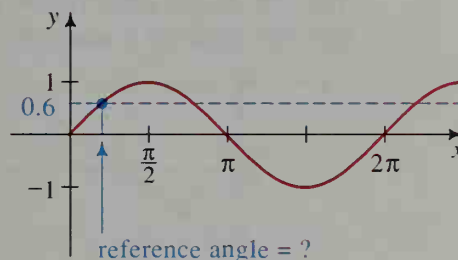
$$x = \sin^{-1} 0.6 \approx 0.6435.$$

**Step 2** 0.6435 is the reference angle for other solutions. Since  $\sin x$  is positive, a Quadrant II angle also satisfies the equation.

$$x = \pi - 0.6435 \approx 2.4981. \text{ Therefore, } x \approx 0.6435, 2.4981.$$

Note that if you had been asked to find *all* values of  $x$  for which  $\sin x = 0.6$ , then your answer would be

$$x \approx 0.6435 + 2\pi n \text{ and } x \approx 2.4981 + 2\pi n, \text{ for any integer } n.$$



To solve an equation involving a single trigonometric function, we first transform the equation so that the function is alone on one side of the equals sign. Then we follow the same procedure used in Example 1.

▶ The hour of sunset varies with the time of year; tides advance and recede in a pattern that is determined by the moon. Studied since ancient times, these natural cycles can be modeled by sine and cosine curves.

**Example 2**

To the nearest tenth of a degree, solve  $3 \cos \theta + 9 = 7$  for  $0^\circ \leq \theta < 360^\circ$ .

**Solution**

$$3 \cos \theta + 9 = 7$$

$$3 \cos \theta = -2$$

$$\cos \theta = -\frac{2}{3}$$

Since  $\cos \theta < 0$ , the solutions are between  $90^\circ$  and  $180^\circ$ , and  $180^\circ$  and  $270^\circ$ . (See graph.) One way to find these solutions is to first find the reference angle by ignoring the negative sign.

The reference angle is:

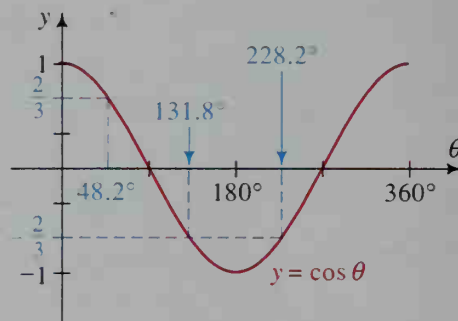
$$\cos^{-1} \left( \frac{2}{3} \right) \approx 48.2^\circ$$

The first solution is:

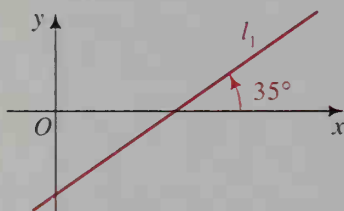
$$\theta \approx 180^\circ - 48.2^\circ = 131.8^\circ$$

The second solution is:

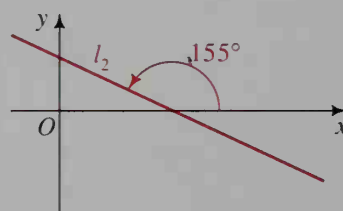
$$\theta \approx 180^\circ + 48.2^\circ = 228.2^\circ$$

**Inclination and Slope**

The **inclination** of a line is the angle  $\alpha$ , where  $0^\circ \leq \alpha < 180^\circ$ , that is measured from the positive  $x$ -axis to the line. The line at the left below has inclination  $35^\circ$ . The line at the right below has inclination  $155^\circ$ . The theorem that follows states that the slope of a nonvertical line is the tangent of its inclination.



$$\text{slope of } l_1 = \tan 35^\circ \approx 0.7002$$



$$\text{slope of } l_2 = \tan 155^\circ \approx -0.4663$$

**Theorem**

For any line with slope  $m$  and inclination  $\alpha$ ,

$$m = \tan \alpha \quad \text{if } \alpha \neq 90^\circ.$$

If  $\alpha = 90^\circ$ , then the line has no slope. (The line is vertical.)

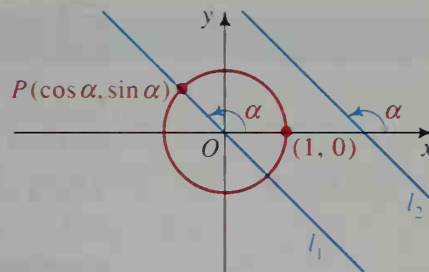
**Proof:**

Refer to the diagram on the next page. Suppose that  $l_1$  passes through the origin and has inclination  $\alpha$ . Let  $P$  be the point where  $l_1$  intersects the unit circle.

Then  $P = (\cos \alpha, \sin \alpha)$ . The slope of  $l_1$  is

$$m = \frac{\sin \alpha - 0}{\cos \alpha - 0} = \tan \alpha.$$

If the line  $l_2$  has inclination  $\alpha$  and does not contain the origin, it is parallel to  $l_1$ . Therefore, the slope of  $l_2$  also equals  $\tan \alpha$ .



### Example 3

To the nearest degree, find the inclination of the line  $2x + 5y = 15$ .

### Solution

Rewrite the equation as  $y = -\frac{2}{5}x + 3$ .

$$\text{slope} = -\frac{2}{5} = \tan \alpha$$

$$\alpha = \tan^{-1}\left(-\frac{2}{5}\right) \approx -21.8^\circ \text{ (The reference angle is } 21.8^\circ\text{.)}$$

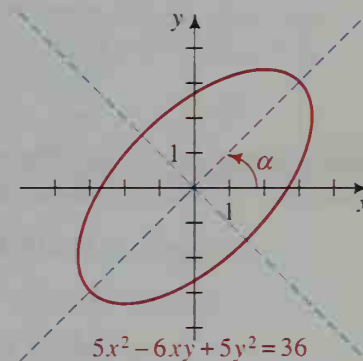
Since  $\tan \alpha$  is negative and  $\alpha$  is a positive angle,  $90^\circ < \alpha < 180^\circ$ .

The inclination is  $180^\circ - 21.8^\circ \approx 158.2^\circ$ .

In Section 6-7, you learned to graph conic sections whose equations have no  $xy$ -term. That is, equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where  $B = 0$ . The graph at the right shows a conic section with center at the origin whose equation has an  $xy$ -term ( $B \neq 0$ ). Conics like this have one of their two axes inclined at an angle  $\alpha$  to the  $x$ -axis. To find this *direction angle*  $\alpha$ , use the formula below.



$$\alpha = \frac{\pi}{4} \quad \text{if } A = C$$

$$\tan 2\alpha = \frac{B}{A - C} \quad \text{if } A \neq C, \text{ and } 0 < 2\alpha < \pi$$

The direction angle  $\alpha$  is useful in finding the equations of the axes of these conic sections. This is shown in Method 1 of Example 4 on the next page.

### Example 4

Identify the graph of the equation, find the direction angle  $\alpha$ , and sketch the curve  $x^2 - 2xy + 3y^2 = 1$ .

### Solution

$A \neq C$  and  $B^2 - 4AC = (-2)^2 - 4(1)(3) = -8 < 0$ , so the graph is an ellipse.

$$\tan 2\alpha = \frac{B}{A - C} = \frac{-2}{1 - 3} = 1 \quad \text{Thus, } 2\alpha = \frac{\pi}{4} \text{ and } \alpha = \frac{\pi}{8}.$$

(Solution continues on the next page.)





To sketch the conic, we can use a graphing calculator or a computer, as shown in Method 1, or we can use substitution and our knowledge of the particular conic, as shown in Method 2.

### Method 1

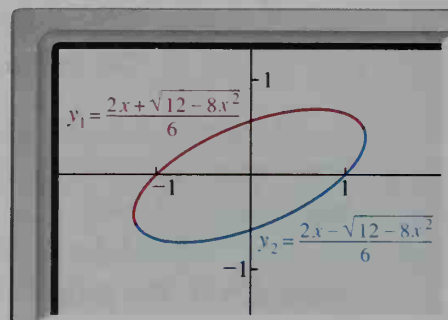
With a graphing calculator, it is often necessary to rewrite the equation so that it is quadratic in  $y$ . Then the equation can be solved for  $y$  by using the quadratic formula.

$$\begin{aligned}x^2 - 2xy + 3y^2 &= 1 \\3y^2 + (-2x)y + (x^2 - 1) &= 0 \\y &= \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 3(x^2 - 1)}}{2 \cdot 3} \\y &= \frac{2x \pm \sqrt{12 - 8x^2}}{6}\end{aligned}$$

Rewrite the equation above as two equations:

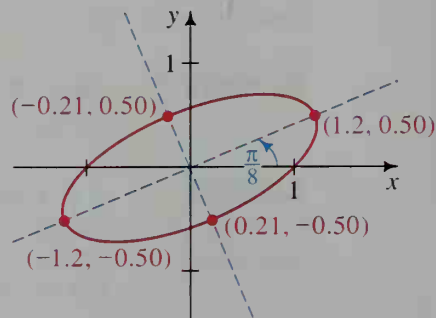
$$\begin{aligned}y_1 &= \frac{2x + \sqrt{12 - 8x^2}}{6} \quad \text{and} \\y_2 &= \frac{2x - \sqrt{12 - 8x^2}}{6}\end{aligned}$$

Entering these two functions into a graphing calculator or computer gives the graph shown at the right.



### Method 2

Use the direction angle found on page 297 to find equations of the axes of the ellipse. Since  $\alpha = \frac{\pi}{8}$ , the slope of one axis is  $\tan \frac{\pi}{8} \approx 0.41$ , and the slope of the other axis is  $-\frac{1}{0.41} \approx -2.4$ . Thus, equations of the two axes are  $y = 0.41x$  and  $y = -2.4x$ . By substituting for  $y$  into  $x^2 - 2xy + 3y^2 = 1$ , we find that these axes intersect the ellipse at points  $(1.2, 0.50)$ ,  $(-1.2, -0.50)$ ,  $(0.21, -0.50)$ , and  $(-0.21, 0.50)$ . Also, by substituting  $y = 0$  and  $x = 0$ , we find that the graph intersects the  $x$ -axis at  $(\pm 1, 0)$  and intersects the  $y$ -axis at  $(0, \pm \frac{\sqrt{3}}{3})$ . We can use these points to sketch the graph of the ellipse, as shown at the right.



## CLASS EXERCISES

Solve for  $0^\circ \leq \theta < 360^\circ$  without using tables or a calculator.

1.  $\cos \theta = \frac{1}{2}$       2.  $\sin \theta = -4$       3.  $\csc \theta = 2$       4.  $\cot \theta = -1$

Solve for  $0 \leq x < 2\pi$  without using tables or a calculator.

5.  $\cos x = -\frac{\sqrt{3}}{2}$       6.  $\cot x = 1$       7.  $\tan x = -\sqrt{3}$       8.  $\sec x = \frac{1}{2}$

Solve for  $\theta$  in degrees, giving *all* solutions.

9.  $\cos \theta = -1$       10.  $\sin \theta = -\frac{\sqrt{2}}{2}$       11.  $\tan \theta = 1$

12. The inclination of a line is  $140^\circ$ . Find its slope.

13. The slope of a line is  $\frac{3}{5}$ . Find its inclination.

14. Find to the nearest degree a direction angle  $\alpha$  for the conic whose equation is  $5x^2 + 2xy - y^2 = 6$ .

15. **Discussion** Comment on why you think it may be necessary to enter two equations when graphing a conic using a graphing calculator, as mentioned in Method 1 of Example 4.

## WRITTEN EXERCISES

**A** Solve for  $0^\circ \leq \theta < 360^\circ$ . Give answers to the nearest tenth of a degree.

1.  $\sin \theta = -0.7$       2.  $\cos \theta = 0.42$       3.  $\tan \theta = 1.2$   
4.  $\cot \theta = -0.3$       5.  $\sec \theta = -5$       6.  $\csc \theta = 14$   
7.  $3 \cos \theta = 1$       8.  $4 \sin \theta = 3$       9.  $5 \sec \theta + 6 = 0$   
10.  $2 \tan \theta + 1 = 0$       11.  $6 \csc \theta - 9 = 0$       12.  $4 \cot \theta - 5 = 0$

Solve for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth of a radian.

13.  $\tan x = -1.5$       14.  $\sec x = 2.5$       15.  $\csc x = -1.4$   
16.  $\cos x = -0.8$       17.  $\cot x = 6$       18.  $3 \sin x + 2 = 4$   
19.  $8 = 9 \cos x + 2$       20.  $\frac{5 \csc x}{3} = \frac{9}{4}$       21.  $\frac{3 \cot x}{4} + 1 = 0$

Find the slope and equation of each line described. Sketch the line.

22. inclination =  $45^\circ$   
y-intercept = 4      23. inclination =  $120^\circ$   
contains (2, 3)      24. inclination =  $158^\circ$   
contains (-3, 5)

Find the inclination of each line. Give your answers to the nearest degree.

25. The line  $3x + 5y = 8$

26. The line  $x - 4y = 7$

27. The line joining  $(-1, 2)$  and  $(4, 1)$

28. The line joining  $(-1, 1)$  and  $(4, 2)$

29. A line perpendicular to  $4x + 3y = 12$

30. A line parallel to  $2x + 3y = -6$

Solve for  $0 \leq x < 2\pi$  without using tables or a calculator.

**B** 31.  $|\csc x| = 1$

32.  $|\sec x| = \sqrt{2}$

33.  $\log_2(\sin x) = 0$

34.  $\log_2(\cos x) = -1$

35.  $\log_3(\tan x) = \frac{1}{2}$

36.  $\log_{\sqrt{3}}(\cot x) = 1$



Identify the graph of each equation, find its direction angle  $\alpha$ , and sketch the curve. You may wish to use a graphing calculator or computer.

37.  $x^2 + xy + y^2 = 1$

38.  $x^2 - 2xy - y^2 = 4$

39.  $x^2 - xy + y^2 = 1$

40.  $x^2 - xy + 2y^2 = 2$

41.  $x^2 - 3xy + y^2 = -6$

42.  $y = x - \frac{1}{x}$

43.  $x^2 - 2xy + y^2 - 4\sqrt{2}x - 4\sqrt{2}y = 0$

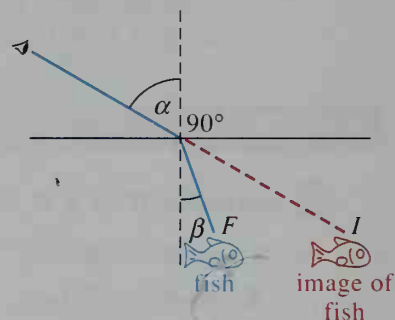
44.  $x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 0$



You may find it helpful to have a computer or a graphing calculator to complete part (b) of Exercise 45. Use degree measure for  $\alpha$  and  $\beta$ .

45. **Optics** The rays of light that are reflected from a fish in a pond to your eyes travel at different speeds through water and through air. Consequently, these rays bend at the water's surface. Your mind, which expects light to travel in a straight line, perceives the fish to be at location  $I$  when the fish is actually at location  $F$ . According to Snell's law,

$$\frac{\text{speed of light in air}}{\text{speed of light in water}} = \frac{\sin \alpha}{\sin \beta}$$



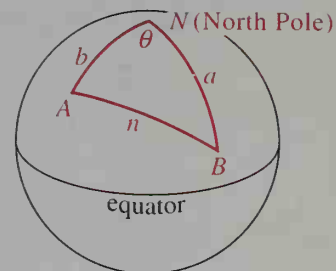
- a. Using the fact that the speed of light in air is  $3.00 \times 10^8$  m/s and the speed of light in water is  $2.25 \times 10^8$  m/s, show that  $\beta = \sin^{-1}(0.75 \sin \alpha)$ . Give an appropriate domain and range for this function.

- b. Use the graphs of  $y = \beta$  and  $y = 0.5\alpha$  to decide if there is a positive angle  $\alpha$  for which

$$\beta = \frac{1}{2}\alpha.$$



46. **Navigation** A *great circle* on Earth's surface is a circle whose center is at Earth's center. The shortest distance measured in degrees between points  $A$  and  $B$  on Earth's surface is along the arc of a great circle. To find this shortest distance, we can work with the spherical triangle  $ABN$  shown in the diagram, whose sides are arcs of great circles. The important measurements in this triangle are:



great circle arc  $AB = n^\circ$

great circle arc  $AN = b^\circ = 90^\circ - \text{latitude of } A$

great circle arc  $BN = a^\circ = 90^\circ - \text{latitude of } B$

$\theta = \text{difference in longitudes of } A \text{ and } B$

Use the formula  $\cos n = \cos a \cos b + \sin a \sin b \cos \theta$  and the following data to find the great circle distance measured in degrees between Rome and Boston. Give your answer to the nearest tenth of a degree.

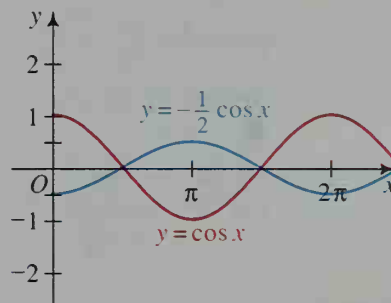
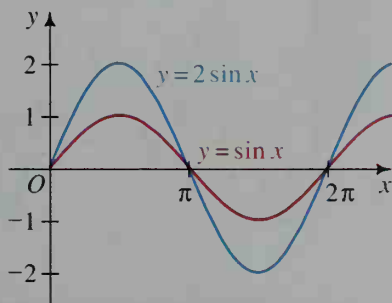
	Latitude	Longitude
Rome	$41.53^\circ\text{N}$	$12.30^\circ\text{E}$
Boston	$42.20^\circ\text{N}$	$71.05^\circ\text{W}$

## 8-2 Sine and Cosine Curves

### Objective

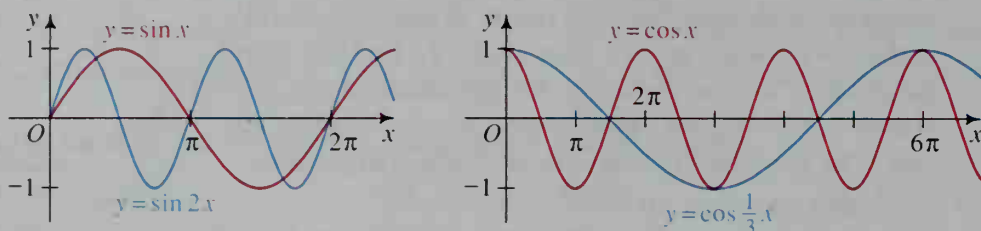
To find equations of different sine and cosine curves and to apply these equations.

Recall from Section 4-4 that the graph of  $y = cf(x)$  can be obtained by vertically stretching or shrinking the graph of  $y = f(x)$ . This is illustrated by the sine and cosine curves below.



Notice that  $y = -\frac{1}{2} \cos x$  is the graph of  $y = \frac{1}{2} \cos x$  that has been reflected in the  $x$ -axis. You can see that the graph of  $y = 2 \sin x$  has amplitude 2 and that the graph of  $y = -\frac{1}{2} \cos x$  has amplitude  $\frac{1}{2}$ .

Also in Section 4-4, you learned that the graph of  $y = f(cx)$  is obtained by horizontally stretching or shrinking the graph of  $y = f(x)$ . This is illustrated below.



The period of  $y = \sin x$  and  $y = \cos x$  is  $2\pi$ . (In this chapter, when we refer to the period  $p$  of a function, we mean the *fundamental period* of the function.) You can see from above that the period of  $y = \sin 2x$  is  $\frac{2\pi}{2}$ , or  $\pi$ , and that the period of  $y = \cos \frac{1}{3}x$  is  $\frac{2\pi}{\frac{1}{3}}$ , or  $6\pi$ .

In general, we can determine useful information about the graphs of  $y = A \sin Bx$  and  $y = A \cos Bx$  by analyzing the factors  $A$  and  $B$ .

### Period and Amplitude of Sine and Cosine Curves

For functions  $y = A \sin Bx$  and  $y = A \cos Bx$  ( $A \neq 0$  and  $B > 0$ ):

$$\text{amplitude} = |A| \quad \text{period} = \frac{2\pi}{B}$$

#### Example 1

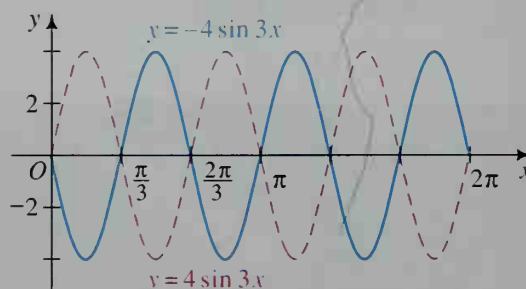
Give the amplitude and period of the function  $y = -4 \sin 3x$ . Then sketch at least one cycle of its graph.

#### Solution

$$\text{amplitude} = |-4| = 4$$

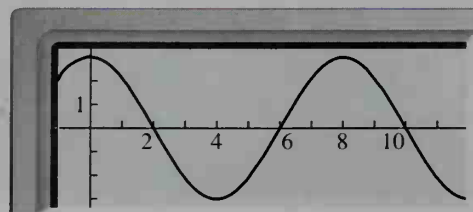
$$\text{period} = \frac{2\pi}{B} = \frac{2\pi}{3}$$

Reflect the graph of  $y = 4 \sin 3x$  in the  $x$ -axis to get the graph of  $y = -4 \sin 3x$ .



#### Example 2

Give the amplitude, period, and an equation of the curve shown at the right.





**Solution**

Use the cosine curve,  $y = A \cos Bx$ , with amplitude 3 and period 8. Thus,  $8 = \frac{2\pi}{B}$ , which gives  $B = \frac{\pi}{4}$ . The equation is

$$y = A \cos Bx = 3 \cos \frac{\pi}{4}x.$$



Example 3 shows two methods of solving a trigonometric equation. Method 1 is an algebraic solution, as developed in Section 8-1. Method 2 is a graphical solution in which a graphing calculator or a computer can be used.

**Example 3**

Solve the equation  $6 \sin 2x = 5$  for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth of a radian.

**Solution**

**Method 1** Transform the equation as follows:

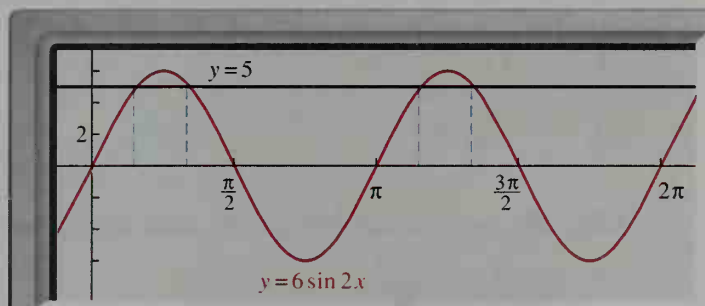
$$6 \sin 2x = 5$$

$$\sin 2x = \frac{5}{6}$$

$$2x \approx 0.99, 2.16, 7.27, 8.44 \quad \leftarrow \text{If } 0 \leq x < 2\pi, \text{ then}$$

$$x \approx 0.49, 1.08, 3.63, 4.22 \quad 0 \leq 2x < 4\pi.$$

**Method 2** Use a graphing calculator or computer to sketch the graphs of  $y = 5$  and  $y = 6 \sin 2x$  on the same set of axes. Then find the  $x$ -coordinates of the intersection points over the interval  $0 \leq x < 2\pi$ .



Using the zoom feature to get more accuracy, you will be able to see that the line  $y = 5$  intersects the curve  $y = 6 \sin 2x$  at  $x \approx 0.49, 1.08, 3.63$ , and  $4.22$ .

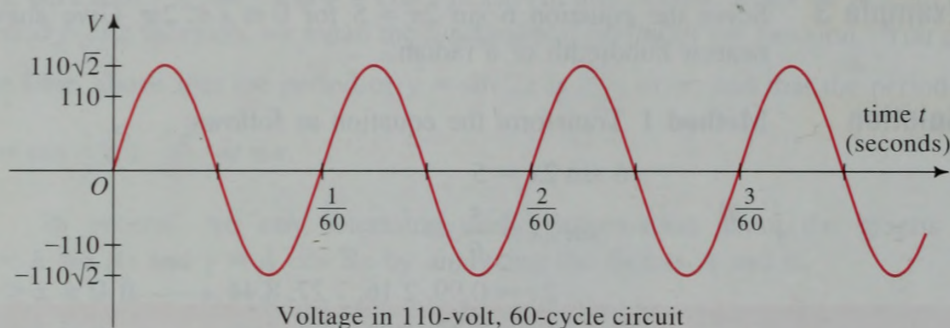
## Applications to Electricity

Most household circuits in the United States are 60-cycle alternating current (AC) circuits. This means that the voltage oscillates like the sine curve at a frequency of 60 cycles per second. In other words, 1 cycle is completed every  $\frac{1}{60}$  of a second, which is the period. The period of a sine or cosine curve is always the reciprocal of the frequency.

In the following diagram, the frequency is 60, and the period is  $\frac{1}{60}$ . Thus,  $\frac{1}{60} = \frac{2\pi}{B}$ , and so  $B = 120\pi$ . Most household circuits are called 110-volt circuits because they deliver energy at the same rate as a direct current of 110 volts. In actuality, maximum voltage in a 110-volt circuit is  $110\sqrt{2}$  volts, and so  $A = 110\sqrt{2}$ . The voltage equation is:

$$V = A \sin Bx$$

$$V = 110\sqrt{2} \sin 120\pi t \quad (t \text{ in seconds})$$



## CLASS EXERCISES

Give the period and amplitude of each function.

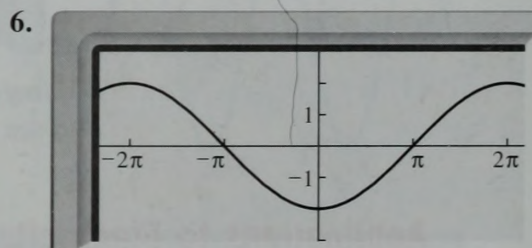
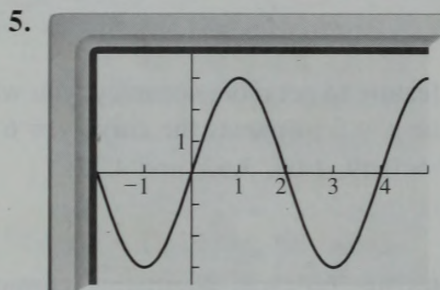
1.  $y = 4 \cos 2x$

2.  $y = 3 \sin \frac{1}{2}x$

3.  $y = 5 \sin \frac{2\pi}{7}x$

4.  $y = 6 \cos \frac{2\pi}{3}t$

Give the period, the amplitude, and an equation for each curve.



7. **Visual Thinking** Draw a quick sketch to tell how many solutions each equation has between 0 and  $2\pi$ .

a.  $\sin x = 1$

b.  $\sin 2x = \frac{1}{2}$

c.  $\sin 3x = -1$

## WRITTEN EXERCISES

**A** For Exercises 1–4, sketch and label the graphs given on a single set of axes.

1.  $y = \cos x$ ,  $y = 3 \cos x$ ,  $y = \frac{1}{3} \cos x$

2.  $y = \sin x$ ,  $y = 4 \sin x$ ,  $y = -4 \sin x$

3.  $y = \sin x$ ,  $y = \sin \frac{1}{2}x$

4.  $y = \cos x$ ,  $y = \cos 3x$

Give the amplitude and period of each function. Then sketch its graph.

5.  $y = 2 \sin 3x$

6.  $y = 4 \cos 2x$

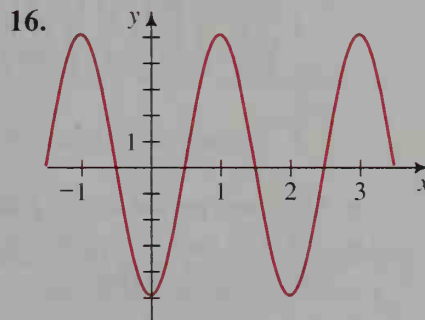
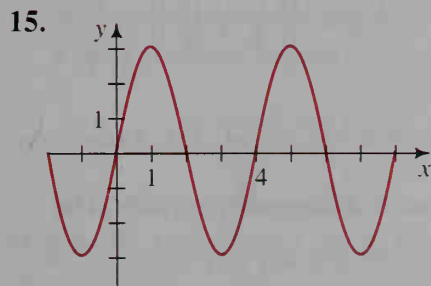
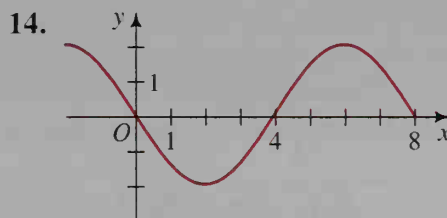
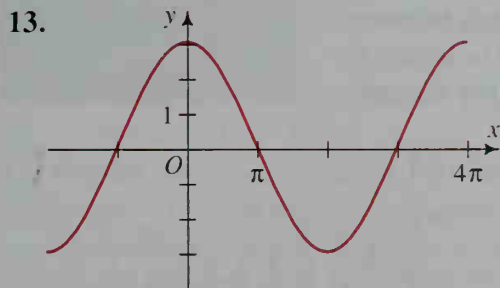
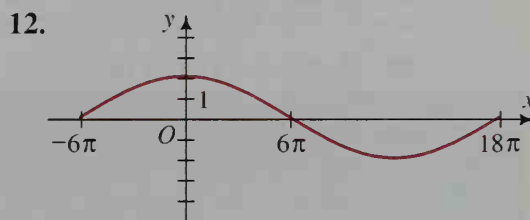
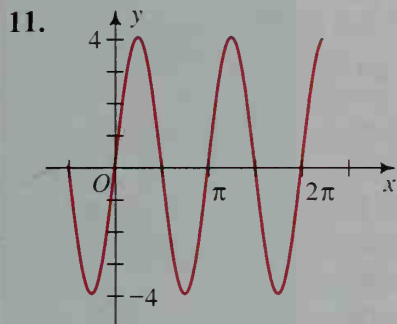
7.  $y = -2 \cos 2t$

8.  $y = -4 \sin \frac{t}{3}$

9.  $y = \frac{1}{2} \cos 2\pi t$

10.  $y = 1.5 \sin \frac{\pi}{2}x$

Give the amplitude, period, and an equation for each curve.



17. A sine curve varies between 4 and  $-4$  with period 12. Find its equation.  
 18. A cosine curve varies between  $-9$  and  $9$  with period 5. Find its equation.



Solve each equation for  $0 \leq x < 2\pi$  either algebraically or graphically using a computer or graphing calculator. Give answers to the nearest hundredth of a radian.

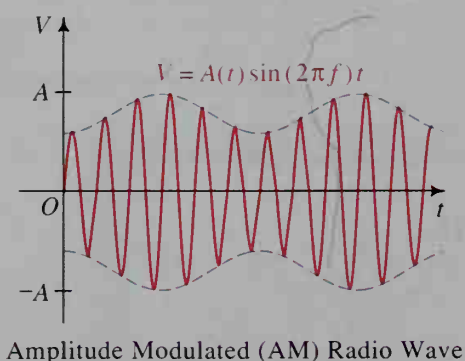
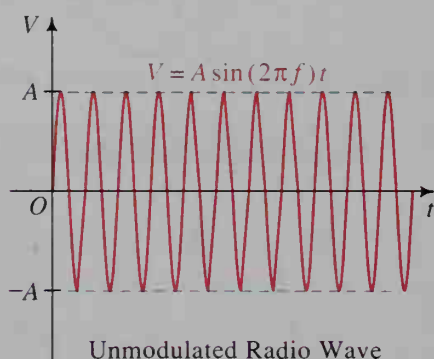
19. a.  $\cos x = -1$                       b.  $\cos 2x = -1$                       c.  $\cos 3x = -1$   
 20. a.  $2 \sin x = 1$                       b.  $2 \sin 2x = 1$                       c.  $2 \sin \frac{x}{2} = 1$   
**B** 21.  $8 \cos 2x = 1$                       22.  $5 \sin 3x = -2$                       23.  $3 \sin \frac{x}{2} = -1$   
 24.  $1.5 \cos \frac{x}{2} = \frac{1}{2}$                       25.  $4 \sin \frac{\pi}{2}x = 1$                       26.  $-3 \cos \frac{\pi}{4}x = 1$

27. Sketch the graphs of  $y = \tan 2x$  and  $y = \tan \frac{1}{2}x$ .

28. Sketch the graphs of  $y = \sec 2x$  and  $y = \sec \frac{1}{2}x$ .

29. **Writing** Write a paragraph to explain why it does not make sense to talk about the amplitude of the graph of  $y = \tan x$  and of the graph of  $y = \sec x$ .

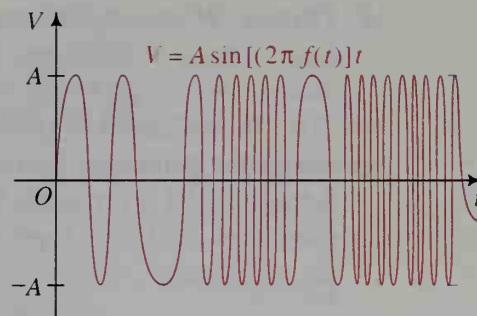
30. **Electronics** An AM radio wave means that the *amplitude is modulated*. Thus, instead of a constant amplitude  $A$ , there is a varying amplitude  $A(t)$  as shown below. Each AM radio station in the United States is assigned a frequency between 540 kHz and 1700 kHz. (A hertz (Hz) is one cycle per second, so 1 kHz is 1000 cycles per second.)



- a. Give the period of an AM radio wave with a frequency of 800 kHz.  
 b. An AM radio wave has equation  $V(t) = A(t) \sin 1,850,000\pi t$ . What is its frequency in kHz?



31. **Electronics** The initials FM stand for *frequency modulation*. In FM broadcasting, information is communicated by varying the frequency; that is, the constant frequency  $f$  is replaced by a variable frequency  $f(t)$ , which is a function of time. An exaggerated graph of this situation appears at the right. A complication of FM broadcasting is that  $f(t)$  must remain near the radio station's assigned frequency. FM frequencies are given in MHz (millions of cycles per second).



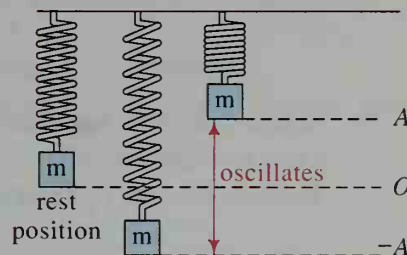
Frequency Modulated (FM) Radio Wave

- Suppose that an FM radio wave has an equation of the following form:  $V(t) = A \sin [2\pi(200,000,000 + 10,000 \sin 500\pi t)]t$ . Thus, the varying frequency is  $f(t) = 200,000,000 + 10,000 \sin 500\pi t$ . What is the assigned frequency in MHz?
  - Give the maximum and minimum frequencies of this FM radio wave over a long period of time.
32. **Research** In a reference book find information about two of the following:
- the range of assigned frequencies of the FM broadcasting channels in the United States
  - the frequency ranges for VHF (very high frequency) and UHF (ultra high frequency) television
  - the frequencies of ultraviolet light and infrared light
  - the frequency of the musical pitch called "middle C"



33. **Physics** Suppose a weight with mass  $m$  grams hangs on a spring. If you pull the weight  $A$  centimeters downward and let go of it, the weight will oscillate according to the formula below. Let  $d$  represent the displacement in centimeters  $t$  seconds after the initial displacement. Let  $k$  be a constant measuring the spring's stiffness.

$$d = -A \cos \left[ \left( \sqrt{\frac{k}{m}} \right) t \right]$$

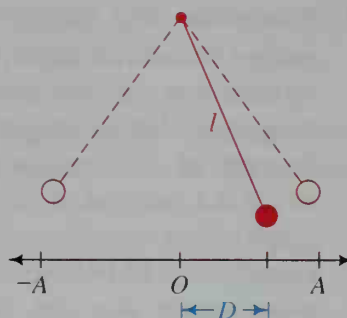


- Show that the period of the motion is  $2\pi\sqrt{\frac{m}{k}}$ .
- Suppose you put a weight with known mass 100 g on the spring and you time its period to be  $t = 1.1$  s. Find the value of the spring constant  $k$ .
- Suppose you put a weight with unknown mass  $m$  on the spring and time its period to be  $t = 1.4$  s. Find  $m$ . (Use the value of  $k$  found in part (b).)



34. **Physics** When a pendulum swings back and forth through a small arc, its horizontal displacement is given by the formula below. Let  $D$  be the horizontal displacement in centimeters of the pendulum  $t$  seconds after passing through  $O$ , let  $A$  represent the maximum displacement, and let  $l$  represent the length of the pendulum.

$$D \approx A \sin \left[ \left( \sqrt{\frac{980}{l}} \right) t \right]$$



- a. If the length  $l$  of the pendulum is 100 cm, find the earliest time  $t$  for which the displacement is maximum.
- b. How long is a clock pendulum that has a period of 1 second?
- C** 35. On the same set of axes, sketch the graphs of  $y = 2^{-x}$ ,  $y = -2^{-x}$ , and  $y = 2^{-x} \sin x$  for  $x > 0$ .
36. **Music** When the note called *concert A* is sounded on a piano, the piano string vibrates with a frequency of 440 Hz. The equation that gives the displacement of a point on the vibrating piano string is

$$D = B(2^{-t}) \sin 880\pi t,$$

where  $t$  is the number of seconds after the string is struck and  $B$  is a constant that depends on how hard the string is hit and the point's position on the string. (This formula applies only for the first few seconds.)

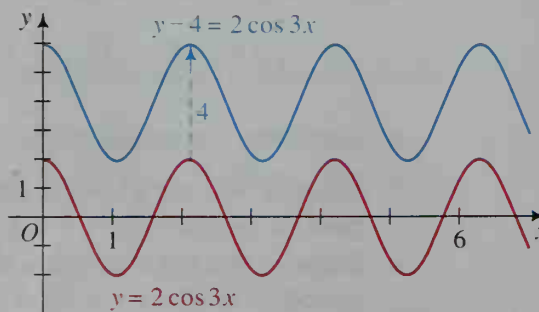
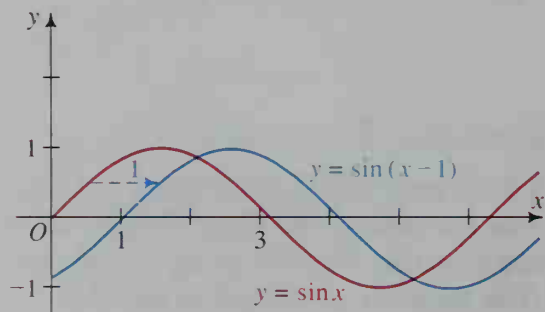
- a. What part of the equation models the gradual dying out of the vibrations?
- b. What is the earliest time  $t$  for which the displacement is maximum?
- c. The piano string for the A an octave lower than concert A has a frequency of 220 Hz. Write an equation for its displacement.

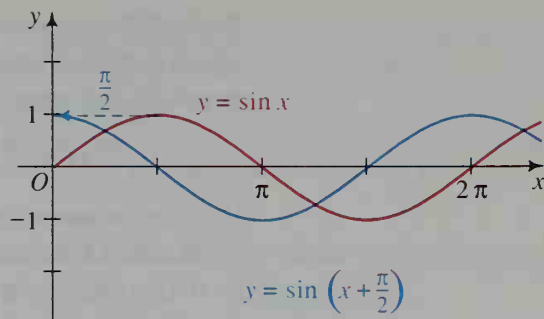
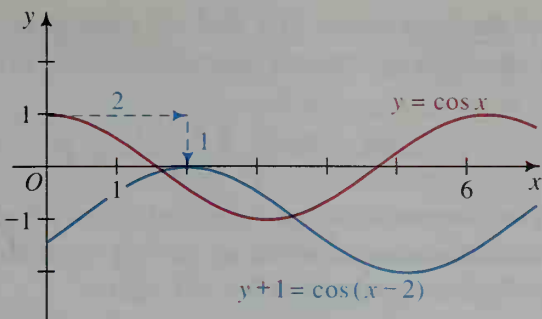
## 8-3 Modeling Periodic Behavior

### Objective

To use trigonometric functions to model periodic behavior.

When a graph is translated  $h$  units horizontally and  $k$  units vertically, then  $x$  and  $y$  must be replaced by  $(x - h)$  and  $(y - k)$ , respectively. This idea was introduced in Section 4-4 and is illustrated below and on the next page.





Notice that the last graph, whose equation is  $y = \sin(x + \frac{\pi}{2})$ , is also the graph of  $y = \cos x$ . Thus, the cosine curve is the sine curve that has been shifted  $\frac{\pi}{2}$  units to the left. Since the cosine curve is congruent to the sine curve, we refer to sine and cosine curves as *sine waves*.

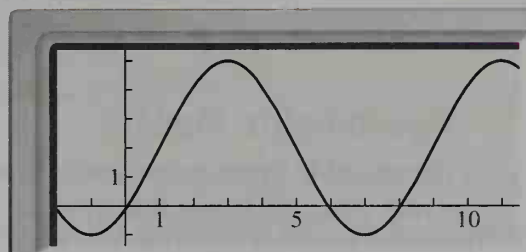
## General Sine Waves

If the graphs of  $y = A \sin Bx$  and  $y = A \cos Bx$  are translated horizontally  $h$  units and vertically  $k$  units, then the resulting graphs have equations

$$y - k = A \sin B(x - h) \quad \text{and} \quad y - k = A \cos B(x - h).$$

### Example 1

Give an equation of the sine wave shown at the right.



### Solution

The graph suggests a translation of  $y = A \sin Bx$  or  $y = A \cos Bx$ . To find  $A$  and  $B$ , reason as follows:

$$\text{amplitude: } A = \frac{\text{Max} - \text{min}}{2} = \frac{5 - (-1)}{2} = 3$$

$$\text{period: } p = \text{horizontal distance between successive maximums} \\ = 11 - 3 = 8$$

$$\text{Since } 8 = \frac{2\pi}{B}, B = \frac{\pi}{4}.$$

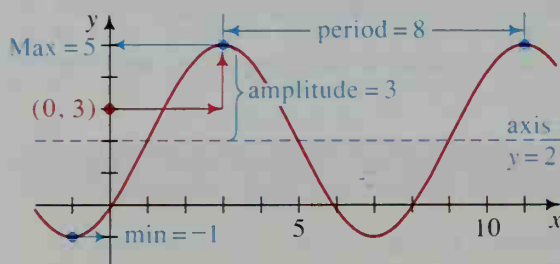
The given sine wave is a translation of  $y = 3 \sin \frac{\pi}{4}x$  or of  $y = 3 \cos \frac{\pi}{4}x$ .

(Solution continues on the next page.)

To find the amounts of the translation, first find the *axis of the wave*, which is the horizontal line midway between the maximum and minimum points of the curve.

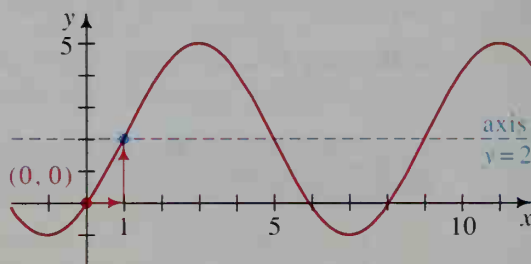
$$\text{axis of wave: } y = \frac{\text{Max} + \text{min}}{2} = \frac{5 + (-1)}{2} = 2$$

- (1) If the equation is to be in terms of cosine, select a highest point on the curve. Determine the translation amounts in moving from the point  $(0, A)$  to this point. See the diagram at the left below.
- (2) If the equation is to be in terms of sine, select a point where the curve intersects its axis. Determine the translation amounts in moving from the point  $(0, 0)$  to this point. See the diagram at the right below.



The graph of  $y = 3 \cos \frac{\pi}{4} x$  is translated 3 units right and 2 units up.

$$y - 2 = 3 \cos \frac{\pi}{4} (x - 3)$$



The graph of  $y = 3 \sin \frac{\pi}{4} x$  is translated 1 unit right and 2 units up.

$$y - 2 = 3 \sin \frac{\pi}{4} (x - 1)$$

Thus, an equation of the given graph is

$$y - 2 = 3 \cos \frac{\pi}{4} (x - 3) \quad \text{or} \quad y - 2 = 3 \sin \frac{\pi}{4} (x - 1).$$

## Trigonometric Models

Trigonometric functions are useful in solving many problems that involve periodic behavior, such as the motion of the tide. In many applications, the variable might represent something other than angles. For example, an application might involve  $\sin t$  or  $\cos t$ , where  $t$  represents time. In this case, you evaluate  $\sin t$  or  $\cos t$  as if  $t$  were in radians.



As the example on the next page shows, a sketch of a graph showing given information is very helpful in analyzing a problem in order to find an equation of the trigonometric model. Then you could use a graphing calculator or a computer to draw quickly the functions involved in the equation, and then zoom in on the intersection points to find the time values.



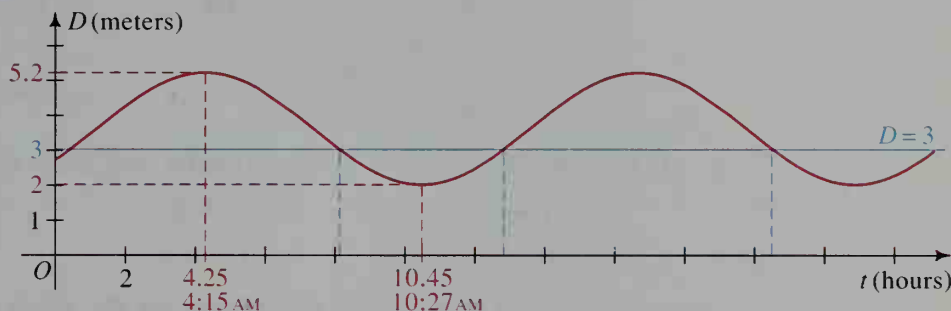
**Example 2**

The depth of water at the end of a pier varies with the tides throughout the day. Today the high tide occurs at 4:15 A.M. with a depth of 5.2 m. The low tide occurs at 10:27 A.M. with a depth of 2.0 m.

- Find a trigonometric equation that models the depth of the water  $t$  hours after midnight.
- Find the depth of the water at noon.
- A large boat needs at least 3 m of water to moor at the end of the pier. During what time period after noon can it safely moor?

**Solution**

- First make a sketch, noting the given information.



The amplitude is  $A = \frac{5.2 - 2.0}{2} = 1.6$

The period is  $p = 2 \cdot (\text{time of low tide} - \text{time of high tide})$   
 $= 2(10.45 - 4.25)$   
 $= 12.4$

Since  $12.4 = \frac{2\pi}{B}$ ,  $B \approx 0.507$ .

The axis of the wave has the equation  $y = \frac{5.2 + 2.0}{2} = 3.6$ .

Thus, an equation that models the depth  $D$  at time  $t$  hours after midnight is:

$$D - 3.6 = 1.6 \cos [0.507(t - 4.25)], \text{ or}$$

$$D = 1.6 \cos [0.507(t - 4.25)] + 3.6$$

- Substitute  $t = 12$  into the equation above to find the depth at noon.

$$D = 1.6 \cos [0.507(12 - 4.25)] + 3.6$$

$$D \approx 2.47 \text{ m}$$

- Using a graphing calculator or a computer, first enter the equation  $D - 3.6 = 1.6 \cos [0.507(t - 4.25)]$  to get the curve shown above. Then graph the line  $D = 3$  and find that it intersects the curve at  $t = 8.1$ ,  $t = 12.8$ , and  $t = 20.5$  (the times when the depth is 3 m).

Without a graphing calculator, you can find these values of  $t$  algebraically, as shown on the next page.

$$D = 1.6 \cos [0.507(t - 4.25)] + 3.6$$

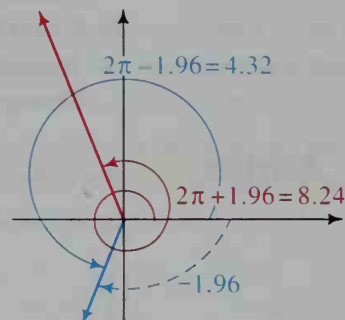
$$3 = 1.6 \cos [0.507(t - 4.25)] + 3.6$$

$$-0.6 = 1.6 \cos [0.507(t - 4.25)]$$

$$-0.375 = \cos [0.507(t - 4.25)]$$

$$\cos^{-1}(-0.375) = 0.507(t - 4.25)$$

Use a scientific calculator to find that  $\cos^{-1}(-0.375) \approx 1.96$ . Any radian value coterminal with 1.96 or with  $-1.96$  would also have  $-0.375$  as its cosine value. The diagram shows that 4.32 and 8.24 are such possible radian values. Substituting these values gives:



$$1.96 = 0.507(t - 4.25) \quad 4.32 = 0.507(t - 4.25) \quad 8.24 = 0.507(t - 4.25)$$

$$t \approx 8.1 \text{ h}$$

$$t \approx 12.8 \text{ h}$$

$$t \approx 20.5 \text{ h}$$

$$t = 8:06 \text{ A.M.}$$

$$t = 12:48 \text{ P.M.}$$

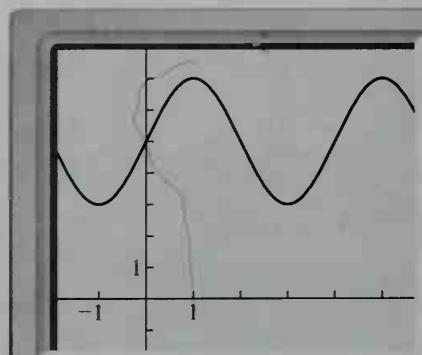
$$t = 8:30 \text{ P.M.}$$

You should locate these three times on the graph shown on page 311. Looking at the graph, you can see that the curve is above the line  $D = 3$  when  $12.8 < t < 20.5$ . Therefore, the boat can safely moor between 12:48 P.M. and 8:30 P.M.

## CLASS EXERCISES

For Exercises 1–5, refer to the sine wave shown.

- What is the amplitude?
- What is the period?
- What is the axis of the wave?
- If the wave is considered to be the translation of the graph of  $y = A \sin Bx$ , what are the horizontal and vertical translation amounts?
  - What is this equation?
- If the wave is considered to be the translation of the graph of  $y = A \cos Bx$ , what are the translation amounts?
  - What is this equation?
- The equation of a sine wave is  $y - 6 = 4 \sin 3(x + 5)$ . What do the numbers 3, 4, 5, and 6 tell you about the wave?

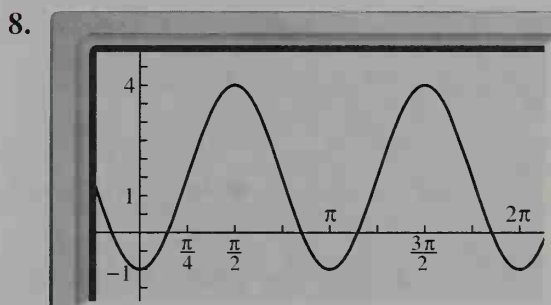
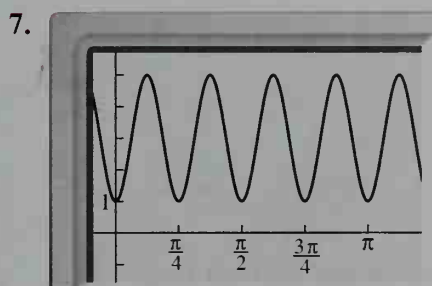
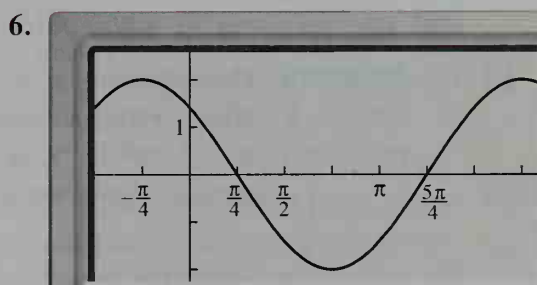
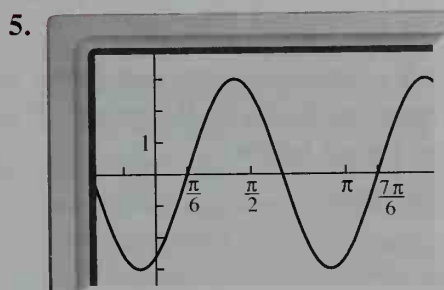
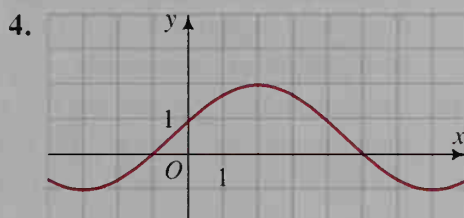
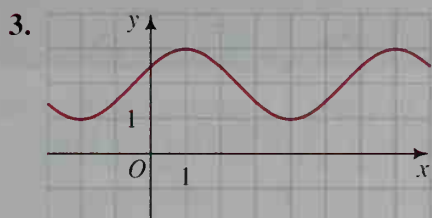
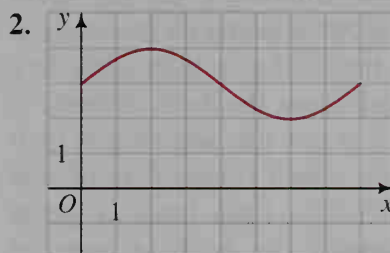
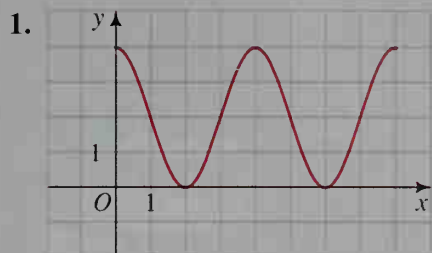




## WRITTEN EXERCISES

Give an equation for each trigonometric graph.

**A**



Sketch the graph of each equation.

9.  $y = 3 + 5 \sin 2x$

10.  $y = -2 + 2 \cos \frac{1}{2}x$

11.  $y = -3 \sin\left(x - \frac{\pi}{6}\right)$

12.  $y = 4 - 4 \cos 2(x - \pi)$

13.  $y - 3 = 2 \cos \frac{\pi}{2}(x - 1)$

14.  $y + 3 = 6 \sin \frac{\pi}{4}(x + 2)$

15. **Oceanography** The Bay of Fundy is an inlet of the Atlantic Ocean bounded by Maine and New Brunswick on the north and Nova Scotia on the south. It is famous for its high tides. At a dock there, the depth of water is 2 ft at low tide and 58 ft at high tide, which occurs 6 h 12 min after low tide. Draw a graph showing the depth of water at the dock as a function of the time since high tide occurs. Find an equation of your graph.

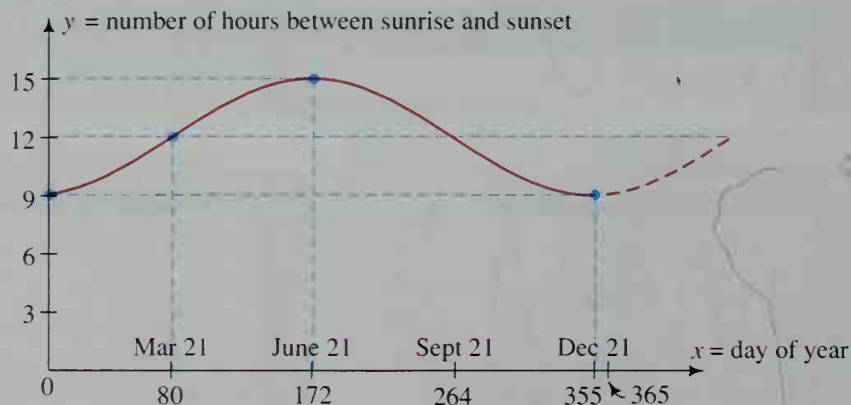


16. **Oceanography** Rework the tide problem in Example 2 assuming that the first high tide today occurs at 3:00 A.M. with a depth of 4.0 m, and the first low tide occurs at 9:24 A.M. with a depth of 1.8 m.



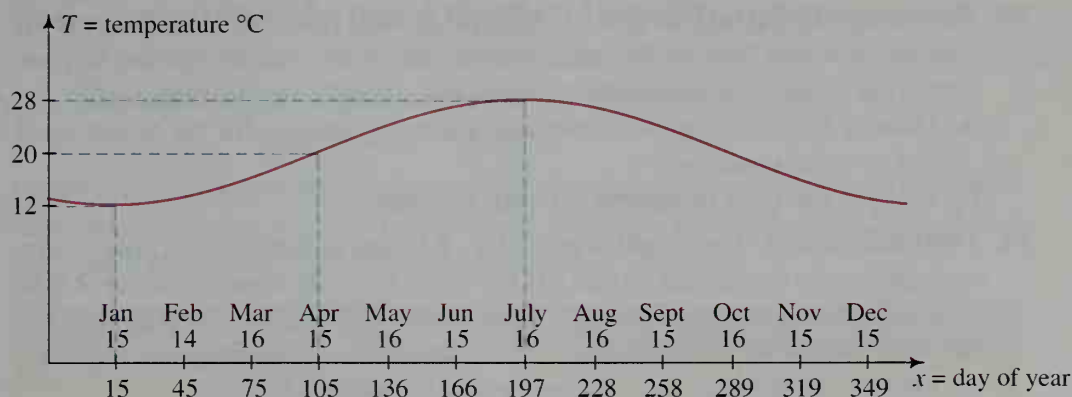
For some parts of the following exercises, you may wish to use a graphing calculator or a computer. Be sure to use radian measure.

- B** 17. **Astronomy** The approximate number of hours between sunrise and sunset in Denver, Colorado, varies throughout the year as shown in the graph. This variation is approximated by the sine wave shown below.

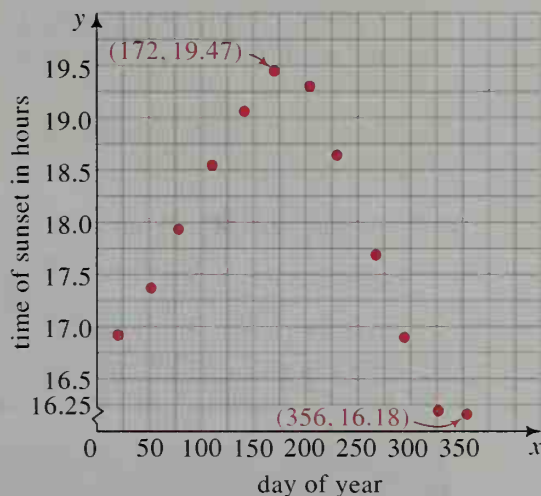
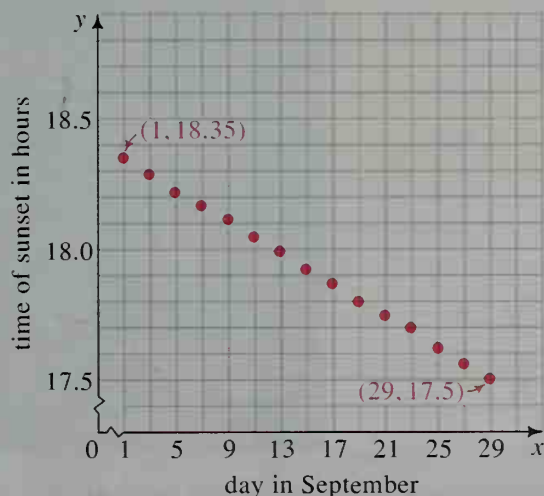


- Give the period, amplitude, and equation of the daylight-hours graph.
  - Find the amount of daylight in Denver on January 1 and on July 4.
  - Over the course of a year, during what period of time is the amount of daylight in Denver at least 14 hours?
  - If you were to draw a daylight-hours graph for Seattle, Washington, which is north of Denver, do you think its amplitude would be less than or greater than that for the Denver curve?
18. **Astronomy** The graph given in Exercise 17 applies to Denver and to all other locations at latitude  $39^{\circ}44'N$ . Modify the graph for a location at  $39^{\circ}44'S$ . Give an equation of your modified graph.

19. **Meteorology** Average monthly temperatures for New Orleans are plotted for the middle of each month. These points are connected to give a smooth curve, shown below, that gives an approximation of the average daily temperatures.

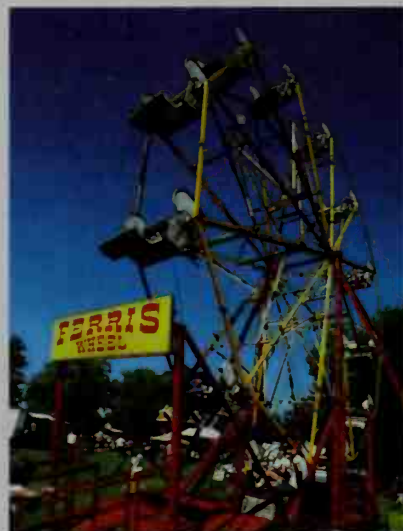
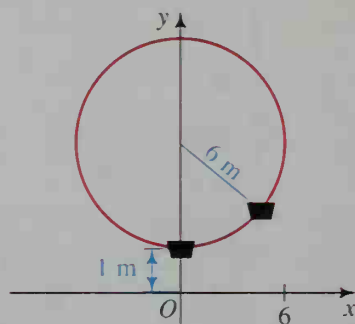


- Find an equation of the average-daily-temperature graph.
  - Over the course of a year, during what periods of time is the average daily temperature in New Orleans no more than  $15^\circ\text{C}$ ?
20. **Meteorology** The average maximum and minimum temperatures of two cities are given below. For each location, sketch a temperature sine wave like that in Exercise 19. Then give an equation of the curve.
- Winnipeg, Canada:  $26^\circ\text{C}$  (July 16) and  $-14^\circ\text{C}$  (January 15)
  - Rio de Janeiro, Brazil:  $28^\circ\text{C}$  (January 15) and  $22^\circ\text{C}$  (July 16)
21. **Astronomy** The graph at the left below shows the time of sunsets occurring every other day during September in Exeter, New Hampshire. The graph at the right shows the time of sunsets on either the 21st or 22nd day of each month for the entire year. All times are given in Eastern Standard Time.



- Find a linear function that models the data in the graph at the left.
- Find a trigonometric function that models the data in the graph at the right.
- Use the function in part (b) to find the period of time, over the course of a year, during which the sun sets after 7:00 P.M. EST.

22. **Research** Refer to Exercise 20. Consult a local almanac or newspaper to find the average maximum and minimum temperatures for the area where you live. Sketch a temperature sine wave and give an equation of the curve.
23. **Research** Refer to Exercise 21. Consult a local almanac or newspaper to find the date and time of the latest sunrise and of the earliest sunrise in your area. (For simplicity, use standard time, not daylight-saving time.)
- Draw a graph that approximates the times of sunrises for the whole year and give its equation.
  - What is the time of sunrise on your birthday?
24. **Thermodynamics** On a cold winter day, a house is heated until it is warm enough for the thermostat to turn off the heat. Then the house cools until it is cool enough for the thermostat to turn on the heat again. Assume that this periodic change in temperature can be modeled by a trigonometric function.
- Find an equation of this function given the following data:
    - Thermostat turns heat on at 10:15 A.M. when house temperature is  $18^{\circ}\text{C}$ .
    - Thermostat turns heat off at 10:30 A.M. when house temperature is  $20^{\circ}\text{C}$ .
  - Writing** Write a paragraph explaining why this trigonometric function might not be a good model. Think about how long it takes for the heating and cooling parts of the cycle. Think about the effects of temperature changes outdoors.
- C** 25. **Physics** Suppose that after you are loaded into a Ferris wheel car, the wheel begins turning at 4 rpm. The wheel has diameter 12 m and the bottom seat of the wheel is 1 m above the ground. (See the diagram below.) Express the height  $h$  of the seat above the ground as a function of time  $t$  seconds after it begins turning. (*Suggestion:* Begin your solution by sketching the graph of height versus time. Label the times at which you reach the lowest and highest points.)



26. **Physics** A reflector is fastened to the front wheel of a bicycle 20 cm from the center of the wheel. The diameter of the wheel and inflated tire is 70 cm. If the bike is traveling at 10 km/h, express the height of the reflector above the ground as a function of time. Assume that at time  $t = 0$  seconds, the reflector is at its highest point. (*Hint:* The hardest part is determining the period.)

# Identities and Equations

## 8-4 Relationships Among the Functions

**Objective** To simplify trigonometric expressions and to prove trigonometric identities.

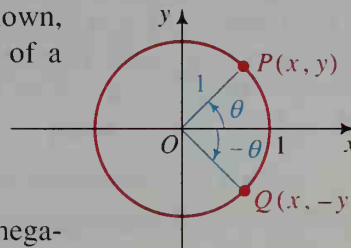
In this section, we will investigate some of the relationships among the trigonometric functions. Some of these relationships you have seen before. In Section 7-5, you saw that certain functions are reciprocals of others.

### Reciprocal Relationships

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

Recall that if  $P$  is a point on the unit circle as shown, then  $x = \cos \theta$  and  $y = \sin \theta$ . From the symmetry of a circle, the following relationships are true.

$$\begin{aligned}\sin(-\theta) &= -y = -\sin \theta \\ \cos(-\theta) &= x = \cos \theta\end{aligned}$$



Using the reciprocal relationships, we can find negative relationships for the other functions.

### Relationships with Negatives

$$\begin{array}{lll}\sin(-\theta) = -\sin \theta & \text{and} & \cos(-\theta) = \cos \theta \\ \csc(-\theta) = -\csc \theta & \text{and} & \sec(-\theta) = \sec \theta \\ \tan(-\theta) = -\tan \theta & \text{and} & \cot(-\theta) = -\cot \theta\end{array}$$

We can find other relationships using the unit circle. Again, refer to the diagram above. Since  $P(\cos \theta, \sin \theta)$  is on the unit circle,

$$(\sin \theta)^2 + (\cos \theta)^2 = x^2 + y^2 = 1.$$

This is the first of the three Pythagorean relationships that follow. You are asked to explore the other two relationships in Activity 2 and in Exercises 25 and 26.

### Pythagorean Relationships

$$\sin^2 \theta + \cos^2 \theta = 1 \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad 1 + \cot^2 \theta = \csc^2 \theta$$



The sine and cosine are called **cofunctions**, as are the tangent and cotangent, and the secant and cosecant. There is a special relationship between a function and its cofunction, as you will discover in the following activity.

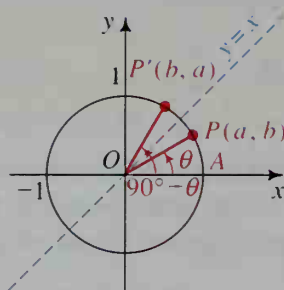
### Activity 1

- Use a calculator to evaluate the following.
  - $\sin 50^\circ$ ,  $\cos 40^\circ$
  - $\sin 25^\circ$ ,  $\cos 65^\circ$
  - $\cos 11^\circ$ ,  $\sin 79^\circ$
  - $\sin 83^\circ$ ,  $\cos 7^\circ$
- Complete each of the following.
  - $\sin 18^\circ = \cos (?^\circ)$
  - $\cos 89^\circ = \sin (?^\circ)$
  - $\sin \theta = \cos (?)$
  - $\cos \theta = \sin (?)$

The reason for the cofunction relationships can be seen from the diagram at the right. If the sum of the measures of  $\angle POA$  and  $\angle P'OA$  is  $90^\circ$ , then  $P$  and  $P'$  are symmetric with respect to the line  $y = x$ . Hence, if  $P = (a, b)$ , then  $P' = (b, a)$ . Consequently,

$$\begin{aligned}\sin \theta &= y\text{-coordinate of } P \\ &= x\text{-coordinate of } P' \\ &= \cos (90^\circ - \theta)\end{aligned}$$

$$\begin{aligned}\cos \theta &= x\text{-coordinate of } P \\ &= y\text{-coordinate of } P' \\ &= \sin (90^\circ - \theta)\end{aligned}$$



You should convince yourself that this argument remains valid if the diagram is changed so that  $\theta$  is not in Quadrant I. In general,

function of  $\theta$  = cofunction of the complement of  $\theta$ .

The cofunction relationships are summarized below.

#### Cofunction Relationships

$\sin \theta = \cos (90^\circ - \theta)$	and	$\cos \theta = \sin (90^\circ - \theta)$
$\tan \theta = \cot (90^\circ - \theta)$	and	$\cot \theta = \tan (90^\circ - \theta)$
$\sec \theta = \csc (90^\circ - \theta)$	and	$\csc \theta = \sec (90^\circ - \theta)$

### Identities

Each of the trigonometric relationships given is true for all values of the variable for which *each side of the equation is defined*. Such relationships are called **trigonometric identities**, just as  $(a + b)^2 = a^2 + 2ab + b^2$  is called an *algebraic identity*. The following activity will help you understand identities.

## Activity 2



Use a graphing calculator or computer to answer each question. Be sure you use radian measure. If your calculator does not have a particular function, use its reciprocal function. For example, you can use  $\frac{1}{\cos x}$  instead of  $\sec x$ .

1.
  - a. On the same set of axes, graph  $y = 1 + \tan^2 x$  and  $y = \sec^2 x$ .
  - b. Compare the two graphs. Would you say that the equation  $1 + \tan^2 x = \sec^2 x$  is an identity? Explain.
2.
  - a. On the same set of axes, graph  $y = \sin 2x$  and  $y = 2 \sin x$ .
  - b. Compare the two graphs. Would you say that the equation  $\sin 2x = 2 \sin x$  is an identity? Explain.
3.
  - a. Graph  $y = \sec x - \sin x \tan x$ . What is the domain of the sine function? of the secant function? of the tangent function?
  - b. On the same set of axes, graph  $y = \cos x$ . What is the domain of the cosine function?
  - c. If you eliminate those values of  $x$  for which any of these functions are undefined, what can you say about the comparison of the two graphs for the remaining values of  $x$ ? Does the screen display suggest an identity?

In the following examples, we will use trigonometric identities to simplify expressions and to prove other identities.

**Example 1** Simplify  $\sec x - \sin x \tan x$ .

**Solution** Express each function in terms of  $\sin x$  and  $\cos x$ .

$$\begin{aligned}\sec x - \sin x \tan x &= \frac{1}{\cos x} - \sin x \cdot \frac{\sin x}{\cos x} \\ &= \frac{1 - \sin^2 x}{\cos x} \\ &= \frac{\cos^2 x}{\cos x} \\ &= \cos x\end{aligned}$$

**Example 2** Prove:  $\frac{\cot A(1 + \tan^2 A)}{\tan A} = \csc^2 A$

**Solution**

Use identities to simplify the expression on either the left or right side of the equals sign. Usually, it's a good idea to choose the more complicated expression. Here, start simplifying the left side by replacing the expression  $1 + \tan^2 A$  with  $\sec^2 A$ , as shown on the next page.

$$\begin{aligned}
\frac{\cot A (1 + \tan^2 A)}{\tan A} &= \frac{\cot A \cdot \sec^2 A}{\tan A} \quad \leftarrow 1 + \tan^2 A = \sec^2 A \\
&= \cot^2 A \cdot \sec^2 A \quad \leftarrow \frac{1}{\tan A} = \cot A \\
&= \frac{\cos^2 A}{\sin^2 A} \cdot \frac{1}{\cos^2 A} \quad \leftarrow \cot A = \frac{\cos A}{\sin A} \text{ and} \\
&= \frac{1}{\sin^2 A} \quad \sec A = \frac{1}{\cos A} \\
&= \csc^2 A
\end{aligned}$$

Since the result gives the expression on the other side of the equals sign, the given equation is an identity.

Sometimes when simplifying an expression, you may not see which identity applies. If this happens, try expressing the functions involved in terms of sine and cosine only. Usually this method takes longer, but it can be effective if all else fails.

## CLASS EXERCISES

- Since  $\sin^2 \theta + \cos^2 \theta = 1$ : a.  $1 - \sin^2 \theta = \underline{\quad ? \quad}$  b.  $1 - \cos^2 \theta = \underline{\quad ? \quad}$
- Since  $1 + \tan^2 \theta = \sec^2 \theta$ : a.  $\sec^2 \theta - 1 = \underline{\quad ? \quad}$  b.  $\sec^2 \theta - \tan^2 \theta = \underline{\quad ? \quad}$
- Since  $1 + \cot^2 \theta = \csc^2 \theta$ : a.  $\csc^2 \theta - 1 = \underline{\quad ? \quad}$  b.  $\csc^2 \theta - \cot^2 \theta = \underline{\quad ? \quad}$

Simplify each expression.

- a.  $\tan \theta \cdot \cos \theta$  b.  $\tan (90^\circ - A)$  c.  $\cos \left( \frac{\pi}{2} - x \right)$
- a.  $(1 - \sin x)(1 + \sin x)$  b.  $\sin^2 x - 1$  c.  $(\sec x - 1)(\sec x + 1)$
- a.  $\tan A \cdot \cot A$  b.  $\cot y \cdot \sin y$  c.  $\cot^2 x - \csc^2 x$
- Evaluate each expression.
  - $\sin^2 \frac{5\pi}{6} + \cos^2 \frac{5\pi}{6}$
  - $\sec^2 \pi - \tan^2 \pi$
  - $\csc^2 135^\circ - \cot^2 135^\circ$
- Discussion** Tell how you would simplify each complex fraction. The result of the first fraction given should help you simplify the second fraction.

$$\text{a. } \frac{t + \frac{1}{t}}{t} \quad \frac{\tan A + \frac{1}{\tan A}}{\tan A}$$

$$\text{b. } \frac{\frac{a}{1}}{\frac{1}{a}} - \frac{\frac{b}{1}}{\frac{1}{b}} \quad \frac{\sec \theta}{\cos \theta} - \frac{\tan \theta}{\cot \theta}$$

$$\text{c. } \frac{\frac{y}{x} + \frac{x}{y}}{\frac{1}{xy}} \quad \frac{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}}{\frac{1}{\cos \theta \sin \theta}}$$

$$\text{d. } \frac{1}{x + \frac{y^2}{x}} \quad \frac{1}{\cos \theta + \frac{\sin^2 \theta}{\cos \theta}}$$

## WRITTEN EXERCISES

**Simplify.**

- A**
1. a.  $\cos^2 \theta + \sin^2 \theta$       b.  $(1 - \cos \theta)(1 + \cos \theta)$       c.  $(\sin \theta - 1)(\sin \theta + 1)$
  2. a.  $1 + \tan^2 \theta$       b.  $(\sec x - 1)(\sec x + 1)$       c.  $\tan^2 x - \sec^2 x$
  3. a.  $1 + \cot^2 A$       b.  $(\csc A - 1)(\csc A + 1)$       c.  $\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A}$
  4. a.  $\frac{1}{\cos(90^\circ - \theta)}$       b.  $1 - \frac{\sin^2 \theta}{\tan^2 \theta}$       c.  $\frac{1}{\cos^2 \theta} - \frac{1}{\cot^2 \theta}$
  5. a.  $\cos \theta \cot(90^\circ - \theta)$       b.  $\csc^2 x (1 - \cos^2 x)$       c.  $\cos \theta (\sec \theta - \cos \theta)$
  6. a.  $\cot A \sec A \sin A$       b.  $\cos^2 A (\sec^2 A - 1)$       c.  $\sin \theta (\csc \theta - \sin \theta)$
  7.  $\sin A \tan A + \sin(90^\circ - A)$       8.  $\csc A - \cos A \cot A$
  9.  $(\sec B - \tan B)(\sec B + \tan B)$       10.  $(1 - \cos B)(\csc B + \cot B)$
  11.  $(\csc x - \cot x)(\sec x + 1)$       12.  $(1 - \cos x)(1 + \sec x) \cos x$

**Simplify each expression.**

13.  $\frac{\sin x \cos x}{1 - \cos^2 x}$
14.  $\frac{\tan x + \cot x}{\sec^2 x}$
15.  $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$
16.  $(\sec^2 \theta - 1)(\csc^2 \theta - 1)$
17.  $\frac{\cot^2 \theta}{1 + \csc \theta} + \sin \theta \csc \theta$
18.  $\frac{\tan^2 \theta}{\sec \theta + 1} + 1$
19.  $\cos^3 y + \cos y \sin^2 y$
20.  $\frac{\sec y + \csc y}{1 + \tan y}$
21.  $\frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta}$
22.  $\frac{\sin \theta \cot \theta + \cos \theta}{2 \tan(90^\circ - \theta)}$
23.  $\frac{\sin^4 \theta - \cos^4 \theta}{\sin^2 \theta - \cos^2 \theta}$
24.  $\frac{\sin^2 \theta}{1 + \cos \theta}$  (*Hint: See Exercise 1(b).*)
25. Use the equation  $\sin^2 \theta + \cos^2 \theta = 1$  to prove that  $\tan^2 \theta + 1 = \sec^2 \theta$ .
26. Use the equation  $\sin^2 \theta + \cos^2 \theta = 1$  to prove that  $\cot^2 \theta + 1 = \csc^2 \theta$ .



**In Exercises 27 and 28, use a graphing calculator or a computer to graph the given functions.**

27. Graph the function  $y = \sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x$ . What is the domain of this function? What other function could have this same graph? Use trigonometric relationships to verify the suggested identity.
28. Graph the function  $y = (\sin x \div \cos x) \div \tan x$ . What is the domain of this function? What other function could have this same graph? Use trigonometric relationships to verify the suggested identity.

In Exercises 29–36, prove the given identity.

29.  $\cot^2 \theta + \cos^2 \theta + \sin^2 \theta = \csc^2 \theta$

30.  $\frac{\cot \theta - \tan \theta}{\sin \theta \cos \theta} = \csc^2 \theta - \sec^2 \theta$

31.  $\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta} = \sin \theta \csc \theta$

32.  $\frac{1 - \sin^2 \theta}{1 + \cot^2 \theta} = \sin^2 \theta \cos^2 \theta$

33.  $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

34.  $\frac{\tan^2 x}{1 + \tan^2 x} = \sin^2 x$

35.  $\frac{\sin \theta}{\sin \theta + \cos \theta} = \frac{\tan \theta}{1 + \tan \theta}$

36.  $\frac{\tan x}{1 + \sec x} + \frac{1 + \sec x}{\tan x} = 2 \csc x$

37.  $\overline{AB}$  is tangent to the unit circle at  $B(1, 0)$ .

a. Why is  $\triangle OPQ \sim \triangle OAB$ ?

b. Use part (a) to explain why

$$\frac{PQ}{OQ} = \frac{AB}{OB} \text{ and } \frac{OP}{OQ} = \frac{AO}{BO}.$$

c. Use part (b) to show that  $AB = \tan \theta$  and  $AO = \sec \theta$ .

d. **Visual Thinking** Use the diagram to explain why the name *tangent* is given to the expression  $\frac{\sin x}{\cos x}$  and the

name *secant* is given to the expression  $\frac{1}{\cos x}$ .

e. Use right triangle  $AOB$  to prove that  $\sec^2 \theta = 1 + \tan^2 \theta$ .

f. Extend  $\overline{AO}$  to intersect the circle at  $C$ . A theorem from geometry states that  $(AB)^2 = AP \cdot AC$ . Use this fact to prove  $\tan^2 \theta = (\sec \theta - 1)(\sec \theta + 1)$ .

38.  $\overline{CD}$  is tangent to the unit circle at  $D(0, 1)$ . Show that

$$CD = \cot \theta \text{ and } CO = \csc \theta.$$

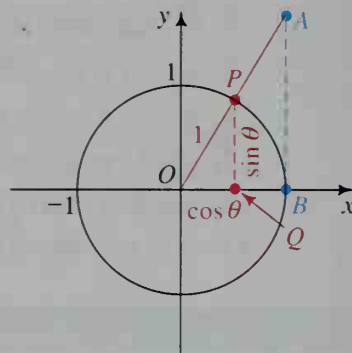
(Hint: See Exercise 37.)

39. **Writing** Jon expected that the graph of  $y = \sqrt{1 + \tan^2 x}$  would be the same as the graph of  $y = \sec x$ , but his graphing calculator showed that this was not the case. Write a paragraph explaining why this happened.

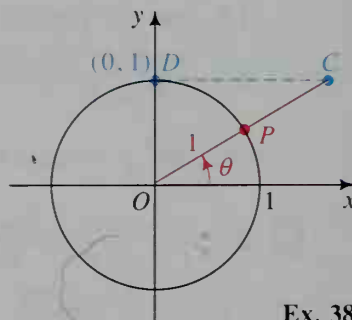
40. Express  $\tan \theta$  in terms of  $\cos \theta$  only.

41. Express  $\sec \theta$  in terms of  $\sin \theta$  only.

**C** 42. Prove  $\sqrt{\frac{1 - \sin x}{1 + \sin x}} = |\sec x - \tan x|$ . For what values of  $x$  is this identity true?



Ex. 37



Ex. 38

## //////COMPUTER EXERCISE

Imagine that your computer can calculate only the sine function. Write a program for which the input is any  $x$ , where  $0 \leq x \leq \frac{\pi}{2}$ , and the outputs are the six trigonometric functions of  $x$ . (Hint: See Exercise 41.)





**Amalie Emmy Noether (1882–1935)**

Emmy Noether was born and educated in Erlangen, Germany. She lectured at the University of Göttingen from 1915 until 1933 when she and other Jewish mathematicians were denied the right to teach. She emigrated to the United States, where she became a visiting professor at Bryn Mawr College and lectured at the Institute of Advanced Study at Princeton.

Noether's contributions centered on invariants and on noncommutative algebras. Her work on invariants culminated in a theorem, known to physicists as Noether's theorem, which is basic to the general theory of relativity. Her impact, however, extended far beyond her own work. Her insight, advice, and encouragement affected the research and publications of many colleagues and students.

## 8-5 Solving More Difficult Trigonometric Equations

### Objective

To use trigonometric identities or technology to solve more difficult trigonometric equations.

Many trigonometric equations can be solved in the same way that algebraic equations are solved.

### Example 1

Solve  $2 \sin^2 \theta - 1 = 0$  for  $0^\circ \leq \theta < 360^\circ$ .

### Solution

First solve for  $\sin^2 \theta$ .

$$2 \sin^2 \theta - 1 = 0$$

$$2 \sin^2 \theta = 1$$

$$\sin^2 \theta = \frac{1}{2}$$

$$\sin \theta = \pm \frac{\sqrt{2}}{2} \quad \leftarrow \sin^2 \theta = \frac{1}{2} \text{ implies that } \sin \theta = \pm \sqrt{\frac{1}{2}}.$$

$$\theta = 45^\circ, 135^\circ, 225^\circ, \text{ and } 315^\circ$$

Notice that there are four solutions because  $\sin \theta$  can be either positive or negative, and so we need to look at all four quadrants.

Compare the solving of the quadratic equation  $x^2 - 3x - 4 = 0$  with that of  $\cos^2 \theta - 3 \cos \theta - 4 = 0$ :

$$\begin{aligned}x^2 - 3x - 4 &= 0 \\(x + 1)(x - 4) &= 0 \\x + 1 = 0 \quad \text{or} \quad x - 4 &= 0 \\x = -1 \quad \text{or} \quad x &= 4\end{aligned}$$

$$\begin{aligned}\cos^2 \theta - 3 \cos \theta - 4 &= 0 \\(\cos \theta + 1)(\cos \theta - 4) &= 0 \\\cos \theta = -1 \quad \text{or} \quad \cos \theta &= 4 \\\theta = 180^\circ + n \cdot 360^\circ\end{aligned}$$

(Since the range of the cosine function is all real numbers between  $-1$  and  $1$ ,  $\cos \theta = 4$  has no solution.)

Some trigonometric equations that are not quadratic can be transformed into equations that have the quadratic form.

**Example 2** Solve  $\sin^2 x - \sin x = \cos^2 x$  for  $0 \leq x < 2\pi$ .

**Solution** To get an equation involving only  $\sin x$ , substitute  $1 - \sin^2 x$  for  $\cos^2 x$ .

$$\begin{aligned}\sin^2 x - \sin x &= \cos^2 x \\\sin^2 x - \sin x &= 1 - \sin^2 x \\2 \sin^2 x - \sin x - 1 &= 0 \\(2 \sin x + 1)(\sin x - 1) &= 0 \\\sin x = -\frac{1}{2} \quad \text{or} \quad \sin x &= 1 \\x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}\end{aligned}$$

**Example 3** Solve  $\sin x \tan x = 3 \sin x$  for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth of a radian.

**Solution**

$$\begin{aligned}\sin x \tan x &= 3 \sin x \\\sin x \tan x - 3 \sin x &= 0 \\\sin x(\tan x - 3) &= 0 \\\sin x = 0 \quad \text{or} \quad \tan x &= 3 \\x \approx 0, 3.14 \quad \quad \quad x &\approx 1.25, 4.39\end{aligned}$$

The solutions are  $0$ ,  $1.25$ ,  $3.14$ , and  $4.39$ .

Notice that in Example 3 we did not divide both sides of the equation by the factor  $\sin x$ . Doing so would have caused us to lose a root. (See the discussion concerning losing roots on page 32.) In the next example, there is no common factor for both sides of the equation. In this case, there is no difficulty in dividing both sides by  $\sin \theta$ , as long as  $\sin \theta \neq 0$ , of course. We do not lose a root since values of  $\theta$  for which  $\sin \theta = 0$  are clearly not solutions to the original equation.

**Example 4** Solve  $2 \sin \theta = \cos \theta$  for  $0^\circ \leq \theta < 360^\circ$ .

**Solution**

$$2 \sin \theta = \cos \theta$$

$$2 = \frac{\cos \theta}{\sin \theta} \quad \leftarrow \text{Divide both sides by } \sin \theta.$$

$$2 = \cot \theta \quad \leftarrow \text{You can use } \tan \theta = \frac{1}{2} \text{ instead.}$$

$$\theta \approx 26.6^\circ, 206.6^\circ$$



The next example shows two strategies for solving a trigonometric equation. One method is to write a simpler equation by using identities and then solve the equation algebraically. Another method is to use a graphing calculator or computer to draw graphs and then zoom in on the intersection points.

**Example 5** Solve  $2 \sin \theta = \cos \theta + 1$  for  $0^\circ \leq \theta < 360^\circ$ .

**Solution**

**Method 1**

Try to rewrite the equation so that there is only one function. You can replace  $\sin \theta$  by  $\pm\sqrt{1 - \cos^2 \theta}$  since  $\sin^2 \theta + \cos^2 \theta = 1$ . Then the equation is in terms of  $\cos \theta$  only.

$$2 \sin \theta = \cos \theta + 1$$

$$2(\pm\sqrt{1 - \cos^2 \theta}) = \cos \theta + 1$$

$$[2(\pm\sqrt{1 - \cos^2 \theta})]^2 = (\cos \theta + 1)^2$$

$$4(1 - \cos^2 \theta) = \cos^2 \theta + 2 \cos \theta + 1$$

$$5 \cos^2 \theta + 2 \cos \theta - 3 = 0$$

$$(5 \cos \theta - 3)(\cos \theta + 1) = 0$$

$$\cos \theta = 0.6 \quad \text{or} \quad \cos \theta = -1$$

$$\theta \approx 53.1^\circ, 306.9^\circ \quad \theta = 180^\circ$$

Since we squared the original equation, it is possible that we may have gained a root. Therefore, we must check each of these solutions in the *original equation*.

$$2 \sin 53.1^\circ \stackrel{?}{=} \cos 53.1^\circ + 1$$

$$1.6 = 1.6$$

$$2 \sin 306.9^\circ \stackrel{?}{=} \cos 306.9^\circ + 1$$

$$-1.6 \neq 1.6$$

$$2 \sin 180^\circ \stackrel{?}{=} \cos 180^\circ + 1$$

$$0 = 0$$

Thus,  $306.9^\circ$  is an extraneous root and must be rejected.

The solutions are  $53.1^\circ$  and  $180^\circ$ .

(Solution continues on the next page.)

## Method 2

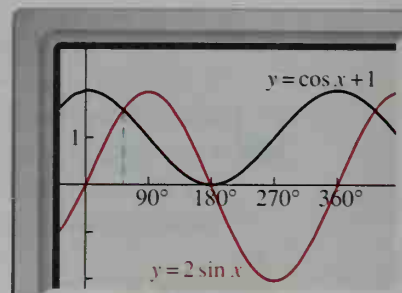
Use a computer or graphing calculator to graph the equations

$$y = 2 \sin x \text{ and } y = \cos x + 1.$$

(If you can't use your computer in the degree mode, use the radian mode and then convert your answers to degrees.) For the graphs, choose the intervals

$$0^\circ \leq x < 360^\circ \text{ and } -2 \leq y \leq 2.$$

The graphs are shown at the right. Notice that for values of  $x$  between  $0^\circ$  and  $360^\circ$ , the graphs intersect in two points. By using a zoom (or trace) feature, you can find that the solutions are about  $53.1^\circ$  and  $180^\circ$ .



## WRITTEN EXERCISES

Solve for  $0^\circ \leq \theta < 360^\circ$ . Give answers to the nearest tenth of a degree.

- |  |  |
|--|--|
| <b>A</b> 1. $\sec^2 \theta = 9$              | 2. $\tan^2 \theta = 1$                   |
| 3. $1 - \csc^2 \theta = -3$                  | 4. $8 \cos^2 \theta - 3 = 1$             |
| 5. $6 \sin^2 \theta - 7 \sin \theta + 2 = 0$ | 6. $2 \tan^2 \theta = 3 \tan \theta - 1$ |
| 7. $6 \sin^2 \theta = 7 - 5 \cos \theta$     | 8. $\cos^2 \theta - 3 \sin \theta = 3$   |

Solve for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth of a radian when necessary.

- |                             |                                |
|-----------------------------|--------------------------------|
| 9. $\cos x \tan x = \cos x$ | 10. $\sec x \sin x = 2 \sin x$ |
| 11. $\sin^2 x = \sin x$     | 12. $\tan^2 x = \tan x$        |
| 13. $2 \cos^2 x = \cos x$   | 14. $3 \sin x = \cos x$        |
| 15. $\sin x + \cos x = 0$   | 16. $\sec x = 2 \csc x$        |



Solve each equation for  $0 \leq x < 2\pi$  algebraically by using identities or graphically by using a graphing calculator or computer. Give answers to the nearest hundredth of a radian when necessary.

- |   |  |
|---|--|
| <b>B</b> 17. $\tan^2 x = 2 \tan x \sin x$ | 18. $2 \sin x \cos x = \tan x$                 |
| 19. $2 \csc^2 x = 3 \cot^2 x - 1$         | 20. $2 \sec^2 x + \tan x = 5$                  |
| 21. $\sin^2 x + \sin x - 1 = 0$           | 22. $\cos^2 x - 2 \cos x - 1 = 0$              |
| 23. $3 \cos x \cot x + 7 = 5 \csc x$      | 24. $2 \sin^3 x - \sin^2 x - 2 \sin x + 1 = 0$ |
| 25. $2 \cos^2 x - \cos x = 2 - \sec x$    | 26. $\csc^2 x - 2 \csc x = 2 - 4 \sin x$       |

27.  $2 \cos x = 2 + \sin x$

28.  $1 - \sin x = 3 \cos x$



Solve each equation for  $0 \leq x < 2\pi$  by using a graphing calculator or computer. Give answers to the nearest hundredth of a radian.

29.  $\cos x = x$

30.  $\tan x = x$

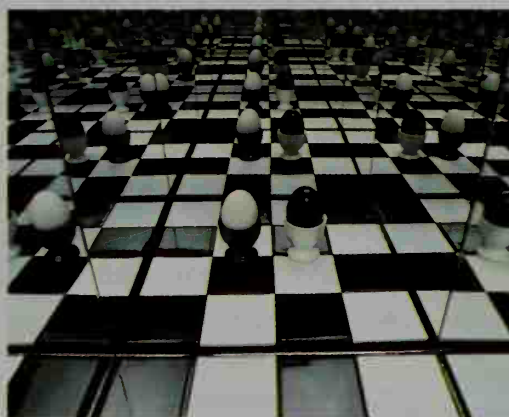
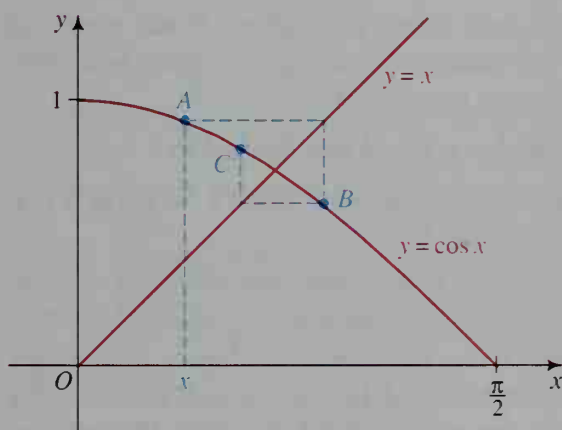
31.  $\sin x = \frac{x}{2}$

32.  $\sin x = 2x$

33.  $\cos x = 2x - 3$

34.  $\sin x = x^2$

- C** 35. **Investigation** Consider the graphs of  $y = \cos x$  and  $y = x$  shown at the left below. Suppose you choose an arbitrary value of  $x$  between 0 and  $\frac{\pi}{2}$ . From this  $x$ -value move directly up to point  $A$  on the cosine curve. To get to point  $B$ , move horizontally to the line  $y = x$  and from there move vertically back to the cosine curve. To get to point  $C$ , move horizontally and vertically again.



A set of mirrors iterates the basic image of two eggs to create a striking photograph.

- The  $x$ -coordinate of point  $A$  is obviously  $x$ . Find the  $x$ -coordinates of points  $B$  and  $C$  in terms of  $x$ .
- If you continue to move from point to point on the cosine curve as described above, what appears to happen?
- Based on your answers to parts (a) and (b), what relationship exists between the  $x$ -coordinates of the points  $A, B, C, \dots$  and the solution of the equation  $\cos x = x$ ?
- Describe a method for approximating the solution of the equation  $\cos x = x$  using a scientific calculator.
- Implement your method of part (d) to find the solution of the equation  $\cos x = x$  to the nearest thousandth of a radian.
- Determine whether the initial choice of  $x$  needs to be limited to the interval from 0 to  $\frac{\pi}{2}$ .



## Chapter Summary

1. Techniques for solving simple *trigonometric equations* are illustrated in Examples 1 and 2 of Section 8-1.
2. If a line has *inclination*  $\alpha$  and *slope*  $m$ , then  $m = \tan \alpha$ .
3. If  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  is the equation of a non-degenerate conic section, one of its axes has *direction angle*  $\alpha$ , where  $\alpha$  is an angle formed by the positive  $x$ -axis and this axis of the conic. Also:

$$\alpha = \frac{\pi}{4} \text{ (if } A = C \text{) and } \tan 2\alpha = \frac{B}{A - C} \text{ (if } A \neq C, 0 < 2\alpha < \pi \text{)}$$

Example 4 on pages 297–298 shows how to graph such a conic.

4. The functions  $y = A \sin Bx$  and  $y = A \cos Bx$  have amplitude  $|A|$  and period  $\frac{2\pi}{B}$ , where  $A \neq 0$  and  $B > 0$ .
5. Sine curves and cosine curves are both called *sine waves*. If the graphs of  $y = A \sin Bx$  and  $y = A \cos Bx$  are translated horizontally  $h$  units and vertically  $k$  units, then the resulting graphs have equations

$$y - k = A \sin B(x - h) \text{ and } y - k = A \cos B(x - h).$$

Information about how to find the translation amounts  $h$  and  $k$  for either a sine curve or a cosine curve is given on page 310.

6. The graphs of the sine waves have many applications, including the description of tides, radio waves, sunrises, musical tones, and the motion of springs.
7. A *trigonometric identity* is an equation that is true for all values of the variable for which both sides of the equation are defined. In proving such identities, the following relationships are helpful: the reciprocal relationships, the negative relationships, the Pythagorean relationships, and the cofunction relationships. These relationships are summarized on pages 317–318.
8. More advanced techniques for solving various trigonometric equations either algebraically or graphically are discussed in Examples 1–5 of Section 8-5.

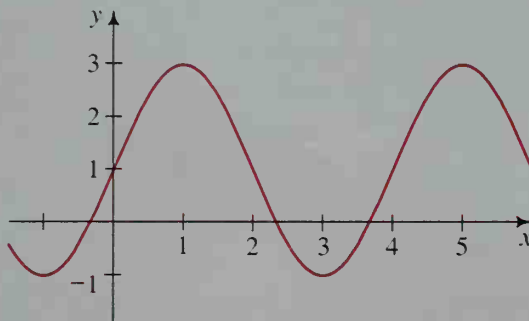
### Key vocabulary and ideas

inclination of a line, direction angle of a conic (pp. 296–297)  
period and amplitude of sine and cosine curves (p. 302)  
equations of general sine waves (p. 309)  
trigonometric identity (p. 318)

## Chapter Test

1. Find the inclination of the line  $x - 3y = 9$  to the nearest degree. **8-1**
2. Solve  $5 \cos \theta = -1$  for  $0^\circ \leq \theta < 360^\circ$ . Give answers to the nearest tenth of a degree.

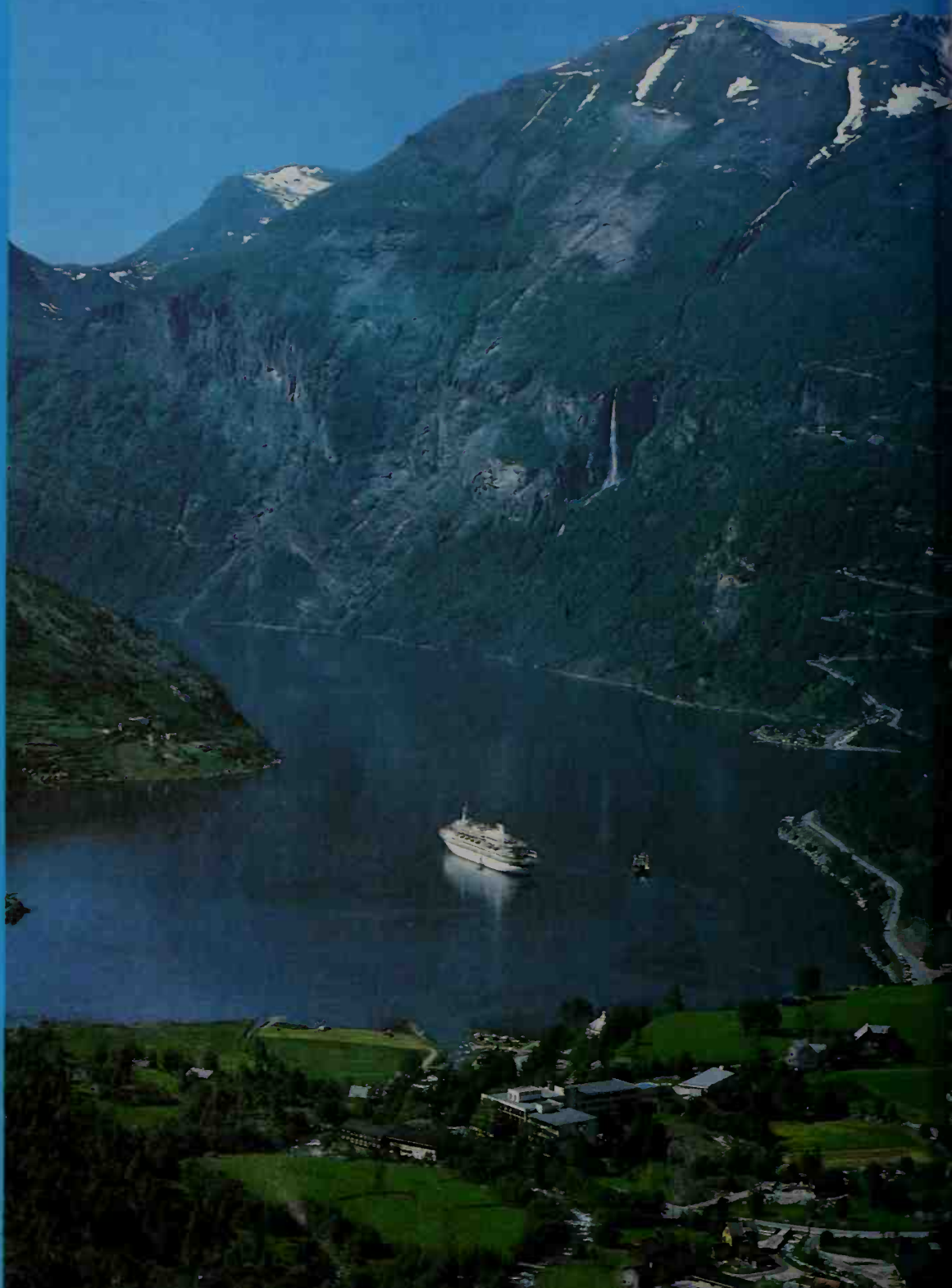
3. Solve  $3 - \csc x = 7$  for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth of a radian.
4. Solve  $2 \sin 3x = \sqrt{2}$  for  $0 \leq x < 2\pi$ . 8-2
5. Graph  $y = 3 \sin \frac{x}{2}$ , and give its period and amplitude.
6. Find an equation for the trigonometric function whose graph is shown below. 8-3



7. At Ocean Tide Dock the first high tide today occurs at 2:00 A.M. with depth 5 m, and the first low tide occurs at 8:30 A.M. with depth 2.2 m.
  - a. Sketch and label a graph showing the depth of the water at the dock as a function of time after midnight.
  - b. **Writing** Write a paragraph or two to describe how you would use the given information about the high and low tides to find a trigonometric equation of this function.
8. Simplify each expression: 8-4
  - a.  $\cot A (\sec A - \cos A)$
  - b.  $\frac{\cot \theta}{\sin (90^\circ - \theta)}$
  - c.  $(\sec x + \tan x) (1 - \sin x)$
  - d.  $\frac{\cot \alpha + \tan \alpha}{\csc^2 \alpha}$
9. Solve each equation for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth of a radian when necessary. 8-5
  - a.  $2 \cos x = \sin x$
  - b.  $\sin x = \csc x$
10. Solve each equation for  $0^\circ \leq \theta < 360^\circ$ . Give answers to the nearest tenth of a degree.
  - a.  $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$
  - b.  $\cos \theta \cot \theta = 2 \cos \theta$

# 9

## *Triangle Trigonometry*



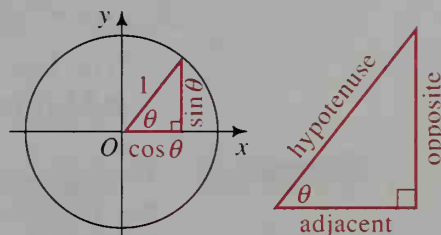
## 9-1 Solving Right Triangles

### Objective

To use trigonometry to find unknown sides or angles of a right triangle.

In Chapter 7, we defined the trigonometric functions in terms of coordinates of points on a circle. In this chapter, our emphasis shifts from circles to triangles. When certain parts (sides and angles) of a triangle are known, you will see that trigonometric relationships can be used to find the unknown parts. This is called **solving a triangle**. For example, if you know the lengths of the sides of a triangle, then you can find the measures of its angles. In this section, we will consider how trigonometry can be applied to right triangles.

The right triangles shown in the diagrams at the right both have an acute angle of measure  $\theta$  and are, therefore, similar. Consequently, the lengths of corresponding sides are proportional, and we have the equations shown below:



$$\frac{\sin \theta}{1} = \frac{\text{opposite}}{\text{hypotenuse}} \quad \frac{\cos \theta}{1} = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opposite}}{\text{adjacent}}$$

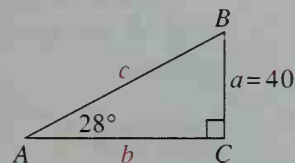
By the reciprocal relationships (see page 282), we also have:

$$\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} \quad \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} \quad \cot \theta = \frac{\text{adjacent}}{\text{opposite}}$$

Applications of these equations are given in the following examples. In Example 1, notice the convention of using a capital letter to denote an angle and the corresponding lower-case letter to denote the length of the side opposite that angle.

### Example 1

For the right triangle  $ABC$  shown at the right, find the value of  $b$  to three significant digits.



### Solution

To find the value of  $b$ , use either  $\tan 28^\circ$  or  $\cot 28^\circ$ .

Using  $\tan 28^\circ$ :

$$\begin{aligned} \tan 28^\circ &= \frac{\text{opposite}}{\text{adjacent}} = \frac{40}{b} \\ b &= \frac{40}{\tan 28^\circ} \approx 75.2 \end{aligned}$$

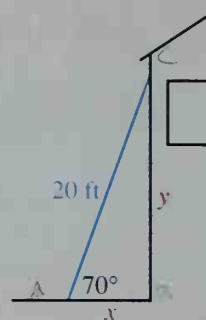
Using  $\cot 28^\circ$ :

$$\begin{aligned} \cot 28^\circ &= \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{40} \\ b &= 40 \cot 28^\circ \approx 75.2 \end{aligned}$$

◀ Triangle trigonometry is especially useful in navigation. The pilot of the ship shown can use it to find the height of the mountains around the ship, to measure the width of the channel ahead, and to plot the safest course along that channel.

**Example 2**

The safety instructions for a 20 ft ladder indicate that the ladder should not be inclined at more than a  $70^\circ$  angle with the ground. Suppose the ladder is leaned against a house at this angle, as shown. Find (a) the distance  $x$  from the base of the house to the foot of the ladder and (b) the height  $y$  reached by the ladder.

**Solution**

$$\text{a. } \cos 70^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{20}$$

$$x = 20 \cos 70^\circ \approx 6.84$$

The foot of the ladder is about 6.84 ft from the base of the house.

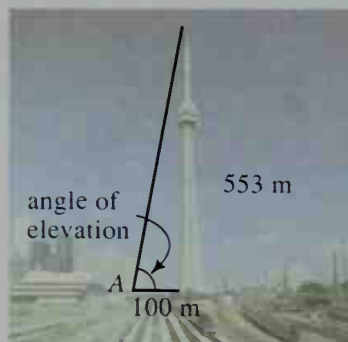
$$\text{b. } \sin 70^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{20}$$

$$y = 20 \sin 70^\circ \approx 18.8$$

The ladder reaches about 18.8 ft above the ground.

**Example 3**

The highest tower in the world is in Toronto, Canada, and is 553 m high. An observer at point A, 100 m from the center of the tower's base, sights the top of the tower. The *angle of elevation* is  $\angle A$ . Find the measure of this angle to the nearest tenth of a degree.

**Solution**

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{553}{100} = 5.53$$

$$\angle A = \tan^{-1} 5.53 \approx 79.7^\circ$$

{  $\tan A$  is an abbreviation for the tangent of  $\angle A$ .

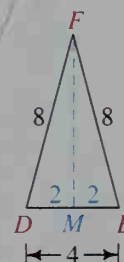
Because we can divide an isosceles triangle into two congruent right triangles, we can apply trigonometry to isosceles triangles, as shown in the following example.

**Example 4**

A triangle has sides of lengths 8, 8, and 4. Find the measures of the angles of the triangle to the nearest tenth of a degree.

**Solution**

By drawing the altitude to the base of isosceles triangle  $DEF$ , we get two congruent right triangles. In  $\triangle DMF$ , we have:



$$\cos D = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{8} = 0.25$$

$$\angle D = \cos^{-1} 0.25 \approx 75.5^\circ$$

Thus,  $\angle E = \angle D \approx 75.5^\circ$ , and  $\angle F \approx 180^\circ - 2(75.5^\circ) = 29.0^\circ$ .



When calculators or trigonometric tables are used, only approximate results are possible in most cases. Throughout this chapter, therefore, you should round angle measures to the nearest tenth of a degree and give lengths accurate to three significant digits.

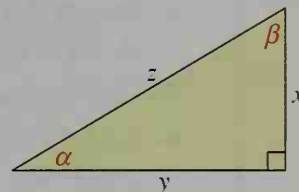
## CLASS EXERCISES

1. Match each element of row A with an element in row B.

A:	sine	cosine	tangent	cotangent	secant	cosecant
B:	$\frac{\text{opposite}}{\text{adjacent}}$	$\frac{\text{opposite}}{\text{hypotenuse}}$	$\frac{\text{adjacent}}{\text{opposite}}$	$\frac{\text{hypotenuse}}{\text{opposite}}$	$\frac{\text{adjacent}}{\text{hypotenuse}}$	$\frac{\text{hypotenuse}}{\text{adjacent}}$

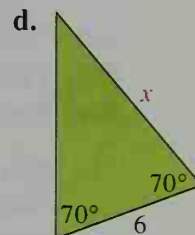
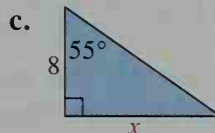
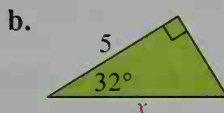
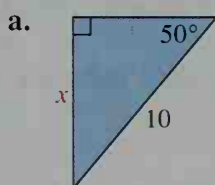
2. Refer to the diagram at the right.

- Express the sine, cosine, and tangent of  $\alpha$  in terms of  $x$ ,  $y$ , and  $z$ .
- Express the sine, cosine, and tangent of  $\beta$  in terms of  $x$ ,  $y$ , and  $z$ .
- Tell whether each of the following statements is true or false.

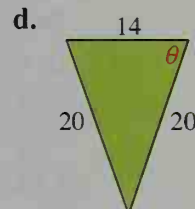
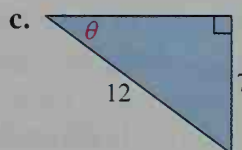
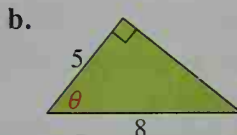
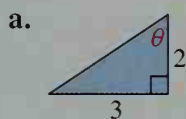


(1)  $\sin \alpha = \cos \beta$       (2)  $\tan \alpha = \cot \beta$       (3)  $\sec \alpha = \csc \beta$

3. **Reading** In the solution of Example 1, suppose a calculator is not available. Would you prefer to find the value of  $b$  using  $\tan 28^\circ$  or  $\cot 28^\circ$ ? Explain.
4. a. For right triangle  $ABC$  in Example 1, use trigonometry to find the value of  $c$  to three significant digits.  
b. Knowing the values of  $a$  and  $b$  in Example 1, how could you find the value of  $c$  without using trigonometry?
5. For each triangle, state two equations (using reciprocal trigonometric functions) that can be used to find the value of  $x$ .



6. For each triangle, state two equations (using reciprocal trigonometric functions) that can be used to find the measure of  $\theta$ .



## WRITTEN EXERCISES

Throughout the exercises, give angle measures to the nearest tenth of a degree and lengths to three significant digits.

**A**

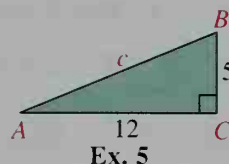
1. In  $\triangle ABC$ ,  $\angle A = 90^\circ$ ,  $\angle B = 25^\circ$ , and  $a = 18$ . Find  $b$  and  $c$ .
2. In  $\triangle PQR$ ,  $\angle P = 90^\circ$ ,  $\angle Q = 64^\circ$ , and  $p = 27$ . Find  $q$  and  $r$ .
3. In  $\triangle DEF$ ,  $\angle D = 90^\circ$ ,  $\angle E = 12^\circ$ , and  $e = 9$ . Find  $d$  and  $f$ .
4. In  $\triangle XYZ$ ,  $\angle X = 90^\circ$ ,  $\angle Y = 37^\circ$ , and  $z = 25$ . Find  $x$  and  $y$ .
5. Use the diagram at the right to find:
 

a. $\sin A$	b. $\cos B$	c. $\tan A$
d. $\cot B$	e. $\sec A$	f. $\csc B$
6. Sketch  $\triangle ABC$  with  $\angle C = 90^\circ$ . What is the relationship between:
 

a. $\sin A$ and $\cos B$ ?	b. $\tan A$ and $\cot B$ ?	c. $\sec A$ and $\csc B$ ?
----------------------------	----------------------------	----------------------------
7. Find the measures of the acute angles of a 3-4-5 right triangle.
8. Find the measures of the acute angles of a right triangle whose legs are 9 cm and 16 cm long.
9. The legs of an isosceles right triangle are 1 unit long.
  - a. Find the length of the hypotenuse in simplest radical form.
  - b. Use part (a) to find the exact value of each of the following.
 

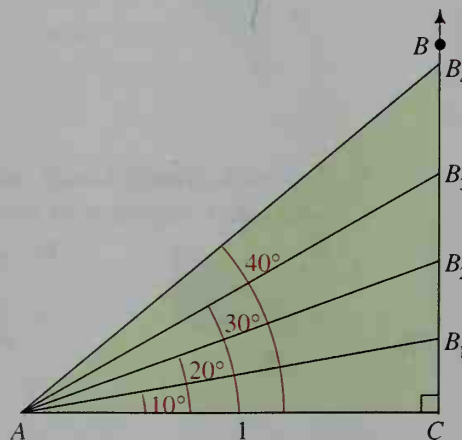
(1) $\tan 45^\circ$	(2) $\sin 45^\circ$	(3) $\cos 45^\circ$
---------------------	---------------------	---------------------
  - c. Convert the answers in part (b) to decimal form. Compare these with the values of  $\tan 45^\circ$ ,  $\sin 45^\circ$ , and  $\cos 45^\circ$  obtained from the calculator.
10. The hypotenuse of a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle is 2 units long.
  - a. Find the lengths of the legs in simplest radical form.
  - b. Use part (a) to find the exact value of each of the following.
 

(1) $\sin 30^\circ$	(2) $\sin 60^\circ$	(3) $\tan 30^\circ$	(4) $\tan 60^\circ$
---------------------	---------------------	---------------------	---------------------
  - c. Convert the answers in part (b) to decimal form. Compare these with the values of  $\sin 30^\circ$ ,  $\sin 60^\circ$ ,  $\tan 30^\circ$ , and  $\tan 60^\circ$  obtained from the calculator.



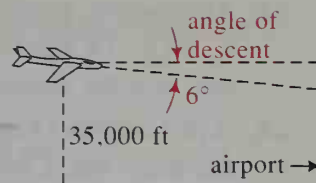
For Exercise 11, you may wish to use a computer with software that draws and measures geometric figures.

11. Draw a horizontal segment  $AC$  of length 1 unit and a vertical ray  $CB$ , as shown. Then draw segments that make angles of  $10^\circ$ ,  $20^\circ$ ,  $\dots$ ,  $80^\circ$  with  $\overline{AC}$ . Measure the lengths  $CB_1$ ,  $CB_2$ ,  $\dots$ ,  $CB_8$ . What is the significance of these lengths?
12. In the diagram that you drew for Exercise 11, note that  $\angle CAB_2$  is twice as large as  $\angle CAB_1$ . Is  $\overline{CB_2}$  twice as long as  $\overline{CB_1}$ ? Is  $\overline{CB_4}$  four times as long as  $\overline{CB_1}$ ?

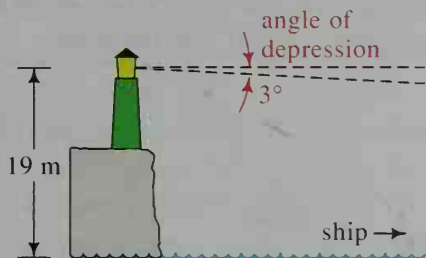


13. **Aviation** An airplane is at an elevation of 35,000 ft when it begins its approach to an airport. Its *angle of descent* is  $6^\circ$ .

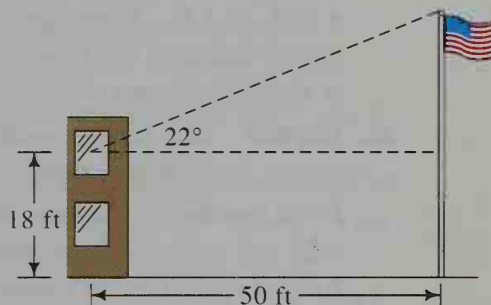
- a. What is the distance between the airport and the point on the ground directly below the airplane?
- b. What is the approximate air distance between the plane and the airport?  
What assumptions did you make in finding this distance?



14. **Navigation** A lighthouse keeper observes that there is a  $3^\circ$  *angle of depression* between the horizontal and the line of sight to a ship. If the keeper is 19 m above the water, how far is the ship from shore?

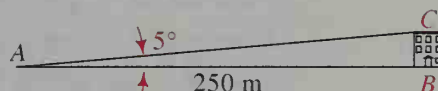


Ex. 14



Ex. 15

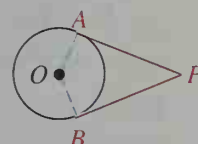
15. A student looks out of a second-story school window and sees the top of the school flagpole at an angle of elevation of  $22^\circ$ . The student is 18 ft above the ground and 50 ft from the flagpole. Find the height of the flagpole.
16. For an observer at point A, 250 m from a building, the angle of elevation of the top of the building is  $5^\circ$ . In Chapter 7, we said that  $\triangle ABC$  is about the same as a sector with central angle A.



- a. Use the arc length formula  $s = r\theta$  to approximate  $BC$ . (Remember to express  $\theta$  in radians.)
- b. Use right-triangle trigonometry to find  $BC$  more accurately. Compare your answers.

17. Find the measures of the angles of an isosceles triangle whose sides are 6, 6, and 8. Also find the area of the triangle.
18. The legs of an isosceles triangle are each 21 cm long and the angle between them has measure  $52^\circ$ . What is the length of the third side?

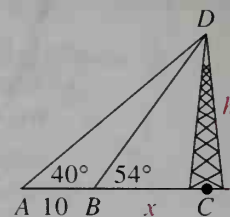
- B** 19. In the figure at the right,  $\overline{PA}$  and  $\overline{PB}$  are tangents to a circle with radius  $OA = 6$ . If the measure of  $\angle APB$  is  $42^\circ$ , find  $PA$  and  $PB$ .



Ex. 19

20. Sketch the circle  $(x - 6)^2 + (y - 8)^2 = 9$  and the two tangents to the circle from the origin. Find the measure of the angle between the tangents.

21. An isosceles trapezoid has sides whose lengths are in the ratio  $5 : 8 : 5 : 14$ . Find the measure of the angle between one of the legs and the shorter base. (*Hint*: Draw an altitude.)
22. A rectangle is 14 cm wide and 48 cm long. Find the measure of the acute angle between its diagonals.
23. From points  $A$  and  $B$ , 10 m apart, the angles of elevation of the top of a tower are  $40^\circ$  and  $54^\circ$ , respectively, as shown at the right. Find the tower's height. (*Hint*: Write two equations in the unknowns  $x$  and  $h$ .)

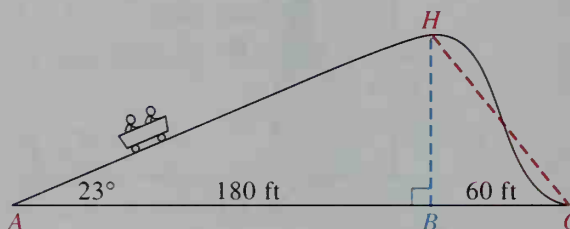


Ex. 23

24. **Navigation** From a ship off-shore, the angle of elevation of a hill is  $1.1^\circ$ . After the ship moves inland at 4.5 knots for 20 min, the angle of elevation is  $1.4^\circ$ . How high is the hill? (1 knot = 1 nautical mile per hour  $\approx$  6080 feet per hour)



25. **Physics** The roller coaster car shown in the diagram takes 23.5 s to go up the  $23^\circ$  incline  $\overline{AH}$  and only 2.8 s to go down the drop from  $H$  to  $C$ . The car covers horizontal distances of 180 ft on the incline and 60 ft on the drop.

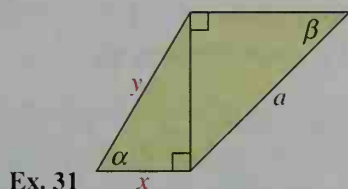
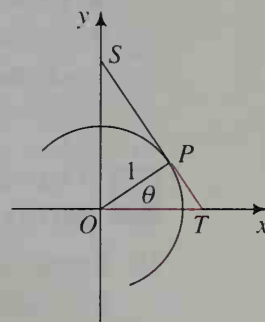


- How high is the roller coaster above point  $B$ ?
  - Find the distances  $AH$  and  $HC$ .
  - How fast (in ft/s) does the car go up the incline?
  - What is the approximate average speed of the car as it goes down the drop? (Assume that the car travels along  $\overline{HC}$ . Since the actual path is longer than  $\overline{HC}$ , is your approximate answer too big or too small?)
26. **Space Science** From a point 1.5 mi from a launch pad at Cape Canaveral, an observer sights a space shuttle at an angle of elevation of  $10^\circ$  moments after it is launched. After 10 s, the angle of elevation is  $50^\circ$ .
- How far does the space shuttle travel vertically during the 10 s interval?
  - What is the average speed of the space shuttle during that time interval?

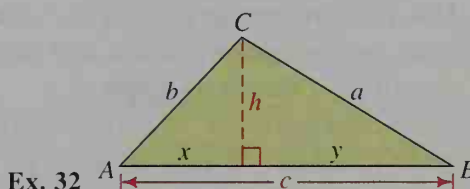


27. a. Draw  $\triangle ABC$  with  $\angle C = 90^\circ$  and  $\sin A = \frac{5}{13}$ . Without using a calculator or table, find  $\sec A$  and  $\csc A$ .
- b. Given that  $\sin A = x$ , find  $\sec A$  in terms of  $x$ .

28. a. Draw  $\triangle PQR$  with  $\angle R = 90^\circ$  and  $\tan P = \frac{1}{4}$ . Without using a calculator or table, find  $\sin P$  and  $\cos Q$ .  
 b. Given that  $\tan P = x$ , find  $\sin P$  in terms of  $x$ .
29. A line tangent to the circle  $x^2 + y^2 = 1$  at  $P$  intersects the axes at  $T$  and  $S$ , as shown in the diagram.
- a. Show that  $PT = \tan \theta$  and  $OT = \sec \theta$ . (This may explain the use of the names *tangent* and *secant*. Another interpretation is given in Exercise 37 on page 322.)  
 b. Show that  $\sec^2 \theta = 1 + \tan^2 \theta$ .  
 c. **Visual Thinking** Describe what happens to point  $T$  as  $\theta$  increases from  $0^\circ$  to  $90^\circ$ , from  $90^\circ$  to  $180^\circ$ , from  $180^\circ$  to  $270^\circ$ , and from  $270^\circ$  to  $360^\circ$ . Your answers should agree with the graphs of  $y = \tan \theta$  and  $y = \sec \theta$ .
30. a. Use the diagram in Exercise 29 to show that  $PS = \cot \theta$  and  $OS = \csc \theta$ .  
 b. Show that  $\csc^2 \theta = 1 + \cot^2 \theta$ .  
 c. **Visual Thinking** Describe what happens to point  $S$  as  $\theta$  increases from  $0^\circ$  to  $90^\circ$ , from  $90^\circ$  to  $180^\circ$ , from  $180^\circ$  to  $270^\circ$ , and from  $270^\circ$  to  $360^\circ$ . Your answers should agree with the graphs of  $y = \cot \theta$  and  $y = \csc \theta$ .
31. Use the diagram at the left below. Express  $x$  and  $y$  in terms of  $\alpha$ ,  $\beta$ , and  $a$ .

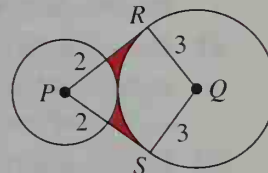
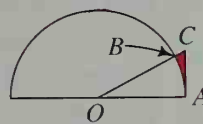


Ex. 31



Ex. 32

32. Use the diagram at the right above. Show that:  
 a.  $h = a \sin B = b \sin A$     b. area of  $\triangle ABC = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$
33. In the diagram at the right,  $\overline{AC}$  is tangent to a semicircle with radius  $OA = 12$  cm. If the measure of  $\angle AOB$  is  $28^\circ$ , find to the nearest square centimeter the area of the region that is inside  $\triangle AOC$  and outside the circle.
34. As shown, two circles with radii 2 and 3 and centers  $P$  and  $Q$ , respectively, are externally tangent. From  $P$ , tangents  $\overline{PR}$  and  $\overline{PS}$  are drawn to the larger circle.
- a. Find the measures of  $\angle RPS$  and  $\angle RQS$  to the nearest hundredth of a radian.  
 b. Find the area of the shaded region to the nearest hundredth of a square unit.
35. Each leg of an isosceles triangle is 8 cm long. Express the measure of the triangle's vertex angle as a function of the length of the triangle's base. Give an appropriate domain and range for this function.
36. The diagonals of a rectangle are 20 cm long. Express the measure of the acute angle formed by the diagonals as a function of the length of the rectangle's shorter sides. Give an appropriate domain and range for this function.



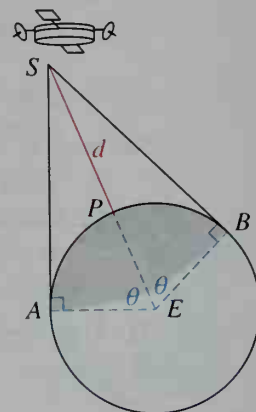


37. **Aviation** An airplane at an elevation of 30,000 ft begins to descend toward the runway on which it will land. Let  $d$  be the horizontal distance between the plane and the runway. Let  $\theta$  be the plane's angle of descent (see Exercise 13).
- Express  $\theta$  as a function of  $d$ .
  - Suppose that the horizontal distance between the airplane and the runway is 60 mi. Find to the nearest tenth the angle at which the airplane must descend in order to land on the runway. (Recall that 1 mi = 5280 ft.)

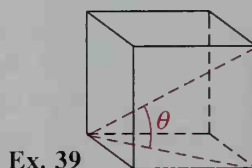


You may wish to use a computer or a graphing calculator to complete part (c) of Exercise 38.

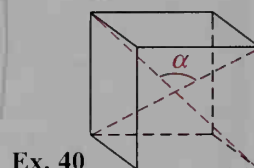
38. **Space Science** A telecommunications satellite  $S$  orbits above the earth. Suppose the shaded region, determined by tangents such as  $\overline{SA}$  and  $\overline{SB}$ , is the portion of the earth's surface that can send and receive signals via the satellite. Assume point  $E$  to be the center of the earth and the radius of the earth to be 4000 mi.
- Arc  $APB$  is an arc of a great circle (see Exercise 46 on page 301). The length of arc  $APB$  is the maximum distance  $a$  between two cities that can communicate via the satellite. Express  $a$  in terms of the central angle  $\theta$ , and express  $\theta$  in terms of the satellite's distance  $d$  above the earth's surface.
  - Use your answers in part (a) to express  $a$  as a function of  $d$ .
  - Give an appropriate domain and range for your function in part (b) and sketch its graph.
  - Many satellites are in *geostationary* orbits about 22,300 mi above the earth's surface. At this distance, the satellite's speed matches that of the earth's rotation, causing the satellite to appear stationary to an observer on the earth. If satellite  $S$  is in a geostationary orbit, can it transmit signals from Lima, Peru, to Cairo, Egypt, which is about 7726 mi away?



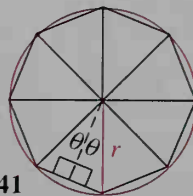
- C** 39. Find the measure of the angle  $\theta$  formed by a diagonal of a cube and a diagonal of one of the faces of the cube.
40. Find the measure of the angle  $\alpha$  formed by two diagonals of a cube.
41. Derive a formula for the area  $A$  of a regular polygon of  $n$  sides inscribed in a circle of radius  $r$ .
42. Derive a formula for the area  $K$  of a regular polygon of  $n$  sides circumscribed about a circle of radius  $r$ .



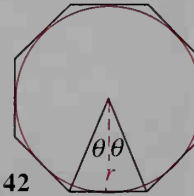
Ex. 39



Ex. 40



Ex. 41



Ex. 42

## //////COMPUTER EXERCISE

Use the answers from Exercises 41 and 42 to write a program that prints the values of  $A$ ,  $K$ , and  $K - A$  for  $r = 1$  and  $n = 10, 20, 30, \dots, 100$ . Interpret the results.

## 9-2 The Area of a Triangle

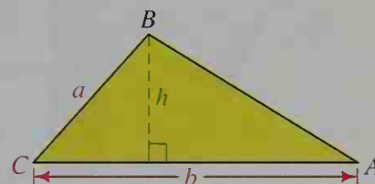
### Objective

To find the area of a triangle given the lengths of two sides and the measure of the included angle.

When the lengths of two sides of a triangle and the measure of the included angle are known, the triangle is uniquely determined. This fact is a consequence of the side-angle-side (SAS) condition for congruence. We now consider the problem of expressing the area of the triangle in terms of these measurements.

Suppose that we are given the lengths  $a$  and  $b$  and the measure of  $\angle C$  in  $\triangle ABC$ . If the length of the altitude from  $B$  is  $h$ , then the area of the triangle is given by

$$K = \frac{1}{2}bh.$$



By right-triangle trigonometry, we know that  $\sin C = \frac{h}{a}$ , or

$$h = a \sin C.$$

Substituting  $a \sin C$  for  $h$ , we find that

$$K = \frac{1}{2}ab \sin C.$$

If some other pair of sides and the included angle of  $\triangle ABC$  were known, we could repeat the procedure for finding the area and thereby obtain two other area formulas. All three formulas are stated below.

### The Area of a Triangle

The area  $K$  of  $\triangle ABC$  is given by:

$$K = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$$

Notice that the area formulas all have the basic pattern:

$$K = \frac{1}{2} \cdot (\text{one side}) \cdot (\text{another side}) \cdot (\text{sine of included angle})$$

### Example 1

Two sides of a triangle have lengths 7 cm and 4 cm. The angle between the sides measures  $73^\circ$ . Find the area of the triangle.

### Solution

$$K = \frac{1}{2} \cdot 7 \cdot 4 \cdot \sin 73^\circ \approx 13.4$$

Thus, the area is about  $13.4 \text{ cm}^2$ .

**Example 2**

The area of  $\triangle PQR$  is 15. If  $p = 5$  and  $q = 10$ , find all possible measures of  $\angle R$ .

**Solution**

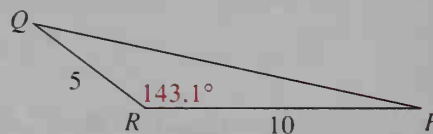
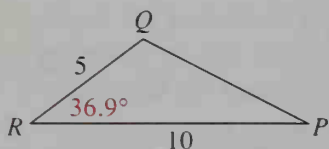
Since  $K = \frac{1}{2}pq \sin R$ ,

$$15 = \frac{1}{2} \cdot 5 \cdot 10 \cdot \sin R = 25 \sin R.$$

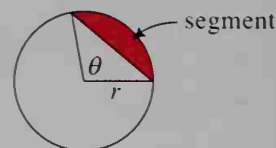
Thus,  $\sin R = \frac{15}{25} = 0.6$ . Since  $\angle R$  could be acute or obtuse,

$$\angle R = \sin^{-1} 0.6 \approx 36.9^\circ \quad \text{or} \quad \angle R \approx 180^\circ - 36.9^\circ = 143.1^\circ$$

The diagrams below show the two possibilities for  $\triangle PQR$ .



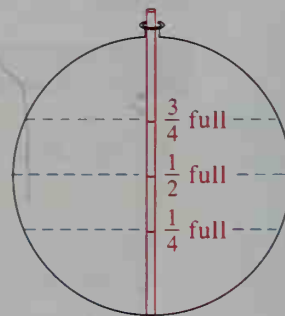
As shown at the right, a **segment** of a circle is the region bounded by an arc of the circle and the chord connecting the endpoints of the arc.



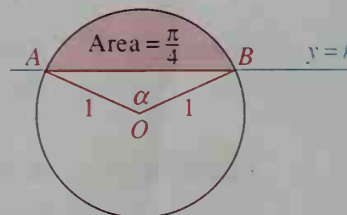
Next we use a computer or graphing calculator to solve an equation based on the area of a segment.

**Example 3**

The diagram shows an end view of a cylindrical oil tank with radius 1 m. Through a hole in the top, a vertical rod is lowered to touch the bottom of the tank. When the rod is removed, the oil level in the tank can be read from the oil mark on the rod. Where on the rod should marks be put to show that the tank is  $\frac{3}{4}$  full,  $\frac{1}{2}$  full, and  $\frac{1}{4}$  full?

**Solution**

Since the radius of the tank is  $r = 1$  m, we can use the unit circle  $x^2 + y^2 = 1$ , shown at the right. To find where to put the " $\frac{3}{4}$  full" mark, we need to find the horizontal line  $y = k$  that cuts off a segment having  $\frac{1}{4}$  of the circle's area, or



$$\frac{1}{4} \pi r^2 = \frac{1}{4} (\pi \cdot 1^2) = \frac{\pi}{4}.$$

Area of segment = area of sector  $AOB$  - area of  $\triangle AOB$

$$\frac{\pi}{4} = \frac{1}{2} \cdot 1^2 \cdot \alpha - \frac{1}{2} \cdot 1^2 \cdot \sin \alpha$$

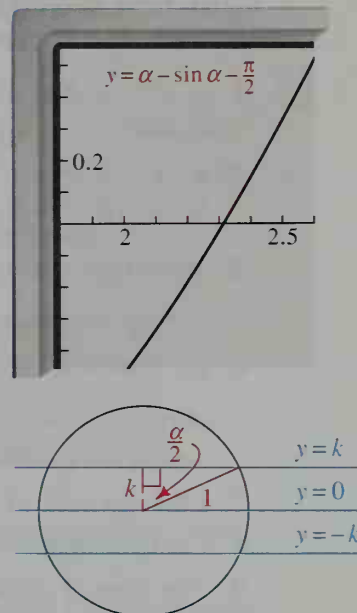
$$\frac{\pi}{4} = \frac{1}{2}(\alpha - \sin \alpha)$$

$$\frac{\pi}{2} = \alpha - \sin \alpha$$

$$0 = \alpha - \sin \alpha - \frac{\pi}{2}$$

To solve the last equation, we graph  $y = \alpha - \sin \alpha - \frac{\pi}{2}$  using a computer or graphing calculator. From the graph at the right, we see that  $y = 0$  when  $\alpha \approx 2.3$ . Using this value of  $\alpha$  and the diagram at the right, we have:

$$\begin{aligned} k &= \cos \frac{\alpha}{2} \\ &\approx \cos \frac{2.3}{2} \approx 0.4 \end{aligned}$$

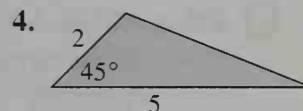
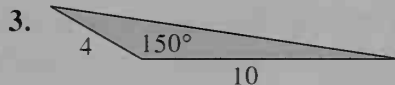
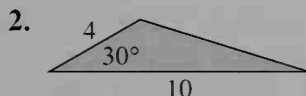


From the symmetry of the circle we deduce that the horizontal lines dividing the circle into four parts with equal areas have equations  $y \approx 0.4$ ,  $y = 0$ , and  $y \approx -0.4$ . Thus, if one end of a vertical rod is at the bottom of the tank, then marks should be placed 0.6 m, 1 m, and 1.4 m from that end to indicate when the tank is  $\frac{1}{4}$  full,  $\frac{1}{2}$  full, and  $\frac{3}{4}$  full, respectively.

## CLASS EXERCISES

- Two adjacent sides of a triangle have lengths 5 cm and 8 cm.
  - If these sides form a  $30^\circ$  angle, what is the area of the triangle?
  - If these sides form a  $150^\circ$  angle, what is the area of the triangle?

Find the area of each triangle.



- A triangle with area  $5 \text{ cm}^2$  has two sides of lengths 4 cm and 5 cm.
  - Find the sine of the angle included between these sides.
  - Find the two possible measures of the included angle.
- Find the area of a segment of a circle with radius 2 if the measure of the central angle of the segment is  $\frac{\pi}{6}$ .

## WRITTEN EXERCISES

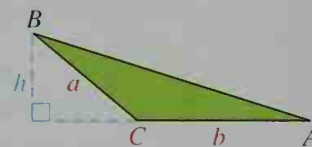
Throughout the exercises, give areas in radical form or to three significant digits. Give lengths to three significant digits and angle measures to the nearest tenth of a degree. In Exercises 1–4, find the area of each  $\triangle ABC$ .

- A**
1. a.  $a = 4, b = 5, \angle C = 30^\circ$   
b.  $a = 4, b = 5, \angle C = 150^\circ$
  2. a.  $b = 3, c = 8, \angle A = 120^\circ$   
b.  $b = 3, c = 8, \angle A = 60^\circ$
  3. a.  $a = 6, c = 2, \angle B = 45^\circ$   
b.  $a = 6, c = 2, \angle B = 135^\circ$
  4. a.  $a = 10, b = 20, \angle C = 70^\circ$   
b.  $a = 10, b = 20, \angle C = 110^\circ$

5. What does the formula  $K = \frac{1}{2}ab \sin C$  become when

$\angle C$  is a right angle? Draw a sketch to illustrate.

6. As shown in the diagram at the right,  $\angle C$  in  $\triangle ABC$  is obtuse. Show that the formula  $K = \frac{1}{2}ab \sin C$  gives the area  $K$  of  $\triangle ABC$ .



Ex. 6

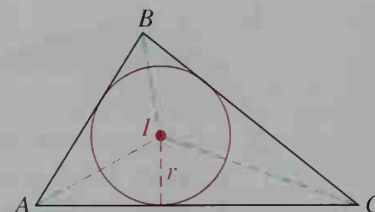
7. Find the area of  $\triangle XYZ$  if  $x = 16, y = 25$ , and  $\angle Z = 52^\circ$ .
8. Find the area of  $\triangle RST$  if  $\angle S = 125^\circ, r = 6$ , and  $t = 15$ .
9. The area of  $\triangle ABC$  is 15. If  $a = 12$  and  $b = 5$ , find the measure(s) of  $\angle C$ .
10. The area of  $\triangle PQR$  is 9. If  $q = 4$  and  $r = 9$ , find the measure(s) of  $\angle P$ .
11. Find the area of a regular octagon inscribed in a circle of radius 40 cm.
12. Find the area of a regular 12-sided polygon inscribed in a circle of radius 8 cm.
13. Adjacent sides of a parallelogram have lengths 6 cm and 7 cm, and the measure of the included angle is  $30^\circ$ . Find the area of the parallelogram.
14. Sketch a parallelogram with sides of lengths  $a$  and  $b$  and with an acute angle  $\theta$ . Express the area of the parallelogram in terms of  $a, b$ , and  $\theta$ .
15. Suppose a triangle has two sides of lengths 3 cm and 4 cm and an included angle  $\theta$ . Express the area of the triangle as a function of  $\theta$ . State the domain and range of the function and sketch its graph.
16. Suppose a triangle has two sides of lengths  $a$  and  $b$ . If the angle between these sides varies, what is the maximum possible area that the triangle can attain? What can you say about the minimum possible area?

- B**
17. a. Given  $\triangle ABC$  with an inscribed circle as shown at the right, show that the radius  $r$  of the circle is given by:

$$r = \frac{2(\text{area of } \triangle ABC)}{\text{perimeter of } \triangle ABC}$$

(Hint: If  $I$  is the center of the inscribed circle, then: area of  $\triangle ABI$  + area of  $\triangle ACI$  + area of  $\triangle BCI$  = area of  $\triangle ABC$ .)

- b. Find the radius of the inscribed circle if  $AB = AC = 10$  and  $BC = 16$ .







Part (b) of Exercise 18 requires the use of a computer or graphing calculator. Give your answer to the nearest tenth.

18. In isosceles triangle  $ABC$ ,  $AB = BC = 10$  and  $AC = 2x$ .

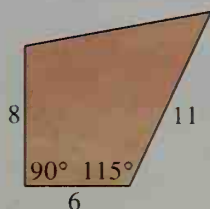
a. Use Exercise 17 to show that the radius of the inscribed circle is:

$$r = \frac{x\sqrt{100 - x^2}}{10 + x}$$

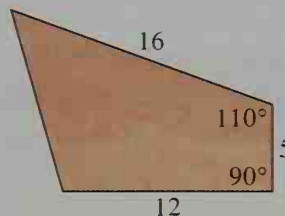
b. Use a computer or graphing calculator to find the value of  $x$  that maximizes  $r$ . Interpret your answer.

Find the area of each quadrilateral to the nearest square unit.

19.



20.



21. Find the area of a segment of a circle of radius 5 if the measure of the central angle of the segment is 2 radians.
22. Find the area of the segment formed by a chord 24 cm long in a circle of radius 13 cm.



Exercises 23 and 24 require the use of a computer or graphing calculator. Give answers to the nearest tenth.

23. In a circle of radius 10, there is a segment with area 95. Use a computer or graphing calculator to find the measure of the central angle of the segment.
24. For the cylindrical oil tank described in Example 3 on pages 340 and 341, use a computer or graphing calculator to determine where on a measuring rod marks should be put to show that the tank is  $\frac{1}{3}$  full and  $\frac{2}{3}$  full.
25. **Visual Thinking** In Example 3 on pages 340 and 341, suppose the cylindrical oil tank sits upright on one of its circular ends (with the opening for the measuring rod at the other end). Describe how the problem of measuring the amount of oil in the tank changes.
26. **Research** Find out and report on what the typical shape of a car's fuel tank is and how the amount of fuel in the tank is measured and indicated on the car's fuel gauge. Is the measuring instrument designed to give a truly accurate measurement of fuel? If not, why not?

Graph the region satisfying both inequalities and find its area.

27.  $x^2 + y^2 \leq 36$ ,  $y \geq 3$

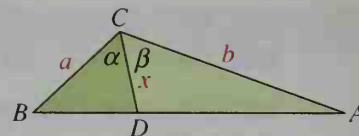
28.  $x^2 + y^2 \leq 9$ ,  $x \geq 1$

**C** 29.  $x^2 + y^2 \leq 9$ ,  $x^2 + y^2 - 10x + 9 \leq 0$

30.  $x^2 + y^2 - 8y \leq 0$ ,  $x^2 + y^2 \leq 16$

31. In the diagram at the right,  $\alpha = 60^\circ$  and  $\beta = 60^\circ$ .

- Express the areas of  $\triangle BCD$  and  $\triangle ACD$  in terms of  $a$ ,  $b$ , and  $x$ .
- Express the area of  $\triangle ABC$  in terms of  $a$  and  $b$ .
- Show that  $x = \frac{ab}{a+b}$ .
- Does part (c) agree with what you know from geometry if  $a = b$ ? Explain.



Exercises 32–36 require the use of a computer or graphing calculator. Give answers to the nearest hundredth.

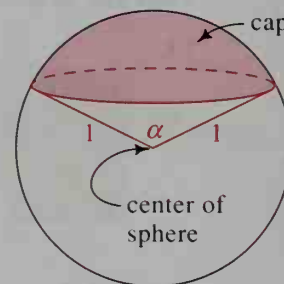
32. **Engineering** Suppose an oil tank is a sphere with radius 1 unit, as shown at the right.

- Find the exact volume of a sphere with radius 1.
- If the volume of the spherical cap formed by an angle  $\alpha$  (as shown in the diagram) is given by

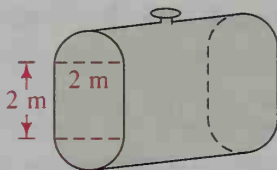
$$\frac{2\pi}{3} - \pi \cos \frac{\alpha}{2} + \frac{\pi}{3} \cos^3 \frac{\alpha}{2},$$

use a computer or graphing calculator to find the value of  $\alpha$  for which the spherical tank is  $\frac{3}{4}$  full.

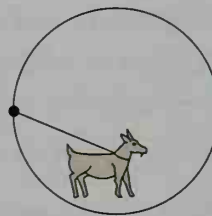
- Where on a measuring rod should marks be placed to show that the tank is  $\frac{3}{4}$  full,  $\frac{1}{2}$  full, and  $\frac{1}{4}$  full?



33. **Engineering** A fuel tank has a cross section whose shape is a  $2 \text{ m} \times 2 \text{ m}$  square capped at the top and bottom by semicircles, as shown at the left below. Use a computer or graphing calculator to determine how to mark a measuring rod to show that the tank is only 10% full.



Ex. 33

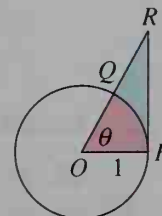


Ex. 34

34. A goat is tethered to a stake at the edge of a circular field with radius 1 unit, as shown at the right above. Use a computer or graphing calculator to determine how long the rope should be so that the goat can graze over half the field.

35.  $\overline{RP}$  is a tangent to a circle with radius 1, as shown at the right. If the area of the region shaded blue equals the area of the region shaded red, use a computer or graphing calculator to find  $\theta$  in radians.

36. Use the diagram for Exercise 35 and a computer or graphing calculator to find the value of  $\theta$  for which  $\overline{QR}$  and arc  $QP$  have the same length.



Exs. 35, 36

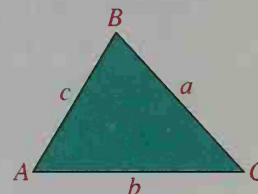
## 9-3 The Law of Sines

**Objective** To use the law of sines to find unknown parts of a triangle.

When there are several methods for solving a problem, a comparison of the solutions can lead to new and useful results. From Section 9-2, we know three ways to find the area  $K$  of  $\triangle ABC$ , depending on which pair of sides is known.

$$K = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

If each of these expressions is divided by  $\frac{1}{2}abc$ , we obtain the *law of sines*.



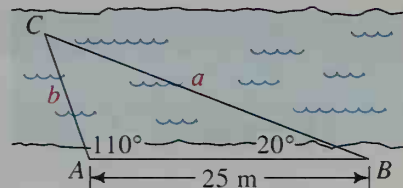
### The Law of Sines

$$\text{In } \triangle ABC, \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

If we know two angles and a side of a triangle, then we can use the law of sines to find the other sides, as shown in the following example.

#### Example 1

A civil engineer wants to determine the distances from points  $A$  and  $B$  to an inaccessible point  $C$ , as shown. From direct measurement the engineer knows that  $AB = 25$  m,  $\angle A = 110^\circ$ , and  $\angle B = 20^\circ$ . Find  $AC$  and  $BC$ .



#### Solution

First find the measure of  $\angle C$ :

$$\angle C = 180^\circ - (110^\circ + 20^\circ) = 50^\circ$$

Then, by the law of sines:

$$\frac{\sin 110^\circ}{a} = \frac{\sin 20^\circ}{b} = \frac{\sin 50^\circ}{25}$$

Therefore:

$$a = \frac{25 \sin 110^\circ}{\sin 50^\circ} \approx 30.7 \quad \text{and} \quad b = \frac{25 \sin 20^\circ}{\sin 50^\circ} \approx 11.2$$

Thus,  $BC \approx 30.7$  m and  $AC \approx 11.2$  m.

In the activity at the top of the next page, you are given the lengths of two sides of a triangle and the measure of a nonincluded angle. You are then asked to investigate whether it is possible to construct the triangle and, if so, whether the triangle is unique.

## Activity 1

For this activity, use a ruler, compass, and protractor. Draw  $\angle A$  with measure  $30^\circ$ . Along one ray of  $\angle A$ , locate point  $C$  10 cm from point  $A$ . For each of the following compass settings, draw a large arc. Then tell whether the arc crosses the other ray of  $\angle A$  and, if so, in how many points.

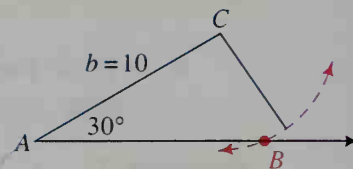
- Compass at  $C$  and opened to 4 cm
- Compass at  $C$  and opened to 5 cm
- Compass at  $C$  and opened to 6 cm

Activity 1 shows that when you are given the lengths of two sides of a triangle and the measure of a nonincluded angle (SSA), it may be possible to construct no triangle, one triangle, or two triangles. For this reason the SSA situation is called the *ambiguous case*.

## Activity 2

Show that your answers to Activity 1 agree with what the law of sines would give in each of the following SSA situations.

- If  $\angle A = 30^\circ$ ,  $b = 10$ , and  $a = 4$ , find  $\angle B$ .
- If  $\angle A = 30^\circ$ ,  $b = 10$ , and  $a = 5$ , find  $\angle B$ .
- If  $\angle A = 30^\circ$ ,  $b = 10$ , and  $a = 6$ , find  $\angle B$ .



## Example 2

In  $\triangle RST$ ,  $\angle S = 126^\circ$ ,  $s = 12$ , and  $t = 7$ . Determine whether  $\angle T$  exists. If so, find all possible measures of  $\angle T$ .

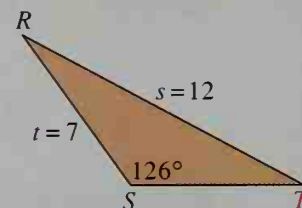
## Solution

Make a sketch, as shown. By the law of sines:

$$\frac{\sin T}{7} = \frac{\sin 126^\circ}{12}$$

$$\sin T = \frac{7 \sin 126^\circ}{12} \approx 0.4719$$

$$\angle T \approx 28.2^\circ \quad \text{or} \quad \angle T \approx 151.8^\circ$$



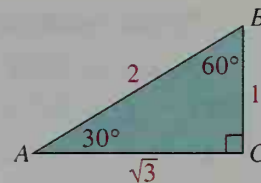
It seems that  $\angle T$  exists and that there are two possible measures of  $\angle T$ . A triangle cannot have two obtuse angles, however, so we must reject  $\angle T \approx 151.8^\circ$ . Thus,  $\angle T \approx 28.2^\circ$ .

## CLASS EXERCISES

- Reading** Rephrase the law of sines in words rather than symbols:

In any triangle,   ?   is constant.

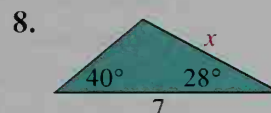
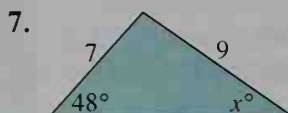
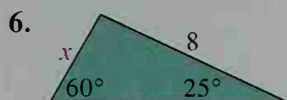
2. The diagram at the right shows a  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle with a hypotenuse of length 2. Find the values of the ratios  $\frac{\sin A}{a}$ ,  $\frac{\sin B}{b}$ , and  $\frac{\sin C}{c}$ . Are the ratios equal?



In Exercises 3–5, consider  $\triangle ABC$ .

3. If  $\angle A \geq 90^\circ$ , what can you conclude about the measure of  $\angle B$ ? Explain.
4. If  $\angle B$  has a greater measure than  $\angle C$ , what must be true of  $b$  and  $c$ ? Why?
5. If  $a > b$ , what must be true of  $\angle A$  and  $\angle B$ ? Why?

In Exercises 6–8, state an equation that you can use to solve for  $x$ .

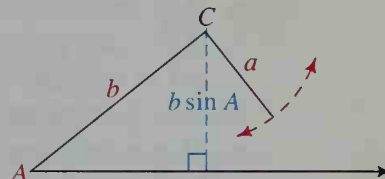


9. If  $a = 8$  and  $b = 6$ , draw a diagram to show that  $\triangle ABC$  is uniquely determined for each of the given measures of  $\angle A$ : a.  $45^\circ$  b.  $90^\circ$  c.  $120^\circ$
10. Use the law of sines to show that  $\triangle ABC$  in part (a) of Exercise 9 is unique.
11. Use the law of sines to show that there is no  $\triangle XYZ$  with  $\angle X = 30^\circ$ ,  $x = 3$ , and  $y = 8$ .

## WRITTEN EXERCISES

**A**

1. The purpose of this exercise is to determine whether  $\triangle ABC$  can be constructed when two lengths  $a$  and  $b$  and the measure of an acute angle  $A$  are given. As shown at the right, first  $\angle A$  and then the side of length  $b$  are constructed; finally a circular arc is drawn with center  $C$  and radius  $a$ . Given that the distance from  $C$  to the side opposite  $C$  is  $b \sin A$ , determine for each of the following conditions whether 0, 1, or 2 triangles can be formed.
  - a.  $a < b \sin A$
  - b.  $a = b \sin A$
  - c.  $b \sin A < a < b$
  - d.  $a \geq b$
2. Use Exercise 1 or the law of sines to determine whether there are 0, 1, or 2 triangles possible for each of the following sets of measurements.
  - a.  $a = 2$ ,  $b = 4$ ,  $\angle A = 22^\circ$
  - b.  $b = 3$ ,  $c = 6$ ,  $\angle B = 30^\circ$
  - c.  $a = 7$ ,  $c = 5$ ,  $\angle A = 68^\circ$
  - d.  $b = 4$ ,  $c = 3$ ,  $\angle C = 76^\circ$



In Exercises 3–14, solve each  $\triangle ABC$ . Give angle measures to the nearest tenth of a degree and lengths in simplest radical form or to three significant digits. Be alert to problems with no solution or two solutions.

3.  $\angle A = 45^\circ$ ,  $\angle B = 60^\circ$ ,  $a = 14$
4.  $\angle B = 30^\circ$ ,  $\angle C = 45^\circ$ ,  $b = 9$
5.  $\angle B = 30^\circ$ ,  $\angle A = 135^\circ$ ,  $b = 4$
6.  $\angle A = 60^\circ$ ,  $\angle B = 75^\circ$ ,  $c = 10$



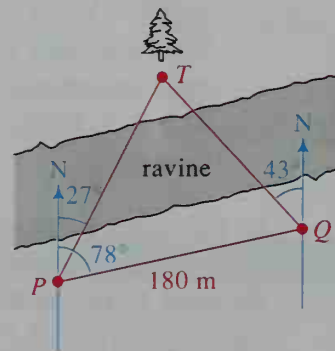
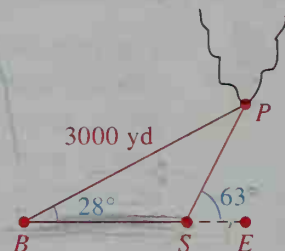
7.  $\angle C = 25^\circ$ ,  $b = 3$ ,  $c = 2$   
 9.  $\angle A = 76^\circ$ ,  $a = 12$ ,  $b = 4$   
 11.  $\angle C = 88^\circ$ ,  $b = 7$ ,  $c = 7$   
 13.  $\angle B = 40^\circ$ ,  $a = 12$ ,  $b = 6$   
 8.  $\angle B = 36^\circ$ ,  $a = 10$ ,  $b = 8$   
 10.  $\angle B = 130^\circ$ ,  $b = 15$ ,  $c = 11$   
 12.  $\angle A = 95^\circ$ ,  $a = 13$ ,  $c = 10$   
 14.  $\angle C = 112^\circ$ ,  $c = 5$ ,  $a = 7$

15. In  $\triangle RST$ ,  $\angle R = 140^\circ$  and  $s = \frac{3}{4}r$ . Find the measures of  $\angle S$  and  $\angle T$ .  
 16. In  $\triangle DEF$ ,  $\angle F = 120^\circ$  and  $f = \frac{4}{3}e$ . Find the measures of  $\angle D$  and  $\angle E$ .  
 17. A fire tower at point  $A$  is 30 km north of a fire tower at point  $B$ . A fire at point  $F$  is observed from both towers. If  $\angle FAB = 54^\circ$  and  $\angle ABF = 31^\circ$ , find  $AF$ .  
 18. From lighthouses  $P$  and  $Q$ , 16 km apart, a disabled ship  $S$  is sighted. If  $\angle SPQ = 44^\circ$  and  $\angle SQP = 66^\circ$ , find the distance from  $S$  to the nearer lighthouse.



19. In  $\triangle ABC$ ,  $\tan A = \frac{3}{4}$ ,  $\tan B = 1$ , and  $a = 10$ . Find  $b$  in simplest radical form.  
 20. In  $\triangle ABC$ ,  $\cos A = \frac{1}{2}$ ,  $\cos B = -\frac{1}{4}$ , and  $a = 6$ . Find  $b$  in simplest radical form.

- B** 21. **Navigation** A ship passes by buoy  $B$  which is known to be 3000 yd from peninsula  $P$ . The ship is steaming east along line  $BE$  and  $\angle PBE$  is measured as  $28^\circ$ . After 10 min, the ship is at  $S$  and  $\angle PSE$  is measured as  $63^\circ$ .  
 a. How far from the peninsula is the ship when it is at  $S$ ?  
 b. If the ship continues east, what is the closest it will get to the peninsula?  
 c. How fast (in yd/min) is the ship traveling?  
 d. Ship speeds are often given in knots, where 1 knot = 1 nautical mile per hour  $\approx$  6080 feet per hour. Convert your answer in part (c) to knots.  
 22. **Surveying** From points  $P$  and  $Q$ , 180 m apart, a tree at  $T$  is sighted on the opposite side of a deep ravine. From point  $P$ , a compass indicates that the angle between the north-south line and line of sight  $\overline{PT}$  is  $27^\circ$  and that the angle between the north-south line and  $\overline{PQ}$  is  $78^\circ$ . From point  $Q$ , the angle between the north-south line and  $\overline{QT}$  is  $43^\circ$ .  
 a. How far from  $P$  is the tree?  
 b. How far from  $P$  is the point on  $\overline{PQ}$  that is closest to the tree?



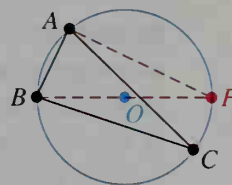
23. The purpose of this exercise is to show that in  $\triangle ABC$  the three equal ratios  $\frac{a}{\sin A}$ ,  $\frac{b}{\sin B}$ , and  $\frac{c}{\sin C}$  are each equal to the diameter of the circumscribed circle of  $\triangle ABC$ .

a. Let the circle with center  $O$  be the circumscribed circle and let  $\overline{BP}$  be the diameter through  $B$ . Show that  $\angle P$  and  $\angle C$  have the same measure.

b. Show that  $\frac{AB}{BP} = \sin P$ .

c. Use parts (a) and (b) to show that

$$\text{diameter} = \frac{c}{\sin C} = \frac{b}{\sin B} = \frac{a}{\sin A}.$$



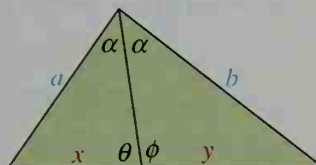
24. A triangle with angles  $50^\circ$ ,  $60^\circ$ , and  $70^\circ$  has all three vertices on a circle of radius 8 cm. Find the lengths of the three sides. (*Hint:* See Exercise 23.)

25. a. Prove that the area of  $\triangle ABC$  is given by  $K = \frac{1}{2} \left( \frac{\sin B \sin C}{\sin A} \right) a^2$ .

b. State two other formulas for  $K$ , one involving  $b$  and one involving  $c$ .

26. Use the results of Exercises 23 and 25 to show that the ratio of the area of  $\triangle ABC$  to the area of its circumscribed circle is  $\frac{2}{\pi} \sin A \sin B \sin C$ .

27. The purpose of this exercise is to use the law of sines to prove that an angle bisector in a triangle divides the opposite side in the ratio of the two adjacent sides: that is,  $\frac{x}{y} = \frac{a}{b}$  in the diagram at the right.



a. Show that  $\sin \theta = \sin \phi$ . ( $\phi$  is the Greek letter *phi*.)

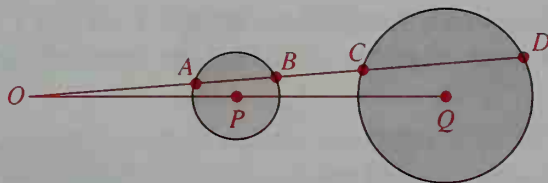
b. Show that  $\frac{a}{x} = \frac{\sin \theta}{\sin \alpha}$  and  $\frac{b}{y} = \frac{\sin \phi}{\sin \alpha}$ .

c. Prove the stated theorem.

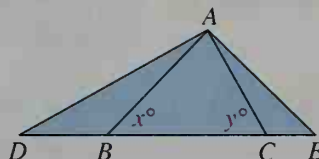
- C** 28. In the diagram at the left below, the two circles with centers  $P$  and  $Q$  have radii 1 and 2, respectively.  $P$  is the midpoint of  $\overline{OQ}$ . An arbitrary line through  $O$  intersects the circles at  $A$ ,  $B$ ,  $C$ , and  $D$ .

a. Use the law of sines to prove that  $A$  is the midpoint of  $\overline{OC}$ .

b. What special property do you think  $B$  has?



Ex. 28



Ex. 29

29. In the diagram at the right above,  $\angle DAC = \angle BAE = 90^\circ$ . Prove that  $DE = BC \tan x \tan y$ .

## 9-4 The Law of Cosines

**Objective** To use the law of cosines to find unknown parts of a triangle.

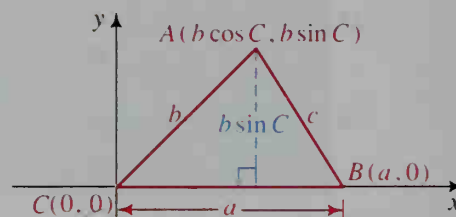
In Section 9-2 we mentioned that by the SAS condition for congruence, a triangle is uniquely determined if the lengths of two sides and the measure of the included angle are known. By the side-side-side (SSS) condition for congruence, a triangle is also uniquely determined if the lengths of three sides are known. The *law of cosines* can be used to solve a triangle in either of these two cases.

### The Law of Cosines

$$\text{In } \triangle ABC, c^2 = a^2 + b^2 - 2ab \cos C.$$

To derive this law, we introduce a coordinate system by placing the  $x$ -axis along  $\overline{BC}$  of  $\triangle ABC$  and the  $y$ -axis at  $C$  so that  $C = (0, 0)$ ,  $B = (a, 0)$ , and  $A = (b \cos C, b \sin C)$ , as shown. Using the distance formula, we have:

$$\begin{aligned} c^2 &= (AB)^2 = (b \cos C - a)^2 + (b \sin C - 0)^2 \\ &= b^2 \cos^2 C - 2ab \cos C + a^2 + b^2 \sin^2 C \\ &= b^2(\cos^2 C + \sin^2 C) - 2ab \cos C + a^2 \\ &= b^2 \cdot 1 - 2ab \cos C + a^2 \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$



To help you remember the law of cosines, notice that it has the basic pattern:

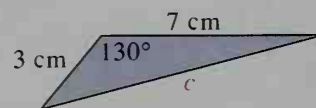
$$\left( \begin{array}{c} \text{side} \\ \text{opposite} \\ \text{angle} \end{array} \right)^2 = \left( \begin{array}{c} \text{side} \\ \text{adjacent} \\ \text{to angle} \end{array} \right)^2 + \left( \begin{array}{c} \text{other side} \\ \text{adjacent} \\ \text{to angle} \end{array} \right)^2 - 2 \left( \begin{array}{c} \text{one} \\ \text{adjacent} \\ \text{side} \end{array} \right) \left( \begin{array}{c} \text{other} \\ \text{adjacent} \\ \text{side} \end{array} \right) \cos (\text{angle})$$

Note that when  $\angle C = 90^\circ$  the law of cosines reduces to  $a^2 + b^2 = c^2$ . Therefore the law of cosines includes the Pythagorean theorem as a special case and is, consequently, more flexible and useful than the Pythagorean theorem. When  $\angle C$  is acute,  $c^2$  is less than  $a^2 + b^2$  by the amount  $2ab \cos C$ ; when  $\angle C$  is obtuse,  $\cos C$  is negative and so  $c^2$  is greater than  $a^2 + b^2$ .

**Example 1** Suppose that two sides of a triangle have lengths 3 cm and 7 cm and that the angle between them measures  $130^\circ$ . Find the length of the third side.

**Solution** Make a sketch, as shown.

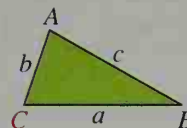
$$\begin{aligned} c^2 &= 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cdot \cos 130^\circ \\ &\approx 85.0 \end{aligned}$$



Thus, the length of the third side is about  $\sqrt{85.0}$  cm, or about 9.22 cm.

If we solve the law of cosines for  $\cos C$ , we obtain:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



In this form, the law of cosines can be used to find the measures of the angles of a triangle when the lengths of the three sides are known. The basic pattern for this form of the law of cosines is:

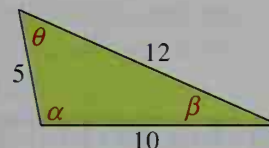
$$\cos(\text{angle}) = \frac{(\text{adjacent})^2 + (\text{adjacent})^2 - (\text{opposite})^2}{2 \cdot (\text{adjacent}) \cdot (\text{adjacent})}$$

### Example 2

The lengths of the sides of a triangle are 5, 10, and 12. Solve the triangle.

### Solution

Make a sketch of the triangle, as shown.



$$1. \cos \alpha = \frac{5^2 + 10^2 - 12^2}{2 \cdot 5 \cdot 10} = -0.19$$

$$\alpha \approx 101.0^\circ$$

$$2. \cos \beta = \frac{12^2 + 10^2 - 5^2}{2 \cdot 12 \cdot 10} = 0.9125$$

$$\beta \approx 24.1^\circ$$

$$3. \theta \approx 180^\circ - (101.0^\circ + 24.1^\circ) = 54.9^\circ$$

Using the law of cosines, we can easily identify acute and obtuse angles. For instance, in Example 2, since  $\cos \alpha$  is negative and  $\cos \beta$  is positive, we know that  $\alpha$  is an obtuse angle and  $\beta$  is an acute angle. The law of sines does not distinguish between acute and obtuse angles, however, because both types of angle have positive sine values.

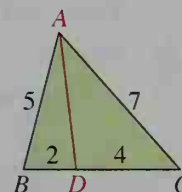
### Example 3

In the diagram at the right,  $AB = 5$ ,  $BD = 2$ ,  $DC = 4$ , and  $CA = 7$ . Find  $AD$ .

### Solution

First we apply the law of cosines to  $\triangle ABC$ :

$$\cos B = \frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6} = 0.2$$



Then we apply the law of cosines to  $\triangle ABD$ :

$$(AD)^2 = 5^2 + 2^2 - 2 \cdot 5 \cdot 2 \cdot 0.2 = 25$$

$$\text{Thus, } AD = \sqrt{25} = 5.$$

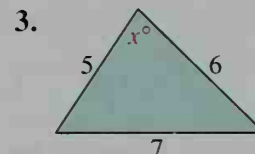
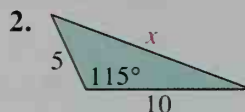
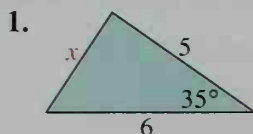
In this section and the preceding section, we have seen various applications of the laws of sines and cosines. The situations in which these laws can be used are summarized at the top of the next page. The summary assumes that once the measures of two angles of a triangle are known, the measure of the third angle is found using the geometric fact that the sum of the measures of the angles is  $180^\circ$ .

Given:	Use:	To find:
SAS	law of cosines	the third side and then one of the remaining angles.
SSS	law of cosines	any two angles.
ASA or AAS	law of sines	the remaining sides.
SSA	law of sines	an angle opposite a given side and then the third side. (Note that 0, 1, or 2 triangles are possible.)

When you use the law of sines, remember that every acute angle and its supplement have the same sine value. Class Exercises 4–7 show how you can tell which angle is correct for a given triangle.

## CLASS EXERCISES

In Exercises 1–3, state an equation that you can use to solve for  $x$ .



In Exercises 4–7, consider  $\triangle XYZ$ , where  $x = 4$ ,  $y = 8$ , and  $\angle Z = 50^\circ$ .

- Use the law of cosines to find  $z$  to the nearest hundredth.
- Use the law of sines to find the measure of  $\angle Y$  to the nearest tenth of a degree. Then find the measure of  $\angle X$ .
- Since  $x < z < y$ , what can you say about the measures of  $\angle X$ ,  $\angle Y$ , and  $\angle Z$ ?
- Do your answers to Exercises 5 and 6 agree? If not, find your error.

## WRITTEN EXERCISES

Solve each triangle. Give lengths to three significant digits and angle measures to the nearest tenth of a degree.

**A**

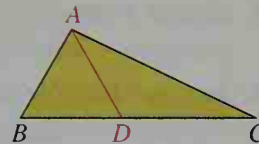
- $a = 8$ ,  $b = 5$ ,  $\angle C = 60^\circ$
- $t = 16$ ,  $s = 14$ ,  $\angle R = 120^\circ$
- $x = 9$ ,  $y = 40$ ,  $z = 41$
- $a = 6$ ,  $b = 10$ ,  $c = 7$
- $p = 3$ ,  $q = 8$ ,  $\angle R = 50^\circ$
- $d = 5$ ,  $e = 9$ ,  $\angle F = 115^\circ$
- $a = 8$ ,  $b = 7$ ,  $c = 13$
- $x = 10$ ,  $y = 11$ ,  $z = 12$



In Exercises 9 and 10, use the method of Example 3, page 351, to find  $AD$  in the diagram at the right.

9.  $AB = 8$ ,  $BD = 7$ ,  $DC = 5$ ,  $AC = 10$

10.  $AB = 5$ ,  $BD = 5$ ,  $DC = 3$ ,  $AC = 7$



In Exercises 11 and 12, find the length of the median from  $A$  in the given  $\triangle ABC$ . Give your answers in simplest radical form.

11.  $a = 8$ ,  $b = 4$ ,  $c = 6$

12.  $a = 12$ ,  $b = 13$ ,  $c = 5$

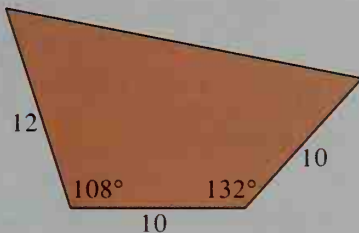
13. A parallelogram has a  $70^\circ$  angle and sides 6 cm and 10 cm long. How long are its diagonals?

14. An isosceles trapezoid has a height of 4 cm and bases 3 cm and 7 cm long. How long are its diagonals?

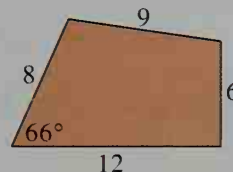
Find the area of each quadrilateral to the nearest square unit.

**B**

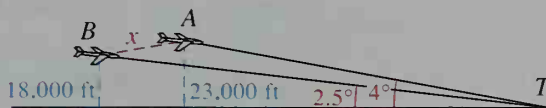
15.



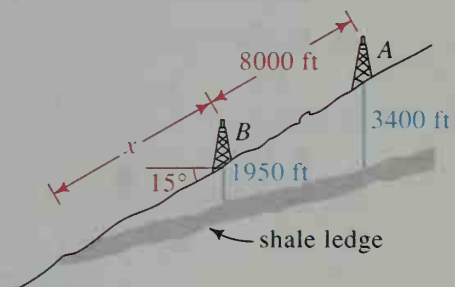
16.



17. **Aviation** Two airplanes, at points  $A$  and  $B$  in the diagram at the left below, have elevations of 23,000 ft and 18,000 ft, respectively. Both are flying east toward an airport control tower at  $T$ . From  $T$ , the angle of elevation of the airplane at  $A$  is  $4^\circ$ , and the angle of elevation of the airplane at  $B$  is  $2.5^\circ$ . How far apart (in mi) are the airplanes? (5280 ft = 1 mi)



Ex. 17



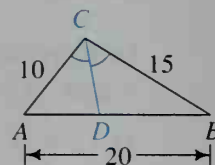
Ex. 18

18. **Geology** In the diagram at the right above, an oil well at  $A$  was drilled 3400 ft before it hit a ledge of shale. The same ledge was only 1950 ft deep when drilled from a well at  $B$ , which is 8000 ft directly downhill from  $A$ . The hill is inclined at  $15^\circ$  to the horizontal.

- If you assume that the ledge lies in a plane, how far down the hill from  $B$  would you expect the shale ledge to emerge?
- What is the angle between the ledge and the hill?

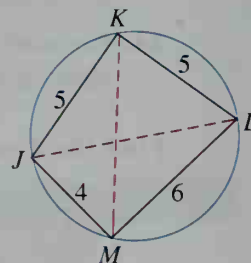
19. **Visual Thinking** Give a geometric interpretation of the law of cosines for the “triangle” with two sides  $a$  and  $b$  and “included angle”  $C = 0^\circ$ .
20. **Visual Thinking** Give a geometric interpretation of the law of cosines for the “triangle” with two sides  $a$  and  $b$  and “included angle”  $C = 180^\circ$ .

21. In  $\triangle ABC$ ,  $AB = 20$ ,  $BC = 15$ ,  $AC = 10$ , and  $\overline{CD}$  bisects  $\angle ACB$ , as shown at the right.



- Use the results of Exercise 27, page 349, to find  $AD$  and  $DB$ .
- Use the method of Example 3, page 351, to find  $CD$ .

22. Quadrilateral  $JKLM$  is inscribed in a circle as shown. Find the length of each diagonal. (*Hint: Opposite angles of an inscribed quadrilateral are supplementary. Express  $\cos L$  in terms of  $\cos J$  in order to find  $KM$ .*)



Ex. 22

- C** 23. a. Prove that in  $\triangle ABC$  the length  $x$  of the median from  $C$  is given by:

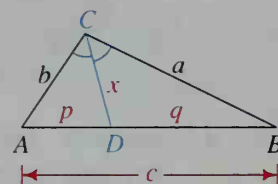
$$x = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

- What happens to this formula when  $\angle C = 90^\circ$ ?
  - State the theorem from geometry that part (b) justifies.
24. In the diagram at the right,  $\overline{CD}$  bisects  $\angle ACB$ .

- a. Use the method of part (a) of Exercise 21 to show:

$$p = \frac{bc}{a+b} \text{ and } q = \frac{ac}{a+b}$$

- Prove that  $x^2 = ab - pq$ .
- Use part (b) to find  $CD$  in Exercise 21.



25. A parallelogram has two adjacent sides of lengths  $a$  and  $b$  and diagonals of lengths  $x$  and  $y$ . Show that  $x^2 + y^2 = 2a^2 + 2b^2$ .
26. In this exercise, you are to derive *Hero's formula* for the area  $K$  of  $\triangle ABC$ :

$$K = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{a+b+c}{2}$$

- a. In the partial derivation of Hero's formula shown below, use the law of cosines to show that the second expression in blue is equal to the first.

$$\begin{aligned} \sin^2 C &= 1 - \cos^2 C \\ &= (1 + \cos C)(1 - \cos C) \\ &= \frac{(a+b)^2 - c^2}{2ab} \cdot \frac{c^2 - (a-b)^2}{2ab} \\ &= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2b^2} \\ &= \frac{(2s)(2s-2c)(2s-2b)(2s-2a)}{4a^2b^2} \end{aligned}$$

- b. Use the formula  $K = \frac{1}{2}ab \sin C$  to complete the derivation in part (a).

## ////COMPUTER EXERCISE

Write a program that accepts the input of the measures of any three parts of a triangle and then prints the measures of the remaining three parts. You might want to work in a group of five students and write a structured program that begins with a main routine like the one below (written in BASIC).

```
10 PRINT ''INDICATE THE PROBLEM TYPE.''
20 INPUT ''(ENTER SSS, SAS, ASA, AAS, OR SSA): '' ; T$
30 IF T$ = ''SSS'' THEN GOSUB 100
40 IF T$ = ''SAS'' THEN GOSUB 200
50 IF T$ = ''ASA'' THEN GOSUB 300
60 IF T$ = ''AAS'' THEN GOSUB 400
70 IF T$ = ''SSA'' THEN GOSUB 500
80 END
```

Each member of the group can write one of the subroutines called by the main routine. Since many programming languages use radians instead of degrees, you may need to convert between the two types of angle measurement. Also, since many programming languages do not have inverse sine and inverse cosine as built-in functions but do have inverse tangent as a built-in function, you may need to use the following relationships, which are valid for  $|x| < 1$ :

$$\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}} \quad \text{and} \quad \cos^{-1} x = \frac{\pi}{2} - \tan^{-1} \frac{x}{\sqrt{1-x^2}}$$

## MIXED TRIGONOMETRY EXERCISES

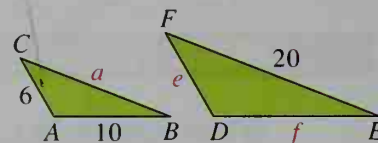
Where appropriate, give angle measures to the nearest tenth of a degree and lengths of sides in simplest radical form or to three significant digits.

- A**
1. Find the area of  $\triangle PQR$  if  $q = 6$ ,  $r = 7$ , and  $\angle P = 50^\circ$ . Also find  $p$ .
  2. Find the area of  $\triangle PQR$  if  $p = 7$ ,  $q = 10$ , and  $\angle R = 130^\circ$ . Also find  $r$ .
  3. Find the measure of the largest angle in a triangle with sides having lengths  $3\sqrt{6}$ ,  $6\sqrt{3}$ , and  $9\sqrt{2}$ .
  4. Find the measure of the smallest angle in a triangle with sides having lengths 7, 12, and 13.
  5. In  $\triangle RST$ ,  $\angle R = 75^\circ$ ,  $\angle S = 45^\circ$ , and  $t = 3$ . Find  $r$  and  $s$ .
  6. In  $\triangle RST$ ,  $\angle S = 100^\circ$ ,  $\angle T = 30^\circ$ , and  $r = 8$ . Find  $s$  and  $t$ .
  7. Three measurements in  $\triangle ABC$  are given as  $\angle A = 60^\circ$ ,  $a = 4$ , and  $b = 5$ . Show that at least one of the measurements is incorrect.
  8. A regular polygon with 180 sides is inscribed in a circle with radius 1. Find its area. Compare your answer with  $\pi$ .
  9. In  $\triangle ABC$ ,  $\cos A = -0.6$ . Find  $\sin A$  and  $\tan A$ .
  10. In  $\triangle ABC$ ,  $a = 17$ ,  $b = 10$ , and  $c = 21$ . Find  $\cos A$  and  $\sin A$ .

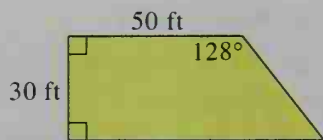
11. **Civil Engineering** A wheelchair ramp must rise 30 in. to meet the front door of a public library. If the ramp's angle of elevation is not to exceed  $8^\circ$ , what is the minimum horizontal length of the ramp?
12. **Navigation** A submarine dives at an angle of  $16^\circ$  with the horizontal. If it takes 4 min to dive from the surface to a depth of 300 ft, how fast does it move along its sloping path downward? Give your answer in feet per minute. Then convert it to nautical miles per hour. (Note: 1 nautical mile per hour  $\approx$  6080 feet per hour)



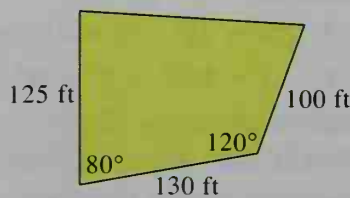
13. In  $\triangle XYZ$ ,  $\angle X = 21.1^\circ$ ,  $x = 6$ , and  $y = 9$ . Find the measure(s) of  $\angle Y$ .
14. In parallelogram  $ABCD$ ,  $\angle A = 60^\circ$ ,  $AB = 5$ , and  $AD = 8$ .  
 a. Find the area of  $ABCD$ .      b. Find the lengths of both diagonals.
15. A triangle has area  $21 \text{ cm}^2$  and two of its sides are 9 cm and 14 cm long. Find the possible measures of the angle formed by these sides.
16. In  $\triangle DEF$ ,  $\angle D = 36^\circ$ ,  $\angle E = 64^\circ$ , and  $f = 8$ . Find  $d$  and  $e$ .
17. In the diagram at the right,  $\triangle ABC$  is similar to  $\triangle DEF$  and  $\angle A = 120^\circ$ .  
 a. Find the lengths  $a$ ,  $e$ , and  $f$ .  
 b. Find the ratio of the areas of the triangles.



18. The diagonals of a parallelogram have lengths 8 and 14 and they meet at a  $60^\circ$  angle. Find the area and the perimeter of the parallelogram.
19. An obtuse triangle with area 12 has two sides of lengths 4 and 10. Find the length of the third side. (There are two answers.)
20. The perimeter of a regular decagon (10 sides) is 240. Find its area.
21. If fencing costs \$2.50 per foot, how much will it cost to buy fencing to go around the plot of land shown at the left below?



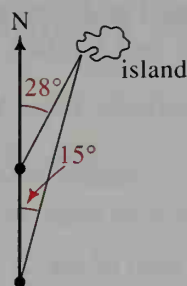
Ex. 21



Ex. 22

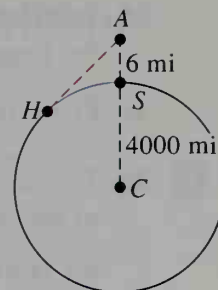
22. In the township of Madison, rural undeveloped land is taxed at a rate of \$115 per acre. Find the tax on the plot of land shown at the right above. (Note: 1 acre =  $43,560 \text{ ft}^2$ )

23. **Navigation** A ship is steaming north at 6 knots (6 nautical miles per hour) when the captain sights a small island at an angle of  $15^\circ$  to the east of the ship's course, as shown at the right. After 10 min, the angle is  $28^\circ$ . How far away is the island at this moment?



Ex. 23

24. **Aviation** An airplane at  $A$  is flying at a height of 6 mi above Earth's surface at  $S$ , as shown at the far right.



Ex. 24

- Find the distance to the nearest tenth of a mile from  $A$  to the horizon  $H$ . (The radius of Earth is about 4000 mi.)
- Find the curved distance to the nearest tenth of a mile from  $S$  along Earth's surface to  $H$ .

25. In  $\triangle ABC$ ,  $\tan A = 1$ ,  $\tan B = \frac{3}{4}$ , and  $b = 18$ . Find  $\sin A$ ,  $\sin B$ , and  $a$ .
26. In  $\triangle DEF$ ,  $\sec F = -\sqrt{2}$ . Find the measure of  $\angle F$  and  $\tan F$ .
27. If  $180^\circ < x < 360^\circ$  and  $\tan x = -\frac{1}{5}$ , find  $\sin x$  and  $\cos x$ .
28. The area of  $\triangle PQR$  is 84. If  $r = 14$  and  $q = 13$ , find  $\sin P$ . Use a trigonometric identity to find two possible values of  $\cos P$  and two possible values of  $p$ .

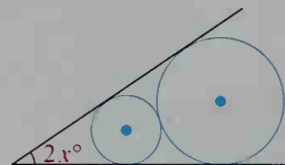
In geometry you can prove two triangles congruent by SSS. This means that when the lengths of the three sides of a triangle are given, its shape is completely determined. Exercises 29 and 30 below illustrate this principle. Exercises 31 and 32 illustrate the principle for SAS and ASA.

- B** 29. In  $\triangle ABC$ ,  $a = 5$ ,  $b = 8$ , and  $c = 7$ . (SSS)
- Solve  $\triangle ABC$ .
  - Find the area of  $\triangle ABC$ . Use the formula  $K = \frac{1}{2}ab \sin C$  or Hero's formula (Exercise 26, page 354).
  - Find the length of the altitude to  $\overline{AC}$ .
  - Find the length of the median from  $B$  to  $\overline{AC}$  (Exercise 23, page 354).
  - Find the length of the angle bisector from  $B$  to  $\overline{AC}$  (Exercise 24, page 354).
  - Find the radius  $R$  of the circumscribed circle (Exercise 23, page 349).
  - Find the radius  $r$  of the inscribed circle (Exercise 17, page 342).
30. Repeat Exercise 29 if  $a = 13$ ,  $b = 14$ , and  $c = 15$ . (SSS)
31. Repeat Exercise 29 if  $a = 9$ ,  $b = 10$ , and  $\cos C = -\frac{3}{5}$ . (SAS)
32. Repeat Exercise 29 if  $\sin A = \frac{8}{17}$ ,  $c = 21$ , and  $\sin B = \frac{\sqrt{2}}{2}$ . (ASA) Assume that  $\angle B$  is obtuse.
33. a. The consecutive sides of a quadrilateral inscribed in a circle have lengths 1, 4, 3, and 2. Find the length of each diagonal, using the fact that opposite angles are supplementary.
- b. Check your answer to part (a) by using Ptolemy's theorem: If  $ABCD$  is inscribed in a circle, then  $AC \cdot BD = AB \cdot DC + AD \cdot BC$ .



34.  $\triangle DEF$  is inscribed in a circle of radius 3. If  $\angle D = 120^\circ$  and  $\angle E = 15^\circ$ , find the lengths  $d$ ,  $e$ , and  $f$ . (Hint: Find the measures of arc  $DE$  and arc  $DF$ .)
35. Prove that the altitude from  $A$  in  $\triangle ABC$  has length  $h = \frac{a}{\cot B + \cot C}$ .

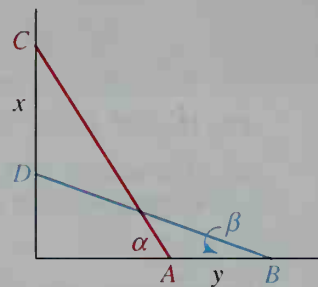
- C** 36. Two circles are externally tangent. Common tangents to the circles form an angle of measure  $2x$ . Prove that the ratio of the radii of the circles is  $\frac{1 - \sin x}{1 + \sin x}$ . (Hint: Express  $\sin x$  in terms of the two radii.)



37. Prove that  $\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4(\text{area of } \triangle ABC)}$  in any  $\triangle ABC$ .

38. Given  $\triangle ABC$  with  $c^2 = \frac{a^3 + b^3 + c^3}{a + b + c}$ , find the measure of  $\angle C$ .

39. As shown at the right, a ladder,  $\overline{AC}$ , leans against the side of a house at an angle  $\alpha$  with the ground. Suppose that the foot of the ladder slides  $y$  units from  $A$  to  $B$ , the top of the ladder slides  $x$  units from  $C$  to  $D$ , and the ladder makes an angle  $\beta$  with the ground. Prove:

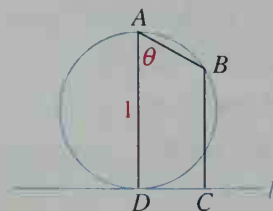


$$x = \frac{y(\sin \alpha - \sin \beta)}{\cos \beta - \cos \alpha}$$

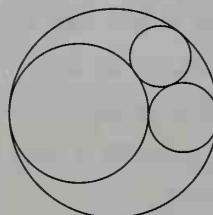
40. A sailor at the seashore watches a ship with a smokestack 30 m above water level as the ship steams out to sea. The sailor's eye level is 4 m above water level. About how far is the ship from shore when the stack disappears from the sailor's view? (The radius of Earth is about 6400 km.)
41. In  $\triangle XYZ$ ,  $\overline{YR}$  and  $\overline{ZS}$  are altitudes. Prove that  $RS = x \cos X$ .
42. a. As shown in the diagram at the left below,  $\overline{AD}$  is a diameter of the circle and is tangent to line  $l$  at  $D$ . If  $AD = 1$  and  $\overline{BC} \perp l$ , show that:

$$AB + BC = 1 + \cos \theta - \cos^2 \theta$$

- b. What value of  $\theta$  makes the sum  $AB + BC$  a maximum?



Ex. 42



Ex. 43

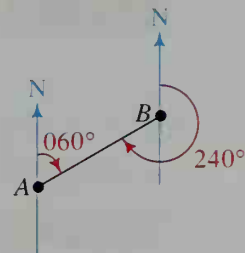
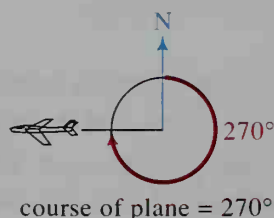
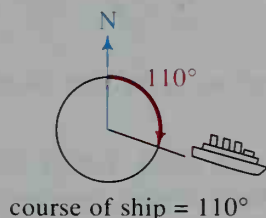
43. Each circle in the diagram at the right above is tangent to the other three circles. The largest three circles have radii of 1, 2, and 3, respectively. Find the radius of the smallest circle.

## 9-5 Applications of Trigonometry to Navigation and Surveying

### Objective

To use trigonometry to solve navigation and surveying problems.

As shown below, the *course* of a ship or plane is the angle, measured clockwise, from the north direction to the direction of the ship or plane.



bearing of B from A = 060°  
bearing of A from B = 240°

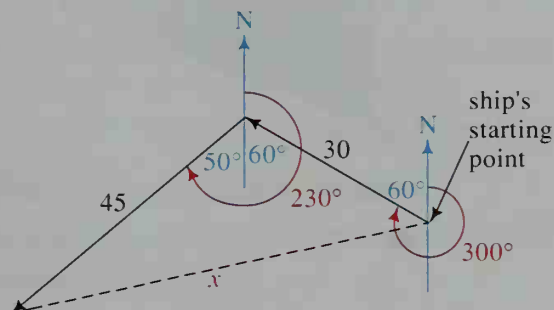
As shown at the right, the *compass bearing* of one location from another is measured in the same way. Note that compass bearings and courses are given with three digits, such as 060° rather than 60°.

### Example 1

A ship proceeds on a course of 300° for 2 hours at a speed of 15 knots (1 knot = 1 nautical mile per hour). Then it changes course to 230°, continuing at 15 knots for 3 more hours. At that time, how far is the ship from its starting point?

### Solution

Make a diagram, as shown below.

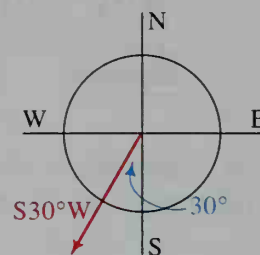
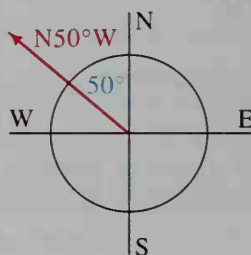
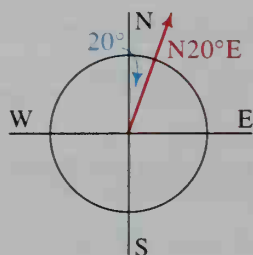


The ship travels first along a path of length  $2 \cdot 15 = 30$  nautical miles and then along a path of length  $3 \cdot 15 = 45$  nautical miles. The angle between the two paths is  $110^\circ$ . (You can find this angle by drawing north-south lines and using geometry.) To find  $x$ , the distance of the ship from its starting point, use the law of cosines:

$$x^2 = 30^2 + 45^2 - 2 \cdot 30 \cdot 45 \cdot \cos 110^\circ \approx 3848$$

Thus,  $x \approx \sqrt{3848} \approx 62.0$  nautical miles.

In surveying, a compass reading is usually given as an acute angle from the north-south line toward the east or west. A few examples are shown below.



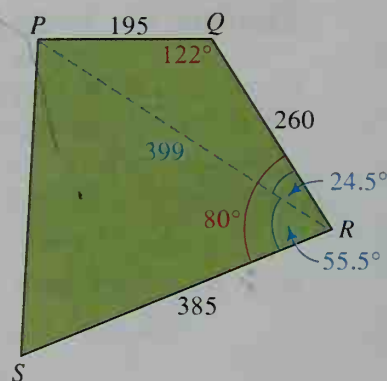
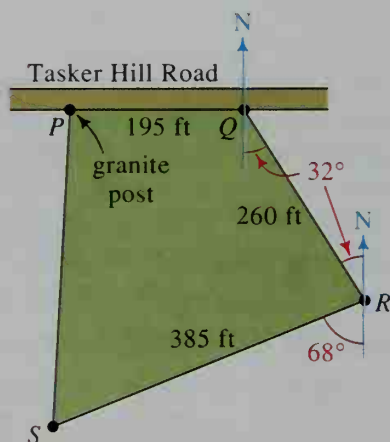
## Example 2

Very often a plot of land is taxed according to its area. Sketch the plot of land described. Then find its area.

From a granite post, proceed 195 ft east along Tasker Hill Road, then along a bearing of S32°E for 260 ft, then along a bearing of S68°W for 385 ft, and finally along a line back to the granite post.

## Solution

We first sketch the plot of land, one side at a time and in the sequence described. (See the diagram at the left below.)



From the bearings given, we deduce that:

$$\angle PQR = 90^\circ + 32^\circ = 122^\circ$$

$$\angle QRS = 180^\circ - (32^\circ + 68^\circ) = 80^\circ$$

To find the area of  $PQRS$ , we divide the quadrilateral into two triangles by introducing  $\overline{PR}$ . (See the diagram at the right above.) We can find the area of  $\triangle PQR$  directly:

$$\begin{aligned} \text{Area of } \triangle PQR &= \frac{1}{2} \cdot PQ \cdot QR \cdot \sin Q \\ &= \frac{1}{2} \cdot 195 \cdot 260 \cdot \sin 122^\circ \approx 21,500 \text{ ft}^2 \end{aligned}$$

To find the area of  $\triangle PRS$ , we must first find  $PR$  and  $\angle PRS$ .

To find  $PR$ , we use the law of cosines:

$$PR^2 = 195^2 + 260^2 - 2 \cdot 195 \cdot 260 \cdot \cos 122^\circ \\ \approx 159,000$$

Therefore,  $PR \approx \sqrt{159,000} \approx 399$  ft.

To find  $\angle PRS$ , we find  $\angle PRQ$  by the law of sines:

$$\frac{\sin PRQ}{195} = \frac{\sin 122^\circ}{399}$$

$$\sin PRQ = \frac{195 \sin 122^\circ}{399} \approx 0.4145$$

$$\angle PRQ \approx 24.5^\circ$$

Therefore,  $\angle PRS = \angle QRS - \angle PRQ \approx 80^\circ - 24.5^\circ = 55.5^\circ$ .

Knowing that  $PR \approx 399$  ft and  $\angle PRS \approx 55.5^\circ$ , we have:

$$\text{Area of } \triangle PRS = \frac{1}{2} \cdot PR \cdot RS \cdot \sin PRS \\ \approx \frac{1}{2} \cdot 399 \cdot 385 \cdot \sin 55.5^\circ \approx 63,300 \text{ ft}^2$$

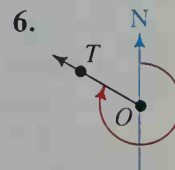
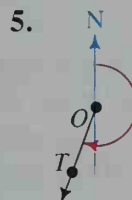
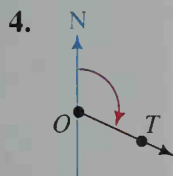
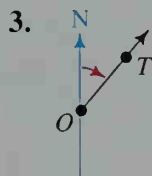
Thus, we have:

$$\begin{aligned} \text{Area of quadrilateral } PQRS &= \text{area of } \triangle PQR + \text{area of } \triangle PRS \\ &\approx 21,500 + 63,300 \\ &= 84,800 \text{ ft}^2 \end{aligned}$$

## CLASS EXERCISES

- Reading** In the solution of Example 1, justify the fact that the angle opposite the side of length  $x$  in the diagram has a measure of  $110^\circ$ .
- Suppose land is taxed at a rate of \$75 per acre. Determine the approximate tax on the land in Example 2. (1 acre = 43,560 ft<sup>2</sup>)

**Visual Thinking** In each diagram, a north-south line is given. If  $\overrightarrow{OT}$  represents the path of a ship, estimate its course.



- Visual Thinking** Suppose point  $X$  is directly east of point  $Y$ .
  - Give the bearing of  $X$  from  $Y$ .
  - Give the bearing of  $Y$  from  $X$ .

For Exercises 8–11, make a sketch of each compass reading.

8.  $N70^{\circ}E$

9.  $N10^{\circ}W$

10.  $S15^{\circ}E$

11.  $S40^{\circ}W$

12. Match each compass direction with a course.

Direction	Course
northeast	$135^{\circ}$
southeast	$225^{\circ}$
northwest	$315^{\circ}$
southwest	$045^{\circ}$

13. A famous Alfred Hitchcock movie is *North by Northwest*. This direction is midway between north and northwest. What course corresponds to this direction?

14. From one corner of a triangular plot of land, a surveyor determines the directions to the other two corners to be  $N32^{\circ}E$  and  $S76^{\circ}E$ . What is the measure of the angle formed by the edges of the plot of land at the corner where the surveyor is?



## WRITTEN EXERCISES

In Exercises 1–4, draw a diagram like those in Class Exercises 3–6 to show the path of a ship proceeding on each given course.

**A**

1.  $070^{\circ}$

2.  $150^{\circ}$

3.  $340^{\circ}$

4.  $225^{\circ}$

5. **Navigation** Ship A sights ship B on a compass bearing of  $080^{\circ}$ . Make a sketch and give the compass bearing of ship A from ship B.

6. **Navigation** Ship X sights ship Y on a bearing of  $308^{\circ}$ . What is the bearing of X from Y?

7. **Aviation** An airplane flies on a course of  $110^{\circ}$  at a speed of 1200 km/h. How far east of its starting point is it after 2 h?

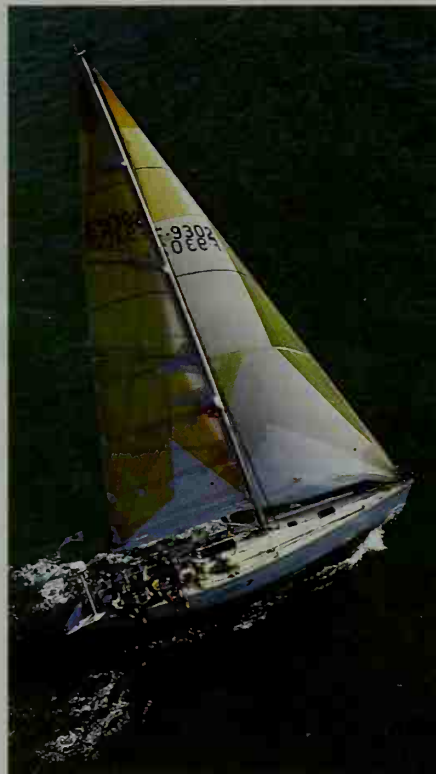
8. A hunter walks east for 1 h and then north for  $1\frac{1}{2}$  h. What course should the hunter take to return to his starting point? What assumptions do you make to answer the question?

9. Point B is 10 km north of point A, and point C is 10 km from B on a bearing of  $060^{\circ}$  from B. Find the bearing and distance of C from A.





10. Point  $S$  is 4 km west of point  $R$ , and point  $T$  is 4 km southwest of  $S$ . Find the bearing and distance of  $R$  from  $T$ .
11. **Navigation** Traveling at a speed of 10 knots, a ship proceeds south from its port for  $1\frac{1}{2}$  h and then changes course to  $130^\circ$  for  $\frac{1}{2}$  h. At this time, how far from port is the ship?
12. **Navigation** A sailboat leaves its dock and proceeds east for 2 mi. It then changes course to  $205^\circ$  until it is due south of its dock. How far south is this?
13. **Navigation** Two ships,  $A$  and  $B$ , leave port at the same time. Ship  $A$  proceeds at 12 knots on a course of  $040^\circ$ , while ship  $B$  proceeds at 9 knots on a course of  $115^\circ$ . After 2 h, ship  $A$  loses power and radios for help. How far and on what course must ship  $B$  travel to reach ship  $A$ ?
14. **Navigation** A ship leaves port and sails northwest for 1 h and then northeast for 2 h. If it does not change speed, find what course the ship should take to return directly to port. Also find how long this return will take.



In Exercises 15–18, sketch each plot of land described and find its area.

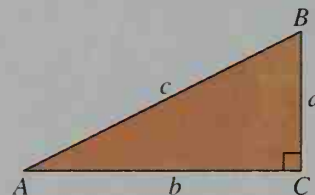
- B** 15. **Surveying** From an iron post, proceed 500 m northeast to the brook, then 300 m east along the brook to the old mill, then 200 m  $S15^\circ E$  to a post on the edge of Wiggin's Road, and finally along Wiggin's Road back to the iron post.
16. **Surveying** From a cement marker, proceed 260 m southwest to the river, then 240 m south along the river to the bridge, then 280 m  $N40^\circ E$  to a sign on the edge of Sycamore Lane, and finally along Sycamore Lane back to the cement marker.
17. **Surveying** From the southeast corner of the cemetery on Burnham Road, proceed  $S78^\circ W$  for 250 m along the southern boundary of the cemetery until a granite post is reached, then  $S15^\circ E$  for 180 m to Allard Road, then  $N78^\circ E$  along Allard Road until it intersects Burnham Road, and finally  $N30^\circ E$  along Burnham Road back to the starting point.
18. **Surveying** From the intersection of Simpson's Road and Mulberry Lane, proceed  $N32^\circ W$  for 320 m along Simpson's Road, then  $S56^\circ W$  for 280 m to the old oak tree, then  $S22^\circ E$  until Mulberry Lane is reached, and finally  $N68^\circ E$  along Mulberry Lane back to the starting point.

19. **Research** There is a difference between magnetic north, the direction in which a compass needle points, and true north. Research what this difference is and what the phrase “compass variation” means. In what parts of the world is the variation “east” and in what parts is it “west”? Also explain what is meant by the mariner’s rhyme:

Error west, compass best.  
Error east, compass least.

## Chapter Summary

1. If you know the lengths of two sides of a right triangle, or the length of one side and the measure of one acute angle, then you can find the measures of the remaining sides and angles using the trigonometric functions, whose definitions are given below. (The definitions are based on the diagram at the right.)



$$\begin{aligned}\sin A &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c} & \csc A &= \frac{1}{\sin A} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{a} \\ \cos A &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c} & \sec A &= \frac{1}{\cos A} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{b} \\ \tan A &= \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} & \cot A &= \frac{1}{\tan A} = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}\end{aligned}$$

2. Knowing the lengths  $a$  and  $b$  of two sides of a triangle and the measure of the included angle  $C$ , you can obtain the area  $K$  of the triangle using the formula:

$$K = \frac{1}{2}ab \sin C$$

3. In any  $\triangle ABC$ , the following relationships hold:

$$\text{Law of sines: } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\text{Law of cosines: } c^2 = a^2 + b^2 - 2ab \cos C$$

$$\text{Law of cosines (alternate form): } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

4. You can use the law of sines and the law of cosines to solve triangles, as summarized on page 352. It is important to remember that the SSA (side-side-nonincluded angle) case may result in 0, 1, or 2 triangles.
5. The application of trigonometry, particularly the laws of sines and cosines, to navigation and surveying is discussed in Section 9-5. In navigation, north represents  $0^\circ$  and the *course* or *compass bearing* of a ship or plane is measured by a clockwise rotation from north. In surveying, angles are given acute measures east or west of a north-south line.

### Key vocabulary and ideas

right triangle definitions of the trigonometric functions (p. 331)  
solving a triangle (p. 331)    area of a triangle (p. 339)    law of cosines (p. 350)  
angle of elevation (p. 332)    law of sines (p. 345)    course, bearing (p. 359)

## Chapter Test

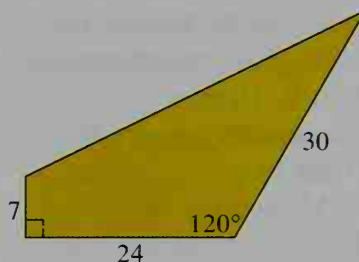
Where appropriate, give angle measures to the nearest tenth of a degree and lengths of sides in simplest radical form or to three significant digits.

1. The sides of an isosceles triangle have lengths 5, 10, and 10. What are the measures of its angles? **9-1**

2. At a distance of 100 m, the angle of elevation to the top of a fir tree is  $28^\circ$ . About how tall is the tree?

3. A regular pentagon is inscribed in a circle of radius 4 in. Find the area of the pentagon. **9-2**

4. Find the area of the quadrilateral shown at the right.

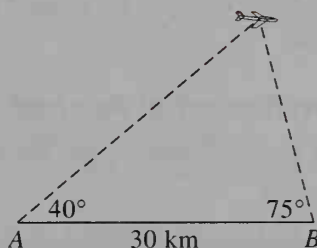


Ex. 4

5. How many different triangles  $PQR$  can be constructed using the given information?

- a.  $p = 5$ ,  $q = 4$ ,  $\angle Q = 74^\circ$   
b.  $p = 9$ ,  $q = 8$ ,  $\angle P = 23^\circ$

6. Observers at points  $A$  and  $B$ , 30 km apart, sight an airplane at angles of elevation of  $40^\circ$  and  $75^\circ$ , respectively, as shown in the diagram. How far is the plane from each observer?



Ex. 6

7. A triangle has sides of length 5, 8, and 10. What is the measure of its largest angle?

8. Two hikers follow a trail that splits into two forks. Each hiker takes a different fork. The forks diverge at an angle of  $67^\circ$  and both hikers walk at a speed of 3.5 mi/h. How far apart are the hikers after 1 h?

9. **Writing** When only three parts of a triangle are known, the law of sines and the law of cosines can be used to find the unknown parts. Discuss the specific circumstances for using each law.

10. After leaving an airport, a plane flies for 1.5 h at a speed of 200 km/h on a course of  $200^\circ$ . Then, on a course of  $340^\circ$ , the plane flies for 2 h at a speed of 250 km/h. At this time, how far from the airport is the plane? **9-5**

## The People of Mathematics

Why do you think people choose to become mathematicians? What events in their lives steer them in the mathematical direction? How has mathematics changed over the centuries? Is mathematics discovered or created? Maybe you've thought about questions like these before. The answers, though, may actually surprise you. Here's a chance to do a little research in some areas of mathematics that perhaps you haven't yet formally explored.

Mathematics is such an extensive field, it's easy to get lost in all of the available information. You'll probably find it helpful, then, to narrow your study by choosing from the following list of suggested topics. Your goal will be to summarize your findings in a report of at least six pages in length.

### **Mathematical Essays and Autobiographies**    **Famous Problems and Paradoxes**

- *Adventures of a Mathematician*  
by Stanislaw M. Ulam
- *I Want to Be a Mathematician*  
by Paul R. Halmos
- "Mathematical Creation"  
by Henri Poincaré
- "The Mathematician"  
by John von Neumann
- *A Mathematician's Apology*  
by G. H. Hardy
- "Olga Taussky-Todd: An Autobiographical Essay" by Olga Taussky-Todd

- The Chinese remainder theorem
- The Cantor set
- Goldbach's conjecture
- Russell's paradox
- The Möbius strip
- Chaotic motion and the Lorenzian waterwheel

### **A Few Masters of Discovery**

- Benjamin Baneker
- János Bolyai
- Leonhard Euler
- Sophie Germain
- Sofia Kovalevskaja  
(also known as Sonya Kovalevsky)
- George Polya
- Srinivasa Ramanujan

### **The Development of Numbers**

- The mathematics of the Rhind papyrus
- The Mayan numeration system and the representation of zero
- The Babylonian (or Sumerian) numeration system
- The Hindu-Arabic numeration system
- Irrational numbers and the school of Pythagoras
- Calculators of  $\pi$ : Archimedes (Greece), Zǔ Chōngzhī and Zǔ Gěng (China), Aryabhata (India), al-Kashi (Persia), Viète (France), Kanada (Japan), Chudnovsky brothers (United States)

### **People and Computational Machines**

- The abacus, including the Chinese *suan pan* and the Japanese *soroban*
- Napier's rods
- Charles Babbage's difference engine
- Ada Byron Lovelace and the art of programming
- Alan Turing and the Turing machine
- Early computers: Mark I, ENIAC, EDSAC, and UNIVAC I



## Writing Your Report

A report on any mathematical topic should be more than just a list of facts. It should also convey the elements of discovery and excitement. Imagine what it must have felt like to be among the first to experience the development of a new idea or theory. Try to reconstruct each episode. Include worked-out examples and diagrams or photographs. Investigate the social and historical events surrounding your subjects. Also investigate the economic or scientific needs that might have promoted each discovery. Finally, convince your readers that the mathematicians you mention are or were real people. Describe their personalities and traits as well as their accomplishments. Use quotations from their writings whenever possible.

### Some Resources

- Albers, Donald J., and G. L. Alexanderson, eds. *Mathematical People: Profiles and Interviews*. Boston: Birkhäuser, 1985.
- Bell, E. T. *Men of Mathematics*. New York: Simon and Schuster, 1937.
- Boyer, Carl B. *A History of Mathematics*. New York: John Wiley and Sons, 1968.
- Bunch, Bryan H. *Mathematical Fallacies and Paradoxes*. New York: Van Nostrand Reinhold, 1982.
- Campbell, Douglas M., and John C. Higgins, eds. *Mathematics: People, Problems, Results*. Belmont, CA: Wadsworth, 1984.
- Davis, Philip J., and Reuben Hersh. *The Mathematical Experience*. Boston: Houghton Mifflin, 1982.
- Evans, Christopher. *The Making of the Micro: A History of the Computer*. New York: Van Nostrand Reinhold, 1981.
- Gleick, James. *Chaos: Making a New Science*. New York: Penguin Books, 1988.
- Haber, Louis. *Black Pioneers of Science and Invention*. New York: Harcourt, Brace, and World, 1970.
- Hoffman, Paul. *Archimedes' Revenge: The Joys and Perils of Mathematics*. New York: W. W. Norton, 1988.
- Ifrah, Georges. *From One to Zero: A Universal History of Numbers*. New York: Viking, 1985.
- Newman, James R. *The World of Mathematics*. New York: Simon and Schuster, 1956.
- Osen, Lynn. *Women in Mathematics*. Cambridge, MA: MIT Press, 1974.
- Perl, Teri. *Math Equals: Biographies of Women Mathematicians + Related Activities*. Menlo Park, CA: Addison Wesley, 1978.
- Peterson, Ivars. *Islands of Truth: A Mathematical Mystery Cruise*. New York: W. H. Freeman, 1990.
- Yǎn, Lǐ, and Dù Shírán. *Chinese Mathematics: A Concise History*. Oxford: Clarendon Press, 1987.



A Mayan Pyramid



# 10

## *Trigonometric Addition Formulas*

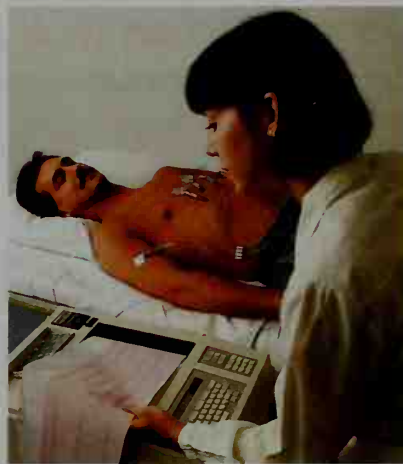


## 10-1 Formulas for $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$

### Objective

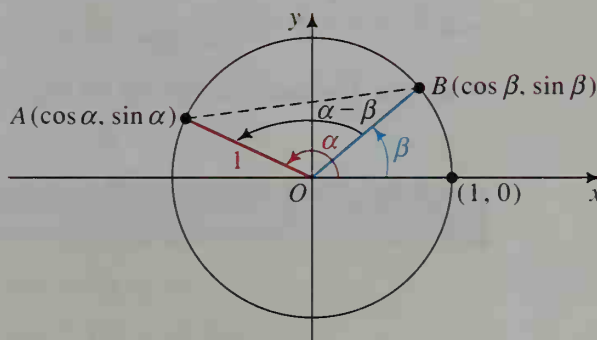
To derive and apply formulas for  $\cos(\alpha \pm \beta)$  and for  $\sin(\alpha \pm \beta)$ .

The photograph at the right shows a monitor (an electrocardiograph) recording the heartbeats of a healthy athlete. The monitor shows a wave that depicts the occurrence of each heartbeat over a period of time. The equation of this wave involves sines and cosines of  $\alpha$ ,  $2\alpha$ ,  $3\alpha$ , and larger multiples of  $\alpha$ . Our goal in this chapter is to gain experience working with expressions like  $\sin 2\alpha$  and  $\cos 3\alpha$ . We will derive formulas showing, for example, how  $\cos 2\alpha$  is related to the cosine of  $\alpha$  and how  $\cos(\alpha + \beta)$  is related to the sine and cosine of  $\alpha$  and  $\beta$ .



### Formulas for $\cos(\alpha \pm \beta)$

To find a formula for  $\cos(\alpha - \beta)$ , let  $A$  and  $B$  be points on the unit circle with coordinates as shown in the diagram at the right. Then the measure of  $\angle AOB$  is  $\alpha - \beta$ . The distance  $AB$  can be found by using either the law of cosines or the distance formula.



Using the law of cosines:

$$(AB)^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos(\alpha - \beta) = 2 - 2 \cos(\alpha - \beta)$$

Using the distance formula:

$$\begin{aligned} (AB)^2 &= (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 \\ &= \cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta \\ &= (\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= 2 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \end{aligned}$$

Therefore,

$$2 - 2 \cos(\alpha - \beta) = 2 - 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta).$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (1)$$

◀ The sound of every stringed instrument can be represented by the graph of a trigonometric function. The curve is determined by unique patterns of tone and overtone, which translate into sums of trigonometric functions.

To obtain a formula for  $\cos(\alpha + \beta)$ , we can use the formula for  $\cos(\alpha - \beta)$  and replace  $\beta$  with  $-\beta$ . Recall that  $\cos(-\beta) = \cos \beta$  and  $\sin(-\beta) = -\sin \beta$ .

$$\begin{aligned}\cos(\alpha - (-\beta)) &= \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta) \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}\quad (2)$$

## Formulas for $\sin(\alpha \pm \beta)$

To find a formula for  $\sin(\alpha + \beta)$ , we use the cofunction relationship

$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right).$$

Let  $\theta = \alpha + \beta$ . Then:

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] \\ &= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] \\ &= \cos\left(\frac{\pi}{2} - \alpha\right) \cos \beta + \sin\left(\frac{\pi}{2} - \alpha\right) \sin \beta\end{aligned}$$

Since  $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$  and  $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$ :

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (3)$$

If we use the formula for  $\sin(\alpha + \beta)$  and replace  $\beta$  with  $-\beta$ , we get:

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (4)$$

## Sum and Difference Formulas for Cosine and Sine

$$\begin{aligned}\cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\ \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta\end{aligned}$$

There are two main purposes for the addition formulas: finding *exact* values of trigonometric expressions and simplifying expressions to obtain other identities. The following examples illustrate how to use these formulas.

**Example 1** Find the exact value of  $\sin 15^\circ$ .

**Solution** Use formula (4) to get the exact value.

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

**Example 2**Find the exact value of: **a.**  $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ$ 

**b.**  $\sin \frac{5\pi}{12} \cos \frac{\pi}{12} + \cos \frac{5\pi}{12} \sin \frac{\pi}{12}$

**Solution**

The two given expressions have the patterns shown in formulas (2) and (3), respectively.

**a.**  $\cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos (\alpha + \beta)$

$$\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ = \cos (50^\circ + 10^\circ)$$

$$= \cos 60^\circ$$

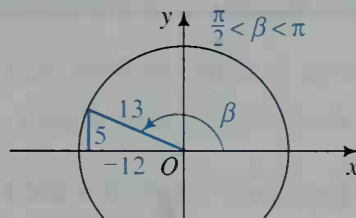
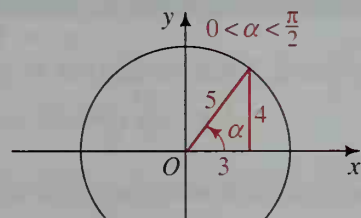
$$= \frac{1}{2}$$

**b.**  $\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin (\alpha + \beta)$

$$\sin \frac{5\pi}{12} \cos \frac{\pi}{12} + \cos \frac{5\pi}{12} \sin \frac{\pi}{12} = \sin \left( \frac{5\pi}{12} + \frac{\pi}{12} \right)$$

$$= \sin \frac{\pi}{2}$$

$$= 1$$

**Example 3**Suppose that  $\sin \alpha = \frac{4}{5}$  and  $\sin \beta = \frac{5}{13}$ , where  $0 < \alpha < \frac{\pi}{2}$  and $\frac{\pi}{2} < \beta < \pi$ . Find  $\cos (\alpha + \beta)$ .**Solution**Sketch right-triangle diagrams to help find  $\cos \alpha$  and  $\cos \beta$ .If  $\sin \alpha = \frac{4}{5}$  and  $0 < \alpha < \frac{\pi}{2}$ , then  $\cos \alpha = \frac{3}{5}$ .If  $\sin \beta = \frac{5}{13}$  and  $\frac{\pi}{2} < \beta < \pi$ , then  $\cos \beta = -\frac{12}{13}$ .

Thus,

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos (\alpha + \beta) = \left( \frac{3}{5} \right) \left( -\frac{12}{13} \right) - \left( \frac{4}{5} \right) \left( \frac{5}{13} \right) = -\frac{56}{65}$$

The sum or difference formulas can be used to verify many identities that we have seen, such as  $\sin (90^\circ - \theta) = \cos \theta$ , and also to derive new identities.

**Example 4** Show that  $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$ .

**Solution** By formula (4),

$$\begin{aligned}\sin\left(\frac{3\pi}{2} - x\right) &= \sin\left(\frac{3\pi}{2}\right) \cos x - \cos\left(\frac{3\pi}{2}\right) \sin x \\ &= (-1) \cos x - (0) \sin x \\ &= -\cos x\end{aligned}$$

Sometimes a problem involving a sum can be more easily solved if the sum can be expressed as a product. The formulas given below are derived from the addition formulas in Exercises 39–41.

### Rewriting a Sum or Difference as a Product

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

## CLASS EXERCISES

Find an example to show that, in general, each statement is true.

1.  $\sin(x+y) \neq \sin x + \sin y$

2.  $\cos(a-b) \neq \cos a - \cos b$

Are there any values of  $a$  and  $b$  for which each statement is true? If so, give an example.

3.  $\sin(a-b) = \sin a - \sin b$

4.  $\sin(a+b) = \sin a + \sin b$

Simplify each expression. Do not evaluate.

5.  $\sin 1^\circ \cos 2^\circ + \cos 1^\circ \sin 2^\circ$

6.  $\sin 20^\circ \cos 15^\circ - \cos 20^\circ \sin 15^\circ$

7.  $\cos \frac{\pi}{4} \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \sin \frac{\pi}{4}$

8.  $\cos 75^\circ \cos 25^\circ + \sin 75^\circ \sin 25^\circ$

9. **Discussion** Explain how the identity  $\sin(-\alpha) = -\sin \alpha$  is a special case of the difference formula for sine.



## WRITTEN EXERCISES

Simplify the given expression.

- A**
- |  |  |
|--|--|
| 1. $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$                           | 2. $\cos 105^\circ \cos 15^\circ + \sin 105^\circ \sin 15^\circ$                     |
| 3. $\cos \frac{5\pi}{12} \cos \frac{\pi}{12} - \sin \frac{5\pi}{12} \sin \frac{\pi}{12}$ | 4. $\sin \frac{4\pi}{3} \cos \frac{\pi}{3} - \cos \frac{4\pi}{3} \sin \frac{\pi}{3}$ |
| 5. $\sin 3x \cos 2x - \cos 3x \sin 2x$   | 6. $\cos 2x \cos x + \sin 2x \sin x$   |

In Exercises 7–10, prove that the given equation is an identity.

- |   |   |
|---|---|
| 7. $\sin(x + \pi) = -\sin x$                      | 8. $\cos(\pi + x) = -\cos x$                      |
| 9. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$ | 10. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ |
11. **Visual Thinking** Translate the graph of  $y = \sin x$  to the right by  $\pi$  units, and then reflect this curve about the  $x$ -axis. What graph results?
12. **Visual Thinking** Translate the graph of  $y = \cos x$  to the left by  $\frac{\pi}{2}$  units, and then reflect this curve about the  $x$ -axis. What graph results?

Find the exact value of each expression.

- |                       |                        |                            |                             |
|-----------------------|------------------------|----------------------------|-----------------------------|
| 13. $\cos 75^\circ$   | 14. $\cos 15^\circ$    | 15. $\cos 105^\circ$       | 16. $\sin 105^\circ$        |
| 17. $\sin(-15^\circ)$ | 18. $\cos(-165^\circ)$ | 19. $\sin \frac{7\pi}{12}$ | 20. $\sin \frac{11\pi}{12}$ |

Simplify the given expression.

- |   |   |
|---|---|
| 21. $\sin(30^\circ + \theta) + \sin(30^\circ - \theta)$                         | 22. $\cos(30^\circ + \theta) + \cos(30^\circ - \theta)$                       |
| 23. $\cos\left(\frac{\pi}{3} + x\right) + \cos\left(\frac{\pi}{3} - x\right)$   | 24. $\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right)$ |
| 25. $\cos\left(\frac{3\pi}{2} + x\right) + \cos\left(\frac{3\pi}{2} - x\right)$ | 26. $\sin(\pi + x) + \sin(\pi - x)$   |

- B**
27. Suppose that  $\sin \alpha = \frac{3}{5}$  and  $\sin \beta = \frac{24}{25}$ , where  $0 < \alpha < \frac{\pi}{2} < \beta < \pi$ . Find  $\sin(\alpha + \beta)$ .
28. Suppose that  $\sin \alpha = \frac{4}{5}$  and  $\sin \beta = \frac{1}{2}$ , where  $\frac{\pi}{2} < \beta < \alpha < \pi$ . Find  $\sin(\alpha - \beta)$ .
29. Suppose that  $\tan \alpha = \frac{4}{3}$  and  $\tan \beta = \frac{12}{5}$ , where  $0 < \alpha < \beta < \frac{\pi}{2}$ . Find  $\cos(\alpha - \beta)$ .
30. Suppose that  $\sec \alpha = \frac{5}{4}$  and  $\tan \beta = -1$ , where  $0 < \alpha < \frac{\pi}{2} < \beta < \pi$ . Find  $\cos(\alpha + \beta)$ .



Use a graphing calculator set in radian mode to sketch each graph. Explain how the graph is related to a familiar graph and why.

31.  $y = \sin x \cos 1 + \cos x \sin 1$

32.  $y = \cos x \cos 2 + \sin x \sin 2$

Simplify the given expression.

33.  $\frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{\cos \alpha \cos \beta}$

34.  $\frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{\sin \alpha \sin \beta}$

35.  $\cos x \cos y (\tan x + \tan y)$

36.  $\sin x \sin y (\cot x \cot y - 1)$

37.  $\sin(x + y) \sec x \sec y$

38.  $\cos\left(x + \frac{\pi}{3}\right) + \sin\left(x - \frac{\pi}{6}\right)$

39. a. Derive the formula  $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$ .

(Hint: Show that  $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$ , and then substitute  $\alpha = \frac{x+y}{2}$  and  $\beta = \frac{x-y}{2}$ .)

b. Use the formula in part (a) to show that  $\sin 40^\circ + \sin 20^\circ = \cos 10^\circ$ .

c. Use the formula in part (a) to derive the formula for  $\sin x - \sin y$ .

40. a. Derive the formula  $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$ . (Simplify

$\cos(\alpha + \beta) + \cos(\alpha - \beta)$ . Then substitute  $\alpha = \frac{x+y}{2}$  and  $\beta = \frac{x-y}{2}$ , as was done in Exercise 39.)

b. Find the exact value of  $\cos 105^\circ + \cos 15^\circ$ .

41. a. Derive the formula  $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$ . (Hint: See Exercise 39.)

b. Find the exact value of  $\cos 75^\circ - \cos 15^\circ$ .

**C** 42. Evaluate  $\sin\left(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}\right)$  without using a calculator or tables.

43. Evaluate  $\cos\left(\tan^{-1} \frac{1}{2} + \tan^{-1} 2\right)$  without using a calculator or tables.

44. Ptolemy's theorem states that if  $ABCD$  is inscribed in a circle, then

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

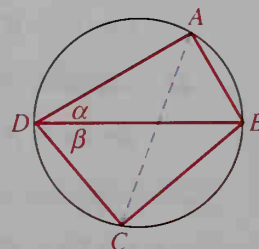
Consider a special case of this theorem in which  $\overline{BD}$  is a diameter with length 1.

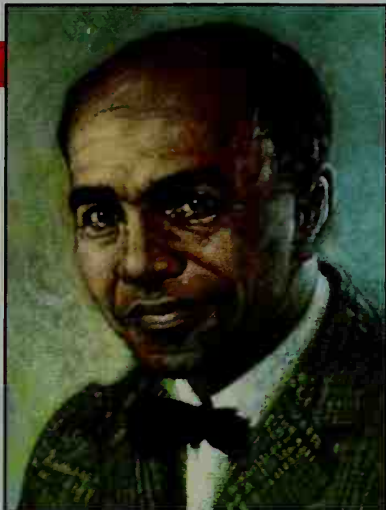
a. Show that  $AB = \sin \alpha$  and  $AD = \cos \alpha$ .

b. Find  $BC$  and  $CD$  in terms of  $\beta$ .

c. Show that  $AC = \sin(\alpha + \beta)$ .

d. Use parts (a), (b), and (c), and Ptolemy's theorem to derive the formula for  $\sin(\alpha + \beta)$ .





**David Blackwell (1919– )**

David Blackwell expected to become a school-teacher like his grandfather, but his interest in statistics led to a Ph.D. at the age of 22. After teaching at Howard University for ten years, he became a professor of statistics at the University of California at Berkeley in 1954.

Blackwell has made contributions to many fields of mathematics, including probability theory, game theory, set theory, and dynamic programming. In 1965, Blackwell was the first African American mathematician elected to the National Academy of Sciences.

## 10-2 Formulas for $\tan (\alpha \pm \beta)$

**Objective** To derive and apply formulas for  $\tan (\alpha \pm \beta)$ .

To derive a formula expressing  $\tan (\alpha + \beta)$  in terms of  $\tan \alpha$  and  $\tan \beta$ , we use the formulas for  $\sin (\alpha + \beta)$  and  $\cos (\alpha + \beta)$ .

$$\begin{aligned}\tan (\alpha + \beta) &= \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}\end{aligned}$$

Dividing the numerator and the denominator by  $\cos \alpha \cos \beta$ , we obtain:

$$\tan (\alpha + \beta) = \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}}$$

Therefore, we have the following formula.

### Sum Formula for Tangent

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad (5)$$

This identity is valid for all values of  $\alpha$  and  $\beta$  for which  $\tan \alpha$ ,  $\tan \beta$ , and  $\tan (\alpha + \beta)$  are defined.

To derive a formula for  $\tan(\alpha - \beta)$ , simply replace  $\beta$  with  $-\beta$  in formula (5) and use the fact that  $\tan(-\beta) = -\tan \beta$ . Thus, we have the following formula.

### Difference Formula for Tangent

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \quad (6)$$

**Example 1** Suppose  $\tan \alpha = \frac{1}{3}$  and  $\tan \beta = \frac{1}{2}$ .

- a. Find  $\tan(\alpha + \beta)$ .      b. Show that  $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} = \frac{\pi}{4}$ .

**Solution**

a.  $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{3} \cdot \frac{1}{2}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$

- b. Let  $\alpha = \tan^{-1} \frac{1}{3}$  and  $\beta = \tan^{-1} \frac{1}{2}$ . Since  $\alpha$  and  $\beta$  are both between 0 and  $\frac{\pi}{2}$ ,  $\alpha + \beta$  is between 0 and  $\pi$ . From part (a), we know that  $\tan(\alpha + \beta) = 1$ . Thus, since  $\frac{\pi}{4}$  is the only angle between 0 and  $\pi$  whose tangent is 1,

$$\alpha + \beta = \frac{\pi}{4}.$$

That is,

$$\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} = \frac{\pi}{4}.$$

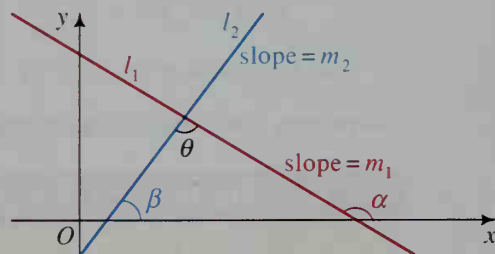
### Angles between Two Lines

The formula for  $\tan(\alpha - \beta)$  can be used to find an angle between two intersecting lines. For example, consider the angle  $\theta$  between lines  $l_1$  and  $l_2$  with slopes  $m_1$  and  $m_2$ . Suppose the inclinations of these lines are  $\alpha$  and  $\beta$ , respectively. Recall from Section 8-1 that the slope of a nonvertical line is the tangent of its inclination. Thus,

$$\tan \alpha = m_1 \text{ and } \tan \beta = m_2.$$

Since the measure of an exterior angle of a triangle equals the sum of the measures of the opposite interior angles,  $\alpha = \beta + \theta$ . Therefore,

$$\theta = \alpha - \beta \quad \text{and} \quad \tan \theta = \tan(\alpha - \beta).$$



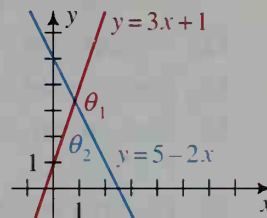
Substituting in  $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$  gives the following formula:

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

**Example 2** Find an angle between the lines  $y = 3x + 1$  and  $y = 5 - 2x$ .

**Solution**

The lines have slopes 3 and  $-2$ . Thus, we can either let  $m_1 = 3$  and  $m_2 = -2$  or let  $m_1 = -2$  and  $m_2 = 3$ . These two possibilities give us the two supplementary angles,  $\theta_1$  and  $\theta_2$ , that are formed by the lines.



(1) The first case gives

$$\tan \theta_1 = \frac{3 - (-2)}{1 + 3(-2)} = -1. \text{ So, } \theta_1 = 135^\circ.$$

(2) The second case gives

$$\tan \theta_2 = \frac{-2 - 3}{1 + (-2)3} = 1. \text{ So, } \theta_2 = 45^\circ.$$

## CLASS EXERCISES

- Suppose  $\tan \alpha = 2$  and  $\tan \beta = 3$ . Find (a)  $\tan(\alpha + \beta)$  and (b)  $\tan(\alpha - \beta)$ .
- Find the exact value: a.  $\frac{\tan 15^\circ + \tan 30^\circ}{1 - \tan 15^\circ \tan 30^\circ}$  b.  $\frac{\tan 85^\circ - \tan 25^\circ}{1 + \tan 85^\circ \tan 25^\circ}$
- Discussion** Interpret the formula for finding an angle between two lines when  $1 + m_1 m_2 = 0$ .

## WRITTEN EXERCISES

In Exercises 1 and 2, find (a)  $\tan(\alpha + \beta)$  and (b)  $\tan(\alpha - \beta)$ .

- A** 1.  $\tan \alpha = \frac{2}{3}$  and  $\tan \beta = \frac{1}{2}$       2.  $\tan \alpha = 2$  and  $\tan \beta = -\frac{1}{3}$

In Exercises 3–6, find the exact value of the given expression.

- $\frac{\tan 75^\circ - \tan 30^\circ}{1 + \tan 75^\circ \tan 30^\circ}$
- $\frac{\tan 100^\circ + \tan 50^\circ}{1 - \tan 100^\circ \tan 50^\circ}$
- $\frac{\tan \frac{2\pi}{3} + \tan \frac{\pi}{12}}{1 - \tan \frac{2\pi}{3} \tan \frac{\pi}{12}}$
- $\frac{\tan \frac{4\pi}{3} - \tan \frac{\pi}{12}}{1 + \tan \frac{4\pi}{3} \tan \frac{\pi}{12}}$



7. Evaluate  $\tan\left(\frac{\pi}{4} + \theta\right)$  when  $\tan \theta = \frac{1}{2}$ .
8. Evaluate  $\tan\left(\frac{3\pi}{4} - \theta\right)$  when  $\tan \theta = \frac{1}{3}$ .
9. Show that  $\tan(-\alpha) = -\tan \alpha$ .
10. **Visual Thinking** Simplify  $\tan(x + \pi)$  and interpret your answer graphically. What does your answer illustrate about the period of the tangent function?
11. Evaluate  $\tan 75^\circ$  and  $\tan 165^\circ$  without using a calculator or tables.
12. Evaluate  $\tan 15^\circ$  and  $\tan 105^\circ$  without using a calculator or tables.
13. Find the two supplementary angles formed by the line  $y = 3x - 5$  and the line  $y = x + 4$ .
14. Find the two supplementary angles formed by the line  $3x + 2y = 5$  and the line  $4x - 3y = 1$ .



Use a graphing calculator set in radian mode to sketch each graph. Explain how the graph is related to the graph of  $y = \tan x$  and why.

**B** 15.  $y = \frac{\tan x + \tan 1}{1 - \tan x \tan 1}$

16.  $y = \frac{\tan x - \tan \frac{\pi}{8}}{1 + \tan x \tan \frac{\pi}{8}}$

17. Suppose  $\tan \alpha = \frac{1}{4}$  and  $\tan \beta = \frac{3}{5}$ , where  $0 < \alpha < \beta < \frac{\pi}{2}$ . Find  $\tan(\alpha + \beta)$ .

Then show that  $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{3}{5} = \frac{\pi}{4}$ .

18. Suppose  $\tan \alpha = 3$  and  $\tan \beta = \frac{1}{2}$ , where  $0 < \beta < \alpha < \frac{\pi}{2}$ . Find  $\tan(\alpha - \beta)$ .

Then show that  $\tan^{-1} 3 - \tan^{-1} \frac{1}{2} = \frac{\pi}{4}$ .

19. Suppose  $\alpha = \tan^{-1} 2$  and  $\beta = \tan^{-1} 3$ . Show that  $\tan(\alpha + \beta) = -1$ .

20. Suppose  $\alpha = \tan^{-1} 5$  and  $\beta = \tan^{-1} \frac{2}{3}$ . Show that  $\alpha - \beta = \frac{\pi}{4}$ .

21. **Investigation** For what values of  $\alpha$  and  $\beta$  does  $\tan(\alpha + \beta) = \tan \alpha + \tan \beta$ ?  
For what values of  $\alpha$  and  $\beta$  does  $\tan(\alpha - \beta) = \tan \alpha - \tan \beta$ ?

22. Given  $A(3, 1)$ ,  $B(14, -1)$ , and  $C(5, 5)$ , find the measure of  $\angle BAC$  by using (a) the slopes of lines  $AB$  and  $AC$  and (b) the law of cosines.

23. Suppose  $\cot \alpha = 2$  and  $\cot \beta = \frac{2}{3}$ , where  $0 < \alpha < \beta < \frac{\pi}{2}$ . Find  $\cot(\alpha + \beta)$ .

24. Suppose  $\cot \alpha = \frac{3}{2}$  and  $\cot \beta = \frac{1}{2}$ , where  $0 < \alpha < \beta < \frac{\pi}{2}$ . Find  $\cot(\alpha - \beta)$ .

25. Suppose  $\sin \alpha = \frac{3}{5}$  and  $\cos \beta = \frac{5}{13}$ , where  $0 < \alpha < \beta < \frac{\pi}{2}$ . Find:

a.  $\sin(\alpha + \beta)$

b.  $\cos(\alpha + \beta)$

c.  $\tan(\alpha + \beta)$

26. Suppose  $\sin \alpha = \frac{4}{5}$  and  $\tan \beta = -\frac{3}{4}$ , where  $\frac{\pi}{2} < \alpha < \beta < \pi$ . Find:

a.  $\sin (\alpha + \beta)$

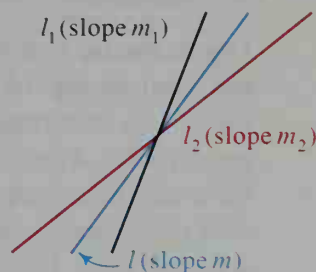
b.  $\cos (\alpha + \beta)$

c.  $\tan (\alpha + \beta)$

- C** 27. a. Line  $l$  bisects  $\theta$ , the angle formed by lines  $l_1$  and  $l_2$ . If the slopes of these three lines are  $m$ ,  $m_1$ , and  $m_2$ , respectively, show that:

$$\frac{m_1 - m}{1 + m_1 m} = \frac{m - m_2}{1 + m_2 m}$$

- b. If  $l_1$  and  $l_2$  have equations  $y = 2x$  and  $y = x$ , find an equation of  $l$ .



28. Verify that in any  $\triangle ABC$ ,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

29. Verify that  $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}$ . (Hint: Let  $\alpha = \tan^{-1} \frac{1}{5}$  and let  $\beta = \tan^{-1} \frac{1}{239}$ . Then find  $\tan (4\alpha - \beta)$ .)

## COMMUNICATION: Discussion

### Small Group Discussion

Often when you are faced with a difficult problem, you can respond better if you have discussed it with others. Working cooperatively in a small group allows each of you to contribute your strengths and skills to solving the problem. Sometimes another person's perspective helps you better understand a problem. Likewise, your perspective may help someone else in the group gain insight. When working in a group, it is important that all members participate. This means that you must speak up when you have ideas and also listen carefully to the ideas of other persons in your group.

Working in groups of three or four, discuss one of the following questions. Then present your results to the entire class.

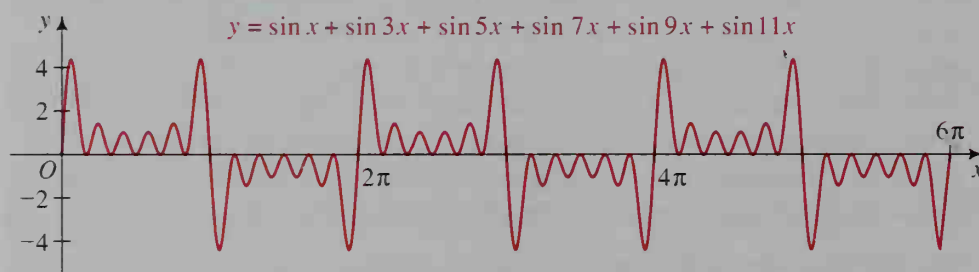
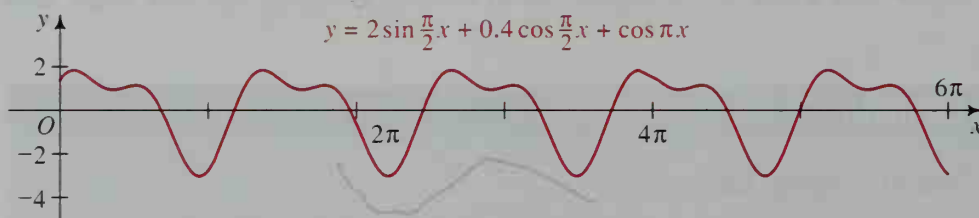
1. The decimal expansion of  $\frac{1}{7}$ , namely  $0.\overline{142857}$ , has a six-digit block of repeating digits. What is the maximum length of the block of repeating digits in the decimal expansion of  $\frac{1}{n}$ ? Explain your answer.
2. Are there more rational numbers or irrational numbers?
3. Is the sum of two sine curves always another sine curve?
4. Why does it make sense that  $b^0 = 1$  if  $b \neq 0$ ? Why is  $0^0$  undefined?

## 10-3 Double-Angle and Half-Angle Formulas

### Objective

To derive and apply double-angle and half-angle formulas.

Trigonometric functions are used in science and engineering to study light and sound waves. An important application is the wave pattern of a vibrating string. Consider the wave of a note sounded by a violin. Not only does the string vibrate as a whole, producing a fundamental tone, but it also vibrates in halves, thirds, and progressively smaller segments, producing overtones (called harmonics). An equation of this type of wave involves sums of sines and cosines of  $x$ ,  $2x$ ,  $3x$ , and greater multiples of  $x$ . The computer-generated graphs below illustrate this.



If you know the value of  $\sin \alpha$ , you do *not* double it to find  $\sin 2\alpha$ . Nor do you halve it to find  $\sin \frac{1}{2}\alpha$ . To see that this is true, complete the following activity.

### Activity

1. On a single set of axes, graph  $y = \sin 2x$  and  $y = 2 \sin x$ . Do the graphs coincide? What does this tell you about the values of  $\sin 2x$  and  $2 \sin x$ ?
2. On a single set of axes, graph  $y = \sin \frac{1}{2}x$  and  $y = \frac{1}{2} \sin x$ . Do the graphs coincide? What does this tell you about the values of  $\sin \frac{1}{2}x$  and  $\frac{1}{2} \sin x$ ?

## Double-Angle Formulas

The following double-angle formulas are special cases of the formulas for  $\sin(\alpha + \beta)$ ,  $\cos(\alpha + \beta)$ , and  $\tan(\alpha + \beta)$ . If we let  $\beta = \alpha$  in these formulas we obtain the following formulas.

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \sin(\alpha + \alpha) &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha \\ \sin 2\alpha &= 2 \sin \alpha \cos \alpha\end{aligned}\quad (7)$$

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \cos(\alpha + \alpha) &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \\ \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha\end{aligned}\quad (8a)$$

Using the fact that  $\sin^2 \alpha + \cos^2 \alpha = 1$ , we can obtain alternative formulas for  $\cos 2\alpha$ :

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha \quad (8b)$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1 \quad (8c)$$

To express  $\tan 2\alpha$  in terms of  $\tan \alpha$ , we again let  $\beta = \alpha$ :

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ \tan(\alpha + \alpha) &= \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha} \\ \tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha}\end{aligned}\quad (9)$$

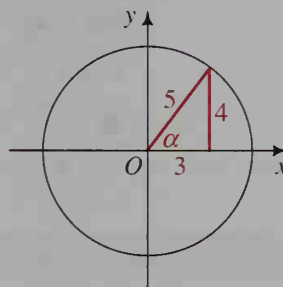
**Example 1** If  $\sin \alpha = \frac{4}{5}$  and  $0 < \alpha < \frac{\pi}{2}$ , find  $\sin 2\alpha$ ,  $\cos 2\alpha$ , and  $\tan 2\alpha$ .

### Solution

From the given information and the diagram, we know that  $\cos \alpha = \frac{3}{5}$  and  $\tan \alpha = \frac{4}{3}$ .

$$\begin{aligned}\sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25} \\ \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}\end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2} = -\frac{24}{7}$$



**Example 2** Derive a formula for  $\sin 4x$  in terms of functions of  $x$ .

**Solution**

$$\begin{aligned}\sin 4x &= \sin [2(2x)] = 2 \sin 2x \cos 2x \\ &= 2(2 \sin x \cos x)(\cos^2 x - \sin^2 x) \\ &= 4 \sin x \cos^3 x - 4 \sin^3 x \cos x\end{aligned}$$

Using another form for  $\cos 2x$  will yield different but equivalent results.

## Half-Angle Formulas

To obtain the sine and cosine half-angle formulas, we use formulas (8b) and (8c), replacing  $\alpha$  with  $\frac{x}{2}$ .

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

$$\cos 2\left(\frac{x}{2}\right) = 1 - 2 \sin^2 \frac{x}{2}$$

$$\cos x = 1 - 2 \sin^2 \frac{x}{2}$$

$$2 \sin^2 \frac{x}{2} = 1 - \cos x$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad (10)$$

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\cos 2\left(\frac{x}{2}\right) = 2 \cos^2 \frac{x}{2} - 1$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$2 \cos^2 \frac{x}{2} = 1 + \cos x$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \quad (11)$$

When you use the half-angle formulas, choose  $+$  or  $-$  depending on the quadrant in which  $\frac{x}{2}$  lies.

**Example 3** Find the exact value of  $\cos \frac{5\pi}{8}$ .

**Solution** Because  $\frac{5\pi}{8}$  is in the second quadrant,  $\cos \frac{5\pi}{8}$  is negative.

Since  $\frac{5\pi}{8} = \frac{1}{2}\left(\frac{5\pi}{4}\right)$ , we can let  $x = \frac{5\pi}{4}$  in formula (11).

$$\begin{aligned}\cos \frac{5\pi}{8} &= -\sqrt{\frac{1 + \cos \frac{5\pi}{4}}{2}} = -\sqrt{\frac{1 + \left(-\frac{\sqrt{2}}{2}\right)}{2}} \\ &= -\sqrt{\frac{2 - \sqrt{2}}{4}} = -\frac{\sqrt{2 - \sqrt{2}}}{2}\end{aligned}$$

To derive a formula for  $\tan \frac{x}{2}$ , divide equation (10) by equation (11):

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad (12)$$



However, the following formulas, which can be derived by simplifying the radical expression in formula (12), may be more useful.

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} \quad (12a)$$

$$\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x} \quad (12b)$$

Notice that these formulas don't need the ambiguous sign  $\pm$ . (Why?) You are asked to prove these formulas in Exercises 37 and 38.

The following table summarizes the double-angle and half-angle formulas.

### Double-Angle and Half-Angle Formulas

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

### CLASS EXERCISES

In Exercises 1–10, simplify the given expression.

1.  $2 \sin 10^\circ \cos 10^\circ$

2.  $\cos^2 15^\circ - \sin^2 15^\circ$

3.  $1 - 2 \sin^2 35^\circ$

4.  $2 \cos^2 25^\circ - 1$

5.  $\frac{2 \tan 50^\circ}{1 - \tan^2 50^\circ}$

6.  $\frac{2 \tan 40^\circ}{1 - \tan^2 40^\circ}$

7.  $1 - \sin^2 x$

8.  $1 - 2 \sin^2 x$

9.  $2 \sin 3\alpha \cos 3\alpha$

10.  $\cos^2 5\theta - \sin^2 5\theta$

11. **Discussion** Given that  $\cos 70^\circ \approx 0.3420$ , explain how you can find  $\cos 35^\circ$ .

### WRITTEN EXERCISES

In Exercises 1–12, simplify the given expression.

**A** 1.  $2 \cos^2 10^\circ - 1$

2.  $2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$

3.  $\frac{4 \tan \beta}{1 - \tan^2 \beta}$

4.  $1 - 2 \sin^2 20^\circ$

5.  $2 \sin 35^\circ \cos 35^\circ$

6.  $\cos^2 4A - \sin^2 4A$

7.  $\frac{2 \tan 25^\circ}{1 - \tan^2 25^\circ}$

8.  $2 \cos^2 3\alpha - 1$

9.  $1 - 2 \sin^2 \frac{x}{2}$

10.  $\cos^2 40^\circ - \sin^2 40^\circ$

11.  $\sqrt{\frac{1 - \cos 80^\circ}{2}}$

12.  $\sqrt{\frac{1 + \cos 70^\circ}{2}}$

In Exercises 13–18, find the exact value of the given expression.

13.  $2 \cos^2 \frac{\pi}{8} - 1$

14.  $\frac{2 \tan \frac{\pi}{8}}{1 - \tan^2 \frac{\pi}{8}}$

15.  $\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12}$

16.  $1 - 2 \sin^2 \frac{7\pi}{12}$

17.  $\sin 15^\circ \cos 15^\circ$

18.  $4 \sin \frac{2\pi}{3} \cos \frac{2\pi}{3}$

In Exercises 19–24,  $\angle A$  is acute.

19. If  $\sin A = \frac{5}{13}$ , find  $\sin 2A$  and  $\cos 2A$ .

20. If  $\tan A = \frac{1}{2}$ , find  $\cos 2A$  and  $\tan 2A$ .

21. If  $\sin A = \frac{3}{5}$ , find  $\sin 2A$  and  $\sin 4A$ .

22. If  $\cos A = \frac{1}{3}$ , find  $\cos 2A$  and  $\cos 4A$ .

23. If  $\cos A = \frac{1}{5}$ , find  $\cos 2A$  and  $\cos \frac{A}{2}$ .

24. If  $\cos A = \frac{1}{4}$ , find  $\sin 2A$  and  $\sin \frac{A}{2}$ .

25. Find  $\cos 105^\circ$  using (a) an addition formula and (b) a half-angle formula.

26. Find  $\sin 75^\circ$  using (a) an addition formula and (b) a half-angle formula.



Use a graphing calculator or computer to sketch the graph of each function. Then give the range and period of the function.

**B** 27.  $y = \sin x + \cos 2x$

28.  $y = \sin 3x + 2 \cos x$

29.  $y = 2 \sin 2x + 4 \cos 4x$

30.  $y = 3 \cos \frac{x}{2} - 2 \cos 3x$

In Exercises 31–38, prove that the given equation is an identity.

31.  $\frac{\sin 2A}{1 - \cos 2A} = \cot A$

32.  $\frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$

33.  $\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2 = 1 + \sin x$

34.  $\sin 4x = 4 \sin x \cos x \cos 2x$

35.  $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$

36.  $\frac{1 + \sin A - \cos 2A}{\cos A + \sin 2A} = \tan A$

37.  $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$

38.  $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$

Simplify the given expression.

39.  $\frac{1 + \cos 2x}{\cot x}$

40.  $\frac{(1 + \tan^2 x)(1 - \cos 2x)}{2}$

41.  $(1 - \sin^2 x)(1 - \tan^2 x)$

42.  $\sin x \tan x + \cos 2x \sec x$

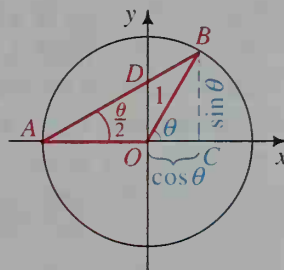
43.  $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$

44.  $\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right) - \sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)$

45. **Visual Thinking** The diagram at the right shows a unit circle, with the measure of  $\angle BOC = \theta$ .

a. Explain why the measure of  $\angle BAO = \frac{\theta}{2}$ .

b. Explain why  $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$ .



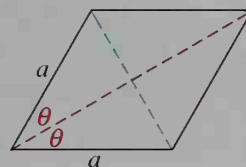
46. Use the rhombus at the right and the steps below to derive the formula for  $\sin 2\theta$  for  $0^\circ < \theta < 90^\circ$ .

a. Show that the area of the rhombus is  $a^2 \sin 2\theta$ .

b. Use the fact that the diagonals of a rhombus are perpendicular to show that its area is

$$2a^2 \sin \theta \cos \theta.$$

c. Use your answers from parts (a) and (b) to obtain a formula for  $\sin 2\theta$ .



47. Find  $\log_2 2 + \log_2 (\sin x) + \log_2 (\cos x)$  when  $x = \frac{\pi}{12}$ .

48. Find  $\frac{4^{2 \cos^2 \theta}}{4}$  when  $\theta = \frac{\pi}{3}$ .

**Prove that the given equation is an identity.**

49.  $\sin 3x = 3 \sin x - 4 \sin^3 x$

50.  $\cos 3x = 4 \cos^3 x - 3 \cos x$

**C** 51. Given:  $QR = QS = 1$ ,  $\angle Q = 36^\circ$ , and  $\overline{RT}$  bisects  $\angle R$ .

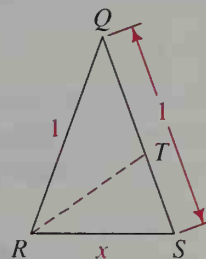
a. Prove that  $\triangle QRS$  is similar to  $\triangle RST$ .

b. Use similar triangles to show that  $\frac{x}{1-x} = \frac{1}{x}$ .

c. Show that  $x = \frac{\sqrt{5}-1}{2}$ .

d. Draw the bisector of  $\angle Q$ . Show that

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ.$$



52. In  $\triangle ABC$ , the measure of  $\angle B$  is twice the measure of  $\angle C$ .

a. Use the law of sines to show that  $b = 2c \cos C$ .

b. Use the law of cosines to show that  $b^2 = c(a + c)$ .

53. The lengths of the sides of a triangle are consecutive integers  $n$ ,  $n + 1$ , and  $n + 2$ , and the largest angle is twice the smallest angle  $\theta$ .

a. Use the law of sines to show that  $\cos \theta = \frac{n+2}{2n}$ .

b. Use the law of cosines to show that  $\cos \theta = \frac{n+5}{2(n+2)}$ .

c. Use parts (a) and (b) to find  $n$ .

## 10-4 Solving Trigonometric Equations

### Objective

To use identities to solve trigonometric equations.

The acceleration of a body falling toward the Earth's surface is called acceleration due to gravity, often denoted by  $g$ . In theoretical physics,  $g$  is usually considered constant. However,  $g$  is not actually constant but varies slightly with latitude. A good approximation to the value of  $g$  can be found by using the following formula, which expresses  $g$  in terms of  $\theta$ , the latitude in degrees.



Gravity acting on water creates this waterfall.

$$g \approx 9.78049(1 + 0.005288 \sin^2 \theta - 0.000006 \sin^2 2\theta)$$

For example, if you live in Chicago, which has a latitude of  $42^\circ\text{N}$ ,  $\sin \theta \approx 0.6691$  and  $\sin 2\theta \approx 0.9945$ . Therefore,  $g \approx 9.8036 \text{ m/s}^2$ .



As you can see from this example, some problems involve trigonometric equations that have multiples of angles or numbers. The following suggests two methods that may be helpful in solving such equations. The first method gives a graphical method using a graphing calculator or computer, and the second method gives ways for solving the equation algebraically.

### Methods for Solving the Trigonometric Equation $f(x) = g(x)$

#### Method 1

Use a graphing calculator or computer to graph  $y = f(x)$  and  $y = g(x)$  on the same set of axes. Use the zoom or trace feature to find the  $x$ -coordinates of any intersection points of the two graphs.

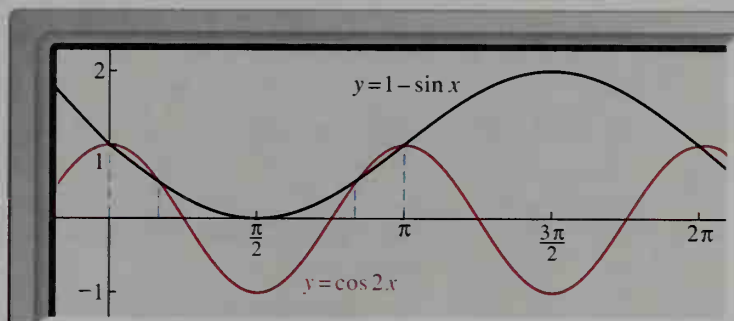
#### Method 2

Use the following guidelines.

- It may be helpful to draw a quick sketch of  $y = f(x)$  and  $y = g(x)$  to see roughly where the solutions are.
- If the equation involves functions of  $2x$  and  $x$ , transform the functions of  $2x$  into functions of  $x$  by using identities.
- If the equation involves functions of  $2x$  only, it is usually better to solve for  $2x$  directly and then solve for  $x$ .
- Be careful not to lose roots when you divide both sides of an equation by a function of the variable. Review the discussion about losing roots on pages 32–33.

**Example 1** Solve  $\cos 2x = 1 - \sin x$  for  $0 \leq x < 2\pi$ .

**Solution** **Method 1** The diagram below shows the graphs of  $y = \cos 2x$  and  $y = 1 - \sin x$  on the same set of axes. There are four solutions in the interval  $0 \leq x < 2\pi$ . From the diagram, you can see that 0 and  $\pi$  are solutions. Using a zoom feature, you can find that  $0.52 \approx \frac{\pi}{6}$  and  $2.62 \approx \frac{5\pi}{6}$  are also solutions.



**Method 2**

$$\begin{aligned}\cos 2x &= 1 - \sin x \\ 1 - 2 \sin^2 x &= 1 - \sin x \\ 2 \sin^2 x &= \sin x\end{aligned}$$

$\sin x = 0$	$2 \sin x = 1$
$x = 0, \pi$	$\sin x = \frac{1}{2}$
	$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Graphs are useful for solving not only trigonometric equations but also trigonometric inequalities. For example, to solve the inequality  $\cos 2x < 1 - \sin x$ , look at the graph shown in Example 1. You can see that the graph of  $y = \cos 2x$  is *below* the graph of  $y = 1 - \sin x$  when  $\frac{\pi}{6} < x < \frac{5\pi}{6}$  and when  $\pi < x < 2\pi$ .

**Example 2** Solve  $3 \cos 2x + \cos x = 2$  for  $0 \leq x < 2\pi$ .

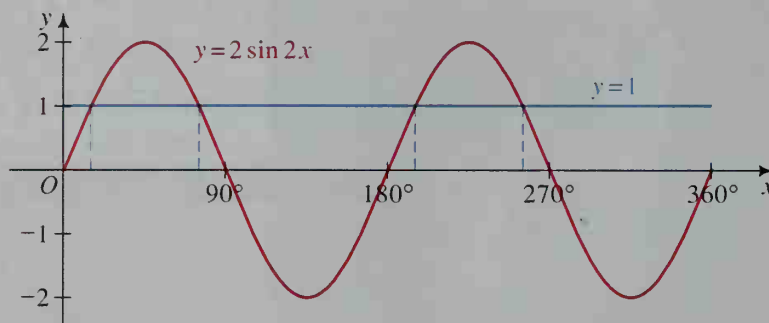
**Solution**

$$\begin{aligned}3 \cos 2x + \cos x &= 2 \\ 3(2 \cos^2 x - 1) + \cos x &= 2 \leftarrow \cos 2x = 2 \cos^2 x - 1 \\ 6 \cos^2 x + \cos x - 5 &= 0 \\ (6 \cos x - 5)(\cos x + 1) &= 0 \\ \cos x &= \frac{5}{6} \quad \text{or} \quad \cos x = -1 \\ x &\approx 0.59, 5.70 \quad \quad \quad x = \pi \\ &\quad \quad \quad x &\approx 3.14\end{aligned}$$



**Example 3**Solve  $2 \sin 2x = 1$  for  $0^\circ \leq x < 360^\circ$ .**Solution**

The graphs of  $y = 2 \sin 2x$  and  $y = 1$  over the interval  $0^\circ \leq x < 360^\circ$  are shown below. You can see that there are four possible solutions.



The most efficient way to solve this equation algebraically is to solve directly for  $2x$ .

$$\sin 2x = \frac{1}{2}$$

$$2x = 30^\circ, 150^\circ, 390^\circ, 510^\circ \quad \leftarrow 0^\circ \leq 2x < 720^\circ$$

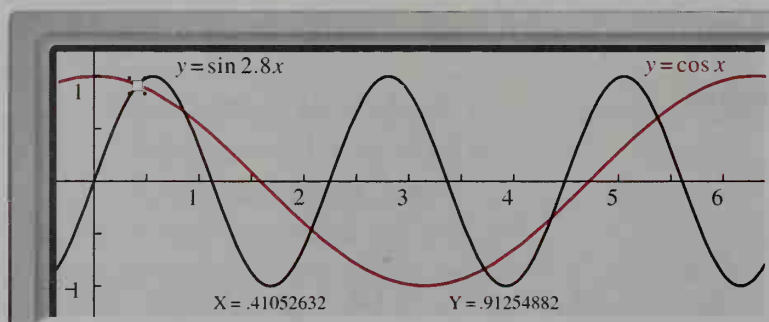
$$x = 15^\circ, 75^\circ, 195^\circ, 255^\circ$$



Examples 1 through 3 show that some trigonometric equations can be solved either algebraically or graphically. Example 4 shows that for some equations the best method is to use a graphing calculator or computer.

**Example 4**Find the smallest positive root of  $\sin 2.8x = \cos x$ .**Solution**

You can make a sketch of the graphs of  $y = \sin 2.8x$  and  $y = \cos x$  on the same set of axes to see approximately where the solutions are, but then finding an algebraic solution would be very difficult. In this case, the most efficient method is to use a graphing calculator or a computer. The diagram below indicates that there are six roots between 0 and  $2\pi$ . You are interested in the smallest one.



Using the zoom or trace feature, you will find that  $x \approx 0.41$ .

## CLASS EXERCISES

Describe the method you would use to solve the following equations for  $0^\circ \leq x < 360^\circ$ . Some of these equations can be solved by methods discussed in Chapter 8.

1.  $\cos 2x = \cos x$
2.  $\sin^2 x = \sin x$
3.  $\sin x = \cos x$
4.  $\sin 2x = \cos 2x$
5.  $\sin 3x = \cos 3x$
6.  $\tan (x - 10^\circ) = 1$
7. **Discussion** When solving  $\sin 4x = \sin 2x$ , would you begin by writing the expressions in terms of functions of  $x$  or  $2x$ ? Why?
8. **Discussion** When solving  $\cos 4x = 1 - 3 \cos 2x$ , would you begin by writing the expressions in terms of functions of  $x$  or  $2x$ ? Why?
9. a. On a single set of axes, sketch the graphs of  $y = \sin x$  and  $y = \cos x$  over the interval  $0 \leq x \leq \frac{\pi}{2}$ .  
b. Find the  $x$ -coordinates of any points of intersection.  
c. Over what interval(s) is  $\sin x > \cos x$ ?  
d. Over what interval(s) is  $\sin x < \cos x$ ?

## WRITTEN EXERCISES

- A** 1–8. Solve the equations in Class Exercises 1–8 for  $0^\circ \leq x < 360^\circ$ .
9. a. On a single set of axes, sketch the graphs of  $y = \sin 2x$  and  $y = \tan x$  for  $0 \leq x < 2\pi$ .  
b. Find the  $x$ -coordinates of the points where the graphs intersect by solving  $\sin 2x = \tan x$ .  
c. Solve  $\sin 2x > \tan x$  for  $0 \leq x < 2\pi$ .
  10. a. On a single set of axes, sketch the graphs of  $y = \cos 2x$  and  $y = 3 \cos x$  for  $0 \leq x < 2\pi$ .  
b. Find the  $x$ -coordinates of the points where the graphs intersect by solving  $\cos 2x = 3 \cos x$ .  
c. Solve  $\cos 2x < 3 \cos x$  for  $0 \leq x < 2\pi$ .



In Exercises 11–16, solve each inequality for  $0 \leq x < 2\pi$ . Give answers to the nearest hundredth. You may wish to use a graphing calculator or computer.

11.  $\cos x > \frac{1}{2} \sin x$
12.  $\tan x < 2 \sin x$
13.  $\sin \left( 3x - \frac{\pi}{2} \right) > 0$
14.  $2 \cos x \leq \cos \left( x - \frac{\pi}{2} \right)$
15.  $\cos x \leq \sin 2x$
16.  $\cos 2x \geq 5 - \cos x$

In Exercises 17–22, solve each equation for  $0^\circ \leq x < 360^\circ$  by using trigonometric identities. Give answers to the nearest tenth when necessary. You may wish to check your answers using a graphing calculator or computer.

**B**

17.  $2 \cos (x + 45^\circ) = 1$

18.  $\cot (x - 20^\circ) = 1$

19.  $\sin (60^\circ - x) = 2 \sin x$

20.  $2 \sin (30^\circ + x) = 3 \cos x$

21.  $\sin x = \sin 2x$

22.  $\tan^2 2x - 1 = 0$

In Exercises 23–32, solve each equation for  $0 \leq x < 2\pi$  by using trigonometric identities. Give answers to the nearest hundredth when necessary. You may wish to check your answers using a graphing calculator or computer.

23.  $\sin x \cos x = \frac{1}{2}$

24.  $\cos (3x + \pi) = \frac{\sqrt{2}}{2}$

25.  $\tan 2x = 3 \tan x$

26.  $\tan 2x + \tan x = 0$

27.  $\cos 2x = 5 \sin^2 x - \cos^2 x$

28.  $\sin 2x \sec x + 2 \cos x = 0$

29.  $3 \sin x = 1 + \cos 2x$

30.  $\sin 2x = 5 \cos^2 x$

31.  $\cos 2x = \sec x$

32.  $\sin x \cos 2x = 1$

33. a. Is it possible to use trigonometric identities to solve  $\sin 2x = x$ ?  
 b. On a single set of axes, sketch the graphs of  $y = \sin 2x$  and  $y = x$ .  
 c. How many solutions are there to the equation  $\sin 2x = x$ ?  
 d. Use a scientific calculator and trial and error to approximate the positive solution(s) to the nearest tenth.
34. **Visual Thinking** On the same set of axes, sketch the graphs of  $y = \sin 2x$  and  $y = \ln (x + 1)$  for  $-2\pi < x < 2\pi$ . Then tell *how many* solutions the equation  $\sin 2x = \ln (x + 1)$  has in the interval  $-2\pi < x < 2\pi$ .
35. **Visual Thinking** On the same set of axes, sketch the graphs of  $y = 2 \cos 2x$  and  $y = 2^{-x}$  for  $-2\pi < x < 2\pi$ . Then tell *how many* solutions the equation  $2 \cos 2x = 2^{-x}$  has in the interval  $-2\pi < x < 2\pi$ .



Dolphins locate objects by means of ultrasonic pulses.



Solve each equation graphically for  $-2\pi \leq x < 2\pi$  by using a graphing calculator or computer. Give answers to the nearest hundredth.

36.  $\sin 1.5x = 2 \cos x$

37.  $\tan 2x = x$

38.  $\sin 2x = \ln(x + 1)$

39.  $2 \cos 2x = 2^{-x}$

Solve each equation for  $0 \leq x < 2\pi$ .



40.  $\sin 3x = \sin 5x + \sin x$  (Hint: Use one of the formulas given on page 372.)

41.  $\cos 3x + \cos x = \cos 2x$

42.  $\sin 3x - \sin x = 2 \cos 2x$

Solve.

43.  $\cos^{-1} 2x = \sin^{-1} x$

44.  $\tan^{-1} 2x = \sin^{-1} x$

## ////COMPUTER EXERCISES

Write a computer program that will print a table of sines and cosines for  $1^\circ, 2^\circ, \dots, 90^\circ$  given only that  $\sin 1^\circ = 0.017452406$ . Hint: Here is how you could begin:

1. Since  $\sin^2 1^\circ + \cos^2 1^\circ = 1$ ,

$$\cos 1^\circ = \sqrt{1 - \sin^2 1^\circ} = \sqrt{1 - (0.017452406)^2},$$

which the computer may evaluate as 0.999847695.

2.  $\sin 2^\circ = \sin (1^\circ + 1^\circ) = \sin 1^\circ \cos 1^\circ + \cos 1^\circ \sin 1^\circ = 0.034899495$ .

$$\cos 2^\circ = \sqrt{1 - \sin^2 2^\circ} = 0.999390827.$$

3.  $\sin 3^\circ = \sin (2^\circ + 1^\circ) = \sin 2^\circ \cos 1^\circ + \cos 2^\circ \sin 1^\circ$ , and so on.

## Chapter Summary

1. The *sum and difference formulas* for sine, cosine, and tangent are as follows:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

2. The formulas for rewriting a trigonometric sum or difference as a product are as follows:

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

3. An angle  $\theta$  formed by intersecting lines  $l_1$  and  $l_2$  with slopes  $m_1$  and  $m_2$  can be found by using the following formula:

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

4. The *double-angle formulas* shown below are derived from the formulas for  $\sin(\alpha + \beta)$ ,  $\cos(\alpha + \beta)$ , and  $\tan(\alpha + \beta)$ .

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

5. The *half-angle formulas* shown below are derived from two of the double-angle formulas for cosine.

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}$$

6. Suggested methods for solving a trigonometric equation either graphically or algebraically are given on page 386.

### Key vocabulary and ideas

sum and difference formulas for cosine and sine (p. 370)  
 formulas for rewriting a sum or difference as a product (p. 372)  
 sum and difference formulas for tangent (pp. 375–376)  
 angle formed by intersecting lines (pp. 376–377)  
 double-angle formulas (p. 381, p. 383)  
 half-angle formulas (pp. 382–383)



## Chapter Test

1. Simplify the given expression. 10-1
  - a.  $\cos 75^\circ \cos 15^\circ + \sin 75^\circ \sin 15^\circ$
  - b.  $\sin 75^\circ \cos 15^\circ + \cos 75^\circ \sin 15^\circ$
  - c.  $\cos (30^\circ + x) + \cos (30^\circ - x)$
  - d.  $\sin (45^\circ - x) - \sin (45^\circ + x)$
2. Find the exact value of  $\cos 15^\circ$ .
3. Find  $\tan \left( \frac{5\pi}{4} - \theta \right)$  when  $\tan \theta = -\frac{1}{3}$ . 10-2
4. If  $\tan \alpha = \frac{4}{3}$  and  $\tan \beta = -\frac{1}{2}$ , show that  $\tan (\alpha + \beta) = \tan (\pi - \beta)$ .
5. **Writing** Consider the lines  $y = 2x + 1$  and  $y = 4 - 3x$ . Write a paragraph explaining how it is possible to find two different angles that are formed by the intersection of these lines. What is the relationship between these angles?
6. Suppose  $\angle A$  is acute and  $\cos A = \frac{4}{5}$ . Find each of the following: 10-3
  - a.  $\sin A$
  - b.  $\cos 2A$
  - c.  $\sin 2A$
  - d.  $\sin 4A$
7. Simplify the given expression.
  - a.  $\frac{\sin 2x}{1 - \cos 2x}$
  - b.  $(1 + \tan^2 y)(\cos 2y - 1)$
  - c.  $\frac{\tan t}{\sec t + 1}$
  - d.  $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$
8. Evaluate the given expression.
  - a.  $2 \cos^2 \frac{\pi}{12} - 1$
  - b.  $4 \sin \frac{\pi}{6} \cos \frac{\pi}{6}$
9. Prove that the given equation is an identity.
  - a.  $(1 + \cot^2 x)(1 - \cos 2x) = 2$
  - b.  $\frac{\sin \theta \sec \theta}{\tan \theta + \cot \theta} = \cos^2 \theta - \cos 2\theta$
10. a. On the same set of axes, sketch the graphs of  $y = \cos 2\theta$  and  $y = \sin \theta$  for  $0^\circ \leq \theta < 360^\circ$ . 10-4  
b. Determine where the graphs intersect by solving  $\cos 2\theta = \sin \theta$ .
11. Solve the equation  $\cos 2x = \cos x + 2$  either graphically or algebraically for  $0 \leq x < 2\pi$ .

# 11

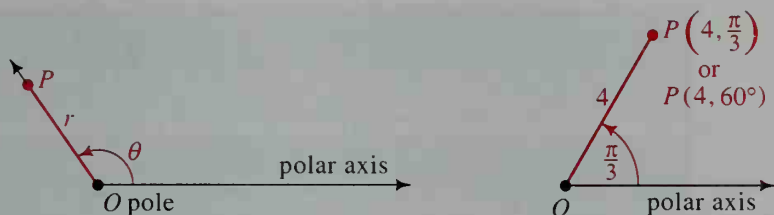
## *Polar Coordinates and Complex Numbers*



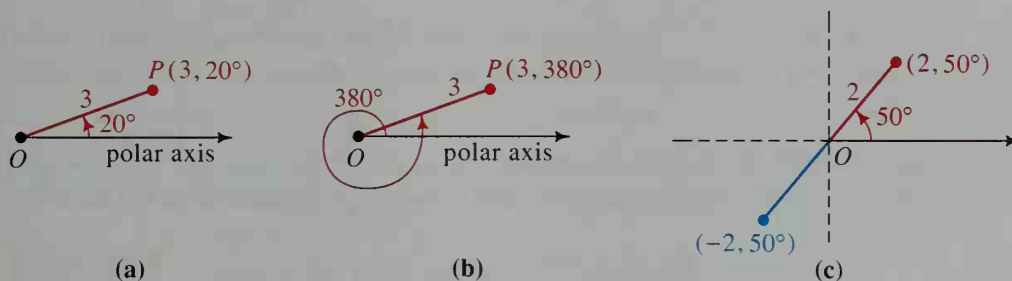
# 11-1 Polar Coordinates and Graphs

**Objective** To graph polar equations.

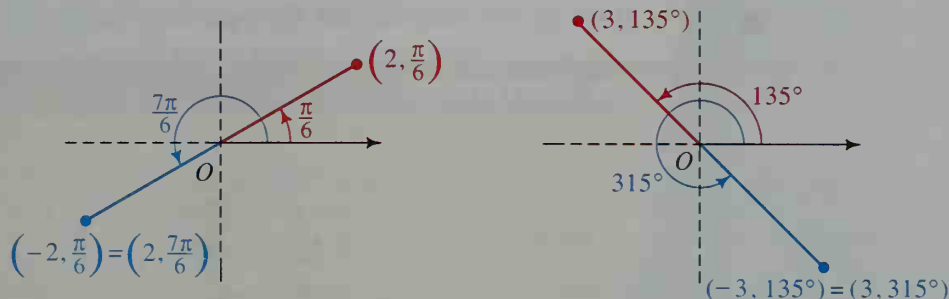
The position of a point  $P$  in the plane can be described by giving its directed distance  $r$  from a fixed point  $O$ , called the **pole**, and the measure of an angle formed by ray  $OP$  and a reference ray, called the **polar axis**.



The number  $r$  and the angle  $\theta$  are called **polar coordinates** of  $P$ , and we write  $P = (r, \theta)$ . The angle  $\theta$  can be measured either in degrees or radians. Figures (a) and (b) show that the same point can be represented by more than one pair of polar coordinates. To find a point when  $r$  is negative, simply find the ray that forms the angle  $\theta$  with the polar axis, and then go  $r$  units in the opposite direction from the ray, as shown in figure (c).



Another way to find a point when  $r < 0$  is to plot the point  $(|r|, \theta + \pi)$  or the point  $(|r|, \theta + 180^\circ)$ . This is illustrated below.



◀ The graceful shape of this modern staircase echoes the lines of the chambered nautilus. The underlying mathematical curve is the *equiangular spiral*, a shape that can be described very simply with polar coordinates.

An equation in  $r$  and  $\theta$  is called a **polar equation**. To sketch its graph, called a **polar graph**, you plot in a *polar coordinate system* several points  $(r, \theta)$  that satisfy the equation, and then draw a smooth curve through them.

**Example 1** Sketch the polar graph of  $r = 2 \cos \theta$ .

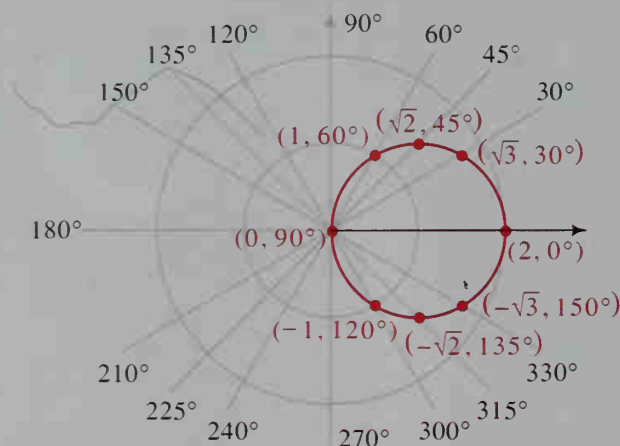
**Solution** Make a table of values to find values of  $r$  for selected values of  $\theta$ .

$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
$r$	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2

$\theta$	$210^\circ$	$225^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$315^\circ$	$330^\circ$	$360^\circ$
$r$	$-\sqrt{3}$	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	$\sqrt{3}$	2

From the table, you can see that the graph is complete for values of  $\theta$  between  $0^\circ$  and  $180^\circ$ . For values of  $\theta$  between  $180^\circ$  and  $360^\circ$ , the same graph is traced again. The figure below suggests that the graph is a circle.



The relationship between polar coordinates and Cartesian (or rectangular) coordinates of any point is given by the diagram and the equations below.

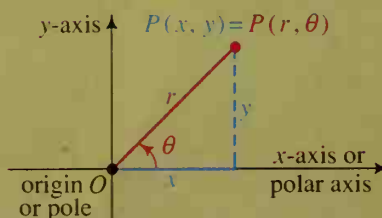
### Polar and Rectangular Conversions

Converting from polar to rectangular coordinates:

$$x = r \cos \theta \quad y = r \sin \theta$$

Converting from rectangular to polar coordinates:

$$r = \pm \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x}$$



The diagram on the previous page illustrates the convention that the pole is placed at the origin with the polar axis coinciding with the positive  $x$ -axis and with positive angles measured in the counterclockwise direction. The given equations can be used as formulas for converting from one coordinate system to another. Many scientific and graphing calculators have built-in conversion operations.

### Example 2

- Give polar coordinates for the point  $(3, 4)$ .
- Give the rectangular coordinates for the point  $(3, 30^\circ)$ .

### Solution

$$\text{a. } r = \pm \sqrt{x^2 + y^2} = \pm \sqrt{3^2 + 4^2} = \pm 5$$

$$\tan \theta = \frac{y}{x} = \frac{4}{3}. \text{ Thus, } \theta \approx 53.1^\circ \text{ is one possible value of the angle.}$$

$(3, 4)$  is in Quadrant I. If  $\theta = 53.1^\circ$ , then the point is on the terminal ray, and so  $r = 5$ . Therefore, polar coordinates are  $(5, 53.1^\circ)$ . Another possible pair of polar coordinates of the same point is  $(-5, 233.1^\circ)$ .

$$\text{b. } x = r \cos 30^\circ = 3 \cos 30^\circ \text{ and } y = r \sin 30^\circ = 3 \sin 30^\circ.$$

$$\text{Thus, rectangular coordinates of the point are } \left( \frac{3\sqrt{3}}{2}, \frac{3}{2} \right).$$



The graph of a polar equation can be drawn by a graphing calculator, as described below, provided the calculator has a polar mode or a parametric mode. Many calculators use  $t$  instead of  $\theta$ .

If the calculator has polar mode, enter the equation:

$$r = f(t)$$

If the calculator has parametric mode, enter the equations:

$$x = r \cos t = f(t) \cos t$$

$$y = r \sin t = f(t) \sin t$$

The two equations for  $x$  and  $y$  are called *parametric equations*. We will study them in greater detail in Chapter 12, where  $t$  often represents time in a practical problem.

### Example 3

Sketch the polar graph of  $r = 2 \sin 2\theta$ .

### Solution

**With a graphing calculator**

In polar mode, enter:

$$r = 2 \sin 2t$$

In parametric mode, enter:

$$x = 2 \sin 2t \cos t$$

$$y = 2 \sin 2t \sin t$$

The following values were used for the graph of  $r = 2 \sin 2t$  shown:

$$0 \leq t \leq 2\pi, \text{ step: } \frac{\pi}{30}$$

$$-2 \leq x \leq 2, \text{ scale: } 1$$

$$-2 \leq y \leq 2, \text{ scale: } 1$$

(Solution continues on the next page.)

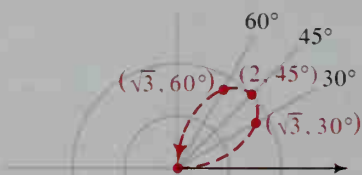




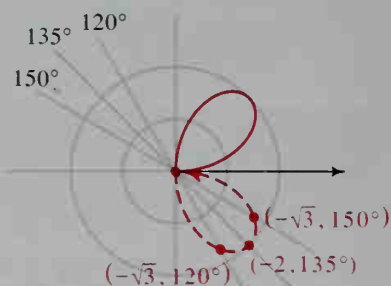
### Without a graphing calculator

Make a table of values and plot the points. The four polar plots below show how the graph evolves as the value of  $\theta$  increases. The completed graph, the last graph shown at the right, is called a *four-leaved rose*.

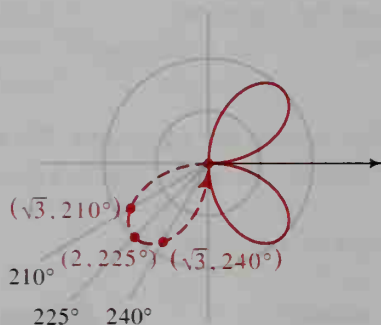
$\theta$	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$
$r$	0	$\sqrt{3}$	2	$\sqrt{3}$	0



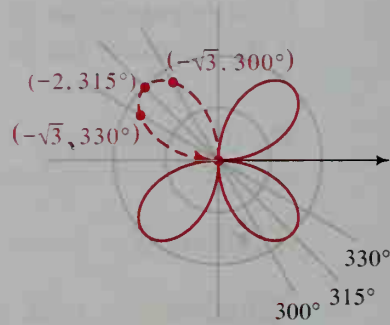
$\theta$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
$r$	$-\sqrt{3}$	-2	$-\sqrt{3}$	0



$\theta$	$210^\circ$	$225^\circ$	$240^\circ$	$270^\circ$
$r$	$\sqrt{3}$	2	$\sqrt{3}$	0



$\theta$	$300^\circ$	$315^\circ$	$330^\circ$	$360^\circ$
$r$	$-\sqrt{3}$	-2	$-\sqrt{3}$	0



#### Example 4

Find a rectangular equation for  $r = 2 \sin 2\theta$ , whose graph was shown in Example 3.

#### Solution

Use the conversion formulas given on page 396.

$$r = 2 \sin 2\theta$$

$$r = 4 \sin \theta \cos \theta$$

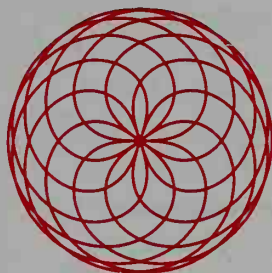
$$r = 4 \cdot \frac{y}{r} \cdot \frac{x}{r}$$

$$r^3 = 4xy$$

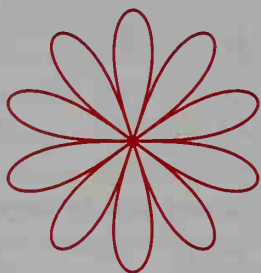
$$(\pm \sqrt{x^2 + y^2})^3 = 4xy$$

$$(x^2 + y^2)^3 = 16x^2y^2$$

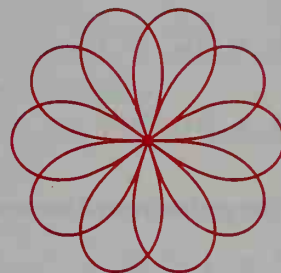
Example 4 illustrates that the polar equation of a curve can be much simpler than its rectangular equation. Sometimes the rectangular equation can be the simpler one. The polar equations of the following curves are all rather simple, but the corresponding rectangular equations are extremely complicated.



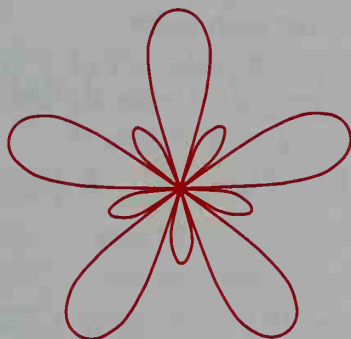
$$r^2 = \sin \frac{5}{8} \theta$$



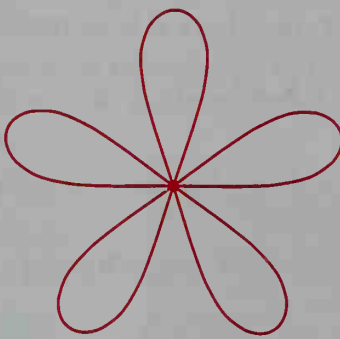
$$r^2 = \sin 5\theta$$



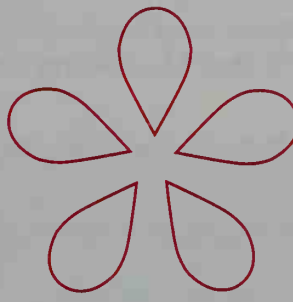
$$r^2 = \sin \frac{5}{2} \theta$$



$$\left(r - \frac{2}{5}\right)^2 = \sin 5\theta$$



$$(r - 1)^2 = \sin 5\theta$$



$$\left(r - \frac{7}{5}\right)^2 = \sin 5\theta$$

These curves were drawn using the *Precalculus Plotter Plus* software. The Computer Exercises on page 402 will ask you to experiment with polar graphs.

## CLASS EXERCISES

- Plot each point whose polar coordinates are given. Then give two other pairs of polar coordinates for the same point.
  - $(6, 50^\circ)$
  - $(-7, 120^\circ)$
  - $\left(3, \frac{\pi}{4}\right)$
  - $\left(-4, \frac{5\pi}{6}\right)$
- Plot each point whose polar coordinates are given. Then give two other pairs of polar coordinates for the same point.
  - $(5, -45^\circ)$
  - $(-2, -60^\circ)$
  - $\left(1, -\frac{\pi}{2}\right)$
  - $\left(-\frac{3}{2}, -3\pi\right)$
- Give polar coordinates for the point  $(1, -1)$ .
- Give the rectangular coordinates for the point  $(-8, 60^\circ)$ .
- Discussion** Find the rectangular equation for  $r = 2 \cos \theta$ , whose polar graph is shown on page 396. What is the graph of the rectangular equation? Does the rectangular equation confirm the polar graph? Explain.

## WRITTEN EXERCISES

- A** Plot each point whose polar coordinates are given. Then give two other pairs of polar coordinates for the point.

- |                        |                                      |                                      |                       |
|------------------------|--------------------------------------|--------------------------------------|-----------------------|
| 1. a. $A(2, 40^\circ)$ | b. $B(3, -100^\circ)$                | c. $C\left(5, \frac{3\pi}{2}\right)$ | d. $D(-4, 10^\circ)$  |
| 2. a. $A(1, 80^\circ)$ | b. $B\left(6, -\frac{\pi}{6}\right)$ | c. $C(2, 490^\circ)$                 | d. $D(-3, 110^\circ)$ |

Give polar coordinates  $(r, \theta)$ , where  $\theta$  is in degrees, for each point.

- |                  |                      |                            |                     |
|------------------|----------------------|----------------------------|---------------------|
| 3. a. $(-2, 2)$  | b. $(5, 0)$          | c. $(\sqrt{2}, -\sqrt{2})$ | d. $(-\sqrt{3}, 1)$ |
| 4. a. $(-4, -4)$ | b. $(3, -3\sqrt{3})$ | c. $(0, 6)$                | d. $(0, 0)$         |

Give polar coordinates  $(r, \theta)$ , where  $\theta$  is in radians, for each point.

- |   |                     |   |              |
|---|---------------------|---|--------------|
| 5. a. $(-1, -1)$  | b. $(0, 12)$        | c. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ | d. $(-2, 0)$ |
| 6. a. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ | b. $(-\sqrt{3}, 3)$ | c. $(0, -4)$                                      | d. $(e, -e)$ |

Give the rectangular coordinates for each point.

- |                        |                     |                                       |                                     |
|------------------------|---------------------|---------------------------------------|-------------------------------------|
| 7. a. $(4, 120^\circ)$ | b. $(-3, 90^\circ)$ | c. $\left(1, \frac{5\pi}{6}\right)$   | d. $\left(2, \frac{3\pi}{4}\right)$ |
| 8. a. $(2, 225^\circ)$ | b. $(6, -30^\circ)$ | c. $\left(10, -\frac{3\pi}{2}\right)$ | d. $\left(-4, \frac{\pi}{3}\right)$ |

Find the rectangular coordinates for each point to three decimal places.

- |                        |                    |             |              |
|------------------------|--------------------|-------------|--------------|
| 9. a. $(1, 20^\circ)$  | b. $(2, 20^\circ)$ | c. $(1, 2)$ | d. $(1, -2)$ |
| 10. a. $(1, 65^\circ)$ | b. $(4, 65^\circ)$ | c. $(2, 3)$ | d. $(2, -3)$ |
11. Find polar coordinates for each point. Give  $r$  to the nearest tenth and  $\theta$  to the nearest tenth of a degree.
- |             |             |              |              |
|-------------|-------------|--------------|--------------|
| a. $(3, 4)$ | b. $(1, 2)$ | c. $(-2, 3)$ | d. $(4, -6)$ |
|-------------|-------------|--------------|--------------|
12. Find polar coordinates for each point. Give  $r$  to the nearest tenth and  $\theta$  to the nearest tenth of a radian.
- |             |              |               |              |
|-------------|--------------|---------------|--------------|
| a. $(5, 2)$ | b. $(8, -6)$ | c. $(-1, -4)$ | d. $(-8, 6)$ |
|-------------|--------------|---------------|--------------|



In Exercises 13–22, sketch the polar graph of each equation. Also, give a rectangular equation of each graph. You may wish to use a computer or a graphing calculator to sketch these graphs.

- |                                      |  |
|--------------------------------------|--|
| 13. $r = \sin \theta$                | 14. $r = \cos \theta$                  |
| 15. $r = -3 \cos \theta$             | 16. $r = -5 \sin \theta$               |
| 17. $r = 1 - \sin \theta$ (cardioid) | 18. $r = 2 + 2 \cos \theta$ (cardioid) |

19.  $r = 1 + 2 \sin \theta$  (limaçon)

20.  $r = 1 - 2 \cos \theta$  (limaçon)

21.  $r = \sec \theta$

22.  $r = 2 \csc \theta$

23. a. Show that the formulas for converting from rectangular to polar coordinates given at the bottom of page 396 come from the definitions of the sine and cosine functions given on page 268.
- b. Use the equations in part (a) and the Pythagorean theorem to derive the formulas for converting from polar to rectangular coordinates.
24. a. The rectangular equations  $x = 4$  and  $y = 2$  have very simple graphs. Sketch them in a rectangular coordinate system.
- b. Do the polar equations  $r = 4$  and  $\theta = 2$  (radians) have simple graphs? Sketch them in a polar coordinate system.
- c. Show that the line with rectangular equation  $x = 3$  has polar equation  $r = 3 \sec \theta$ .
- d. Find a polar equation of the line with rectangular equation  $y = 3$ .



In Exercises 25–34, sketch the polar graph of each equation. Use radian measure for  $\theta$ . You may wish to use a computer or a graphing calculator to sketch these graphs.

**B**

25.  $r = \cos 2\theta$  (four-leaved rose)

26.  $r = \cos 4\theta$  (eight-leaved rose)

27.  $r = \sin 3\theta$  (three-leaved rose)

28.  $r = \sin 5\theta$  (five-leaved rose)

29.  $r = \theta$  (spiral of Archimedes)

30.  $r = \frac{1}{\theta}$  (hyperbolic spiral)

31.  $r = e^\theta$  (logarithmic spiral)

32.  $r^2\theta = 1$  (lituus)

(Hint:  $r = \frac{1}{\sqrt{\theta}}$  and  $r = -\frac{1}{\sqrt{\theta}}$ ).

33.  $r^2 = 4 \sin 2\theta$  (lemniscate)

34.  $r = \cos \frac{1}{2}\theta$



35. **Reading** Look up the definitions of *cardioid* and *limaçon* in a mathematics dictionary. How are these two curves related, and why do you think the name cardioid is given to its curve? Study the equations in Exercises 17–20. How can you identify the type of curve from its equation?
36. **Research** The limaçon was first studied and named by Étienne Pascal (1588?–1651), and so this curve is usually called *Pascal's limaçon*. The lemniscate was first studied by Jacques Bernoulli (1654–1705), and so this curve is frequently called the *lemniscate of Bernoulli*. Write a brief report on these two mathematicians, describing their background and some of their important contributions to mathematics.

- C** 37. **Investigation** In Chapter 4 we discussed how a rectangular equation can be tested to determine if its graph has symmetry. (See symmetry tests on page 134.) Discover some symmetry tests for polar equations. Write your results so that each test is clearly stated and an example of the test is given.
38. A conic section with eccentricity  $e$  has its focus at the origin. Its directrix is the line  $y = -p$ , where  $p > 0$ . Use the eccentricity definition of a conic given on page 247 to show that the polar equation of the conic is:

$$r = \frac{ep}{1 - e \sin \theta}$$

39. a. Find polar coordinates for the point of intersection of the graphs of  $r = 1 + \sin \theta$  and  $r = 2 \sin \theta$ .  
 b. Now sketch the two graphs and note that they have another common point whose polar coordinates do not satisfy both equations. Explain why this happens.
40. Suppose  $P(r, \theta)$  is any point on the circle with center  $C = (a, \alpha)$  and radius  $R$ . Use the law of cosines to show that the polar equation of the circle is:

$$r^2 - 2ar \cos(\theta - \alpha) + a^2 = R^2$$

## //// COMPUTER EXERCISES

You will need to use either computer graphing software or a graphing calculator that graphs polar equations.

1. a. Graph  $r = \sin kt$  or  $r = \cos kt$  for  $k = 1, 3$ , and  $5$ .  
 b. Predict what the graph of  $r = \sin kt$  or  $r = \cos kt$  will look like for larger odd values of  $k$ . Then check your prediction.  
 c. Graph  $r = \sin kt$  or  $r = \cos kt$  for  $k = 2, 4$ , and  $6$ .  
 d. Predict what the graph of  $r = \sin kt$  or  $r = \cos kt$  will look like for larger even values of  $k$ . Then check your prediction.  
 e. Make a generalization about the graph of an equation of the form  $r = a \sin nt$  or  $r = a \cos nt$ , where  $n$  is a positive integer.
2. Graph  $r^2 = k^2 \sin 2t$  and  $r^2 = k^2 \cos 2t$  for  $k = 1, 2$ , and  $3$ . Make a generalization about these graphs, which are all lemniscates.
3. Graph  $r = \sin \frac{a}{b}t$ , where  $a$  and  $b$  are both odd. Describe in a few sentences how the shape of the graph depends on the values of  $a$  and  $b$ .
4. The six equations whose graphs are shown on page 399 are all special cases of the equation  $(r - c)^2 = \sin \frac{a}{b}t$ . Experiment with values of  $b$  and  $c$  to see if you can discover any patterns. You might consider letting  $c = 0$  while gradually varying  $b$ . Then try letting  $b = 1$  while gradually varying  $c$ . Finally, let  $a$  vary while keeping  $b$  and  $c$  fixed. Write a paragraph or two describing your results and conclusions.



# 11-2 Geometric Representation of Complex Numbers

## Objective

To write complex numbers in polar form and to find products in polar form.

In 1806, Jean Robert Argand, a Parisian bookkeeper, published an essay in which complex numbers were represented geometrically. The diagram at the right, known as an **Argand diagram**, shows that the complex number  $3 + 4i$  can be represented by the point  $(3, 4)$  or by an arrow from the origin to  $(3, 4)$ . Similarly, the number  $5 - 2i$  can be represented by a specific point or arrow. These different representations will be discussed again in

Chapter 12. When the plane is used to represent complex numbers, the plane is called the **complex number plane**, or simply the **complex plane**. An important subset of this plane is the real number line. A typical point on this line has coordinates  $(a, 0)$ , and represents the complex number  $a + 0i$ . A point on the vertical axis, or imaginary axis, has coordinates  $(0, b)$  and represents  $0 + bi$ , a pure imaginary number.

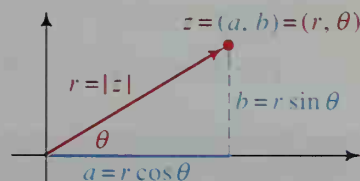
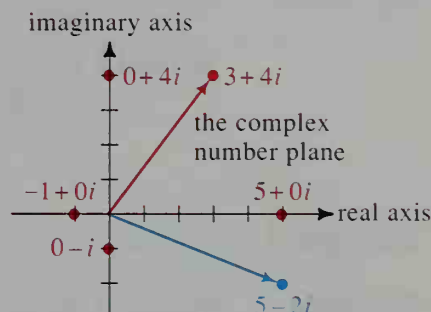
The point representing the complex number  $z = a + bi$  can be given either in rectangular coordinates  $(a, b)$  or in polar coordinates  $(r, \theta)$ . (See the diagram below.) We can specify a complex number in two ways:

rectangular form:  $z = a + bi$

polar form:  $z = r \cos \theta + (r \sin \theta)i$

$$z = r(\cos \theta + i \sin \theta)$$

$$z = r \operatorname{cis} \theta$$



The polar form  $r(\cos \theta + i \sin \theta)$  is often abbreviated as  $r \operatorname{cis} \theta$  (pronounced “*r* siss theta”). For example, we write  $2 \operatorname{cis} 30^\circ$  as the abbreviation for  $2(\cos 30^\circ + i \sin 30^\circ)$ .

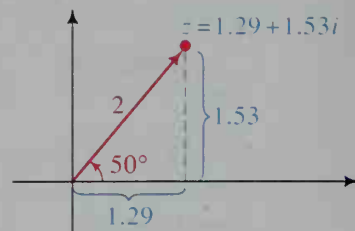
The length of the arrow representing  $z$  is called the **absolute value** of  $z$ , and is defined to be  $|z| = \sqrt{a^2 + b^2}$ . Since  $r = \sqrt{a^2 + b^2} = |z|$ , which is nonnegative, we will see only positive values of  $r$  in polar coordinates of a complex number  $z$ . The angle  $\theta$  is called the **polar angle** of  $z$ . Throughout the rest of this book, we will give final answers using positive radii and positive polar angles.

Many scientific calculators have keys that enable us to convert from rectangular form to polar form and vice versa. These conversion keys allow us to express very easily a complex number in either polar or rectangular form, as discussed in Examples 1 and 2.

**Example 1** Express  $2 \operatorname{cis} 50^\circ$  in rectangular form.

**Solution**

**Method 1** Using the polar to rectangular conversion key on a calculator, we find that  $(2, 50^\circ) \approx (1.29, 1.53)$ . Thus, the rectangular form of  $2 \operatorname{cis} 50^\circ$  is  $1.29 + 1.53i$ .

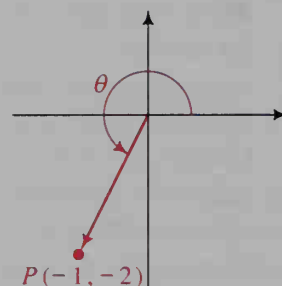


**Method 2** Use the fact that  $a = r \cos \theta$  and  $b = r \sin \theta$ , as follows:  
 $2 \operatorname{cis} 50^\circ = 2(\cos 50^\circ + i \sin 50^\circ)$   
 $= (2 \cos 50^\circ) + (2 \sin 50^\circ)i \approx 1.29 + 1.53i$

**Example 2** Express  $-1 - 2i$  in polar form.

**Solution**

**Method 1** Using the rectangular to polar conversion key on a calculator, we find that  $(-1, -2) \approx (2.24, -117^\circ)$ , or  $(2.24, 243^\circ)$ . Thus, the polar form of  $-1 - 2i$  is  $2.24 \operatorname{cis} 243^\circ$ .



**Method 2**  $r = |-1 - 2i| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5} \approx 2.24$

$$\tan^{-1}\left(\frac{-2}{-1}\right) \approx 63^\circ; \theta \approx 63^\circ + 180^\circ = 243^\circ$$

Thus, the polar form of  $-1 - 2i$  is  $\sqrt{5} \operatorname{cis} 243^\circ$ , or  $2.24 \operatorname{cis} 243^\circ$ .

The theorem below shows how to multiply complex numbers in polar form.

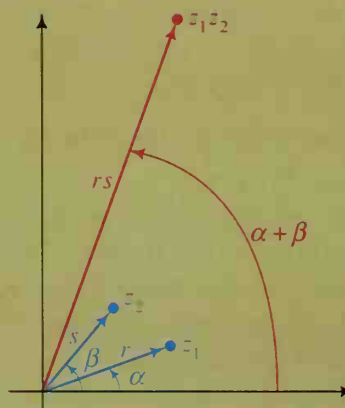
### Product of Two Complex Numbers in Polar Form

To multiply two complex numbers in polar form:

1. Multiply their absolute values.
2. Add their polar angles.

In other words, if  $z_1 = r \operatorname{cis} \alpha$  and  $z_2 = s \operatorname{cis} \beta$ , then:

$$\begin{aligned} z_1 z_2 &= (r \operatorname{cis} \alpha) \cdot (s \operatorname{cis} \beta) \\ &= rs \operatorname{cis}(\alpha + \beta) \end{aligned}$$



**Proof:**

$$\begin{aligned}
 r \operatorname{cis} \alpha \cdot s \operatorname{cis} \beta &= r(\cos \alpha + i \sin \alpha) \cdot s(\cos \beta + i \sin \beta) \\
 &= rs[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + \\
 &\quad i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)] \\
 &= rs[\cos(\alpha + \beta) + i \sin(\alpha + \beta)] \\
 &= rs \operatorname{cis}(\alpha + \beta)
 \end{aligned}$$

**Example 3** Express  $z_1 = 3 + 3i$ ,  $z_2 = 1 + i\sqrt{3}$ , and  $z_1 z_2$  in polar form. Graph each complex number.

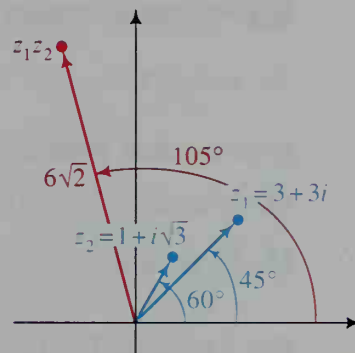
**Solution** Using either method of Example 2:

$$z_1 = 3\sqrt{2} \operatorname{cis} 45^\circ \text{ and}$$

$$z_2 = 2 \operatorname{cis} 60^\circ$$

By the previous theorem:

$$\begin{aligned}
 z_1 z_2 &= 3\sqrt{2} \cdot 2 \operatorname{cis}(45^\circ + 60^\circ) \\
 &= 6\sqrt{2} \operatorname{cis} 105^\circ
 \end{aligned}$$



To see the usefulness of the previous theorem, let's compare the multiplication of two complex numbers in polar form with the multiplication of the same numbers in rectangular form. Consider the numbers:

$$z_1 = 2 \operatorname{cis} 50^\circ \approx 1.286 + 1.532i \quad \text{and} \quad z_2 = 3 \operatorname{cis} 20^\circ \approx 2.819 + 1.026i$$

**Polar Method:**  $(2 \operatorname{cis} 50^\circ)(3 \operatorname{cis} 20^\circ) = 6 \operatorname{cis} 70^\circ$

**Rectangular Method:**  $(1.286 + 1.532i)(2.819 + 1.026i) \approx$   
 $3.625 + 1.319i + 4.319i + 1.572i^2 \approx 2.05 + 5.64i$

You can see that the two answers agree by expressing  $6 \operatorname{cis} 70^\circ$  in rectangular form.

$$6 \operatorname{cis} 70^\circ = (6 \cos 70^\circ) + (6 \sin 70^\circ)i \approx 2.05 + 5.64i$$

## CLASS EXERCISES

Express each complex number in polar form.

1.  $1 + i$

2.  $i$

3.  $-3$

4.  $\sqrt{3} - i$

5. **Reading** What does the expression  $2 \operatorname{cis} \frac{\pi}{4}$  mean?

Express each complex number in rectangular form.

6.  $5 \operatorname{cis} 90^\circ$

7.  $3 \operatorname{cis} \pi$

8.  $4 \operatorname{cis} 45^\circ$

9.  $6 \operatorname{cis} 30^\circ$

Express each product in polar form.

10.  $(4 \operatorname{cis} 25^\circ)(6 \operatorname{cis} 35^\circ)$

11.  $\left(5 \operatorname{cis} \frac{\pi}{4}\right)\left(2 \operatorname{cis} \frac{3\pi}{4}\right)$

## WRITTEN EXERCISES

Express each complex number in polar form. Give angle measures to the nearest degree when necessary.

- A**    1.  $-1 + i$                       2.  $-3i$                       3.  $1 + i\sqrt{3}$                       4.  $-2 - 2i$   
       5.  $-7$                       6.  $2\sqrt{3} + 2i$                       7.  $3 - 4i$                       8.  $5 + 12i$

Express each complex number in rectangular form.

9.  $6 \operatorname{cis} 100^\circ$                       10.  $8 \operatorname{cis} 230^\circ$                       11.  $9 \operatorname{cis} \frac{4\pi}{3}$                       12.  $2 \operatorname{cis} \frac{3\pi}{4}$

Express each product in polar and rectangular form.

13.  $(5 \operatorname{cis} 30^\circ)(2 \operatorname{cis} 60^\circ)$                       14.  $(2 \operatorname{cis} 115^\circ)(3 \operatorname{cis} 65^\circ)$   
 15.  $\left(8 \operatorname{cis} \frac{\pi}{3}\right)\left(\frac{1}{2} \operatorname{cis}\left(-\frac{2\pi}{3}\right)\right)$                       16.  $\left(4 \operatorname{cis} \frac{\pi}{4}\right)\left(3 \operatorname{cis} \frac{\pi}{2}\right)$

For Exercises 17–22:

- (a) Find  $z_1 z_2$  in rectangular form by multiplying  $z_1$  and  $z_2$ .  
 (b) Find  $z_1$ ,  $z_2$ , and  $z_1 z_2$  in polar form. Show that  $z_1 z_2$  in polar form agrees with  $z_1 z_2$  in rectangular form.  
 (c) Show  $z_1$ ,  $z_2$ , and  $z_1 z_2$  in an Argand diagram.

17.  $z_1 = 2 + 2i\sqrt{3}$ ,  $z_2 = \sqrt{3} - i$                       18.  $z_1 = 3 + 3i$ ,  $z_2 = -2i$   
 19.  $z_1 = 2 + 2i$ ,  $z_2 = 2 - 2i$                       20.  $z_1 = -1 + i\sqrt{3}$ ,  $z_2 = -1 - i\sqrt{3}$   
 21.  $z_1 = 6 - 4i$ ,  $z_2 = -5 + 2i$                       22.  $z_1 = -3 - 5i$ ,  $z_2 = 4 + 7i$

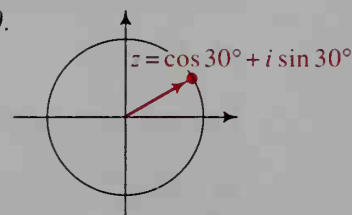
23. **Visual Thinking** Suppose that  $a + bi$  is multiplied by  $1 + i$ . By how many degrees must the arrow from  $(0, 0)$  to  $(a, b)$  be rotated to coincide with the arrow from  $(0, 0)$  to the product?

24. **Visual Thinking** Suppose that  $a + bi$  is multiplied by  $\sqrt{3} + i$ . By how many degrees must the arrow from  $(0, 0)$  to  $(a, b)$  be rotated to coincide with the arrow from  $(0, 0)$  to the product?

Exercises 25–28 should be worked in order because they preview the next section.

25. Restate the theorem on page 404 for the special case when  $z_2 = z_1$ .  
 26. Find  $(\cos \theta + i \sin \theta)^2$ . Show that your result equals  $\cos 2\theta + i \sin 2\theta$ .  
 27. From Exercise 26,  $(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta$ .  
 Make a conjecture about  $(\cos \theta + i \sin \theta)^3$ .

28. The diagram at the right shows the complex number  $z = \cos 30^\circ + i \sin 30^\circ$ . Copy this diagram, and on it show  $z^2$ ,  $z^3$ , and  $z^4$ . (*Hint:* Use the results of Exercises 26 and 27.)



- B** 29. a. If  $z = r(\cos \theta + i \sin \theta)$ , then the conjugate of  $z$  is  $\bar{z} = r(\cos \theta - i \sin \theta)$ . Represent  $z$  and  $\bar{z}$  in an Argand diagram.  
 b. **Visual Thinking** Show that  $\cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$ . Use this fact and your diagram in part (a) to explain why  $\bar{z} = r \operatorname{cis}(-\theta)$  if  $z = r \operatorname{cis} \theta$ .

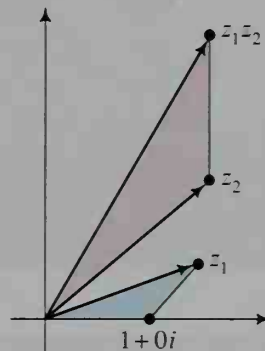
30. If  $z = r \operatorname{cis} \theta = r(\cos \theta + i \sin \theta)$ , show that  $\frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\theta)$ .

31. This exercise shows that to divide two complex numbers, you divide their absolute values and subtract their polar angles. Let  $z_1 = r \operatorname{cis} \alpha$  and  $z_2 = s \operatorname{cis} \beta$ , where  $z_2 \neq 0 + 0i$ . Prove that

$$\frac{z_1}{z_2} = \frac{r}{s} \operatorname{cis}(\alpha - \beta).$$

(Hint: Use the results of Exercise 30.)

- C** 32. (An alternate proof of the theorem on page 404.)  
 a. Let  $z_1 = a + bi$  and  $z_2 = c + di$ . Show  $z_1$  and  $z_2$  in an Argand diagram. Prove that the product is given by  $z_1 z_2 = (ac - bd) + (bc + ad)i$ .  
 b. On your diagram, let  $\alpha$ ,  $\beta$ , and  $\theta$  be the polar angles for  $z_1$ ,  $z_2$ , and  $z_1 z_2$ , respectively. Then  $\tan \alpha = \frac{?}{?}$ ,  $\tan \beta = \frac{?}{?}$ , and  $\tan \theta = \frac{?}{?}$ .  
 c. Use the values of  $\tan \alpha$  and  $\tan \beta$  in part (b) and the formula for  $\tan(\alpha + \beta)$  to show that  $\tan(\alpha + \beta) = \tan \theta$ . Then you can conclude that  $\alpha + \beta = \theta$ .  
 d. By definition,  $|z_1| = \sqrt{a^2 + b^2}$ ,  $|z_2| = \frac{?}{?}$ , and  $|z_1 z_2| = \frac{?}{?}$ . Show that  $|z_1 z_2| = |z_1| \cdot |z_2|$ .



Ex. 33

33. **Writing** Write a paragraph explaining why the two triangles in the diagram at the right are similar.

## 11-3 Powers of Complex Numbers

### Objective

To use De Moivre's theorem to find powers of complex numbers.

The polar form of a complex number makes it easy to find powers of a complex number. To find the square of a complex number, we use the theorem on page 404.

$$(r \operatorname{cis} \alpha)^2 = (r \operatorname{cis} \alpha)(r \operatorname{cis} \alpha) = r \cdot r \operatorname{cis}(\alpha + \alpha) = r^2 \operatorname{cis} 2\alpha$$

To find the cube of  $r \operatorname{cis} \alpha$ , we use the square of  $r \operatorname{cis} \alpha$  and the same theorem.

$$\begin{aligned} (r \operatorname{cis} \alpha)^3 &= (r \operatorname{cis} \alpha)^2 (r \operatorname{cis} \alpha) \\ &= (r^2 \operatorname{cis} 2\alpha)(r \operatorname{cis} \alpha) \\ &= r^2 \cdot r \operatorname{cis}(2\alpha + \alpha) \\ &= r^3 \operatorname{cis} 3\alpha \end{aligned}$$



The pattern just shown can be generalized for any positive integer  $n$ :

$$(r \operatorname{cis} \alpha)^n = r^n \operatorname{cis} n\alpha$$

In fact, this equation holds for *any* integer  $n$  and even for fractional values of  $n$ , as we will see in the next section. This result is known as De Moivre's Theorem.

### De Moivre's Theorem

If  $z = r \operatorname{cis} \theta$ , then  $z^n = r^n \operatorname{cis} n\theta$ .

#### Example

If  $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ , find  $z^2$ ,  $z^3$ ,  $z^4$ ,  $z^5$ , and  $z^6$ . Plot these on an Argand diagram.

#### Solution

First we put  $z$  into polar form, and then use De Moivre's theorem.

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i = 1 \operatorname{cis} 60^\circ$$

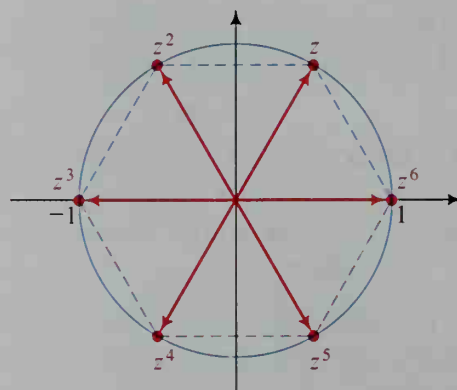
$$z^2 = 1^2 \operatorname{cis}(2 \cdot 60^\circ) = 1 \operatorname{cis} 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z^3 = 1^3 \operatorname{cis}(3 \cdot 60^\circ) = 1 \operatorname{cis} 180^\circ = -1$$

$$z^4 = 1^4 \operatorname{cis}(4 \cdot 60^\circ) = 1 \operatorname{cis} 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z^5 = 1^5 \operatorname{cis}(5 \cdot 60^\circ) = 1 \operatorname{cis} 300^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$z^6 = 1^6 \operatorname{cis}(6 \cdot 60^\circ) = 1 \operatorname{cis} 360^\circ = 1$$



The Argand diagram above shows that these six powers of  $z$  are closely related to the construction of a regular hexagon inscribed in a circle of radius 1.

The path taken by the points corresponding to the powers of a given complex number  $z$  depends greatly on  $z$ . To see a few of the possible paths, complete the activity given on the next page.

## Activity

Express each complex number  $z$  in polar form. Then graph  $z^n$  for  $n = 1, 2, \dots, 6$ . Describe the path taken by the points graphed.

1.  $z = 1 + i$

2.  $z = \frac{1}{2} + \frac{1}{2}i$

3.  $z = \sqrt{3} + i$

The paths discovered in the activity are called *equiangular spirals* and can be found throughout nature. For example, the horns and claws of some animals, the seeds of a sunflower, and the chambers of a nautilus shell form equiangular spirals.



## CLASS EXERCISES

Give the polar form of each of the following.

1.  $(2 \operatorname{cis} 45^\circ)^2$

2.  $(2 \operatorname{cis} 45^\circ)^3$

3.  $(\sqrt{2} \operatorname{cis} (-18^\circ))^4$

4.  $(1 \operatorname{cis} 36^\circ)^{10}$

5.  $\left(4 \operatorname{cis} \frac{\pi}{6}\right)^3$

6.  $\left(\sqrt{3} \operatorname{cis} \frac{5\pi}{6}\right)^6$

7. Let  $z = -1 + i$ .

a. Express  $z$  and  $z^6$  in polar form.

b. Express  $z^6$  in rectangular form.

8. Let  $z = 1 - i$ .

a. Find  $z^{10}$  by writing  $z$  as  $\sqrt{2} \operatorname{cis} 315^\circ$ , applying De Moivre's theorem, and simplifying the result.

b. Find  $z^{10}$  by writing  $z$  as  $\sqrt{2} \operatorname{cis} (-45^\circ)$ , applying De Moivre's theorem, and simplifying the result.

c. **Discussion** Which calculation was easier? Explain why.

## WRITTEN EXERCISES

- A**
1. If  $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ , express  $z$  in polar form. Calculate  $z^2$  and  $z^3$  by using De Moivre's theorem. Show  $z$ ,  $z^2$ , and  $z^3$  in an Argand diagram.
  2. If  $z = i$ , show  $z$ ,  $z^2$ ,  $z^3$ , and  $z^4$  in an Argand diagram.
  3. a. Express  $z = 1 - i$  in polar form. Show  $z^{-1}$ ,  $z^0$ ,  $z$ ,  $z^2$ ,  $z^3$ ,  $z^4$ ,  $z^5$ ,  $z^6$ ,  $z^7$ , and  $z^8$  in an Argand diagram.  
b. Show that  $z^{12} = -64$ .
  4. a. Express  $z = 1 + i\sqrt{3}$  in polar form. Show  $z^{-2}$ ,  $z^{-1}$ ,  $z^0$ ,  $z$ ,  $z^2$ ,  $z^3$ , and  $z^4$  in an Argand diagram.  
b. Show that  $z^{18} = 2^{18}$ .
  5. Compute  $(1 - i\sqrt{3})^3$  by two methods.  
a. Simplify  $(1 - i\sqrt{3})(1 - i\sqrt{3})(1 - i\sqrt{3})$ .  
b. Apply De Moivre's theorem and express your answer in rectangular form.
  6. Compute  $(-1 - i)^3$  by two methods.  
a. Simplify  $(-1 - i)(-1 - i)(-1 - i)$ .  
b. Apply De Moivre's theorem and express your answer in rectangular form.
  7. Refer to the diagram on page 408. Do you see that  $z^7$  simplifies to  $z$ ? Simplify  $z^{14}$ ,  $z^{40}$ ,  $z^{-1}$ , and  $z^{-11}$ .
  8. Let  $z = \cos \theta + i \sin \theta$ . Evaluate  $|z^{100}|$ .

- B**
9. a. Verify that De Moivre's theorem is true when  $n = 0$ .  
b. In Exercise 30 on page 407, you showed that for  $z \neq 0 + 0i$ ,

$$\frac{1}{z} = \frac{1}{r} \operatorname{cis}(-\theta).$$

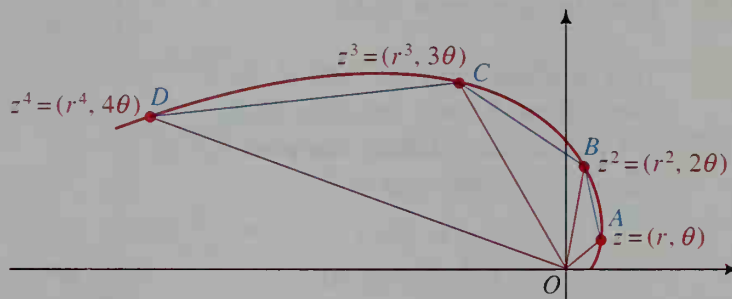
Use this fact to prove De Moivre's theorem for the case when  $n = -1$  by writing  $z^{-1} = \frac{1}{z}$  and then simplifying.

10. In Exercise 9, you proved De Moivre's theorem for  $n = -1$ . Now prove it for  $n = -2$  by writing  $z^{-2} = (z^{-1})^2$ . (A similar proof of De Moivre's theorem can be made for any negative integer  $n$ .)
11. Make an Argand diagram showing the number  $-1$ . On the same diagram, show a number  $z$  whose square is  $-1$ .
12. Make an Argand diagram showing the number  $-i$ . On the same diagram, show a number  $z$  whose cube is  $-i$ .

In Exercises 13–16, use the following identity:  $re^{i\theta} = r \operatorname{cis} \theta$ .

13. Show that  $e^{i\pi} = -1$ .
14. Is  $i^i$  a real number or an imaginary number? (Hint: Let  $r = 1$  and  $\theta = \frac{\pi}{2}$  in the identity  $re^{i\theta} = r(\cos \theta + i \sin \theta)$ , simplify the result, and then raise both sides of the equation to the power  $i$ .) Find  $i^i$ .

15. Prove De Moivre's theorem for all real numbers  $n$ . Assume that the laws of exponents hold for imaginary exponents.
16. Use the laws of exponents to simplify the product  $(re^{i\alpha})(se^{i\beta})$ . Interpret your answer by using the identity  $re^{i\theta} = r \operatorname{cis} \theta$ .
- C** 17. The curve through the powers of a complex number  $z$  is called an **equiangular spiral** because it crosses all lines through the origin at the same angle. You can get a rough idea why this happens by studying the diagram below.



- a. Prove that  $\triangle OAB$ ,  $\triangle OBC$ , and  $\triangle OCD$  are all similar. (It is easiest to use the SAS Similarity theorem. This proof can be extended to include all other triangles formed by the lines through the origin and the points representing the powers of  $z$ .)
  - b. Why are  $\angle OAB$ ,  $\angle OBC$ , and  $\angle OCD$  all congruent? (The fact that these angles are congruent suggests that the lines through the origin meet the curve at the same angle.)
18. Draw the polar graph of  $\ln r = \theta$ . (You may wish to use a computer or graphing calculator.) Your result should look like the equiangular spiral of Exercise 17. Because the polar equation contains  $\ln r$ , the spiral is also called a **logarithmic spiral**.



*René Descartes (1596–1650)*

The philosopher René Descartes was educated in France, and pursued military service in the Netherlands. It is thought that his army work with maps may have led to his notation known today as *Cartesian coordinates*, which is the foundation of analytic geometry.

Descartes modernized algebraic notation. He was the first to use the last letters of the alphabet to represent unknowns and the beginning letters to represent constants. He also introduced the terms *function*, *real*, and *imaginary*.

# 11-4 Roots of Complex Numbers

## Objective

To find roots of complex numbers.

In this section, we use DeMoivre's Theorem to find roots of complex numbers. In general, a nonzero complex number will have two square roots, three cube roots, four fourth roots, and  $k$   $k$ th roots. Here are two examples:

**Example 1** Find the cube roots of  $8i$ .

## Solution

This problem can be solved by thinking of  $8i$  and its cube root,  $z = r \operatorname{cis} \alpha$ , as points in the plane. Since  $z$  is a cube root of  $8i$ ,

$$\begin{aligned} z^3 &= 8i \\ (r \operatorname{cis} \alpha)^3 &= 8 \operatorname{cis} 90^\circ \\ r^3 \operatorname{cis} 3\alpha &= 8 \operatorname{cis} 90^\circ \end{aligned}$$

$$\begin{aligned} \text{Thus, } r^3 &= 8 \quad \text{and} \quad 3\alpha = 90^\circ \text{ or } (90^\circ + \text{a multiple of } 360^\circ) \\ r &= 2 \quad \alpha = 30^\circ \text{ or } (30^\circ + \text{a multiple of } 120^\circ) \\ &\quad \alpha = 30^\circ, 30^\circ + 120^\circ, 30^\circ + 240^\circ \\ &\quad \alpha = 30^\circ, 150^\circ, 270^\circ \end{aligned}$$

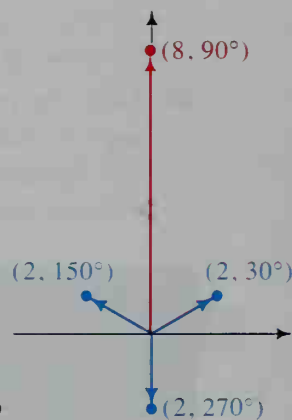
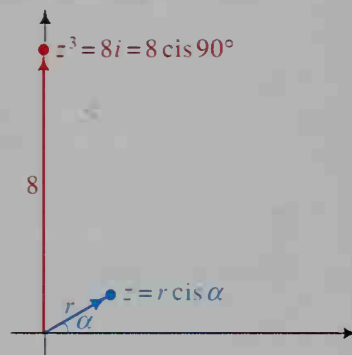
Therefore, the three cube roots of  $8i$  are:

$$\begin{aligned} z_1 &= 2 \operatorname{cis} 30^\circ = \sqrt{3} + i \\ z_2 &= 2 \operatorname{cis} 150^\circ = -\sqrt{3} + i \\ z_3 &= 2 \operatorname{cis} 270^\circ = -2i \end{aligned}$$

As a check, you can use De Moivre's theorem to cube  $z_1$ ,  $z_2$ ,  $z_3$  and see that the result in each case is  $8i$ .

From the analysis above, we see that if  $z = r \operatorname{cis} \theta$ , then:

$$\sqrt[3]{z} = z^{1/3} = r^{1/3} \operatorname{cis} \left( \frac{\theta}{3} + \frac{k \cdot 360^\circ}{3} \right) \text{ for } k = 0, 1, 2$$



**Example 2** Find the four fourth roots of  $-16$ .

## Solution

The complex number  $-16 + 0i$  can be expressed in polar form as  $16 \operatorname{cis} \pi$ . If  $z = r \operatorname{cis} \alpha$  is a fourth root of  $-16$ , then

$$\begin{aligned} z^4 &= -16 \\ (r \operatorname{cis} \alpha)^4 &= 16 \operatorname{cis} \pi \\ r^4 \operatorname{cis} 4\alpha &= 16 \operatorname{cis} \pi \end{aligned}$$



Thus,  $r^4 = 16$  and  $4\alpha = \pi$  or  $(\pi + \text{multiples of } 2\pi)$

$$r = 16^{1/4} \quad \alpha = \frac{\pi}{4} \text{ or } \left(\frac{\pi}{4} + \text{multiples of } \frac{\pi}{2}\right)$$

$$r = 2 \quad \alpha = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

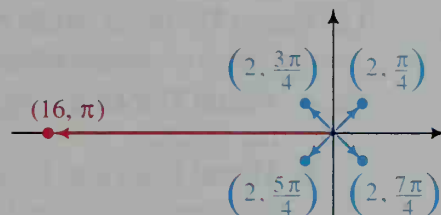
The four fourth roots of  $-16$  are:

$$z_1 = 2 \operatorname{cis} \frac{\pi}{4} = \sqrt{2} + i\sqrt{2}$$

$$z_2 = 2 \operatorname{cis} \frac{3\pi}{4} = -\sqrt{2} + i\sqrt{2}$$

$$z_3 = 2 \operatorname{cis} \frac{5\pi}{4} = -\sqrt{2} - i\sqrt{2}$$

$$z_4 = 2 \operatorname{cis} \frac{7\pi}{4} = \sqrt{2} - i\sqrt{2}$$



The method given in these examples can be generalized as follows:

### The $n$ th Roots of a Complex Number

The  $n$ th roots of  $z = r \operatorname{cis} \theta$  are:

$$\sqrt[n]{z} = z^{1/n} = r^{1/n} \operatorname{cis} \left( \frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right) \text{ for } k = 0, 1, 2, \dots, n-1$$

## WRITTEN EXERCISES

Find the indicated roots of each number.

- A** 1. cube roots of  $i$       2. cube roots of  $-i$       3. cube roots of 8
4. cube roots of  $-8$       5. fourth roots of 16      6. fourth roots of  $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$
7. a. Refer to the answers in Example 1. Show that  $z_1 + z_2 + z_3 = 0$  and that  $z_1 z_2 z_3 = 8i$ .  
 b. Any cube root of  $8i$  must satisfy the equation  $z^3 = 8i$ . Show that the results in part (a) verify Theorem 5 on page 86.
8. a. Refer to the answers in Example 2. Show that  $z_1 + z_2 + z_3 + z_4 = 0$  and that  $z_1 z_2 z_3 z_4 = 16$ .  
 b. Any fourth root of  $-16$  must satisfy  $z^4 = -16$ . Show that the results in part (a) verify Theorem 5 on page 86.
9. The three cube roots of 8 must satisfy the equation  $z^3 - 8 = 0$ . Solve this equation. (Choose one of the methods given in Chapter 2.) Your answers should agree with those of Exercise 3.

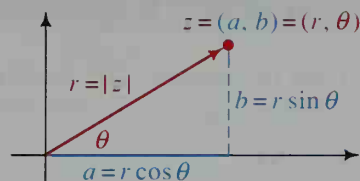
10. The three cube roots of  $-8$  must satisfy the equation  $z^3 + 8 = 0$ . Solve this equation. (Choose one of the methods given in Chapter 2.) Your answers should agree with those of Exercise 4.
- B** 11. a. Factor  $z^4 + 4$  by writing it as  $(z^4 + 4z^2 + 4) - 4z^2$ , which is the difference of two squares.  
 b. Use part (a) to solve  $z^4 = -4$ , and plot the roots of the equation on an Argand diagram.  
 c. Use De Moivre's theorem to verify your answer to part (b).
12. **Visual Thinking** Study the diagrams for Examples 1 and 2, and describe how the roots of complex numbers are related to regular polygons.
13. a. **Visual Thinking** Describe the regular polygon whose vertices are roots of the equation  $z^5 = 100,000$ .  
 b. **Visual Thinking** Describe the regular polygon whose vertices are roots of the equation  $z^9 = -1$ .
14. Find the fifth roots of 32.                      15. Find the cube roots of  $3\sqrt{3} + 3i$ .
16. A prime number of the form  $2^{2^n} + 1$  is called a *Fermat prime*.  
 a. Find the first four Fermat primes by substituting  $n = 0, 1, 2$ , and 3.  
 b. Carl Friedrich Gauss (1777–1855), considered one of the greatest mathematicians of all time, used complex numbers to help determine which regular polygons could be constructed with straightedge and compass. His discovery: A regular polygon of  $N$  sides can be constructed if and only if  $N = 2^j$  ( $j \geq 2$ ) or  $N =$  a product of different Fermat primes  $\times 2^k$  ( $k \geq 0$ ). For example, a polygon with  $3 \times 5 \times 2^3 = 120$  sides is constructible. Decide whether a polygon is constructible if it has 32 sides, 17 sides, 7 sides, 9 sides, 10 sides, 255 sides.  
 c. Explain what the solutions to the equation  $z^{17} = 1$  have to do with a regular polygon of 17 sides.
- C** 17. The roots of  $z^n = 1$  are vertices of a regular polygon. Give the perimeter of an  $n$ -sided regular polygon as a trigonometric function of  $n$ .

## Chapter Summary

- The point  $P(r, \theta)$  is described by its *polar coordinates*,  $r$  and  $\theta$ , where  $r$  is the directed distance from the *pole* to  $P$  and  $\theta$  is the *polar angle* measured from the polar axis to the ray  $OP$ . Although a point has only one pair of rectangular coordinates, it has many pairs of polar coordinates. For example,  $(2, 20^\circ) = (-2, 200^\circ) = (-2, -160^\circ)$  all represent the same point.
- The formulas for converting from polar to rectangular coordinates and from rectangular to polar coordinates are given on page 396.
- The *complex plane* can be represented by an *Argand diagram*. In this diagram, the complex number  $a + bi$  is represented by the point  $(a, b)$  or by an arrow from the origin to  $(a, b)$ .

4. The *absolute value* of a complex number  $z = a + bi$  is  $|z| = \sqrt{a^2 + b^2}$ . Thus,  $|z| = r$ . The diagram at the right illustrates two ways of expressing  $z$ :

rectangular form:  $z = a + bi$   
 polar form:  $z = r \operatorname{cis} \theta$



5. If  $z_1 = r \operatorname{cis} \alpha$  and  $z_2 = s \operatorname{cis} \beta$ , then

$$z_1 z_2 = (r \operatorname{cis} \alpha)(s \operatorname{cis} \beta) = rs \operatorname{cis} (\alpha + \beta).$$

Powers of complex numbers can be found using De Moivre's theorem:

$$\text{If } z = r \operatorname{cis} \theta, \text{ then } z^n = r^n \operatorname{cis} n\theta.$$

6. The  $n$   $n$ th roots of  $z = r \operatorname{cis} \theta$  are:

$$\sqrt[n]{z} = z^{1/n} = r^{1/n} \operatorname{cis} \left( \frac{\theta}{n} + \frac{k \cdot 360^\circ}{n} \right) \text{ for } k = 0, 1, 2, \dots, n-1$$

### Key vocabulary and ideas

polar coordinates (p. 395)	polar form (p. 403)
polar equation, polar graph (p. 396)	absolute value of $z = a + bi$ (p. 403)
Argand diagram (p. 403)	product of two complex numbers (p. 404)
complex plane (p. 403)	De Moivre's theorem (p. 408)
rectangular form (p. 403)	$n$ th roots of a complex number (p. 413)

## Chapter Test

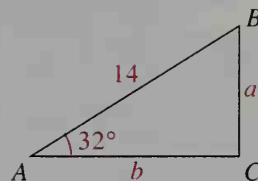
- Give polar or rectangular coordinates for each point, as indicated. **11-1**
  - $(3, 3)$ , polar
  - $(6, 90^\circ)$ , rect.
  - $(0, -2)$ , polar
  - $(8, \pi)$ , rect.
  - $(-2, 60^\circ)$ , rect.
  - $(-1, \sqrt{3})$ , polar
- Sketch the polar graph of  $r = 1 - 2 \cos \theta$ .
  - Give a rectangular equation of this graph.
- Sketch the polar graph of: **a.**  $r = 2$  **b.**  $\theta = 2$  **c.**  $r = \theta, \theta \geq 0^\circ$
- Express  $z_1 = \sqrt{3} - i$ ,  $z_2 = 4 + 4i$ , and  $z_1 z_2$  in polar form. **11-2**
  - Show  $z_1$ ,  $z_2$ , and  $z_1 z_2$  in an Argand diagram.
- Let  $z = 2 \operatorname{cis} 120^\circ$ . Find  $z^2$  in polar form and in rectangular form.
  - Show that  $z^2$  in polar form agrees with  $z^2$  in rectangular form.
- Express  $z = -1 - i$  in polar form. **11-3**
  - Show  $z$ ,  $z^2$ ,  $z^3$ , and  $z^4$  in an Argand diagram.
  - Find  $z^{10}$ .
- Writing** Describe how to find the  $n$   $n$ th roots of a complex number. **11-4**  
 Explain why this method is a special case of De Moivre's theorem.
- Show that  $\sqrt[6]{2} \operatorname{cis} 135^\circ$  is a cube root of  $1 + i$ , and find the other two cube roots.

1. Find two angles, one positive and one negative, that are coterminal with the given angle.
  - a.  $-35^\circ$
  - b.  $\frac{5\pi}{6}$
2.
  - a. Convert  $165^\circ$  to radians. Give the answer in terms of  $\pi$ .
  - b. Convert  $208^\circ$  to radians. Give the answer to the nearest hundredth of a radian.
3.
  - a. Convert  $-\frac{7\pi}{6}$  to degrees.
  - b. Convert 1.8 radians to degrees. Give the answer to the nearest ten minutes or tenth of a degree.
4. A sector of a circle has area  $8.1\pi \text{ cm}^2$  and central angle  $36^\circ$ . Find its radius and arc length.
5. If  $\theta$  is a third-quadrant angle and  $\cos \theta = -\frac{2}{3}$ , find  $\sin \theta$ .
6. Express each of the following in terms of a reference angle.
  - a.  $\cos 236^\circ$
  - b.  $\sin 485^\circ$
  - c.  $\sin (-62^\circ)$
7. Give the exact value of the six trigonometric functions for each angle.
  - a.  $\frac{\pi}{6}$
  - b.  $225^\circ$
  - c.  $-\frac{\pi}{2}$
8. Let  $\cot x = \frac{5}{12}$ , where  $\pi < x < \frac{3\pi}{2}$ . Find the values of the other five trigonometric functions.
9.
  - a. Sketch the graph of  $y = \tan x$  for  $-\pi \leq x \leq 2\pi$ .
  - b. Give the domain, range, and period of the tangent function.
10. Without using a calculator or a table, find the value of each expression. Leave your answers in terms of  $\pi$  whenever appropriate.
  - a.  $\tan^{-1} 1$
  - b.  $\sin^{-1} \frac{\sqrt{3}}{2}$
  - c.  $\cos^{-1} \left(-\frac{\sqrt{3}}{2}\right)$
  - d.  $\tan^{-1} \sqrt{3}$
  - e.  $\sin^{-1} 0$
  - f.  $\sin \left(\cos^{-1} \frac{3}{5}\right)$
  - g.  $\sec \left(\tan^{-1} \frac{5}{12}\right)$
  - h.  $\cos (\sin^{-1} 1)$
11. Solve the equation  $2 \sec x - 5 = 0$  for  $0 \leq x < 2\pi$ . Give your answer to the nearest hundredth of a radian.
12. Find the inclination of the line  $3x - 4y = 6$  to the nearest degree.
13. Sketch the graph of each equation.
  - a.  $y = -2 \sin \frac{x}{2}$
  - b.  $y + 1 = 3 \cos \frac{\pi}{4}(x - 2)$

14. Simplify  $\frac{1 - \cos \theta}{\sin^2 \theta} + \tan^2 \theta - \sec^2 \theta$ .

15. Solve the equation  $\cos x \sin x = \sin^2 x$  for  $0 \leq x < 2\pi$ .

16. For the right triangle shown at the right, find the missing lengths to three significant digits.



17. A parallelogram has sides of lengths 4 cm and 9 cm with an included angle of  $40^\circ$ . Find the area of the parallelogram to the nearest tenth of a square centimeter.

18. Two forest rangers at observation posts  $A$  and  $B$ , 10 mi apart, spot a fire at point  $F$ . The forest rangers know that  $\angle FAB = 110^\circ$  and  $\angle FBA = 25^\circ$ . To the nearest tenth of a mile, how far is the fire from the nearer observation post?

19. Suppose that two sides of a triangle have lengths 8.4 and 7.6 with an included angle of  $82^\circ$ . Find the length of the third side to three significant digits.

20. Two people are standing 100 m apart on the bank of a river that flows due east. If a rock on the opposite bank is along a bearing of  $N25^\circ E$  from one person and along a bearing of  $N15^\circ W$  from the other person, what is the width of the river?

21. Let  $\alpha$  and  $\beta$  be acute angles with  $\sin \alpha = \frac{4}{5}$  and  $\sin \beta = \frac{5}{13}$ . Find  $\cos(\alpha - \beta)$ .

22. Find the exact value of  $\sin 75^\circ$ .

23. Find  $\tan\left(\frac{3\pi}{4} + \theta\right)$  when  $\tan \theta = -\frac{1}{2}$ .

24. Suppose  $\alpha$  is an acute angle and  $\cos \alpha = \frac{1}{3}$ . Find  $\sin 2\alpha$  and  $\cos \frac{1}{2}\alpha$ .

25. Give polar coordinates for the point  $(-1, \sqrt{3})$ . Then give two other pairs of polar coordinates for the same point.

26. Give the rectangular coordinates for the point  $(-6, 30^\circ)$ .

27. a. Sketch the polar graph of  $r = 4 \cos \theta$ .  
b. Give a rectangular equation of this graph.

28. Sketch the polar graph of  $r = 1 - \sin \theta$ . Then identify the graph.

29. Let  $z_1 = 1 - i\sqrt{3}$  and  $z_2 = -1 + i\sqrt{3}$ .

- Express  $z_1 z_2$  in rectangular form.
- Express  $z_1$ ,  $z_2$ , and  $z_1 z_2$  in polar form.
- Show that your answers in parts (a) and (b) agree.

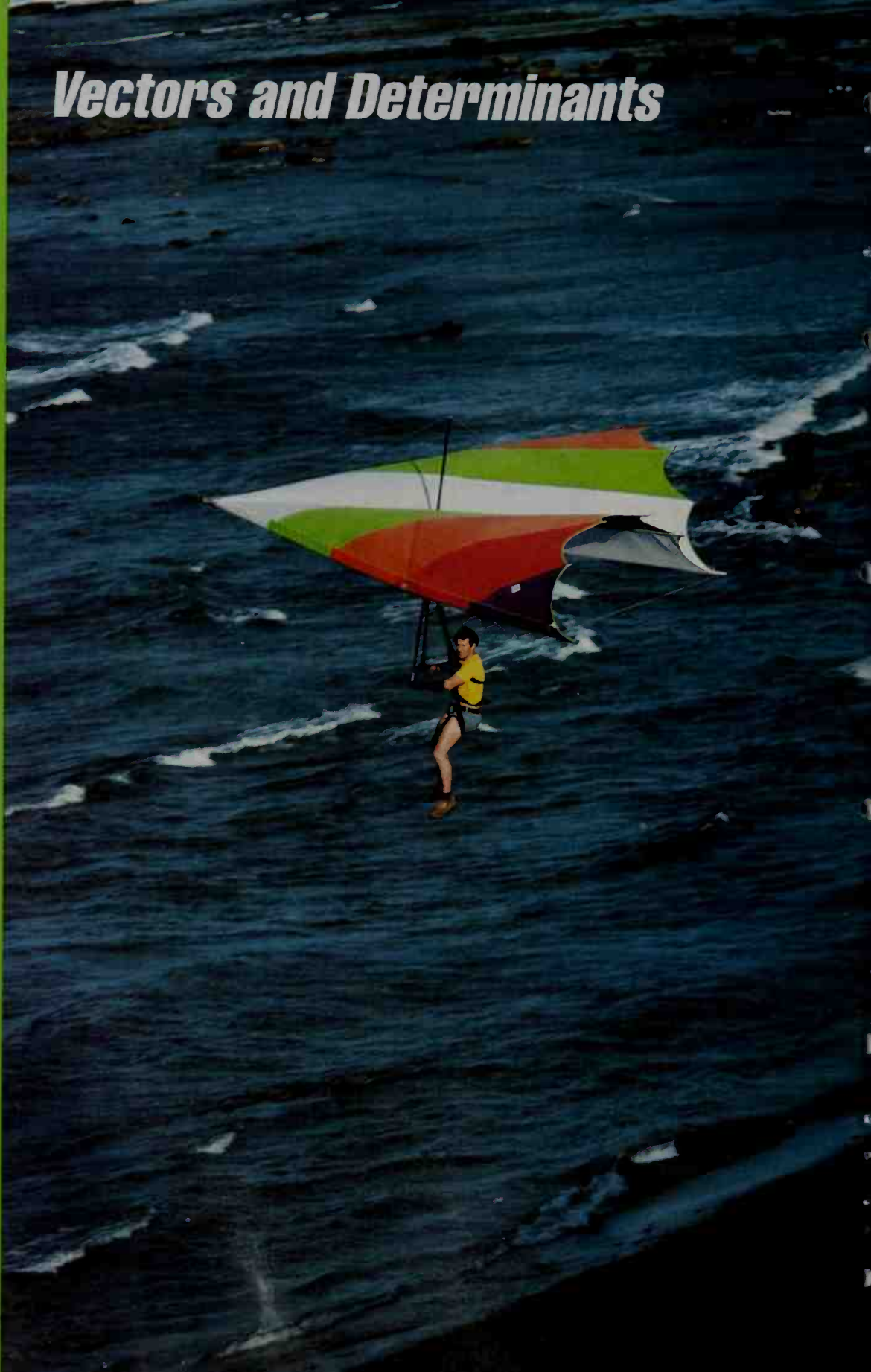
30. Let  $z = -2 - 2i$ . Use DeMoivre's theorem to find  $z^6$ .

31. Find the cube roots of  $1 - i$ . Write your answers in polar form.



# 12

## *Vectors and Determinants*



# ■ Properties and Basic Operations

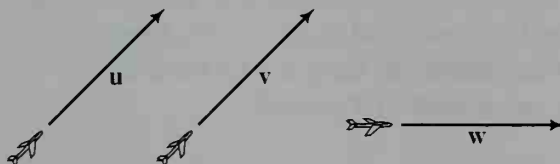
## 12-1 Geometric Representation of Vectors

### Objective

To perform basic operations on vectors.

**Vectors** are quantities that are described by a *direction* and a *magnitude* (size). A force, for example, is a vector quantity because to describe a force, you must specify the direction in which it acts and its strength. Another example of a vector is velocity. The velocity of an airplane is described by its direction and speed.

The velocities of two airplanes each heading northeast at 700 knots are represented by the arrows  $\mathbf{u}$  and  $\mathbf{v}$  in the diagram below. We write  $\mathbf{u} = \mathbf{v}$  to indicate that both planes have the same velocity even though the two arrows are different. In general, any two arrows with the same length and the same direction represent the same vector. The diagram below also shows a third airplane with speed 700 knots, but because it is heading in a different direction, its velocity vector  $\mathbf{w}$  does not equal either  $\mathbf{u}$  or  $\mathbf{v}$ .



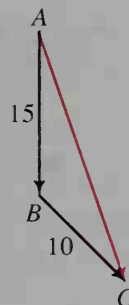
The *magnitude* of a vector  $\mathbf{v}$  (also called the *absolute value* of  $\mathbf{v}$ ) is denoted  $|\mathbf{v}|$ . In the diagram above,  $|\mathbf{v}| = 700$ . Since  $|\mathbf{w}|$  also equals 700, we have that  $|\mathbf{w}| = |\mathbf{v}|$  even though  $\mathbf{w} \neq \mathbf{v}$ .

### Addition of Vectors

If a vector  $\mathbf{v}$  is pictured by an arrow from point  $A$  to point  $B$ , then it is customary to write  $\mathbf{v} = \overrightarrow{AB}$ . In the diagram,  $\overrightarrow{AB}$  represents the motion of an object 15 units south from  $A$  to  $B$ . If the object is then moved 10 units southeast from  $B$  to  $C$ , that motion is represented by  $\overrightarrow{BC}$ . Since the result of moving an object first from  $A$  to  $B$  and then from  $B$  to  $C$  is the same as moving the object directly from  $A$  to  $C$ , it is natural to write

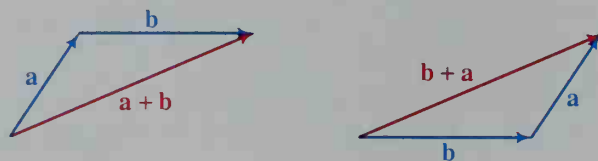
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

We say that  $\overrightarrow{AC}$  is the *vector sum* of  $\overrightarrow{AB}$  and  $\overrightarrow{BC}$ .

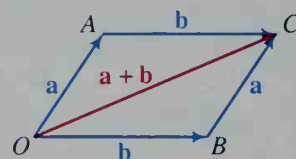


◀ A hang glider gains altitude when the upward vectors of force produced by rising warm air masses, or thermals, exceed the downward vectors of force produced by gravity.

The addition of two vectors is a commutative operation. In other words, the order in which the vectors are added does not make any difference. You can see this in the vector diagrams below where the red arrows denoting  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{b} + \mathbf{a}$  have the same length and direction.



If the two diagrams above are moved together, a parallelogram is formed. This suggests that another way to add  $\mathbf{a}$  and  $\mathbf{b}$  is to draw a parallelogram  $OACB$  with sides  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ . The diagonal  $\overrightarrow{OC}$  of the parallelogram is the sum. This method is frequently used in physics problems involving forces that are combined. The vector sum of the forces is called the *resultant* of the forces. These ideas are illustrated in Example 1 in which the unit of force used is the *newton*. One **newton** (N) is the force needed to bring a 1 kg mass to a speed of 1 meter per second in 1 second.

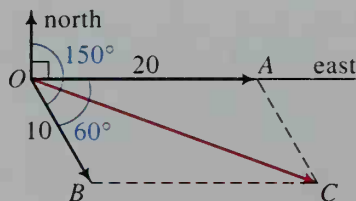


### Example 1

- Make a scale drawing showing a force of 20 N pulling an object east and another force of 10 N pulling the object in the compass direction  $150^\circ$ . Draw the resultant force vector and use your drawing to approximate the magnitude and direction of the resultant force.
- Use trigonometry to find the magnitude (to the nearest hundredth) and the direction (to the nearest tenth of a degree) of the resultant.

### Solution

- The given forces are  $\overrightarrow{OA}$  and  $\overrightarrow{OB}$  and the resultant force is  $\overrightarrow{OC}$ . Note that  $\angle AOB = 150^\circ - 90^\circ = 60^\circ$ .



By measuring with a ruler and protractor, you can determine that the magnitude of  $\overrightarrow{OC}$  is about 26 and that the direction of  $\overrightarrow{OC}$  is about  $110^\circ$ . Therefore,  $\overrightarrow{OC}$  represents a force of 26 N pulling the object in the compass direction  $110^\circ$ .

- b. Use the law of cosines in  $\triangle OBC$ . Since  $\angle B$  is a supplement of  $\angle AOB$ ,  $\angle OBC = 120^\circ$ . Therefore:

$$|\overrightarrow{OC}|^2 = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos 120^\circ = 700$$

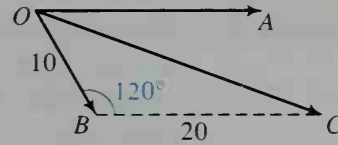
Therefore,  $|\overrightarrow{OC}| \approx 26.46$ .

To find the direction of the resultant, use the law of sines.

$$\frac{\sin C}{10} = \frac{\sin 120^\circ}{26.46}$$

$$\sin C = 0.3273$$

$$\angle C \approx 19.1^\circ$$



Since  $\angle C \approx 19.1^\circ$ ,  $\angle AOC \approx 19.1^\circ$  also. Therefore, the direction of  $\overrightarrow{OC}$  is about  $90^\circ + 19.1^\circ = 109.1^\circ$ .

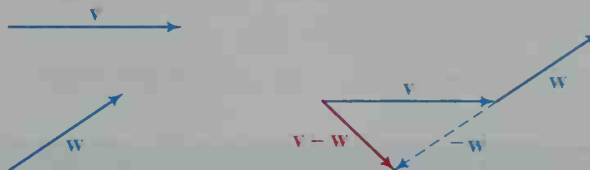
## Vector Subtraction

The *negative* of a vector  $\mathbf{v}$ , denoted  $-\mathbf{v}$ , has the same length as  $\mathbf{v}$  but the opposite direction. The sum of  $\mathbf{v}$  and  $-\mathbf{v}$  is the zero vector  $\mathbf{0}$ . It is best thought of as a point.



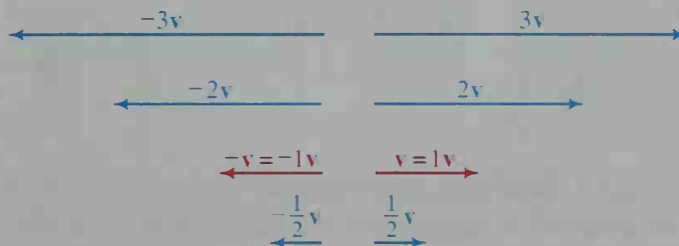
Vectors can be subtracted as well as added.

$\mathbf{v} - \mathbf{w}$  means  $\mathbf{v} + (-\mathbf{w})$ .



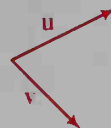
## Multiples of a Vector

The vector sum  $\mathbf{v} + \mathbf{v}$  is abbreviated as  $2\mathbf{v}$ . Likewise,  $\mathbf{v} + \mathbf{v} + \mathbf{v} = 3\mathbf{v}$ . The diagram below shows that the arrows representing  $2\mathbf{v}$  and  $3\mathbf{v}$  have the same direction as the arrow representing  $\mathbf{v}$ , but that they are two and three times as long.



In general, if  $k$  is a positive real number, then  $k\mathbf{v}$  is the vector with the same direction as  $\mathbf{v}$  but with an absolute value  $k$  times as large. If  $k < 0$ , then  $k\mathbf{v}$  has the same direction as  $-\mathbf{v}$  and has an absolute value  $|k|$  times as large. If  $k \neq 0$ , then  $\frac{\mathbf{v}}{k}$  is defined to be equal to the vector  $\frac{1}{k}\mathbf{v}$ .

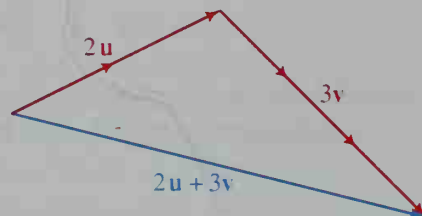
**Example 2** Vectors  $\mathbf{u}$  and  $\mathbf{v}$  are shown at the right. Use these vectors to sketch the following:



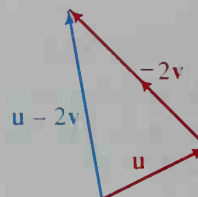
- a.  $2\mathbf{u} + 3\mathbf{v}$
- b.  $\mathbf{u} - 2\mathbf{v}$

**Solution**

a.



b.



When working with vectors, it is customary to refer to real numbers as **scalars**. When this is done, the operation of multiplying a vector  $\mathbf{v}$  by a scalar  $k$  is called **scalar multiplication**. This operation has the following properties. If  $\mathbf{v}$  and  $\mathbf{w}$  are vectors and  $k$  and  $m$  are scalars, then:

$$\left. \begin{aligned} k(\mathbf{v} + \mathbf{w}) &= k\mathbf{v} + k\mathbf{w} \\ (k + m)\mathbf{v} &= k\mathbf{v} + m\mathbf{v} \end{aligned} \right\} \text{Distributive laws}$$

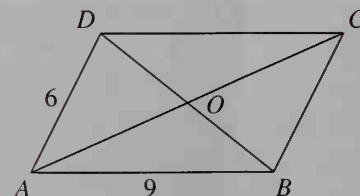
$$k(m\mathbf{v}) = (km)\mathbf{v} \quad \text{Associative law}$$

## CLASS EXERCISES

Quadrilateral  $ABCD$  is the parallelogram shown below. Tell whether each of the following is true or false.

1.  $\overrightarrow{BC} + \overrightarrow{BA} = \overrightarrow{BD}$
2.  $|\overrightarrow{BC}| + |\overrightarrow{BA}| = |\overrightarrow{BD}|$
3.  $\overrightarrow{AO} = \overrightarrow{AC}$
4.  $\overrightarrow{AB} + \overrightarrow{CD} = \mathbf{0}$
5.  $\overrightarrow{AO} = \overrightarrow{OC}$
6.  $\overrightarrow{AO} = \frac{1}{2}\overrightarrow{AC}$

7.  $(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = \overrightarrow{AD}$
8.  $\overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD}) = \overrightarrow{AD}$



9. Exercises 7 and 8 show that vector addition is associative. Is addition of real numbers associative? Is subtraction of real numbers associative?



Complete the following statements. The parallelogram  $ABCD$  is shown at the bottom of the preceding page.

10.  $\overrightarrow{AD} = \underline{\hspace{1cm}}$

11.  $|\overrightarrow{AD}| = \underline{\hspace{1cm}}$

12.  $\frac{1}{2}\overrightarrow{BD} = \underline{\hspace{1cm}}$

13.  $2\overrightarrow{AO} = \underline{\hspace{1cm}}$

14.  $\overrightarrow{AB} + \overrightarrow{AD} = \underline{\hspace{1cm}}$

15.  $\overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB} = \underline{\hspace{1cm}}$

16.  $\overrightarrow{AO} - \overrightarrow{DO} = \overrightarrow{AO} + \underline{\hspace{1cm}} = \underline{\hspace{1cm}}$

17.  $\overrightarrow{BC} - \overrightarrow{BD} = \overrightarrow{BC} + \underline{\hspace{1cm}} = \underline{\hspace{1cm}}$

18. **Reading** Explain the two different uses of the plus sign in Exercises 1 and 2 on page 422.

19. **Reading** Explain the two different uses of the plus sign in the distributive law  $(k + m)\mathbf{v} = k\mathbf{v} + m\mathbf{v}$ .

20. **Reading** In the associative law,  $k(m\mathbf{v}) = (km)\mathbf{v}$ , two different kinds of multiplication are used on the right side of the equation. Explain.

21. **Discussion** Does the statement  $k + \mathbf{v} = \mathbf{v} + k$  make sense? Explain.

## WRITTEN EXERCISES

**A**

1. a. Sketch an arrow representing a velocity  $\mathbf{v}$  of 100 mi/h to the northwest.  
b. Sketch  $2\mathbf{v}$  and tell what velocity it represents.  
c. Sketch  $-\mathbf{v}$  and tell what velocity it represents.

2. a. Sketch an arrow representing a ship's trip  $\mathbf{w}$  of 200 mi on a course of  $200^\circ$ .  
b. Sketch  $3\mathbf{w}$  and tell what trip it represents.  
c. Sketch  $-1.5\mathbf{w}$  and tell what trip it represents.

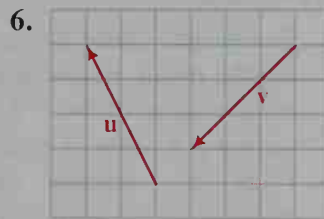
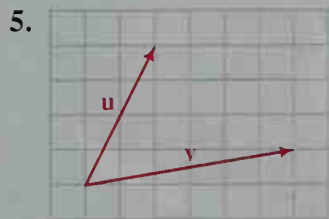
3. Draw rectangle  $PQRS$  with  $PQ = 4$  and  $QR = 3$ . Complete.

a.  $\overrightarrow{PQ} + \overrightarrow{QR} = \underline{\hspace{1cm}}$       b.  $|\overrightarrow{PQ} + \overrightarrow{QR}| = \underline{\hspace{1cm}}$       c.  $\overrightarrow{PQ} + \overrightarrow{RS} = \underline{\hspace{1cm}}$

4. Draw parallelogram  $ABCD$  with  $AB = 10$ ,  $BC = 6$ , and  $\angle A = 60^\circ$ . Complete.

a.  $\overrightarrow{AB} + \overrightarrow{BC} = \underline{\hspace{1cm}}$       b.  $|\overrightarrow{AB} + \overrightarrow{BC}| = \underline{\hspace{1cm}}$       c.  $|\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}| = \underline{\hspace{1cm}}$

In Exercises 5 and 6, copy vectors  $\mathbf{u}$  and  $\mathbf{v}$ . Then sketch  $\mathbf{u} + \mathbf{v}$ ,  $\mathbf{u} + 2\mathbf{v}$ ,  $\mathbf{u} + 3\mathbf{v}$ , and  $\mathbf{u} - \mathbf{v}$ .



7. **Investigation** Draw two vectors  $\mathbf{u}$  and  $\mathbf{v}$ . Sketch  $\mathbf{u} + \mathbf{v}$ ,  $2(\mathbf{u} + \mathbf{v})$ , and  $2\mathbf{u} + 2\mathbf{v}$ . What do you notice?

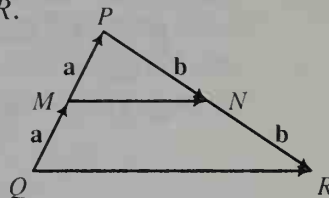
8. **Investigation** Draw two vectors  $\mathbf{a}$  and  $\mathbf{b}$ . Sketch  $\mathbf{a} - \mathbf{b}$ ,  $3(\mathbf{a} - \mathbf{b})$ , and  $3\mathbf{a} - 3\mathbf{b}$ . What do you notice?
9. **Navigation** A ship travels 200 km west from port and then 240 km due south before it is disabled. Illustrate this in a vector diagram. Use trigonometry to find the course that a rescue ship must take from port in order to reach the disabled ship.
10. **Physics** On graph paper, make a diagram that illustrates a force of 8 N north and a force of 6 N west acting on a body. Illustrate the resultant sum of these two forces and estimate its strength. Then, using trigonometry, determine the approximate direction (as a number of degrees west of north) of this force.
11. **Aviation** On graph paper, make a diagram that illustrates the velocity of an airplane heading east at 400 knots. Illustrate a wind velocity of 50 knots blowing toward the northeast. If the airplane encounters this wind, illustrate its resultant velocity. Estimate the resultant speed and direction of the airplane. (The direction is the angle the resultant makes with due north measured clockwise from north.)
12. a. **Sports** A swimmer leaves point A swimming south across a river at 2 km/h. The river is 4 km wide and flows east at the rate of 1 km/h. Make a vector diagram showing her resultant velocity.  
b. Calculate her resultant speed.  
c. How long will it take her to swim across the river? How far east does she swim?
13. **Aviation** On graph paper, make a vector diagram showing an airplane heading southwest at 600 knots and encountering a wind blowing from the west. Show the plane's resultant velocity when the wind blows at (a) 30 knots, (b) 60 knots, and (c) 90 knots.
14. **Navigation** On graph paper, make a vector diagram showing a motorboat heading east at 10 knots. Add to your diagram a vector representing a current moving southeast. Show the boat's resultant velocity if the current moves at (a) 2 knots, (b) 3 knots, and (c) 4 knots.
15. a. **Navigation** Make a diagram showing the result of sailing a ship 3 mi on a course of  $040^\circ$  followed by sailing it 8 mi on a course of  $100^\circ$ .  
b. From your diagram, estimate the distance of the ship from its starting point.  
c. Find the exact distance of the ship from its starting point by using the law of cosines.



16. a. **Physics**  $F_1$  is a force of 3 N pulling an object north and  $F_2$  is a force of 5 N pulling the object in the compass direction  $060^\circ$ . Sketch both force vectors.
- b. Sketch the resultant force vector  $F_1 + F_2$  and use the law of cosines and the law of sines to find its magnitude and direction.
- c. Give the magnitude and direction of the force  $F_3$  such that  $F_1 + F_2 + F_3 = 0$ .

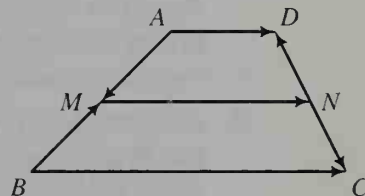
**B** 17. In the diagram,  $M$  and  $N$  are the midpoints of  $\overline{PQ}$  and  $\overline{PR}$ .

- a. Express  $\overrightarrow{MN}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .
- b. Express  $\overrightarrow{QR}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .
- c. Your answers to parts (a) and (b) show that  $\overrightarrow{QR} = 2\overrightarrow{MN}$ . What theorem about the segment joining the midpoints of two sides of a triangle does this equation suggest?



18. a. Complete the following equations given that  $\overrightarrow{MN}$  is the median of trapezoid  $ABCD$ .

$$\begin{aligned} (1) \overrightarrow{AM} + \overrightarrow{MN} + \overrightarrow{ND} &= \underline{\quad ? \quad} \\ (2) \overrightarrow{BM} + \overrightarrow{MN} + \overrightarrow{NC} &= \underline{\quad ? \quad} \\ (3) \overrightarrow{AM} + \overrightarrow{BM} &= \underline{\quad ? \quad} \\ (4) \overrightarrow{ND} + \overrightarrow{NC} &= \underline{\quad ? \quad} \end{aligned}$$



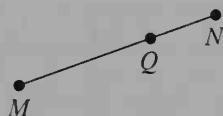
- b. Add equations (1) and (2), and by using (3) and (4), simplify the resulting equation to  $2\overrightarrow{MN} = \overrightarrow{AD} + \overrightarrow{BC}$ . What theorem about the median of a trapezoid does this equation suggest?

19. Point  $Q$  divides  $\overline{MN}$  in the ratio 2:1, that is,  $\overrightarrow{MQ} = 2\overrightarrow{QN}$ . If  $\mathbf{v} = \overrightarrow{MN}$ , express each vector in terms of  $\mathbf{v}$ .

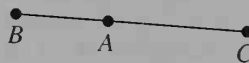
a.  $\overrightarrow{MQ}$

b.  $\overrightarrow{QN}$

c.  $\overrightarrow{NQ}$



Ex. 19



Ex. 20

20. Point  $A$  divides  $\overline{BC}$  in the ratio 2:3, that is,  $\frac{BA}{AC} = \frac{2}{3}$ . If  $\mathbf{v} = \overrightarrow{BA}$ , express each vector in terms of  $\mathbf{v}$ .

a.  $\overrightarrow{CA}$

b.  $\overrightarrow{BC}$

c.  $\overrightarrow{AB}$

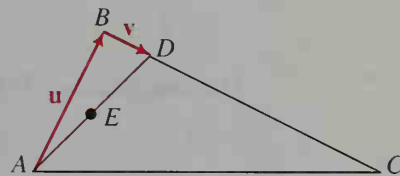
21. In the diagram,  $\mathbf{u} = \overrightarrow{AB}$  and  $\mathbf{v} = \overrightarrow{BD}$ . The midpoint of  $\overline{AD}$  is  $E$  and  $\frac{BD}{DC} = \frac{1}{3}$ . Express each of the following vectors in the form  $r\mathbf{u} + s\mathbf{v}$ , where  $r$  and  $s$  are real numbers. For example,  $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{u} + 4\mathbf{v}$ .

a.  $\overrightarrow{AD}$

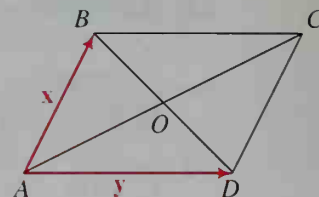
b.  $\overrightarrow{AE}$

c.  $\overrightarrow{BE}$

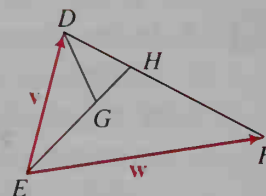
d.  $\overrightarrow{EC}$



22. In the diagram,  $ABCD$  is a parallelogram. If  $\vec{AB} = \mathbf{x}$  and  $\vec{AD} = \mathbf{y}$ , express each vector in terms of  $\mathbf{x}$  and  $\mathbf{y}$ .
- a.  $\vec{BC}$     b.  $\vec{CD}$     c.  $\vec{AC}$     d.  $\vec{AO}$     e.  $\vec{BO}$



23. In the diagram,  $DH:HF = 1:2$  and  $EG:GH = 2:1$ . If  $\mathbf{v} = \vec{ED}$  and  $\mathbf{w} = \vec{EF}$ , express each vector in terms of  $\mathbf{v}$  and  $\mathbf{w}$ .
- a.  $\vec{DF}$     b.  $\vec{DH}$     c.  $\vec{EH}$     d.  $\vec{EG}$     e.  $\vec{DG}$



24. **Navigation** Ship A is 50 nautical miles west of ship B which is heading south at 8 knots (8 nautical miles per hour). If A is to rendezvous with B in 3 h, what should be the course and the speed of ship A? Solve this problem in two ways: (a) by making an accurate vector diagram and estimating the answers by measuring, and (b) by using right triangle trigonometry.
25. **Aviation** Jenny Wu wants to fly from Chicago to Atlanta as quickly as possible. If there were no wind, her course would be  $150^\circ$  and she would fly at her plane's top speed of 450 knots. What course should she plan if there is a 75 knot wind blowing toward the east? Solve this problem in two ways. First, make an accurate vector diagram and measure her course with a protractor. Second, use the law of sines.

## 12-2 Algebraic Representation of Vectors

**Objective** To use coordinates to perform vector operations.

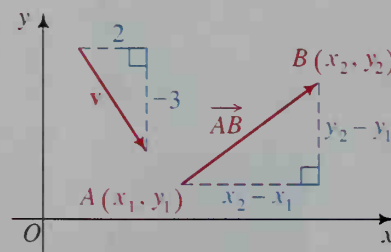
In the coordinate system shown at the right, the vector  $\mathbf{v}$  consists of a 2 unit change in the  $x$ -direction and a  $-3$  unit change in the  $y$ -direction. The numbers 2 and  $-3$  are called the components of  $\mathbf{v}$ . When we write  $\mathbf{v} = (2, -3)$ , we are expressing  $\mathbf{v}$  in **component form**.

The diagram also shows that the components of  $\vec{AB}$  can be found by subtracting the coordinates of points A and B as follows:

$$\vec{AB} = (x_2 - x_1, y_2 - y_1)$$

The magnitude of  $\vec{AB}$  is then found by using the distance formula.

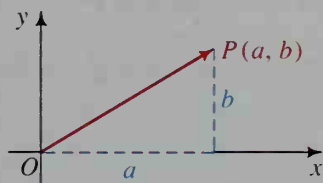
$$|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



**Example 1** Given  $A(4, 2)$  and  $B(9, -1)$ , express  $\overrightarrow{AB}$  in component form. Find  $|\overrightarrow{AB}|$ .

**Solution**  $\overrightarrow{AB} = (9 - 4, -1 - 2) = (5, -3)$   
 $|\overrightarrow{AB}| = \sqrt{5^2 + (-3)^2} = \sqrt{34}$

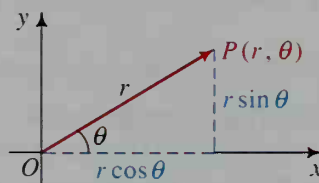
The two diagrams below show that the rectangular coordinates,  $a$  and  $b$ , and the polar coordinates,  $r$  and  $\theta$ , of a point closely tie together three important ideas of mathematics: (1) point, (2) vector, and (3) complex number. Although  $(a, b)$  can represent both the point  $P$  and the vector  $\overrightarrow{OP}$ , you can always tell which meaning is intended from the context.



Point:  $P(a, b)$

Vector:  $\overrightarrow{OP} = (a, b)$

Complex number:  $z = a + bi$



Point:  $P(r, \theta)$

Vector:  $\overrightarrow{OP} = (r \cos \theta, r \sin \theta)$

Complex number:  $z = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$

**Example 2** A force  $\mathbf{F}$  of 10 N acts at an angle of  $130^\circ$  with the positive  $x$ -axis. Find  $\mathbf{F}$  in component form.

**Solution**  $|\mathbf{F}| = 10$ , so  $\mathbf{F} = (10 \cos 130^\circ, 10 \sin 130^\circ) \approx (-6.43, 7.66)$ .

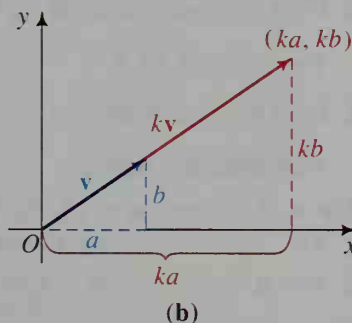
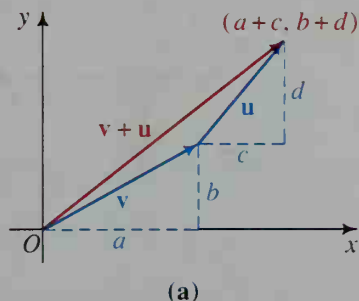
In terms of coordinates, vector operations are defined as below.

### Vector Operations with Coordinates

**Vector Addition:**  $\mathbf{v} + \mathbf{u} = (a, b) + (c, d) = (a + c, b + d)$  Figure (a)

**Vector Subtraction:**  $\mathbf{v} - \mathbf{u} = (a, b) - (c, d) = (a - c, b - d)$

**Scalar Multiplication:**  $k\mathbf{v} = k(a, b) = (ka, kb)$  Figure (b)





**Example 3**If  $\mathbf{u} = (1, -3)$  and  $\mathbf{v} = (2, 5)$ , find:

- a.  $\mathbf{u} + \mathbf{v}$                       b.  $\mathbf{u} - \mathbf{v}$                       c.  $2\mathbf{u} - 3\mathbf{v}$

**Solution**

$$\begin{aligned}\text{a. } \mathbf{u} + \mathbf{v} &= (1, -3) + (2, 5) \\ &= (1 + 2, -3 + 5) = (3, 2)\end{aligned}$$

$$\begin{aligned}\text{b. } \mathbf{u} - \mathbf{v} &= (1, -3) - (2, 5) \\ &= (1 - 2, -3 - 5) = (-1, -8)\end{aligned}$$

$$\begin{aligned}\text{c. } 2\mathbf{u} - 3\mathbf{v} &= 2(1, -3) - 3(2, 5) \\ &= (2, -6) - (6, 15) = (-4, -21)\end{aligned}$$

**Example 4**

If  $A = (0, 4)$  and  $B = (6, 1)$ , find (a) the coordinates of point  $P$  that is  $\frac{2}{3}$  of the way from  $A$  to  $B$ , and (b) the coordinates of point  $Q$  that is  $\frac{5}{6}$  of the way from  $A$  to  $B$ .

**Solution**

a. Refer to the diagram. Notice that:

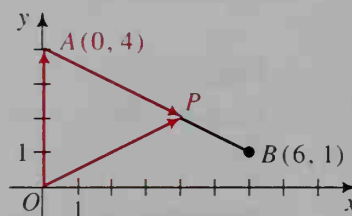
$$\overrightarrow{AB} = (6 - 0, 1 - 4) = (6, -3)$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$$

$$= (0, 4) + \frac{2}{3}(6, -3)$$

$$= (4, 2)$$

The coordinates of  $P$  are  $(4, 2)$ .

b. Similarly,  $\overrightarrow{OQ} = \overrightarrow{OA} + \frac{5}{6}\overrightarrow{AB}$ . The coordinates of  $Q$  are  $(5, 1.5)$ .

**CLASS EXERCISES**

- a. What are the components of  $\overrightarrow{AB}$ ? of  $\overrightarrow{CD}$ ?

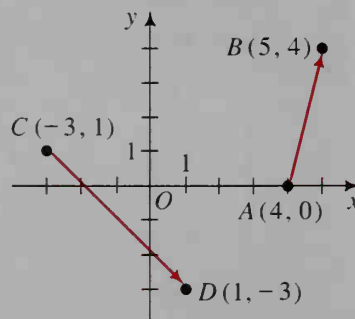
b. Find  $|\overrightarrow{AB}|$  and  $|\overrightarrow{CD}|$ .

c. Find the coordinates of the point  $\frac{1}{4}$  of the way from  $A$  to  $B$ .

d. Find the coordinates of the point  $\frac{3}{4}$  of the way from  $C$  to  $D$ .
- If  $\mathbf{v} = (1, 2)$  and  $\mathbf{u} = (3, 0)$ , find:

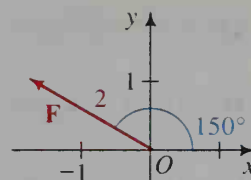
a.  $3\mathbf{v}$                       b.  $\mathbf{v} + \mathbf{u}$

c.  $\mathbf{v} - \mathbf{u}$                       d.  $2\mathbf{v} + 3\mathbf{u}$



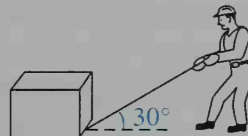
3. **Physics** A force  $\mathbf{F}$  of 2 N is represented in the diagram at the right.

- Express  $\mathbf{F}$  in component form.
- What is  $|\mathbf{F}|$ ?



Ex. 3

4. If  $\mathbf{v} = (3 \cos 40^\circ, 3 \sin 40^\circ)$ , write  $2\mathbf{v}$  and  $-\mathbf{v}$  in component form.
5. a. If  $\overrightarrow{AB} = (3, 2)$  and  $A = (4, 0)$ , find the coordinates of point  $B$ .  
 b. If  $\overrightarrow{CD} = (4, -1)$  and  $D = (8, 8)$ , find the coordinates of point  $C$ .
6. **Physics** Suppose you were moving a heavy box across a floor by pulling on a rope attached near the bottom of the front of the box. If you pull with a force of 80 lb at an angle of  $30^\circ$ , which component (horizontal or vertical) will move the box across the floor? What is the magnitude of this component? Would it be easier to move the box if the angle were smaller?



## WRITTEN EXERCISES

Plot points  $A$  and  $B$ . Give the component form of  $\overrightarrow{AB}$  and find  $|\overrightarrow{AB}|$ .

**A**

- $A(1, -2), B(3, -2)$
- $A(4, 2), B(0, -1)$
- $A(-3, -5), B(-5, 1)$
- $A(7, -2), B(3, -2)$

Polar coordinates of point  $P$  are given and  $O$  is the origin. Draw vector  $\overrightarrow{OP}$  and give its component form.

- $P(6, 72^\circ)$
- $P(8, 140^\circ)$
- $P\left(2, \frac{4\pi}{3}\right)$
- $P\left(8, -\frac{\pi}{4}\right)$

Let  $\mathbf{u} = (3, 1)$ ,  $\mathbf{v} = (-8, 4)$ , and  $\mathbf{w} = (-6, -2)$ . Calculate each expression.

- $\mathbf{u} + \mathbf{v}$
  - $\mathbf{u} - \mathbf{v}$
  - $3\mathbf{u} + \mathbf{w}$
  - $|3\mathbf{u} + \mathbf{w}|$
- $\mathbf{v} + \mathbf{w}$
  - $2(\mathbf{v} + \mathbf{w})$
  - $2\mathbf{v} + 2\mathbf{w}$
  - $|2\mathbf{v} + 2\mathbf{w}|$
- $\mathbf{u} + \frac{1}{2}\mathbf{w}$
  - $\left|\mathbf{u} + \frac{1}{2}\mathbf{w}\right|$
  - $|\mathbf{u}| + \left|\frac{1}{2}\mathbf{w}\right|$
- $\frac{7}{6}\mathbf{v} - \frac{2}{3}\mathbf{w}$
  - $\left(\frac{7}{6} - \frac{2}{3}\right)\mathbf{v}$
  - $\frac{\mathbf{v}}{|\mathbf{v}|}$

In Exercises 13–18, find the coordinates of the point  $P$  described.

- $A(0, 0), B(6, 3); \frac{1}{2}$  of the way from  $A$  to  $B$ .
- $A(1, 4), B(5, -4); \frac{1}{4}$  of the way from  $A$  to  $B$ .

15.  $A(7, -2)$ ,  $B(2, 8)$ ;  $\frac{4}{5}$  of the way from  $A$  to  $B$ .
16.  $A(-3, -4)$ ,  $B(-6, 1)$ ;  $\frac{1}{6}$  of the way from  $A$  to  $B$ .
17.  $A(-7, -4)$ ,  $B(-1, -1)$ ;  $\frac{3}{5}$  of the way from  $A$  to  $B$ .
18.  $A(3, -2)$ ,  $B(5, 1)$ ;  $\frac{2}{3}$  of the way from  $A$  to  $B$ .
19. **Writing** Write a few sentences in which you compare the absolute value of a complex number with the absolute value of a vector.
20. **Visual Thinking** Make a sketch showing the complex number  $z = 1 + i$  represented by the vector  $(1, 1)$ . On your sketch, draw vectors to represent the complex numbers  $z^2$ ,  $z^3$ , and  $z^4$ . Find the components of each vector.
21. **Physics** Suppose that you pull a child in a wagon by pulling a rope that makes a  $60^\circ$  angle with the ground.
- If the pulling force  $\mathbf{F}$  is 40 lb in the direction of the rope, give the horizontal and vertical components of the force.
  - Which component, horizontal or vertical, moves the wagon along the ground?
  - Visual Thinking** If the rope made a  $50^\circ$  angle with the ground rather than a  $60^\circ$  angle, would the wagon move more easily or less easily? Why?
22. **Physics** Suppose that a 50 lb box sits on a ramp that makes a  $20^\circ$  angle with the horizontal.
- Find the component of the 50 lb force that is parallel to the ramp.
  - Find the component of the 50 lb force that is perpendicular to the ramp.
  - The component of force parallel to the ramp tends to push the box down the ramp. What force is needed to counteract that component?
  - If the ramp makes a  $50^\circ$  angle with the horizontal, will the box have a greater tendency to slide down the ramp? What force is needed to keep the box from sliding down the ramp?
23. **Navigation** A ship is on a course of  $108^\circ$  at a speed of 15 knots. What is the north-south component of its velocity vector? What is the east-west component?
24. **Aviation** An airplane is heading on a course of  $240^\circ$  at 700 knots.
- What is the east-west component of its velocity?
  - If the plane encounters a 50 knot wind blowing from the west, what will be the east-west component of the plane's resultant velocity?



25. A bug crawls at a constant speed across the plane so that its position at time  $t$  is  $(x, y) = (3 + t, 1 + 2t)$ . Thus:

$$\text{at time } t = 0 \quad (x, y) = (3 + 0, 1 + 2 \cdot 0) = (3, 1)$$

$$\text{at time } t = 4 \quad (x, y) = (3 + 4, 1 + 2 \cdot 4) = (7, 9)$$

$$\text{at time } t = 8 \quad (x, y) = (3 + 8, 1 + 2 \cdot 8) = (11, 17)$$

Draw a coordinate system and show the bug's position for integral times  $t$  between 0 and 8 inclusive.

- B** 26. **Writing** Given:  $A$  and  $B$  are fixed points. In a few sentences, describe the set of points  $P$  such that  $\overrightarrow{AP} = t\overrightarrow{AB}$ . Consider the following three cases:

$$(1) 0 \leq t \leq 1 \quad (2) t \geq 1 \quad (3) t \leq 0$$

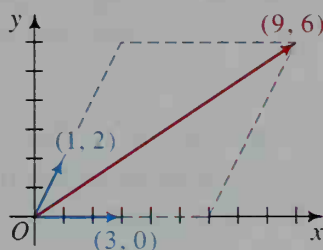
27. a. Make a sketch illustrating that  $|-2\mathbf{v}| = 2|\mathbf{v}|$ .  
 b. Prove that  $|k\mathbf{v}| = |k||\mathbf{v}|$ . (Hint: Let  $\mathbf{v} = (a, b)$ .)
28. a. Make a sketch showing two vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that  $|\mathbf{u} + \mathbf{v}| \neq |\mathbf{u}| + |\mathbf{v}|$ .  
 b. Make a sketch showing two vectors  $\mathbf{a}$  and  $\mathbf{b}$  such that  $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$ .
29. Quadrilateral  $ABCD$  is a parallelogram with vertices  $A = (1, 2)$ ,  $B = (3, 8)$ ,  $C = (9, 10)$ , and  $D = (x, y)$ . Find the values of  $x$  and  $y$ . (Hint:  $\overrightarrow{AB} = \overrightarrow{DC}$ .)
30. Repeat Exercise 29 if  $A = (-2, 1)$ ,  $B = (-5, -3)$ , and  $C = (3, -2)$ .

31. Refer to the diagram at the right.

- a. Find real numbers  $r$  and  $s$  such that  $r(1, 2) + s(3, 0) = (9, 6)$ . (Hint: Solve two simultaneous equations in  $r$  and  $s$ .)

- b. Relate your answer to the diagram.

32. Find real numbers  $r$  and  $s$  such that  $r(1, 2) + s(3, 0) = (-1, 4)$ . Illustrate your solution with a diagram similar to the one in Exercise 31.



Ex. 31

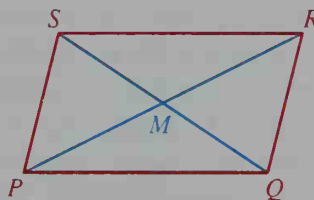
33. Find a scalar  $r$  such that  $|r(3, 4)| = 1$ .
34. Find a vector of length 1 in the same direction as  $(-4, 3)$ .
35. If  $A = (4, 0)$ , describe the set of points  $P$  such that  $|\overrightarrow{AP}| = 2$ .
36. If  $A = (4, 0)$  and  $B = (-4, 0)$ , describe the set of points  $P$  in the plane such that:

a.  $|\overrightarrow{AP}| + |\overrightarrow{BP}| = 10$

b.  $|\overrightarrow{AP} + \overrightarrow{BP}| = 10$

37. Let  $A = (x_1, y_1)$ ,  $B = (x_2, y_2)$ , and  $M$  be the midpoint of  $\overline{AB}$ . Use the method in Example 4 to prove the midpoint formula in Section 1-1.

38. The diagonals of quadrilateral  $PQRS$  have the same midpoint  $M$ . Use Exercise 37 to prove that  $\overrightarrow{SP} = \overrightarrow{RQ}$  and  $\overrightarrow{SR} = \overrightarrow{PQ}$ . What geometry theorem does this prove?



Ex. 38

39. Given: the points with polar coordinates  $P(10, 30^\circ)$  and  $Q(6, -30^\circ)$
- Make a vector diagram that shows  $\vec{OP}$ ,  $\vec{OQ}$ , and  $\vec{OP} + \vec{OQ}$ .
  - Find  $|\vec{OP} + \vec{OQ}|$ .
- C** 40. A theorem from geometry states that the centroid  $G$  of  $\triangle ABC$  is the point  $\frac{2}{3}$  of the way from each vertex to the midpoint of the opposite side.
- Find a formula for the coordinates of  $G$  in terms of the coordinates of  $A$ ,  $B$ , and  $C$ .
  - Given the points  $A(-2, 5)$ ,  $B(3, 7)$  and  $C(5, -3)$ , find the centroid of  $\triangle ABC$ .

## 12-3 Vector and Parametric Equations: Motion in a Plane

### Objective

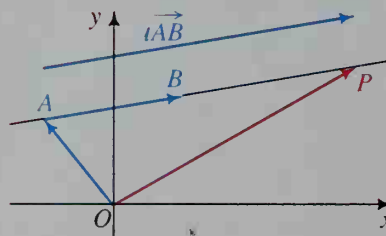
To use vector and parametric equations to describe motion in the plane.

### Vector and Parametric Equations of Lines

Suppose that  $P$  is any point on line  $AB$ . Then as the diagram at the right suggests

$$\vec{OP} = \vec{OA} + t\vec{AB}$$

for some real number  $t$ . This equation is called a **vector equation** of line  $AB$  and  $\vec{AB}$  is called a **direction vector** of the line.



**Example 1** Find a vector equation of the line through points  $A(3, 4)$  and  $B(5, 5)$ .

**Solution** First, find a direction vector of line  $AB$ .

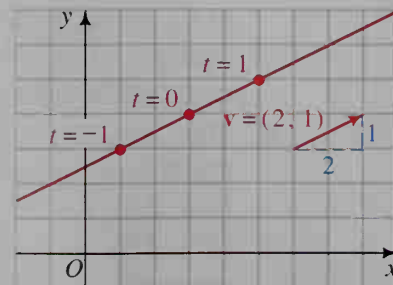
$$\vec{AB} = (5 - 3, 5 - 4) = (2, 1)$$

Then, let  $P = (x, y)$  and substitute into the general vector equation.

$$\vec{OP} = \vec{OA} + t\vec{AB}$$

$$(x, y) = (3, 4) + t(2, 1)$$

In Example 1, the vector equation of line  $AB$  contains the variable  $t$ . If  $t$  is interpreted as time, then you can think of  $P$  as moving along the line, taking various positions at various times  $t$ . The diagram at the right shows this. Notice that during one unit of time,  $P$  moves 2 units right and 1 unit up. In other words,  $P$  moves with constant velocity  $\mathbf{v} = (2, 1)$ . Its speed is  $|\mathbf{v}| = \sqrt{2^2 + 1^2} = \sqrt{5}$ .





It is instructive to compare the diagram at the bottom of the preceding page with the vector equation of the line.

$$(x, y) = (3, 4) + t(2, 1)$$

$\uparrow$                        $\uparrow$   
 position at time  $t = 0$       constant velocity

### Vector Equation for an Object Moving with Constant Velocity

$$(x, y) = (x_0, y_0) + t(a, b)$$

$\uparrow$                        $\uparrow$   
 position at time  $t = 0$       constant velocity

#### Example 2

Suppose an object which is moving with constant velocity is at point  $A(5, 3)$  when the time  $t = 0$  seconds and at point  $B(-4, 15)$  when the time  $t = 3$  seconds.

- Find the velocity and speed of the object.
- Find a vector equation that describes the motion of the object.

#### Solution

- $\overrightarrow{AB} = (-4, 15) - (5, 3) = (-9, 12)$   
 Since in 3 s, the object moves  $(-9, 12)$ ,  
 then in 1 s it moves

$$\frac{1}{3}(-9, 12) = (-3, 4).$$

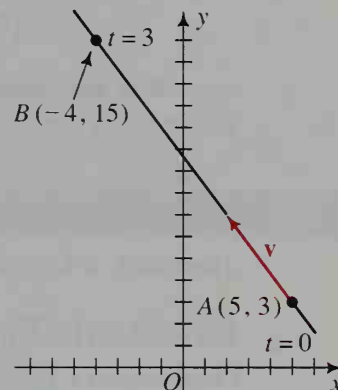
Thus, its velocity is

$$\mathbf{v} = (-3, 4)$$

and its speed is

$$|\mathbf{v}| = \sqrt{(-3)^2 + 4^2} = 5.$$

- $(x, y) = (x_0, y_0) + t(a, b)$   
 $(x, y) = (5, 3) + t(-3, 4)$



From the vector equation of the line in Example 2, we can derive two equations that give the  $x$ - and  $y$ -coordinates of the moving object at time  $t$ . These equations are called the **parametric equations** of the line, and  $t$  is called the **parameter**. They are derived as follows:

$$\begin{aligned} \text{Vector equation: } (x, y) &= (5, 3) + t(-3, 4) \\ &= (5 - 3t, 3 + 4t) \end{aligned}$$

$$\text{Parametric equations: } x = 5 - 3t \text{ and } y = 3 + 4t$$

Parametric equations of a line are useful in finding the times and positions at which an object moving with constant velocity crosses a curve whose equation is known. This situation is illustrated in the next example.

**Example 3**

An object moves along a line in such a way that its  $x$ - and  $y$ -coordinates at time  $t$  are  $x = 1 - t$  and  $y = 1 + 2t$ . When and where does the object cross the circle  $x^2 + (y - 1)^2 = 25$ ?

**Solution**

Substitute the parametric equations in the equation of the circle and solve for  $t$ .

$$\begin{aligned}x^2 + (y - 1)^2 &= 25 \\(1 - t)^2 + (1 + 2t - 1)^2 &= 25 \\1 - 2t + t^2 + 4t^2 &= 25 \\5t^2 - 2t - 24 &= 0 \\(5t - 12)(t + 2) &= 0\end{aligned}$$

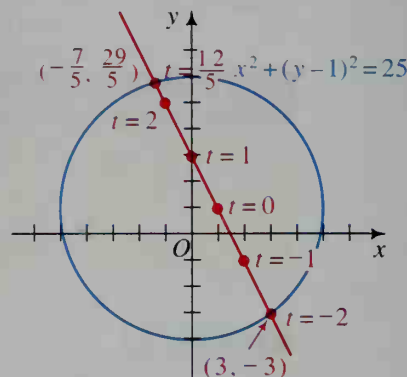
Thus, the times of crossing are  $t = \frac{12}{5}$  and  $t = -2$ .

To find the points of crossing, substitute  $t$  in the parametric equations:

When  $t = \frac{12}{5}$ , we get  $x = 1 - \frac{12}{5} = -\frac{7}{5}$  and  $y = 1 + 2\left(\frac{12}{5}\right) = \frac{29}{5}$ .

When  $t = -2$ , we get  $x = 1 - (-2) = 3$  and  $y = 1 + 2(-2) = -3$ .

Thus, the points of crossing are  $\left(-\frac{7}{5}, \frac{29}{5}\right)$  and  $(3, -3)$ .



It is easy to convert the parametric equations of the line in Example 3 to a single Cartesian equation in  $x$  and  $y$ . We eliminate  $t$ , using the following steps:

Solve the equation  $x = 1 - t$  for  $t$ :  $t = 1 - x$

Substitute for  $t$  in the equation for  $y$ :  $y = 1 + 2t = 1 + 2(1 - x) = 3 - 2x$

Therefore, a Cartesian equation of the line is  $y = -2x + 3$ .

**Parametric Equations of Curves**

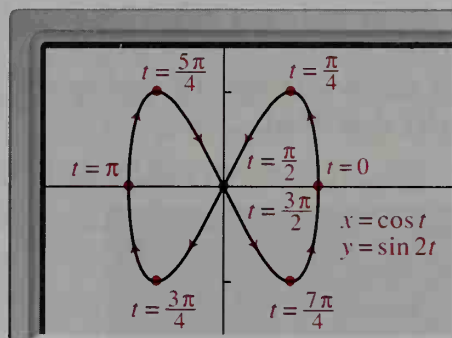
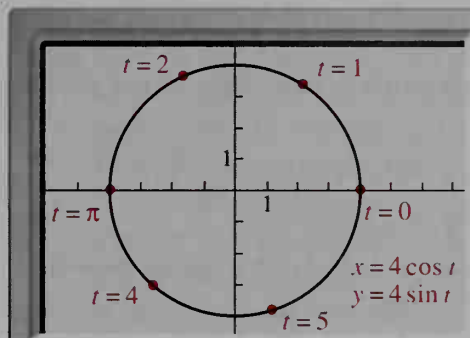
Objects don't always move along a line. In the following Activity, you will use a computer or a graphing calculator to explore the motion of a point along a curve.

**Activity**

Use a computer or a graphing calculator that graphs parametric equations. Be sure to use radian measure for  $t$ , letting the value of  $t$  vary from 0 to 6.28.

- Using a square screen (see page xxvii), graph the parametric equations  $x = 4 \cos t$  and  $y = 4 \sin t$ . Observe the motion of the TRACE cursor as the value of  $t$  varies from 0 to 6.28. Describe the motion. Where does the cursor begin and end, and in what direction does it move?
- Change the equations to  $x = 4 \cos (-t)$  and  $y = 4 \sin (-t)$  and again use the TRACE feature to observe the motion of the cursor for  $0 \leq t \leq 6.28$ . Describe how the change in the equation affects the motion.
- Repeat part (b) using the equations  $x = 4 \cos 3t$  and  $y = 4 \sin 3t$ .

Parametric equations can be used to describe the motion of a point along a variety of curves. In the Activity on the preceding page, you observed the motion of a point as it traveled around the path shown in the diagram at the left below. At the right below, a point moves along a *Lissajous curve*. In each case, as the value of  $t$  varies from 0 to  $2\pi$ , the point makes one complete trip around the curve.



To convert a pair of parametric equations involving trigonometric functions to a single Cartesian equation in  $x$  and  $y$ , we often use the Pythagorean relationship  $\sin^2 t + \cos^2 t = 1$ . For example, to show that the curve at the left above is a circle, use the equations  $x = 4 \cos t$  and  $y = 4 \sin t$  to find the value of  $x^2 + y^2$ :

$$x^2 + y^2 = (4 \cos t)^2 + (4 \sin t)^2 = 16(\cos^2 t + \sin^2 t) = 16$$

Therefore the parametric equations  $x = 4 \cos t$  and  $y = 4 \sin t$  satisfy the Cartesian equation  $x^2 + y^2 = 16$ , which is the equation of a circle.

## CLASS EXERCISES

- A line has vector equation  $(x, y) = (2, -5) + t(1, 3)$ .
  - Name three points on the line.
  - Find a direction vector of the line.
  - Give a pair of parametric equations of the line.
- An object moves with constant velocity along a line from  $A(-3, 1)$  at time  $t = 0$  through  $B(5, 7)$  at time  $t = 2$ . Give a vector equation of line  $AB$ .
- Show that the parametric equations in part (c) of the Activity on page 434 describe the circle with Cartesian equation  $x^2 + y^2 = 16$ .

## WRITTEN EXERCISES

Find vector and parametric equations for each specified line.

**A**

- The line through  $(1, 5)$  with direction vector  $(2, -1)$
- The line with  $x$ -intercept 4 and  $y$ -intercept  $-3$
- The line through  $(1, 0)$  and  $(3, -4)$
- The line through  $(3, 1)$  and  $(-4, -4)$
- The line through  $(-2, 3)$  and  $(5, 1)$
- The line through  $(7, 5)$  with inclination  $45^\circ$
- The horizontal line through  $(\pi, e)$
- The vertical line through  $(\sqrt{2}, \sqrt{3})$

In Exercises 9 and 10, a point moves in the plane so that its position  $P(x, y)$  at time  $t$  is given by the specified vector equation.

- Graph the point's position at the times  $t = 0, 1, 2, 3, -1, -2$ , and  $-3$ .
- Find the velocity and speed of the moving point.
- Find the parametric equations of the moving point.

9.  $(x, y) = (1, 4) + t(3, -2)$

10.  $(x, y) = (-2, 0) + t(1, 3)$

In Exercises 11 and 12, find vector and parametric equations of the moving object described.

- Velocity =  $(3, -1)$  and position at time  $t = 0$  is  $(2, 3)$ .
  - Velocity =  $(1, -1)$  and position at time  $t = 0$  is  $(1, -5)$ .
  - A line has vector equation  $(x, y) = (3, 2) + t(2, 4)$ . Give a pair of parametric equations and a Cartesian equation of the line.
  - A line has parametric equations  $x = 5 - t$  and  $y = 4 + 2t$ . Give a vector equation and a Cartesian equation of the line.
  - Writing** Describe the line having parametric equations  $x = 2$  and  $y = t$ .
    - Give a direction vector of the line.
    - What can you say about the slope of the line?
  - Writing** Describe the line having parametric equations  $x = t$  and  $y = 3$ .
    - Give a direction vector of the line.
    - What is the slope of the line?
  - A line has direction vector  $(2, 3)$ . What is the slope of the line?
    - A line has direction vector  $(4, 6)$ . What is the slope of the line?
    - Explain why the following lines are parallel:  
 $(x, y) = (8, 1) + r(2, 3)$  and  $(x, y) = (2, 5) + s(4, 6)$
    - Find a vector equation of the line through  $(7, 9)$  and parallel to these lines.
  - Find a vector equation of the line through  $(2, 1)$  and parallel to the line  $(x, y) = (-2, 7) + t(3, 5)$ .
  - At time  $t$ , the position of an object moving with constant velocity is given by the parametric equations  $x = 2 - 3t$  and  $y = -1 + 2t$ .
    - What are the velocity and speed of the object?
    - When and where does it cross the line  $x + y = 2$ ?
  - At time  $t$ , the position of an object moving with constant velocity is given by the parametric equations  $x = 1 + 3t$  and  $y = 2 - 4t$ .
    - What are the velocity and speed of the object?
    - When and where does it cross the  $x$ -axis?
- B** 21. An object moves with constant velocity so that its position at time  $t$  is  $(x, y) = (1, 1) + t(-1, 1)$ . When and where does the object cross the circle  $(x - 1)^2 + y^2 = 5$ ? Illustrate with a sketch.
22. An object moves with constant velocity so that its position at time  $t$  is  $(x, y) = (2, 0) + t(1, -1)$ . When and where does the object cross the hyperbola  $x^2 - 2y^2 = 4$ ? Illustrate with a sketch.



In part (b) of Exercises 23 and 24, use a computer or a graphing calculator that will graph two sets of parametric equations simultaneously.

23. A spider and a fly crawl so that their positions at time  $t$  (in seconds) are:  
 spider:  $(x, y) = (-2, 5) + t(1, -2)$       fly:  $(x, y) = (1, 1) + t(-1, 1)$

- Make a sketch showing the position of the spider and the position of the fly at various times. Do their lines of travel meet?
- You can use a computer or a graphing calculator to simulate the movement of the spider and the fly. Graph the parametric equations of their paths simultaneously. Do the spider and the fly appear to meet? If so, use the TRACE feature to determine when and where they meet.
- It is possible to determine algebraically whether the spider meets the fly. Just equate the  $x$ -coordinates of the positions of the bugs and also the  $y$ -coordinates. This gives the following equations:

$$\begin{aligned}x &= -2 + t = 1 - t \\y &= 5 - 2t = 1 + t\end{aligned}$$

The spider will meet the fly if and only if there is a single value of  $t$  satisfying both equations. Is there such a value of  $t$ ?

24. Repeat Exercise 23 for these equations:

$$\begin{aligned}\text{spider: } (x, y) &= (3, -2) + t(2, 1) \\ \text{fly: } (x, y) &= (-1, 6) + t(4, -3)\end{aligned}$$

25. Without graphing, describe the curve with parametric equations  $x = r \cos t$  and  $y = r \sin t$ . (Hint: What is the value of  $x^2 + y^2$ ?)



In Exercises 26–27, you may wish to confirm your answers by using a computer or a graphing calculator that graphs parametric equations.

26. If an object moves so that its position at time  $t$  is given by the parametric equations  $x = 8 \cos t$  and  $y = 8 \sin t$ , then the object travels counterclockwise around the circle of radius 8 centered at the origin. The object begins at the point  $(8, 0)$  and completes one revolution as  $t$  varies from 0 to  $2\pi$ . Give a pair of parametric equations that will cause the object's motion to occur in each of the following ways as  $t$  varies from 0 to  $2\pi$ .
- The object begins at  $(8, 0)$  and completes 2 counterclockwise revolutions.
  - The object begins at  $(8, 0)$  and completes 1 clockwise revolution.
  - The object begins at  $(-8, 0)$  and completes 1 counterclockwise revolution.
  - The object begins at  $(0, -8)$  and completes 2 clockwise revolutions.
27.
  - Before graphing, describe what you think the curve with parametric equations  $x = 2 \cos t$  and  $y = 5 \sin t$  looks like. Then graph the equations.
  - Find a Cartesian equation for the parametric equations given in part (a).
  - An ellipse has Cartesian equation  $4x^2 + y^2 = 36$ . What do you think the parametric equations of the ellipse are? Check your answer by graphing.





In Exercise 28(a), use a computer or a graphing calculator to graph the given parametric equations. Use a square screen (see page xxvii).

28. **a.** Graph  $x = 4 + 3 \cos t$  and  $y = -3 + 3 \sin t$ .  
**b.** Describe the graph in a sentence and give its Cartesian equation.
29. **a.** After the circle  $x^2 + y^2 = 36$  is translated 3 units right and 5 units up, what does its Cartesian equation become?  
**b.** Give parametric equations of the circle before and after it is translated.
30. **a.** Show that the parametric equations  $x = a \cos t$ ,  $y = b \sin t$  satisfy the Cartesian equation of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .  
**b.** Sketch the ellipse with parametric equations  $x = 5 \cos t$ ,  $y = 3 \sin t$ .
31. **a.** Show that the parametric equations  $x = a \sec t$ ,  $y = b \tan t$  satisfy the Cartesian equation of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .  
**b.** A particle moves so that its position  $(x, y)$  at time  $t$  is given by  $x = \sec t$ ,  $y = \tan t$ . Sketch its path and tell what quadrant the particle is located in during each of the following time periods:  
 (1)  $0 < t < \frac{\pi}{2}$     (2)  $\frac{\pi}{2} < t < \pi$     (3)  $\pi < t < \frac{3\pi}{2}$     (4)  $\frac{3\pi}{2} < t < 2\pi$
32. **Discussion** If  $x = \sin t$  and  $y = \sin t$ , then  $y = x$ . However, the graph of the parametric equations is only *part* of the line  $y = x$ . Explain.
33. **Discussion** The graph of  $x = \tan t$ ,  $y = \tan t$  is different from the graph of  $x = \sec t$ ,  $y = \sec t$  even though both sets of parametric equations satisfy the relationship  $y = x$ . Explain.



In Exercises 34–37, use a computer or a graphing calculator to draw the graphs of the given parametric equations.

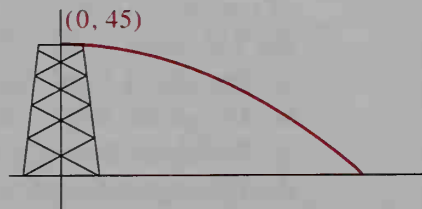
34.  $x = \cos 3t$ ,  $y = \sin 5t$                       35.  $x = \cos t$ ,  $y = \sin 4t$   
 36.  $x = \cos t$ ,  $y = \cos 2t$                       37.  $x = \cos 2t$ ,  $y = \sin t$



In some parts of the following exercises, you will need to use a computer or a graphing calculator that graphs parametric equations.

38. **Physics** The diagram at the right shows a 45 m tower whose top is represented by the point  $(0, 45)$ . If a ball is thrown horizontally from the top of the tower at 30 m/s, then its position  $(x, y)$ ,  $t$  seconds later, is given by

$$x = 30t \text{ and } y = 45 - 4.9t^2.$$

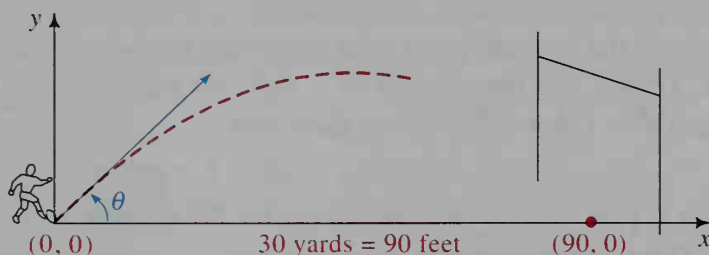


- a.** How far from the base of the tower does the ball land?  
**b.** Use a computer or a graphing calculator to graph the parametric equations given above. Use the TRACE feature to confirm your answer in part (a).  
**c.** Find a single equation in  $x$  and  $y$  that describes the path of the ball.

39. **Physics** From a point 30 yd directly in front of the goal posts, a football is kicked at angle of elevation  $\theta$  with initial velocity  $v$ . If a coordinate system is set up with the ball at  $(0, 0)$  as shown, then the position  $(x, y)$  of the ball  $t$  seconds after it is kicked is given by the parametric equations

$$x = (v \cos \theta)t \text{ and } y = (v \sin \theta)t - 16t^2.$$

- Suppose the initial velocity of the ball is 60 ft/s and the angle of elevation is  $30^\circ$ . Find an equation in  $x$  and  $y$  that describes the path of the ball.
- The goal post crossbar is 10 ft above the ground. Under the conditions stated in part (a), will the ball pass over the goal post crossbar?



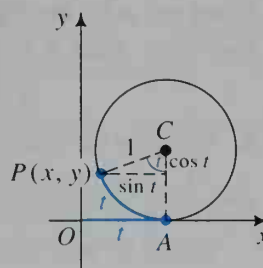
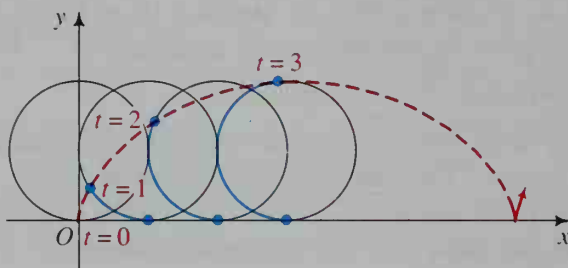
- To simulate the motion of the football, use a computer or a graphing calculator to graph the parametric equations of the ball's path when  $v = 60$  ft/s and  $\theta = 30^\circ$ . What parametric equations can you use to represent the goal post with a vertical line segment 10 units long? (For the most effective simulation, the goal post should be drawn first.)
- Keeping  $v = 60$  ft/s, vary the value of  $\theta$ . To the nearest degree, find the minimum value of  $\theta$  that will allow the ball to pass over the crossbar.

40. **Physics** A projectile is fired from a cannon whose angle of elevation is  $\theta$  and whose muzzle velocity is  $v$ . If the muzzle is at the origin of a coordinate system, the position  $P(x, y)$  of the particle  $t$  seconds later is given by the parametric equations  $x = (v \cos \theta)t$  and  $y = (v \sin \theta)t - 5t^2$ .



- What do these parametric equations give when  $\theta = 90^\circ$ ?
- Show that the projectile will hit the ground ( $y = 0$ ) when  $t = \frac{v \sin \theta}{5}$ . Then show that the  $x$ -value when the projectile hits the ground is  $x = \frac{v^2 \sin 2\theta}{10}$ .
- For what value of  $\theta$  will the cannon fire the longest horizontal distance?
- Graph the projectile's path for  $v = 100$  m/s and  $\theta = 15^\circ, 45^\circ$ , and  $60^\circ$ .
- Find a Cartesian equation of the projectile's path. Is the path parabolic?

41. A circle with radius 1 rolls along the  $x$ -axis at one unit per second. At  $t = 0$ ,  $P$  is at the origin. The path traced by  $P$  as the circle rolls is called a **cycloid**.

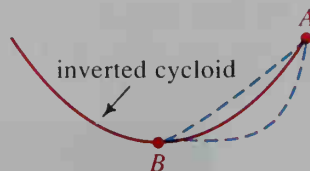


- Refer to the diagram at the right above and explain why at time  $t$ ,  $OA = t$ , arc  $AP = t$ , and the measure of  $\angle ACP$  is  $t$  radians.
- Use  $\vec{OP} = \vec{OA} + \vec{AC} + \vec{CP}$  to show that  

$$(x, y) = (t - \sin t, 1 - \cos t).$$
- Give a pair of parametric equations for the cycloid.

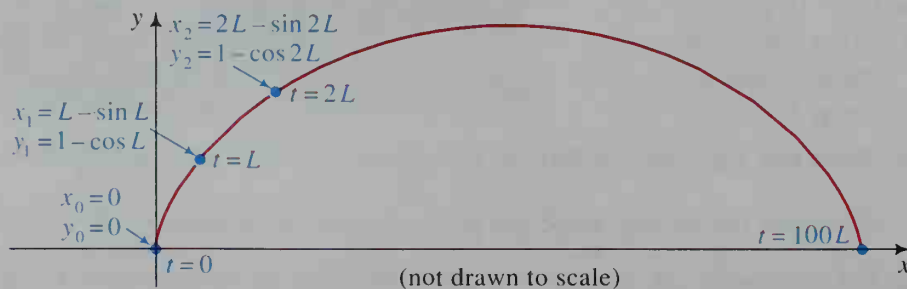
Note: Three of the cycloid's properties are:

- The length of one arch is 8 times the radius of the rolling circle.
- The area under one arch is three times the area of the rolling circle.
- Of all the paths from  $A$  to  $B$  that a sliding particle acting under gravity can take, the fastest is along an inverted cycloid arch.



## ////COMPUTER EXERCISES

- To approximate the length of one arch of a cycloid having parametric equations  $x = t - \sin t$  and  $y = 1 - \cos t$ , subdivide the time interval into 100 subintervals each of length  $L = \frac{2\pi}{100}$ , evaluate  $x$  and  $y$  for  $t = 0, L, 2L, 3L, \dots, 100L$ , and add the lengths of the line segments joining  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $\dots$ ,  $(x_{100}, y_{100})$ . Write a computer program to do this.



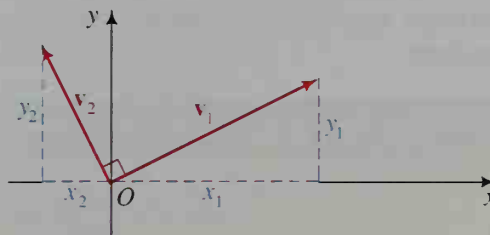
- Use a computer to approximate the area under one arch of the cycloid.

## 12-4 Parallel and Perpendicular Vectors; Dot Product

**Objective** To define and apply the dot product.

Geometrically, two vectors are *parallel* if the lines that contain them are parallel. Algebraically, we say that if  $\mathbf{v}_2 = k\mathbf{v}_1$  for some real number  $k$ , then  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are **parallel**. Similarly, two vectors are *perpendicular* if the lines that contain them are perpendicular. Refer to the diagram below where  $\mathbf{v}_1 = (x_1, y_1)$  and  $\mathbf{v}_2 = (x_2, y_2)$ . If the lines that contain  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are perpendicular, then the product of their slopes is  $-1$ . That is:

$$\begin{aligned}\left(\frac{y_1}{x_1}\right)\left(\frac{y_2}{x_2}\right) &= -1 \\ y_1 y_2 &= -x_1 x_2 \\ x_1 x_2 + y_1 y_2 &= 0\end{aligned}$$



Note that if one vector is horizontal and the other is vertical, then we still have  $x_1 x_2 + y_1 y_2 = 0$ .

The quantity  $x_1 x_2 + y_1 y_2$  occurs often enough in vector work that it is given a special name.

### The Dot Product

If  $\mathbf{v}_1 = (x_1, y_1)$  and  $\mathbf{v}_2 = (x_2, y_2)$ , then the **dot product** of the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , denoted by  $\mathbf{v}_1 \cdot \mathbf{v}_2$ , is defined by

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1 x_2 + y_1 y_2.$$

The dot product is also called the *scalar product*, because  $\mathbf{v}_1 \cdot \mathbf{v}_2$  is a scalar quantity.

#### Remarks

1. Perpendicular vectors are sometimes called **orthogonal** vectors.
2. If  $\mathbf{v}_1 = k\mathbf{v}_2$ , it is customary to call  $\mathbf{v}_1$  parallel to  $\mathbf{v}_2$  even though the arrows representing  $\mathbf{v}_1$  and  $\mathbf{v}_2$  may be collinear.
3. The zero vector is both parallel and perpendicular to all vectors.

#### Example 1

If  $\mathbf{u} = (3, -6)$ ,  $\mathbf{v} = (4, 2)$ , and  $\mathbf{w} = (-12, -6)$ . Find  $\mathbf{u} \cdot \mathbf{v}$  and  $\mathbf{v} \cdot \mathbf{w}$ . Show that  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular and that  $\mathbf{v}$  and  $\mathbf{w}$  are parallel.

#### Solution

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (3, -6) \cdot (4, 2) = 3(4) + (-6)(2) = 0 \\ \mathbf{v} \cdot \mathbf{w} &= (4, 2) \cdot (-12, -6) = 4(-12) + 2(-6) = -60\end{aligned}$$

Since  $\mathbf{u} \cdot \mathbf{v} = 0$ ,  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular.

Since  $\mathbf{w} = (-12, -6) = -3(4, 2) = -3\mathbf{v}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are parallel.

The dot product has many properties that follow from its definition. Proofs of the properties listed below are left as exercises.

### Properties of the Dot Product

1.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2.  $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
3.  $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$
4.  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

The following formula provides an easy way to find the measure of the angle between two nonzero vectors when we know their magnitudes and their dot product.

### The Angle Between Two Vectors

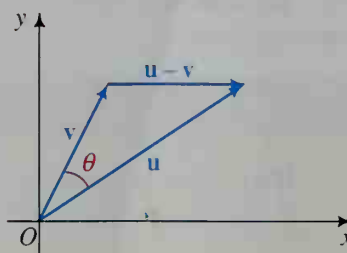
$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}, \text{ where } 0^\circ \leq \theta \leq 180^\circ$$

To prove this formula, we sketch the vector diagram shown at the right. In the diagram,  $\mathbf{u} = (x_1, y_1)$ ,  $\mathbf{v} = (x_2, y_2)$ , and  $\theta$  is the angle between these vectors. Note that the sides of the triangle have lengths:

$$|\mathbf{u}| = \sqrt{x_1^2 + y_1^2}$$

$$|\mathbf{v}| = \sqrt{x_2^2 + y_2^2}$$

$$|\mathbf{u} - \mathbf{v}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



From the law of cosines,

$$\cos \theta = \frac{|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2}{2|\mathbf{u}||\mathbf{v}|},$$

we get:

$$\begin{aligned} \cos \theta &= \frac{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - [(x_1 - x_2)^2 + (y_1 - y_2)^2]}{2|\mathbf{u}||\mathbf{v}|} \\ &= \frac{2x_1x_2 + 2y_1y_2}{2|\mathbf{u}||\mathbf{v}|} \\ &= \frac{x_1x_2 + y_1y_2}{|\mathbf{u}||\mathbf{v}|} \end{aligned}$$

Therefore,  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$ .

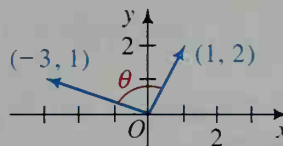


**Example 2**

To the nearest degree, find the measure of  $\theta$ , the angle between the vectors  $(1, 2)$  and  $(-3, 1)$ .

**Solution**

$$\begin{aligned}\cos \theta &= \frac{(1, 2) \cdot (-3, 1)}{|(1, 2)||(-3, 1)|} \\ &= \frac{-1}{\sqrt{5}\sqrt{10}} \approx -0.1414\end{aligned}$$



Therefore,  $\theta \approx 98^\circ$ .

**Example 3**

In  $\triangle PQR$ ,  $P = (2, 1)$ ,  $Q = (4, 7)$ , and  $R = (-2, 4)$ . Find the measure of  $\angle P$  to the nearest tenth of a degree.

**Solution**

First note that  $\angle P$  is formed by  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ .

$$\cos P = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|}$$

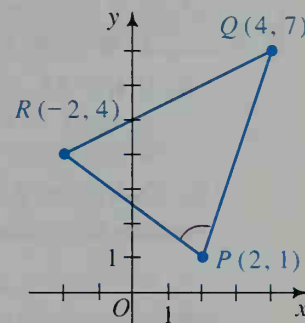
Next find the component form of  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ .

$$\overrightarrow{PQ} = (4 - 2, 7 - 1) = (2, 6)$$

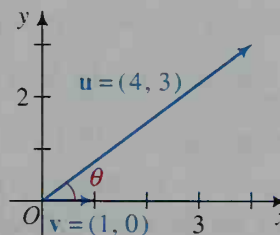
$$\overrightarrow{PR} = (-2 - 2, 4 - 1) = (-4, 3)$$

$$\begin{aligned}\text{Therefore, } \cos P &= \frac{(2, 6) \cdot (-4, 3)}{|(2, 6)||(-4, 3)|} \\ &= \frac{10}{\sqrt{40}\sqrt{25}} \approx 0.3162.\end{aligned}$$

Therefore,  $\angle P \approx 71.6^\circ$ .

**CLASS EXERCISES**

- Which of the following three vectors are parallel and which are perpendicular?  
 $\mathbf{u} = (4, -6)$     $\mathbf{v} = (-2, 3)$     $\mathbf{w} = (9, 6)$
- If  $\mathbf{u} = (3, 4)$  and  $\mathbf{v} = (-2, 2)$ , find:
  - $\mathbf{u} \cdot \mathbf{v}$
  - $2(\mathbf{u} \cdot \mathbf{v})$
  - $(2\mathbf{u}) \cdot \mathbf{v}$
  - $\mathbf{u} \cdot (2\mathbf{v})$
- If  $\mathbf{u} = (3, 4)$ , find:
  - $|\mathbf{u}|$
  - $|\mathbf{u}|^2$
  - $\mathbf{u} \cdot \mathbf{u}$
- If  $\theta$  is the angle between  $\mathbf{u} = (7, 1)$  and  $\mathbf{v} = (5, 5)$ , find:
  - $|\mathbf{u}|$
  - $|\mathbf{v}|$
  - $\mathbf{u} \cdot \mathbf{v}$
  - $\theta$  to the nearest degree
- For the figure shown, find  $\cos \theta$  (a) by using the dot product and (b) by using right-triangle trigonometry. (Your answers should agree.)



Ex. 5

## WRITTEN EXERCISES

- A**
- Find: a.  $(2, 3) \cdot (4, -5)$       b.  $(3, -5) \cdot (7, 4)$
  - Find: a.  $(-3, 0) \cdot (5, 7)$       b.  $\left(\frac{3}{5}, \frac{4}{5}\right) \cdot \left(\frac{1}{2}, -\frac{3}{2}\right)$
  - Find the value of  $a$  if the vectors  $(4, 6)$  and  $(a, 3)$  are (a) parallel, (b) perpendicular.
  - Find the value of  $a$  if the vectors  $(6, -8)$  and  $(4, a)$  are (a) parallel, (b) perpendicular.
  - If  $\mathbf{u} = (-2, 3)$ , find (a)  $\mathbf{u} \cdot \mathbf{u}$  and (b)  $|\mathbf{u}|^2$ .
  - Let  $\mathbf{u} = (a, b)$ . Show that  $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ .
  - If  $\mathbf{u} = (5, -3)$  and  $\mathbf{v} = (3, 7)$ , verify that:  
a.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$       b.  $2(\mathbf{u} \cdot \mathbf{v}) = (2\mathbf{u}) \cdot \mathbf{v}$
  - If  $\mathbf{u} = (1, 3)$  and  $\mathbf{v} = (-4, 2)$ , show that:  
a.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$       b.  $3(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (3\mathbf{v})$

In Exercises 9 and 10, verify that  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$  for the given vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ .

- $\mathbf{u} = (-2, 5)$ ,  $\mathbf{v} = (1, 3)$ , and  $\mathbf{w} = (-1, 2)$
- $\mathbf{u} = (1, -4)$ ,  $\mathbf{v} = (-2, -2)$ , and  $\mathbf{w} = (1, 5)$
- Verify that the angle between  $\mathbf{u} = (1, 3)$  and  $\mathbf{v} = (2, 1)$  is  $45^\circ$ .
- Verify that the angle between  $\mathbf{u} = (2, 3)$  and  $\mathbf{v} = (1, -5)$  is  $135^\circ$ .

In Exercises 13–16, give angle measures to the nearest tenth of a degree.

- Find the measure of the angle between  $\mathbf{u} = (3, -4)$  and  $\mathbf{v} = (3, 4)$ .
- Find the measure of the angle between  $\mathbf{u} = (1, 3)$  and  $\mathbf{v} = (-8, 5)$ .
- Given  $A(1, 5)$ ,  $B(4, 6)$ , and  $C(2, 8)$ , find the measure of  $\angle A$ .
- Given  $A(-5, 1)$ ,  $B(-3, 3)$ , and  $C(2, 2)$ , find the measure of  $\angle A$ .
- Given  $P(0, 3)$ ,  $Q(2, 4)$ , and  $R(3, 7)$ , verify that  $\cos P = \frac{2}{\sqrt{5}}$ .
- Given  $R(0, -3)$ ,  $S(2, 3)$ , and  $T(7, -2)$ , verify that  $\cos R = \frac{1}{\sqrt{5}}$ .
- a. Given  $A(1, -3)$ ,  $B(-1, 3)$ , and  $C(6, 2)$ , find  $\cos C$  and  $\sin C$ .  
b. Use the formula  $\text{Area} = \frac{1}{2}ab \sin C$  to find the area of  $\triangle ABC$ .
- Repeat Exercise 19 for  $A(5, 3)$ ,  $B(3, 0)$ , and  $C(2, 2)$ .
- Given  $A(3, 1)$ ,  $B(14, -1)$ , and  $C(5, 5)$ , find the measure of  $\angle BAC$  (a) by using the dot product and (b) by using angles of inclination. (See page 296 for the inclination formula.)

22. Given  $A(0, 0)$ ,  $B(2, 3)$ , and  $C(1, -5)$ , find the measure of  $\angle BAC$  (a) by using the dot product and (b) by using angles of inclination. (See page 296 for the inclination formula.)

- B** 23. Let  $\mathbf{u} = (x_1, y_1)$ ,  $\mathbf{v} = (x_2, y_2)$ , and  $\mathbf{w} = (x_3, y_3)$ . Show that

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

by expressing both sides of the equation in terms of components.

24. Let  $\mathbf{u} = (x_1, y_1)$ ,  $\mathbf{v} = (x_2, y_2)$ , and let  $k$  be a scalar. Show that

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \quad \text{and} \quad (k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}).$$

25. **Physics** The vector  $\mathbf{s}$  represents the displacement of a sled that is pulled with force  $\mathbf{F}$ . The *work* done in moving the sled in the direction of  $\mathbf{s}$  is defined to be:

component of  $\mathbf{F}$  in the direction of  $\mathbf{s} \times$  distance sled moves

- a. Show that this definition can be written as:

$$\text{Work} = |\mathbf{F}| \cos \theta \times |\mathbf{s}|$$

- b. Use part (a) and the formula for the angle between two vectors to show that:

$$\text{Work} = \mathbf{F} \cdot \mathbf{s}$$

- c. Find the work done if  $\mathbf{F} = (10, 3)$  and  $\mathbf{s} = (35, 0)$ .

26. **Physics** Refer to the diagram in Exercise 25. Write a few sentences to explain what happens to the work done in moving the sled in the direction of  $\mathbf{s}$  if  $|\mathbf{F}|$  remains the same and  $\theta$  increases from  $0^\circ$  to  $90^\circ$ .

27. The purpose of this exercise is to prove the Pythagorean theorem using the dot product. In the right triangle, let  $\mathbf{u} = \overrightarrow{AB}$  and  $\mathbf{v} = \overrightarrow{BC}$ .

- a. Note that  $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$ , so that

$$|\overrightarrow{AC}|^2 = |\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}).$$

Show that this last equation simplifies to

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2.$$

- b. Explain why  $\mathbf{u} \cdot \mathbf{v} = 0$ .

- c. Use part(a) and part(b) to prove that  $|\overrightarrow{AC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$ .

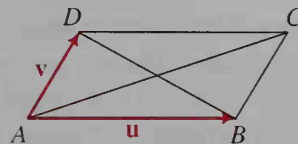
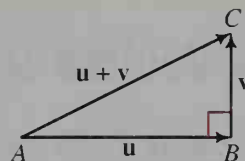
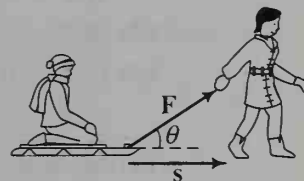
28. Use dot products to show that if  $ABCD$  is a parallelogram, then

$$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = 2|\overrightarrow{AB}|^2 + 2|\overrightarrow{AD}|^2.$$

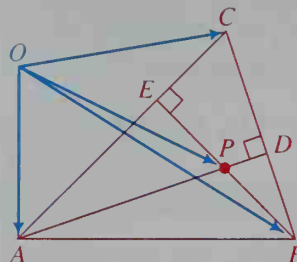
(Hint: Let  $\mathbf{u} = \overrightarrow{AB}$  and  $\mathbf{v} = \overrightarrow{AD}$ . Then use dot products to express each side of the equation in terms of  $\mathbf{u}$  and  $\mathbf{v}$ .)

29. a. Show that  $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$  if and only if  $|\mathbf{u}| = |\mathbf{v}|$ .

- b. Use part (a) and the figure for Exercise 28 to prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus.



- C** 30. The purpose of this exercise is to use vectors to prove the theorem that the lines containing the altitudes of a triangle are concurrent.
- Given:  $\overline{AD}$  and  $\overline{BE}$  are altitudes of  $\triangle ABC$ .



- (1) Show that if line  $AP$  is perpendicular to  $\overline{BC}$ , then:  

$$(\overrightarrow{OP} - \overrightarrow{OA}) \cdot (\overrightarrow{OC} - \overrightarrow{OB}) = 0$$
  - (2) Multiply out the expression on the left side.
  - (3) Prove: If line  $BP$  is perpendicular to  $\overline{CA}$ , then:  

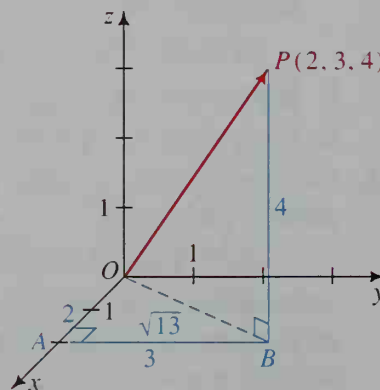
$$(\overrightarrow{OP} - \overrightarrow{OB}) \cdot (\overrightarrow{OA} - \overrightarrow{OC}) = 0$$
  - (4) Multiply out the expression on the left side.
  - (5) Add the equations in (2) and (4) and factor the result.
  - (6) Show how your answer in (5) proves that line  $CP$  is perpendicular to  $\overline{AB}$ .  
 Is the theorem proved?
31. Use  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$  to prove that  $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$ . Then use this result and the result of Exercise 27(a) to prove that  $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ .

## Three Dimensions

### 12-5 Vectors in Three Dimensions

**Objective** To extend vectors to three dimensions and to apply them.

To start our study of vectors in three dimensions, we begin by setting up a three-dimensional coordinate system. In this system, we have three *axes* meeting at the *origin*  $O$  as shown at the right. The plane containing the  $x$ - and  $y$ -axes is called the  **$xy$ -coordinate plane**. The  **$xz$ -coordinate plane** and the  **$yz$ -coordinate plane** are similarly defined. These planes divide space into eight regions called **octants**. We locate points in space by means of ordered triples. In the figure,  $P$  is represented by  $(2, 3, 4)$ . The numbers 2, 3, and 4 are not only the  $x$ -,  $y$ -, and  $z$ -coordinates of point  $P$  but they are also the  $x$ -,  $y$ -, and  $z$ -components of the vector  $\overrightarrow{OP}$ , which is also written  $(2, 3, 4)$ .



To find  $|\vec{OP}|$ , refer to the figure at the bottom of the preceding page. Note that in right  $\triangle OAB$ ,

$$|\vec{OB}| = \sqrt{2^2 + 3^2} = \sqrt{13},$$

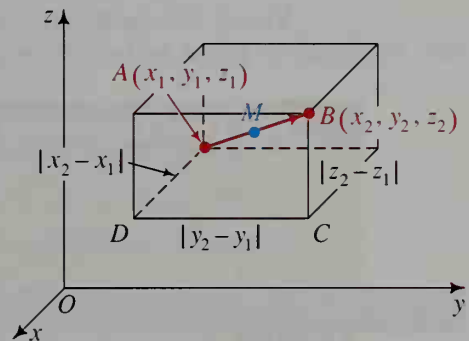
and in right  $\triangle OBP$ ,

$$|\vec{OP}| = \sqrt{(\sqrt{13})^2 + 4^2} = \sqrt{29}.$$

In general, if  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$  are two points in three-dimensional space, then

$$\vec{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

The diagram to the right can be used to derive the following distance and midpoint formulas. (See Written Exercises 29 and 30.) Notice that the distance and midpoint formulas for a three-dimensional coordinate system are generalizations of the two-dimensional formulas.



### The Distance and Midpoint Formulas

$$|\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

= the distance between A and B

$$\text{The midpoint of } \vec{AB} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

#### Example 1

A sphere has points  $A(8, -2, 3)$  and  $B(4, 0, 7)$  as endpoints of a diameter.

- Find the center  $C$  and radius  $r$  of the sphere.
- Find an equation of the sphere.

#### Solution

$$\begin{aligned} \text{a. } C &= \text{midpoint of } \vec{AB} \\ &= \left( \frac{8 + 4}{2}, \frac{-2 + 0}{2}, \frac{3 + 7}{2} \right) = (6, -1, 5) \end{aligned}$$

Use the distance formula to find  $r$ .

$$r = CB = \sqrt{(4 - 6)^2 + (0 - (-1))^2 + (7 - 5)^2} = 3$$

- The distance between  $C(6, -1, 5)$  and  $P(x, y, z)$ , an arbitrary point on the sphere, equals the radius, 3. Thus:

$$PC = \sqrt{(x - 6)^2 + (y + 1)^2 + (z - 5)^2} = 3$$

Equivalently:

$$(x - 6)^2 + (y + 1)^2 + (z - 5)^2 = 9$$



The result of Example 1 can be generalized to give the following. Notice the similarity between the equation of a sphere and the equation of a circle.

### Equation of the Sphere with Center $(x_0, y_0, z_0)$ and Radius $r$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Vector and parametric equations of a line in three dimensions are similar to those of a line in two dimensions.

### The Line Containing $(x_0, y_0, z_0)$ with Direction $(a, b, c)$

Vector equation:  $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$

Parametric equations:  $x = x_0 + at$      $y = y_0 + bt$      $z = z_0 + ct$

#### Example 2

Find vector and parametric equations of the line containing  $A(2, 3, 1)$  and  $B(5, 4, 6)$ .

#### Solution

A direction vector of the line is  $\overrightarrow{AB} = (5 - 2, 4 - 3, 6 - 1) = (3, 1, 5)$ .

A corresponding vector equation is  $(x, y, z) = (2, 3, 1) + t(3, 1, 5)$ .

Parametric equations of the line are:

$$x = 2 + 3t \quad y = 3 + t \quad z = 1 + 5t$$

The preceding discussion and examples show that vector work in three dimensions closely parallels that in two dimensions. The table below summarizes properties of vectors in three dimensions.

### Basic Vector Properties

If  $\mathbf{u} = (x_1, y_1, z_1)$  and  $\mathbf{v} = (x_2, y_2, z_2)$ , then we have the following:

1. Vector addition:  $\mathbf{u} + \mathbf{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
2. Scalar multiplication:  $k\mathbf{u} = k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$
3. The magnitude of  $\mathbf{u}$ :  $|\mathbf{u}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$
4. Dot product:  $\mathbf{u} \cdot \mathbf{v} = x_1x_2 + y_1y_2 + z_1z_2$
5. The angle  $\theta$  between two vectors:  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

Note that perpendicular vectors need not intersect. Also recall that the zero vector is parallel and perpendicular to all vectors.

**Example 3**

Find a vector equation of the line through  $(1, 5, -2)$  and parallel to the line  $L$  with equation  $(x, y, z) = (8, 0, 1) + t(4, 3, 2)$ .

**Solution**

From the equation for  $L$ ,  $(4, 3, 2)$  is a direction vector of  $L$ . Thus,  $(4, 3, 2)$  is a direction vector for any line parallel to  $L$ . Hence, an equation of the line containing  $(1, 5, -2)$  and having direction vector  $(4, 3, 2)$  is  $(x, y, z) = (1, 5, -2) + t(4, 3, 2)$ .

**Example 4**

Find the measure of  $\theta$ , the angle between the vectors  $(4, -5, 3)$  and  $(7, 0, -1)$ .

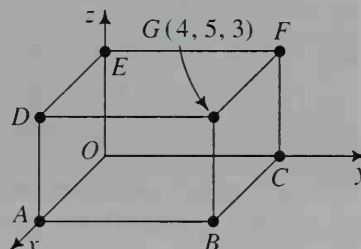
**Solution**

$$\cos \theta = \frac{(4, -5, 3) \cdot (7, 0, -1)}{|(4, -5, 3)|| (7, 0, -1)|} = \frac{28 + 0 - 3}{\sqrt{50}\sqrt{50}} = \frac{25}{50} = \frac{1}{2}$$

Thus,  $\theta = 60^\circ$ .

**CLASS EXERCISES**

- Find the length and midpoint of  $\overline{AB}$ .
  - $A = (0, 0, 0)$  and  $B = (1, -2, 3)$
  - $A = (3, 0, -4)$  and  $B = (5, 4, 2)$
- The figure at the right shows a rectangular box with one vertex at the origin  $O$  of a three-dimensional coordinate system and three edges along the axes as shown.
  - If point  $G = (4, 5, 3)$ , give the coordinates of points  $A, B, C, D, E$ , and  $F$ .
  - Find  $|\overrightarrow{OG}|$ .
- Find an equation of the sphere with radius 5 and center at (a)  $(0, 0, 0)$  and at (b)  $(1, 2, 3)$ .
- Simplify: a.  $(3, 5, -2) + 2(1, 2, 3)$       b.  $(3, 8, 1) \cdot (4, -1, 4)$
- Discussion** How can you tell if two vectors are perpendicular? How can you tell if they are parallel?
- Line  $L$  has vector equation  $(x, y, z) = (2, 5, 1) + t(6, 7, 8)$ .
  - Name two points on  $L$ .
  - Find three parametric equations of  $L$ .
  - Find a direction vector of  $L$ .
  - Explain why  $L$  is parallel to line  $(x, y, z) = (8, -2, 2) + s(6, 7, 8)$ .
  - Explain why  $L$  is perpendicular to line  $(x, y, z) = (2, 5, 1) + s(4, 0, -3)$ .
- Discussion** Explain how to find where the line
 
$$(x, y, z) = (-6, 0, 3) + s(4, 1, -1)$$
 intersects the sphere  $x^2 + y^2 + z^2 = 9$ .
- Discussion** Suppose that  $\cos \theta = \pm 1$ , where  $\theta$  is the angle between two vectors. What can you conclude about the vectors?



## WRITTEN EXERCISES

In Exercises 1–4, find the length and midpoint of  $\overline{AB}$ .

- A** 1.  $A = (2, 5, -3)$  and  $B = (0, 3, 1)$       2.  $A = (2, 8, -5)$  and  $B = (2, 0, -3)$   
3.  $A = (3, -5, 0)$  and  $B = (-1, 1, 2)$       4.  $A = (4, 1, 2)$  and  $B = (-4, -1, -2)$   
5. Refer to Class Exercise 2 and suppose that  $G = (5, 6, 4)$ . What are the coordinates of points  $A, B, C, D, E$ , and  $F$ ? Find  $|\overrightarrow{OG}|$ .  
6. Refer to Class Exercise 2 and suppose that  $G = (6, 7, 3)$ . What are the coordinates of points  $A, B, C, D, E$ , and  $F$ ? Find  $|\overrightarrow{OG}|$ .

In Exercises 7 and 8, simplify each expression.

7. a.  $(3, 8, -2) + 2(4, -1, 2)$       b.  $(1, -8, 6) \cdot (5, 2, 1)$       c.  $|(3, 5, 1)|$   
8. a.  $(8, 7, 4) - (2, 0, 9) + (1, -2, 1)$       b.  $(1, 4, 2) \cdot (3, 1, 2)$       c.  $|(6, 2, 3)|$   
9. Are vectors  $(3, -7, 1)$  and  $(6, 3, 3)$  perpendicular?  
10. Find the value of  $k$  if  $(2, k, -3)$  and  $(4, 2, 6)$  are perpendicular.  
11. Find an equation of the sphere with radius 2 and center at (a) the origin and (b)  $(3, -1, 2)$ .  
12. Find an equation of the sphere with radius 7 and center  $(1, 5, 3)$ . Show that the point  $(7, 7, 6)$  is on the sphere.  
13. Find the center and radius of the sphere with equation  
$$x^2 + y^2 + z^2 + 2x - 4y - 6z = 11.$$
  
(Hint: Complete the squares in  $x, y$ , and  $z$ .)  
14. Find the center and radius of the sphere with equation  
$$x^2 + y^2 + z^2 - 6x + 10y + 2z = 65.$$
  
15. Find the angle between  $(8, 6, 0)$  and  $(2, -1, 2)$  to the nearest tenth of a degree.  
16. Find the angle between  $(2, 2, 1)$  and  $(3, 6, -2)$  to the nearest tenth of a degree.  
17. Let  $A = (1, 3, 4)$ ,  $B = (3, -1, 0)$ , and  $C = (3, 2, 6)$ .  
a. Show that  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are perpendicular.  
b. Find the area of right triangle  $ABC$ .  
18. Repeat Exercise 17 if  $A = (3, 7, -5)$ ,  $B = (5, 9, -4)$ , and  $C = (7, 5, -9)$ .  
19. a. Given  $A(0, 0, 0)$ ,  $B(1, 1, -2)$ , and  $C(0, 1, -1)$ , find  $\cos A$  and  $\sin A$ .  
b. Find the area of  $\triangle ABC$ .  
20. Repeat Exercise 19 given the points  $A(1, 2, 3)$ ,  $B(1, 5, 7)$ , and  $C(3, 3, 1)$ .  
21. Line  $L$  has vector equation  $(x, y, z) = (-2, 0, 1) + t(4, -1, 1)$ .  
a. Find three parametric equations of  $L$ .  
b. Name two points on  $L$ .  
c. Which of the points  $(-10, 2, -1)$ ,  $(18, -2, 6)$ , and  $(14, -5, 4)$  are on  $L$ ?  
d. Write a vector equation of the line containing  $(1, 2, 3)$  and parallel to  $L$ .

22. Repeat Exercise 21 if the vector equation of  $L$  is

$$(x, y, z) = (6, -1, 0) + t(2, -1, 1).$$

23. Write vector and parametric equations for the line containing  $A(4, 2, -1)$  and  $B(6, 3, 2)$ .

24. Write vector and parametric equations for the line containing  $A(2, 3, 1)$  and  $B(4, -1, 3)$ .

**B** 25. Explain why the line  $(x, y, z) = (4, -5, 2) + t(2, 1, 0)$  is parallel to the  $xy$ -plane.

26. Describe the line  $(x, y, z) = (1, 2, 3) + t(0, 0, 1)$ . Where does it intersect the  $xy$ -plane?

27. Given line  $L$  with equation:

$$(x, y, z) = (5, 3, 0) + t(1, 4, -6)$$

a. Find vector and parametric equations of the line through  $(1, 2, 3)$  and parallel to  $L$ .

b. Find vector and parametric equations of a line through  $(5, 3, 0)$  and perpendicular to  $L$ .

28. Given line  $L$  with equation:

$$(x, y, z) = (1, 3, -2) + t(2, 1, 1)$$

a. Find vector and parametric equations of the line through  $(4, 5, 6)$  and parallel to  $L$ .

b. Find vector and parametric equations of a line through  $(1, 3, -2)$  and perpendicular to  $L$ .



Parallel and perpendicular lines dominate the design of this office building.

29. Refer to the diagram on page 447. Let  $A = (x_1, y_1, z_1)$  and  $B = (x_2, y_2, z_2)$ . Prove the distance formula.

30. Refer to the diagram on page 447. Let  $M(x, y, z)$  be the midpoint of  $\overline{AB}$ . Prove the midpoint formula.

In Exercises 31–36,  $O$  is the origin,  $A$  and  $B$  are fixed points, and  $P$  is an arbitrary point. Describe the set of points  $P$  that satisfy the given equation in (a) a two-dimensional plane and (b) three-dimensional space.

31.  $|\overrightarrow{PA}| = |\overrightarrow{PB}|$

32.  $\overrightarrow{PA} \cdot \overrightarrow{AB} = 0$

33.  $\overrightarrow{PA} \cdot \overrightarrow{PB} = 0$

34.  $|\overrightarrow{PA}| = 10$

35.  $|\overrightarrow{PA}| + |\overrightarrow{PB}| = 10$

36.  $\overrightarrow{OP} = \overrightarrow{OA} + t\overrightarrow{AB}, |t| \leq 1$

37. a. If  $P(x, y, z)$  is equidistant from  $A(0, 0, 0)$  and  $B(2, 3, 6)$ , find and simplify an equation that must be satisfied by  $x, y$ , and  $z$ .

b. Verify that the coordinates of the midpoint of  $\overline{AB}$  satisfy the equation.

38. a. Give an equation in  $x, y$ , and  $z$  that satisfies  $\overrightarrow{PA} \cdot \overrightarrow{AB} = 0$  if  $P = (x, y, z)$ ,  $A = (3, 5, 2)$ , and  $B(6, 6, 4)$ .

b. Compare your answer with Exercise 32.

39. a. Give an equation in  $x$ ,  $y$ , and  $z$  that satisfies  $\overrightarrow{PA} \cdot \overrightarrow{PB} = 0$  if  $P = (x, y, z)$ ,  $A = (2, 2, 1)$ , and  $B(-2, -2, -1)$ .  
 b. Compare your answer with Exercise 33.
40. Where does the line  $(x, y, z) = (6, -5, 4) + t(3, -2, 4)$  intersect (a) the  $xy$ -plane, (b) the  $yz$ -plane, and (c) the  $xz$ -plane?
41. Describe the set of points  $S$  in the  $xy$ -plane that are also on the sphere whose equation is  $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$ . Give an equation of  $S$ .
42. Where does the line  $(x, y, z) = (3, 4, -2) + t(2, -1, 2)$  and the sphere whose equation is  $(x - 3)^2 + (y - 4)^2 + (z + 2)^2 = 36$  intersect?
- C** 43. a. Show that the lines with equations  $(x, y, z) = (3, 0, -4) + t(1, 2, 3)$  and  $(x, y, z) = (1, 5, 2) + s(2, 1, 4)$  have no points in common.  
 b. Part(a) shows that the lines do not intersect. Are they parallel?
44. a. Show that the lines with equations  $(x, y, z) = (-1, 5, 0) + t(1, 2, -1)$  and  $(x, y, z) = (0, 1, 3) + s(1, -1, 1)$  intersect.  
 b. Find the coordinates of their point of intersection.
45. a. Explain why the following lines are perpendicular:  
 $L_1: (x, y, z) = (3, -1, 5) + t(2, 3, 4)$  and  $L_2: (x, y, z) = (3, -1, 5) + s(2, 0, -1)$   
 b. Find an equation of a line perpendicular to both  $L_1$  and  $L_2$ .

## 12-6 Vectors and Planes

**Objective** To sketch planes and to find equations of planes.

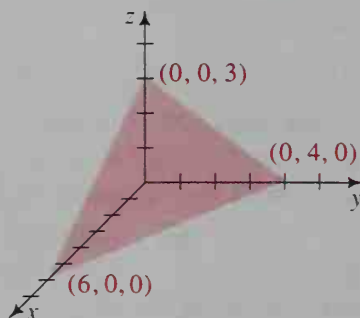
In two-dimensional space, the graph of  $ax + by = c$  is a line provided that  $a$  and  $b$  are not both 0. In three-dimensional space, the graph of  $ax + by + cz = d$  is a *plane* provided that  $a$ ,  $b$ , and  $c$  are not all 0. For example, the graph of  $2x + 3y + 4z = 12$  is a plane. To sketch it, we locate the points where the plane intersects the axes.

Setting  $y$  and  $z = 0$  gives  $2x = 12$ . Thus,  $x = 6$  and 6 is the  $x$ -intercept.

Setting  $x$  and  $z = 0$  gives  $3y = 12$ . Thus,  $y = 4$  and 4 is the  $y$ -intercept.

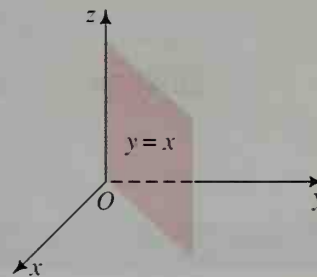
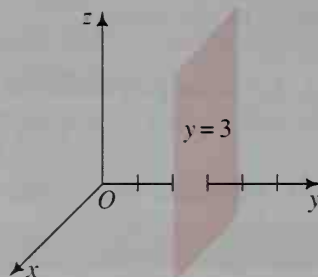
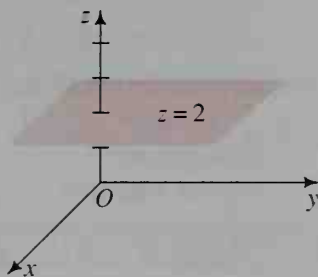
Setting  $x$  and  $y = 0$  gives  $4z = 12$ . Thus,  $z = 3$  and 3 is the  $z$ -intercept.

The diagram below shows that portion of the plane in the *first octant*, the region where all coordinates are positive.





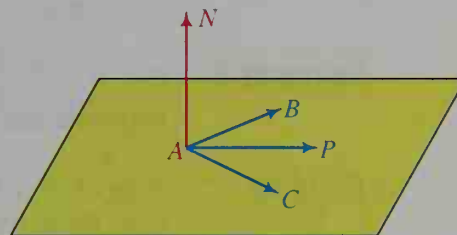
Parts of some other planes are shown below along with their equations.



If a vector  $\vec{AN}$  is perpendicular to a plane containing  $A$ ,  $B$ , and  $C$ , as shown, then it is perpendicular to  $\vec{AB}$ ,  $\vec{AC}$ , and all other vectors in the plane. Thus, if  $P$  is any point in the plane, then

$$\vec{AN} \cdot \vec{AP} = 0.$$

The equation above is called a *vector equation of the plane*.



To derive a Cartesian equation of the plane, let  $A(x_0, y_0, z_0)$  be a fixed point in the plane and let  $P(x, y, z)$  be any point in the plane. Then

$$\vec{AP} = (x - x_0, y - y_0, z - z_0).$$

Since  $\vec{AN} = (a, b, c)$  is perpendicular to the plane,  $\vec{AN}$  is perpendicular to  $\vec{AP}$ . Therefore:

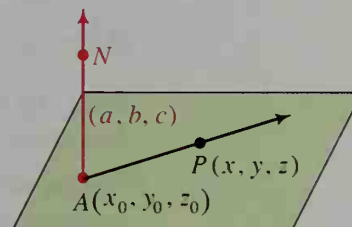
$$\vec{AN} \cdot \vec{AP} = 0$$

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0$$



This result is summarized below.

### The Cartesian Equation of a Plane

If  $(a, b, c)$  is a nonzero vector perpendicular to a plane at the point  $(x_0, y_0, z_0)$ , then an equation of the plane is:

$$ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0.$$

Because the steps of the derivation above are reversible, the converse is also true.

**Example 1**

Vector  $(3, 4, -2)$  is perpendicular to a plane that contains  $A(0, 1, 2)$ . Find an equation of the plane.

**Solution**

Since  $(3, 4, -2)$  is perpendicular to the plane, an equation of the plane is

$$3x + 4y - 2z = d.$$

To find  $d$ , substitute the coordinates of  $A$  into the equation.

$$3(0) + 4(1) - 2(2) = 0 = d$$

An equation of the plane is  $3x + 4y - 2z = 0$ .

(Notice that this plane contains the origin, since  $(0, 0, 0)$  satisfies the equation.)

**Example 2**

If  $A = (1, 0, 2)$  and  $B = (3, -4, 6)$ , find a Cartesian equation of the plane perpendicular to  $\overline{AB}$  at its midpoint.

**Solution**

A quick sketch helps to begin the solution. From the sketch at the right, the plane is perpendicular to  $\overline{AB}$  at its midpoint  $M$ :

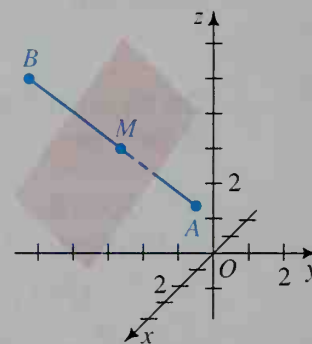
$$\begin{aligned} M &= \left( \frac{1+3}{2}, \frac{0+(-4)}{2}, \frac{2+6}{2} \right) \\ &= (2, -2, 4) \end{aligned}$$

$$\overrightarrow{MB} = (3 - 2, -4 - (-2), 6 - 4) = (1, -2, 2)$$

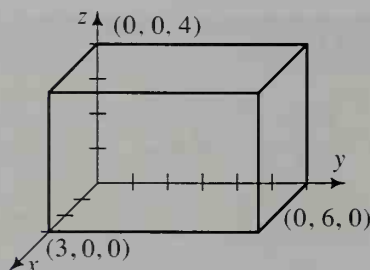
Since  $\overline{AB}$  is perpendicular to the plane,  $\overrightarrow{MB}$  is perpendicular to the plane. Therefore, a Cartesian equation of the plane is  $1x - 2y + 2z = d$ .

To find  $d$ , substitute the coordinates of  $M$  into this equation. Therefore, an equation of the plane is:

$$x - 2y + 2z = 14$$

**CLASS EXERCISES**

- Find the  $x$ -,  $y$ -, and  $z$ -intercepts of the plane  $3x + y + 2z = 6$ .
- A rectangular box is shown. Find an equation of the plane that contains:
  - the top of the box
  - the bottom of the box
  - the front of the box
  - the right side of the box



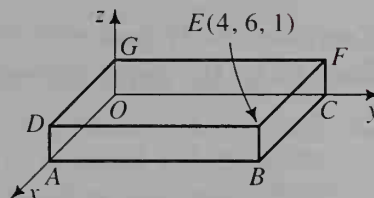
3. Name a vector perpendicular to each plane.
  - a.  $2x + 3y + 4z = 10$
  - b.  $3x - 4z = 12$
  - c.  $z = 5$
4. The vector  $(1, 1, 1)$  is perpendicular to a plane that contains  $(3, 4, 5)$ . What is a Cartesian equation of the plane?
5. The vector  $(6, 7, 8)$  is perpendicular to a plane that contains  $(0, 0, 0)$ . What is a Cartesian equation of the plane?

## WRITTEN EXERCISES

Sketch each plane whose equation is given.

**A**

1.  $2x + 3y + 6z = 12$
2.  $3x + y + 2z = 6$
3.  $5x - 2y + 2z = 10$
4.  $2x + 4y - z = 8$
5.  $z = 2$
6.  $y = 4$
7. Let  $E(4, 6, 1)$  be a vertex of the rectangular box shown. Find a Cartesian equation of each plane.
  - a. plane  $DEFG$
  - b. plane  $ABED$
  - c. plane  $BCFE$
  - d. plane  $OADG$
8. Repeat Exercise 7 with  $E(5, 7, 2)$ .



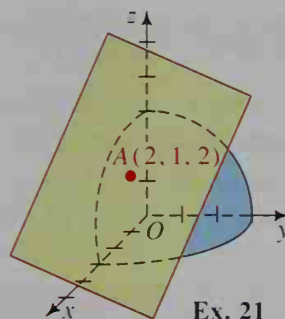
Find a vector perpendicular to the plane whose equation is given.

9.  $3x + 4y + 6z = 12$
10.  $3x - 5y + 4z = 0$
11.  $x + y = 4$
12.  $x - z = 0$
13.  $z = 1$
14.  $y = -3$

In Exercises 15–18, find a Cartesian equation of the plane described.

15. Vector  $(2, 3, 5)$  is perpendicular to the plane that contains point  $A(3, 1, 7)$ .
16. Vector  $(1, -4, 2)$  is perpendicular to the plane that contains point  $A(3, 0, 2)$ .
17. Vector  $(0, 0, 1)$  is perpendicular to the plane that contains point  $A(1, 4, 5)$ .
18. Vector  $(2, 0, 3)$  is perpendicular to the plane that contains point  $A(3, 8, -2)$ .
19. Consider the points  $A(2, 2, 2)$  and  $B(4, 6, 8)$ .
  - a. Find a Cartesian equation of the plane that is perpendicular to  $\overline{AB}$  at its midpoint  $M$ .
  - b. Show that the point  $P(2, 0, 8)$  satisfies your answer to part (a).
  - c. Show that  $PA = PB$ .
20. Consider the points  $A(-2, 4, 1)$  and  $B(0, 2, -1)$ .
  - a. Find a Cartesian equation of the plane that is perpendicular to  $\overline{AB}$  at its midpoint  $M$ .
  - b. Find two points in this plane and show that each of these points is equidistant from  $A$  and  $B$ .

21. a. Show that the point  $A(2, 1, 2)$  is on the sphere  $x^2 + y^2 + z^2 = 9$ .  
 b. Write a Cartesian equation of the plane tangent to the sphere at  $A(2, 1, 2)$ .  
 (Hint:  $\overrightarrow{OA} \perp$  the plane.)



Ex. 21

22. Find an equation of the plane tangent to the sphere

$$(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 49$$

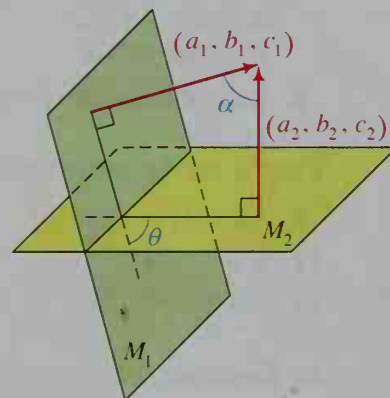
at the point  $(7, -1, 4)$ .

- B** 23. The plane  $z = 3$  intersects the sphere  $x^2 + y^2 + z^2 = 25$  in a circle. Illustrate this with a sketch and find the area of the circle.  
 24. The plane  $y = 5$  intersects the sphere  $x^2 + y^2 + z^2 = 169$  in a circle. Illustrate this with a sketch and find the area of the circle.

Exercises 25–32 deal with the angle  $\theta$  between two planes. This angle is defined to be the angle between vectors perpendicular to the planes.

25. Plane  $M_1$  has equation  $a_1x + b_1y + c_1z = d_1$ .  
 Plane  $M_2$  has equation  $a_2x + b_2y + c_2z = d_2$ .  
 a. Explain why, in the diagram,  $\theta = \alpha$ .  
 b. Explain why

$$\cos \theta = \cos \alpha = \frac{(a_1, b_1, c_1) \cdot (a_2, b_2, c_2)}{\|(a_1, b_1, c_1)\| \|(a_2, b_2, c_2)\|}.$$



26. To the nearest tenth of a degree, find the measure of the angle between the planes  $2x + 2y - z = 3$  and  $x + 2y + z = 5$ . Use the formula in Exercise 25(b).  
 27. To the nearest tenth of a degree, find the measure of the angle between the planes  $2x + 2y - z = 3$  and  $4x - 3y + 2z = 5$ . Use the formula in Exercise 25(b).

In Exercises 28–33, you will need to know that two planes are perpendicular (parallel) if vectors perpendicular to the planes are perpendicular (parallel).

28. a. Are the planes  $3x + 4y + 2z = 5$  and  $2x - y - z = 3$  perpendicular?  
 b. Are the planes  $4x - 5y + 6z = 0$  and  $3x - 2z = 7$  perpendicular?  
 29. a. Name a vector perpendicular to the plane  $2x + 3y - z = 6$ .  
 b. Is the vector in your answer to part (a) perpendicular to the plane  $2x + 3y - z = 12$ ?  
 c. Sketch both planes on a single set of axes. Are the two planes parallel?  
 d. Is the plane  $4x + 6y - 2z = 4$  parallel to the first plane?

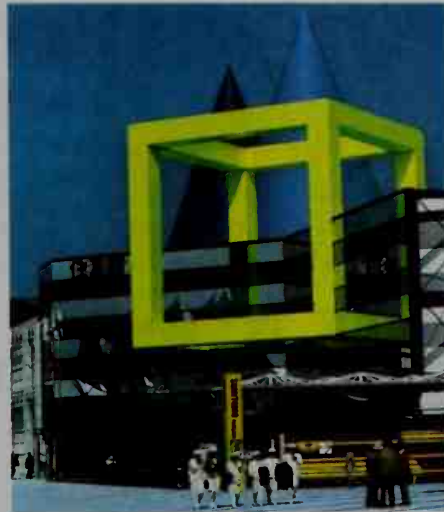
30. Which of the following planes are perpendicular and which are parallel?

$$M_1: 3x + 2y - z = 6$$

$$M_2: 6x + 4y - 2z = 8$$

$$M_3: 4x - 2y + 8z = 7$$

31. Given plane  $M$  with equation  $4x + y - 3z = 10$ , find a Cartesian equation of a plane (a) parallel to  $M$  and (b) perpendicular to  $M$ .
32. Repeat Exercise 31 if plane  $M$  has equation  $5x - 2y + z = 2$ .
33. a. What is a direction vector of the line  $(x, y, z) = (3, 1, 4) + t(4, -5, 2)$ ?  
 b. Explain why this line is parallel to the plane  $2x + 2y + z = 7$ .



Note the clever use of parallel and perpendicular planes in this building.

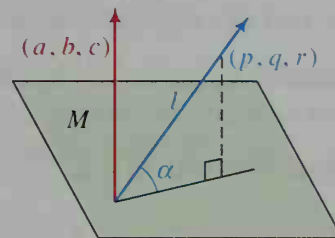
Exercises 34–36 deal with the angle  $\alpha$  between a line and a plane.

34. The angle  $\alpha$  between a line  $l$  and a plane  $M$  is the acute angle between the line and the projection of the line onto the plane.

- a. Suppose that the line has direction vector  $(p, q, r)$  and that  $(a, b, c)$  is perpendicular to the plane. Show that

$$\sin \alpha = \frac{(a, b, c) \cdot (p, q, r)}{|(a, b, c)| |(p, q, r)|}.$$

- b. Find the angle between the line  $(x, y, z) = (2, 1, 1) + t(0, 1, -1)$  and the plane  $x + 2y - 2z = 6$  to the nearest tenth of a degree.
35. a. Explain why the line  $(x, y, z) = (8, 9, 10) + t(3, 4, 5)$  is perpendicular to the plane  $3x + 4y + 5z = 10$ .  
 b. Where does the line intersect the plane?
36. a. Explain why the line  $(x, y, z) = (4, 2, 1) + t(2, 3, 4)$  is perpendicular to the plane  $2x + 3y + 4z = 76$ .  
 b. Where does the line intersect the plane?



Exercises 37–39 deal with the distance between a point and a plane.

37. The purpose of this exercise is to find the distance between the point  $(3, 1, 5)$  and the plane  $2x + 2y + z = 4$ .
- a. Find a vector equation of the line that contains the point  $(3, 1, 5)$  and is perpendicular to the plane  $2x + 2y + z = 4$ .  
 b. Where does the line intersect the plane?  
 c. Find the distance between the point  $(3, 1, 5)$  and your answer to part (b).



- C** 38. Use the method of Exercise 37 to prove that the distance between the point  $(x_0, y_0, z_0)$  and the plane  $ax + by + cz + d = 0$  is given by

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

39. In a plane, you are given point  $(x_0, y_0)$  and the line  $ax + by + c = 0$ . Conjecture a formula for the distance between the point and the line. (Hint: The formula is similar to that in Exercise 38.)

## Determinants and Their Applications

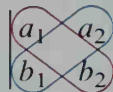
### 12-7 Determinants

**Objective** To define and evaluate determinants.

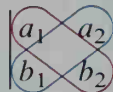
The expression  $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$  is called a  $2 \times 2$  **determinant** (2 rows and 2 columns).

To find the value of this determinant:

take the product of these numbers



then subtract the product of these numbers.



Thus, the value of a  $2 \times 2$  determinant is  $a_1b_2 - a_2b_1$ .

**Example 1**  $\begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix} = (3)(7) - (2)(4) = 13$

The general  $3 \times 3$  determinant is shown at the right. Each *element*, or number, in this determinant is associated with a  $2 \times 2$  determinant called its *minor*. The **minor** of an element is the determinant that remains when you cross out the row and column containing that element.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The minor of element  $a_2$  is  $\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \rightarrow \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$ .

You can evaluate a  $3 \times 3$  determinant by expanding by the minors of the elements in any row or column. Consider the position of each element in the original determinant and find its corresponding position on the checkerboard pattern at the right. This pattern is used in adding and subtracting the product of the row (column) elements with their corresponding minors.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

**Example 2**

Evaluate: 
$$\begin{vmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & -1 & 6 \end{vmatrix}$$

**Solution 1**

Expand by minors of the first row.

$$\begin{vmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & -1 & 6 \end{vmatrix} = 5 \begin{vmatrix} 4 & 1 \\ -1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 7 & 6 \end{vmatrix} + 8 \begin{vmatrix} 3 & 4 \\ 7 & -1 \end{vmatrix}$$

$$= 5(25) - 2(11) + 8(-31)$$

$$= -145$$

**Solution 2**

Alternatively, you can expand by minors of the second column.

$$\begin{vmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & -1 & 6 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 \\ 7 & 6 \end{vmatrix} + 4 \begin{vmatrix} 5 & 8 \\ 7 & 6 \end{vmatrix} - (-1) \begin{vmatrix} 5 & 8 \\ 3 & 1 \end{vmatrix}$$

$$= -2(11) + 4(-26) + 1(-19)$$

$$= -145$$

**Activity**

The determinant in Example 2 was evaluated (1) by minors of the first row and (2) by minors of the second column.

- Expand this determinant by minors of the second row.
- Expand this determinant by minors of the third column.

Examples 1 and 2 have shown how to evaluate  $2 \times 2$  and  $3 \times 3$  determinants with paper and pencil. For  $n \times n$  determinants with  $n \geq 4$ , it is easier to use certain graphing calculators or computer software. Without such technology, pencil and paper calculations can be made easier by transforming the determinant into an equivalent one in which a row (column) has several zero elements. To do this, multiply any row (column) by a constant and add the resulting products to the elements of another row (column).

$$\begin{vmatrix} 2 & 5 & 3 & 7 \\ 4 & 1 & 0 & 2 \\ 3 & 2 & 2 & 1 \\ -2 & 8 & 1 & 0 \end{vmatrix} \begin{array}{l} \text{--- Add } (-3 \times \text{row 4}) \text{ to row 1.} \\ \text{--- Add } (-2 \times \text{row 4}) \text{ to row 3.} \end{array} \longrightarrow \begin{vmatrix} 8 & -19 & 0 & 7 \\ 4 & 1 & 0 & 2 \\ 7 & -14 & 0 & 1 \\ -2 & 8 & 1 & 0 \end{vmatrix}$$

Once you have a row (column) with several zeros, you can use a  $4 \times 4$  checkerboard pattern of (+) and (-) signs. Since column 3 in the determinant at the right above now contains three zeros, we will expand by its minors as shown at the top of the following page.

$$+ 0 \begin{vmatrix} 4 & 1 & 2 \\ 7 & -14 & 1 \\ -2 & 8 & 0 \end{vmatrix} - 0 \begin{vmatrix} 8 & -19 & 7 \\ 7 & -14 & 1 \\ -2 & 8 & 0 \end{vmatrix} + 0 \begin{vmatrix} 8 & -19 & 7 \\ 4 & 1 & 2 \\ -2 & 8 & 0 \end{vmatrix} - 1 \begin{vmatrix} 8 & -19 & 7 \\ 4 & 1 & 2 \\ 7 & -14 & 1 \end{vmatrix}$$

The first three products above are 0, and the  $3 \times 3$  determinant in the fourth product can be evaluated directly as in Example 2. Alternatively, that determinant can be transformed so that it has a row or column with two zeros.

## CLASS EXERCISES

Evaluate each determinant.

1.  $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

2.  $\begin{vmatrix} 8 & 5 \\ 4 & 4 \end{vmatrix}$

3.  $\begin{vmatrix} 3 & -5 \\ 7 & 10 \end{vmatrix}$

4.  $\begin{vmatrix} -5 & 4 \\ -4 & -3 \end{vmatrix}$

5. Complete, expanding by minors of the top row.

$$\begin{vmatrix} 3 & 2 & 4 \\ 5 & 7 & 1 \\ 8 & 6 & 9 \end{vmatrix} = \underline{\quad ? \quad} \begin{vmatrix} 7 & 1 \\ 6 & 9 \end{vmatrix} - \underline{\quad ? \quad} \begin{vmatrix} 5 & 1 \\ 8 & 9 \end{vmatrix} + \underline{\quad ? \quad} \begin{vmatrix} 5 & 7 \\ 8 & 6 \end{vmatrix}$$

6. Complete, expanding by minors of the third column.

$$\begin{vmatrix} 3 & 2 & 4 \\ 5 & 7 & 1 \\ 8 & 6 & 9 \end{vmatrix} = 4 \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} - 1 \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix} + 9 \begin{vmatrix} ? & ? \\ ? & ? \end{vmatrix}$$

7. Evaluate mentally:  $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{vmatrix}$

## WRITTEN EXERCISES

Evaluate each determinant.

**A** 1.  $\begin{vmatrix} 2 & 7 \\ 8 & 4 \end{vmatrix}$

2.  $\begin{vmatrix} -3 & 5 \\ -7 & 2 \end{vmatrix}$

3.  $\begin{vmatrix} 25 & 125 \\ 75 & 250 \end{vmatrix}$

4. Show that  $k \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ kc & kd \end{vmatrix}$ .

5. Show that  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a-c & b-d \\ c & d \end{vmatrix}$ . Then evaluate  $\begin{vmatrix} 387 & 411 \\ 385 & 410 \end{vmatrix}$ .

6. True or false?  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a-kc & b-kd \\ c & d \end{vmatrix}$  for every real number  $k$ .

Evaluate.

$$7. \begin{vmatrix} 4 & -7 & 3 \\ 2 & 0 & 0 \\ 5 & 1 & 6 \end{vmatrix}$$

$$8. \begin{vmatrix} -1 & 3 & 2 \\ 4 & 0 & 1 \\ 1 & 5 & 0 \end{vmatrix}$$

$$9. \begin{vmatrix} 1 & -3 & 4 \\ 0 & 1 & 1 \\ 5 & -2 & 3 \end{vmatrix}$$

$$10. \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 17 & 18 & 19 \end{vmatrix}$$

- B** 11. **Discussion** In Exercise 10, one row is a multiple of another row and the value of the determinant is zero. Is this a coincidence? Justify your answer.

$$12. \text{ Evaluate: } \begin{vmatrix} 2 & -4 & 7 & 3 \\ 0 & 5 & 1 & -2 \\ -2 & 1 & 0 & -3 \\ 0 & -6 & 4 & 2 \end{vmatrix}$$

$$13. \text{ Evaluate: } \begin{vmatrix} -2 & 1 & 5 & 0 \\ 3 & 4 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -3 \end{vmatrix}$$

- C** 14. Consider the linear equations  $a_1x + b_1y = c_1$ , and  $a_2x + b_2y = c_2$ .
- Solve these equations for  $x$  and  $y$  in terms of the constants.
  - Show that your answer can be written as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

## 12-8 Applications of Determinants

### Objective

To use determinants to solve algebraic and geometric problems.

Determinants can be used to solve  $n$  equations in  $n$  variables. The method used is called *Cramer's rule* and is illustrated in Examples 1 and 2. If  $n \geq 4$ , this method involves laborious paper and pencil calculations. As a result, a special calculator or computer is a better tool when  $n \geq 4$ .

### Example 1

Use Cramer's rule to solve:  $3x + 5y = 7$   
 $4x + 9y = 11$

### Solution

**Step 1** Calculate three determinants.

$$D = \begin{vmatrix} 3 & 5 \\ 4 & 9 \end{vmatrix} = 7 \quad D \text{ consists of the coefficients of } x \text{ and } y.$$

$D_x$  is formed from  $D$  by replacing the first column of  $D$  by the constant terms.  $D_y$  is formed from  $D$  by replacing the second column of  $D$  by the constant terms.

$$D_x = \begin{vmatrix} 7 & 5 \\ 11 & 9 \end{vmatrix} = 8 \quad D_y = \begin{vmatrix} 3 & 7 \\ 4 & 11 \end{vmatrix} = 5$$

$$\text{Step 2} \quad x = \frac{D_x}{D} = \frac{8}{7} \text{ and } y = \frac{D_y}{D} = \frac{5}{7}.$$

**Example 2**

Solve the system of equations:

$$\begin{aligned} 3x - y + 2z &= 4 \\ 2x + 3y - z &= 14 \\ 7x - 4y + 3z &= -4 \end{aligned}$$

**Solution**

**Step 1** Write and evaluate four determinants.

$$D = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 7 & -4 & 3 \end{vmatrix} \qquad D_x = \begin{vmatrix} 4 & -1 & 2 \\ 14 & 3 & -1 \\ -4 & -4 & 3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} 3 & 4 & 2 \\ 2 & 14 & -1 \\ 7 & -4 & 3 \end{vmatrix} \qquad D_z = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & 14 \\ 7 & -4 & -4 \end{vmatrix}$$

Notice that the numbers in  $D$  are the coefficients of  $x$ ,  $y$ , and  $z$  in the three equations. To form  $D_x$ , the first column of  $D$  is replaced by the constants on the right sides of the equations. Similarly, the second and third columns of  $D$  are replaced by the constants to get  $D_y$  and  $D_z$ , respectively. Using the methods of Section 12-7 to expand  $D$ ,  $D_x$ ,  $D_y$ , and  $D_z$ , we get:

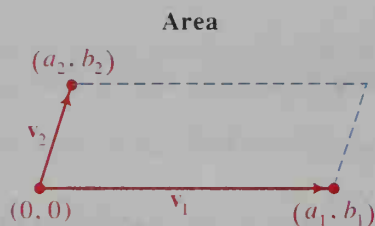
$$D = -30, \quad D_x = -30, \quad D_y = -150, \quad D_z = -90$$

**Step 2** Therefore,

$$x = \frac{D_x}{D} = \frac{-30}{-30} = 1, \quad y = \frac{D_y}{D} = \frac{-150}{-30} = 5, \quad z = \frac{D_z}{D} = \frac{-90}{-30} = 3$$

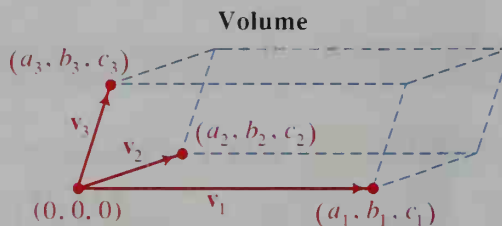
Thus, the solution is  $(x, y, z) = (1, 5, 3)$ .

Determinants are useful in geometry as well as in algebra. We can use determinants to find area and volume, as shown below.



The area of a parallelogram with sides determined by  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is the absolute value of:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$



The volume of a parallelepiped with edges determined by  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  is the absolute value of:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



**Example 3**Find the area of the triangle with vertices  $P(1, 2)$ ,  $Q(3, 6)$ , and  $R(6, 1)$ .**Solution** $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$  determine the sides of triangle  $PQR$  and parallelogram  $PQSR$ .

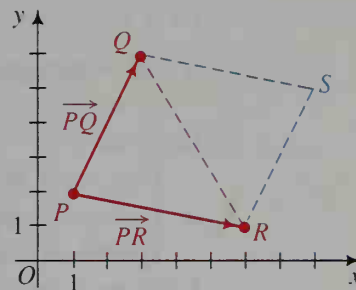
$$\overrightarrow{PQ} = (3, 6) - (1, 2) = (2, 4)$$

$$\overrightarrow{PR} = (6, 1) - (1, 2) = (5, -1)$$

$$\text{Area of } \triangle PQR = \frac{1}{2} \cdot \text{area of } \square PQSR$$

$$\text{Area} = \frac{1}{2} \cdot \text{absolute value of } \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix}$$

$$= \frac{1}{2} |-22| = 11$$



The minus sign in the next-to-last step indicates that  $P$ ,  $Q$ , and  $R$  have a negative (clockwise) orientation.

**CLASS EXERCISES**

- When determinants are used to solve the system of equations  $\begin{cases} 3x + 4y = 7 \\ 5x + 6y = 8 \end{cases}$ , then  $x = \frac{|\begin{smallmatrix} ? \\ ? \end{smallmatrix}|}{|\begin{smallmatrix} ? \\ ? \end{smallmatrix}|}$  and  $y = \frac{|\begin{smallmatrix} ? \\ ? \end{smallmatrix}|}{|\begin{smallmatrix} ? \\ ? \end{smallmatrix}|}$ .
- When determinants are used to solve the system of equations  $\begin{cases} 3x + y - 2z = 4 \\ 2x - y + 4z = 1 \\ x + 2y + 7z = 3 \end{cases}$ , then  $x = \frac{|\begin{smallmatrix} ? \\ ? \\ ? \end{smallmatrix}|}{|\begin{smallmatrix} ? \\ ? \\ ? \end{smallmatrix}|}$ ,  $y = \frac{|\begin{smallmatrix} ? \\ ? \\ ? \end{smallmatrix}|}{|\begin{smallmatrix} ? \\ ? \\ ? \end{smallmatrix}|}$ , and  $z = \frac{|\begin{smallmatrix} ? \\ ? \\ ? \end{smallmatrix}|}{|\begin{smallmatrix} ? \\ ? \\ ? \end{smallmatrix}|}$ .
- What happens when you try to use Cramer's rule to find the solutions for the system of equations  $\begin{cases} 2x + y = 3 \\ 2x + y = 5 \end{cases}$ ?
- What is the area of the parallelogram with sides determined by vectors  $\mathbf{v}_1 = (1, 3)$  and  $\mathbf{v}_2 = (2, 4)$ ?

**WRITTEN EXERCISES**

In Exercises 1–4, solve each system of equations by using Cramer's rule. Then, sketch the graphs of the two equations and label their intersection with the common solution of the equations.

**A**

- $\begin{cases} 5x - 4y = 1 \\ 3x + 2y = 5 \end{cases}$

- $\begin{cases} 3x + 2y = -1 \\ 2x - y = 4 \end{cases}$

- $\begin{cases} 5x - 2y = 11 \\ x + 3y = 9 \end{cases}$

- $\begin{cases} 7x + y = 7 \\ -x + 2y = 14 \end{cases}$

Solve each system of equations by using Cramer's rule.

5.  $ax + by = 1$   
 $bx + ay = 1$

6.  $ax + by = c$   
 $3ax - 2by = 4c$

7. a. Draw the graphs of  $9x - 6y = 3$  and  $6x - 4y = 10$ . Then solve the system of equations by using determinants.  
 b. If  $ax + by = c$  and  $dx + ey = f$  have parallel line graphs, what determinant must be equal to zero?
8. a. Draw the graphs of  $12x + 8y = -4$  and  $6x + 4y = -2$ . Then solve the system of equations by using determinants.  
 b. If  $ax + by = c$  and  $dx + ey = f$  have the same graph, what determinants must be equal to zero?

In Exercises 9–14, find the area of each figure, given points  $P(4, 3)$ ,  $Q(7, -1)$ ,  $R(2, 3)$ ,  $S(-3, 6)$ ,  $T(-5, 4)$ , and  $V(-2, -5)$ .

9.  $\triangle PQR$

10.  $\triangle PQS$

11.  $\triangle RST$

12.  $\triangle PSV$

13.  $\square$  with sides  $\overline{PR}$ ,  $\overline{PS}$

14.  $\square$  with sides  $\overline{PS}$ ,  $\overline{PT}$

15. **Discussion** If the “area” of  $\triangle LMN$  is zero, what can you conclude about points  $L$ ,  $M$ , and  $N$ ?
16. **Discussion** If a parallelepiped with sides  $\overline{DE}$ ,  $\overline{DF}$ , and  $\overline{DG}$  has “zero volume,” what can you say about points  $D$ ,  $E$ ,  $F$ , and  $G$ ?

In Exercises 17–19, solve each system of equations by using Cramer's rule.

**B** 17.  $x - 2y + 3z = 2$   
 $2x - 3y + z = 1$   
 $3x - y + 2z = 9$

18.  $4x - 2y + 3z = 2$   
 $5x - 6y + 2z = -1$   
 $3x + 4y - 5z = 7$

19.  $3x - 2y + z = 7$   
 $2x + y - 3z = 1$   
 $x + 2y + 2z = 4$

In Exercises 20–22, find the volume of each solid, given points  $D(5, 1, 0)$ ,  $E(3, 1, 4)$ ,  $F(0, 2, -1)$ ,  $G(5, 2, 0)$ , and  $H(3, 1, 3)$ .

20. Parallelepiped with edges  $\overline{DE}$ ,  $\overline{DF}$ , and  $\overline{DG}$

21. Parallelepiped with edges  $\overline{EF}$ ,  $\overline{EG}$ , and  $\overline{EH}$

- C** 22. Pyramid with edges  $\overline{DF}$ ,  $\overline{DG}$ , and  $\overline{DH}$  (Hint: The volume of a pyramid is one-sixth the volume of a parallelepiped.)



## 12-9 Determinants and Vectors in Three Dimensions

### Objective

To define and apply the cross product.

The unit vectors in the positive  $x$ ,  $y$ , and  $z$  directions are often denoted by the symbols  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ . This means that if point  $P = (3, 5, 6)$ , then

$$\begin{aligned}\overrightarrow{OP} &= (3, 5, 6) \\ &= 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}.\end{aligned}$$

This representation of a vector is useful in defining a new vector operation, the *cross product* of two vectors.

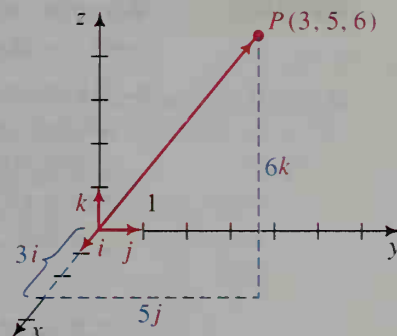
If  $\mathbf{v}_1 = (a_1, b_1, c_1)$  and  $\mathbf{v}_2 = (a_2, b_2, c_2)$ , then the **cross product** of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , denoted by  $\mathbf{v}_1 \times \mathbf{v}_2$ , is found by evaluating:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

For example, if  $\mathbf{v}_1 = (2, 3, 4)$  and  $\mathbf{v}_2 = (1, 0, 5)$ , then:

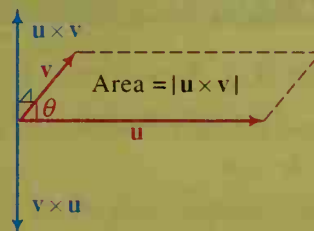
$$\begin{aligned}\mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \mathbf{k} \\ &= 15\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}, \text{ or } (15, -6, -3)\end{aligned}$$

The following list gives some of the major properties of the cross product. The figure at the right below illustrates properties 1–3.



### Properties of the Cross Product

- $\mathbf{u} \times \mathbf{v}$  is perpendicular to  $\mathbf{u}$  and to  $\mathbf{v}$ .
- $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ , that is,  $\mathbf{v} \times \mathbf{u}$  and  $\mathbf{u} \times \mathbf{v}$  have opposite directions.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta$ , where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ . Geometrically, the magnitude of  $\mathbf{u} \times \mathbf{v}$  is the area of the parallelogram formed by  $\mathbf{u}$  and  $\mathbf{v}$ .
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- $\mathbf{u}$  is parallel to  $\mathbf{v}$  if and only if  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ .

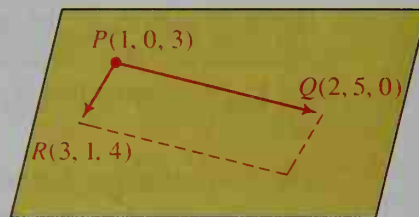


**Example**

- Find a nonzero vector perpendicular to the plane containing the points  $P(1, 0, 3)$ ,  $Q(2, 5, 0)$ , and  $R(3, 1, 4)$ .
- Find an equation of the plane containing  $P$ ,  $Q$ , and  $R$ .
- Find the area of the parallelogram with sides formed by  $\overrightarrow{PQ}$  and  $\overrightarrow{PR}$ .

**Solution**

Without actually plotting the points, make a quick sketch.



- By property 1,  $\overrightarrow{PQ} \times \overrightarrow{PR}$  is a vector perpendicular to the plane containing  $P$ ,  $Q$ , and  $R$ .  
Since  $\overrightarrow{PQ} = (1, 5, -3)$   
and  $\overrightarrow{PR} = (2, 1, 1)$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & -3 \\ 2 & 1 & 1 \end{vmatrix} = 8\mathbf{i} - 7\mathbf{j} - 9\mathbf{k}, \text{ or } (8, -7, -9).$$

- Since  $(8, -7, -9)$  is perpendicular to the plane, an equation of the plane is  $8x - 7y - 9z = d$  (by the theorem in Section 12-6). To find  $d$ , substitute  $P(1, 0, 3)$  (or  $Q$  or  $R$ ) in the equation:

$$\begin{aligned} 8 \cdot 1 - 7 \cdot 0 - 9 \cdot 3 &= d \\ d &= -19 \end{aligned}$$

Thus,  $8x - 7y - 9z = -19$  is an equation of the plane. You can check this by substituting the coordinates of  $P$ ,  $Q$ , and  $R$  into the equation.

- By property 3, the area of the parallelogram is given by:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |(8, -7, -9)| = \sqrt{8^2 + (-7)^2 + (-9)^2} = \sqrt{194}$$

The area is  $\sqrt{194}$  square units, or about 13.9 square units.

If a parallelogram lies in the  $xy$ -plane, its area can be found by using  $2 \times 2$  determinants, as in Section 12-8. Otherwise, the method shown in Example 2(c) above can be used.

**CLASS EXERCISES**

Let  $\mathbf{u} = (1, 2, 3)$ ,  $\mathbf{v} = (4, 5, 6)$ , and  $\mathbf{w} = (7, 8, 9)$ .

- Give the determinant that can be used to evaluate each cross product.
  - $\mathbf{u} \times \mathbf{v}$
  - $\mathbf{v} \times \mathbf{w}$
- Tell whether each of the following is a vector or a scalar.
  - $\mathbf{u} \times \mathbf{v}$
  - $\mathbf{u} \cdot \mathbf{v}$
  - $\mathbf{u} + \mathbf{v}$
  - $\mathbf{u} \cdot (\mathbf{u} + \mathbf{v})$
  - $\mathbf{u} \times (\mathbf{u} + \mathbf{v})$
  - $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$
- Discussion** Why does the following equation make no sense? ( $\times$  denotes the cross product)  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \times (\mathbf{u} \cdot \mathbf{w})$

## WRITTEN EXERCISES

In Exercises 1–8, let  $\mathbf{u} = (4, 0, 1)$ ,  $\mathbf{v} = (5, -1, 0)$ , and  $\mathbf{w} = (-3, 1, -2)$ .

- A**
1. Calculate  $\mathbf{v} \times \mathbf{u}$  and  $\mathbf{u} \times \mathbf{v}$ . Do your results agree with property 2?
  2. Calculate  $\mathbf{v} \times \mathbf{w}$ . Use your answer to find  $\mathbf{w} \times \mathbf{v}$  immediately.
  3. Verify that  $\mathbf{u} \times \mathbf{v}$  is perpendicular to  $\mathbf{u}$  and to  $\mathbf{v}$ .
  4. a. Use a cross product to find a vector perpendicular to the plane of  $\mathbf{u}$  and  $\mathbf{w}$ .  
b. Use a dot product to find a vector perpendicular to  $\mathbf{w}$ .
  5. Find the area of the parallelogram determined by  $\mathbf{u}$  and  $\mathbf{v}$ .
  6. Find the area of the triangle determined by  $\mathbf{u}$  and  $\mathbf{w}$ .
  7. Verify that  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ .
  8. Verify that  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ .

In Exercises 9–12, (a) find a vector perpendicular to the plane determined by  $P$ ,  $Q$ , and  $R$ , (b) find a Cartesian equation of the plane determined by  $P$ ,  $Q$ , and  $R$ , and (c) find the area of  $\triangle PQR$ .

9.  $P(1, 1, 0)$ ,  $Q(-1, 0, 2)$ ,  $R(2, 1, 1)$
10.  $P(0, 0, 0)$ ,  $Q(3, -1, 2)$ ,  $R(4, 5, -2)$
11.  $P(2, 3, -4)$ ,  $Q(2, 1, 0)$ ,  $R(3, 3, -2)$
12.  $P(0, 2, 3)$ ,  $Q(1, 1, 1)$ ,  $R(2, 1, 3)$

- B**
13. Angle  $\theta$  is between vectors  $\mathbf{u} = (1, 2, 2)$  and  $\mathbf{v} = (4, 3, 0)$ .
    - a. Find  $\sin \theta$  by using property 3 of the cross product.
    - b. Find  $\cos \theta$  by using property 5 of the dot product (page 448).
    - c. Verify that  $\sin^2 \theta + \cos^2 \theta = 1$ .
  14. Repeat Exercise 13 for the vectors  $\mathbf{u} = (1, 2, 1)$  and  $\mathbf{v} = (1, 1, 2)$ .
  15. Given  $A(11, 17, 0)$ ,  $B(-8, 12, 0)$ , and  $C(25, 13, 0)$ , find the area of  $\triangle ABC$  by:
    - a. using determinants. (Since all the points lie in the  $xy$ -plane, use the method of Example 3 in Section 12-8.)
    - b. using cross products.
    - c. using Hero's formula. (See page 6.)
    - d. using the dot product and the formula:  $\text{Area} = \frac{1}{2}ab \sin \theta$ .Do your answers in parts (a), (b), (c), and (d) agree?

- C**
16. Let  $\mathbf{u} = (a_1, b_1, c_1)$ ,  $\mathbf{v} = (a_2, b_2, c_2)$ , and  $\mathbf{w} = (a_3, b_3, c_3)$ .
    - a. Prove that  $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ .
    - b. Prove the five properties of the cross product listed on page 465.

## ////COMPUTER EXERCISE

Write a computer program that will print a Cartesian equation of the plane determined by three noncollinear points  $P$ ,  $Q$ , and  $R$ , when you input the coordinates of  $P$ ,  $Q$ , and  $R$ . (*Hint:* Use the cross product.)



## Chapter Summary

1. A *vector* is a quantity that has both magnitude and direction. Vectors can be represented geometrically by arrows or algebraically by ordered pairs of numbers. The ordered pairs can be in *polar form* or in *component form*. Rules for vector operations in components are given on page 427.

2. a. If  $P$  is any point on line  $AB$ , then

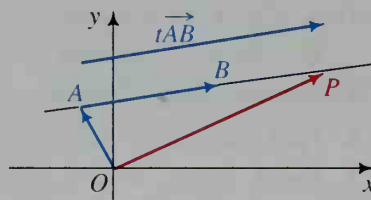
$$\overrightarrow{OP} = \overrightarrow{OA} + t\overrightarrow{AB}$$

is a *vector equation* of line  $AB$ , and  $\overrightarrow{AB}$  is a *direction vector* of the line.

- b. A *vector equation* of an object moving with constant velocity along the line is:

$$\text{new position} = \text{initial position} + t(\text{constant velocity})$$

$$(x, y) = (x_0, y_0) + t(a, b)$$



- c. From the vector equation of the line, we can obtain the two parametric equations  $x = x_0 + ta$  and  $y = y_0 + tb$ . These give the  $x$ - and  $y$ -coordinates of points on the line in terms of the parameter  $t$  ( $t$  is often considered to represent time).

3. a. If  $\mathbf{u} = (x_1, y_1)$  and  $\mathbf{v} = (x_2, y_2)$ , then the *dot product* of  $\mathbf{u}$  and  $\mathbf{v}$  is:

$$\mathbf{u} \cdot \mathbf{v} = x_1x_2 + y_1y_2$$

Some properties of the dot product are listed on page 442.

- b. The angle  $\theta$  between two vectors  $\mathbf{u}$  and  $\mathbf{v}$  is given by the formula

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}.$$

- c. Vectors  $\mathbf{u}$  and  $\mathbf{v}$  are *perpendicular* if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .
- d. Nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are *parallel* if and only if  $\mathbf{u} = k\mathbf{v}$ .
4. Vectors in three-dimensional space are ordered triples of numbers. The basic properties of these vectors, summarized in the box on page 448, are analogous to those of vectors in the plane.
5. The graph of  $ax + by + cz = d$  is a plane (provided  $a, b$  and  $c$  are not all equal to zero). The nonzero vector  $(a, b, c)$  is perpendicular to this plane at the point  $(x_0, y_0, z_0)$ , where  $d = ax_0 + by_0 + cz_0$ .
6. A *determinant* is a square array of numbers evaluated as shown on page 458. The value of an  $n$  by  $n$  determinant can be computed by expanding by the minors of one of its rows or columns.
7. *Cramer's Rule* involves using determinants to solve a system of  $n$  linear equations in  $n$  variables. Determinants can also be used to find areas and volumes.
8. The *cross product* of the vectors  $\mathbf{v}_1 = (a_1, b_1, c_1)$  and  $\mathbf{v}_2 = (a_2, b_2, c_2)$  is defined on page 465. Properties of the cross product are given in the box on page 465. The cross product is useful in finding an equation of a plane, as shown in the example on page 466.

### Key vocabulary and ideas

vector, direction, magnitude (p. 419)

component form (p. 426)

vector equation, direction vector (p. 432)

parametric equations (p. 433)

dot product (p. 441)

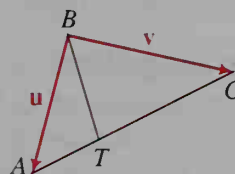
angle between two vectors (p. 442)

Cartesian equation of a plane (p. 453)

Cramer's rule (pp. 461–462)

## Chapter Test

- An object is pulled due south by a force  $\mathbf{F}_1$  of 5 N, and due east by a force  $\mathbf{F}_2$  of 12 N. Find the direction and magnitude of  $\mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2$ . 12-1
- In the diagram,  $AT:TC = 1:2$ . Express each of the following in the form  $ru + sv$ :  
 a.  $\overrightarrow{AB}$       b.  $\overrightarrow{AC}$       c.  $\overrightarrow{AT}$       d.  $\overrightarrow{BT}$  12-2
- A plane is on a course of  $320^\circ$  at a speed of 480 mi/h. What are the north-south and east-west components of its velocity vector? 12-2
- Given the points  $A(1, 3)$ ,  $B(0, -3)$ , and  $C(5, 0)$ , find the coordinates of a point  $D(x, y)$  such that quadrilateral  $ABCD$  is a parallelogram.
- a. Find the velocity and speed of an object that moves with constant velocity so that its position at time  $t$  is  $(x, y) = (-2, -2) + t(1, 4)$ . 12-3  
 b. Find a pair of parametric equations of the path of the object.  
 c. When and where does it intersect the parabola  $y = 4x^2 - 6x$ ?
- Writing** Let  $\mathbf{u}$  and  $\mathbf{v}$  be nonzero vectors such that  $\mathbf{u} \cdot \mathbf{v} = -\|\mathbf{u}\|\|\mathbf{v}\|$ . Describe the relationship between  $\mathbf{u}$  and  $\mathbf{v}$ . Explain your reasoning. 12-4
- Given the points  $A(1, 3)$ ,  $B(-2, 7)$ , and  $C(0, -6)$ , find the measure of  $\angle A$  to the nearest tenth of a degree.
- If  $A = (4, 0, -5)$  and  $B = (-6, -2, 7)$ , find (a) the length of  $\overline{AB}$  and (b) the midpoint of  $\overline{AB}$ . 12-5
- Line  $L$  has equation  $(x, y, z) = (0, -1, 6) + t(4, -1, -3)$ .  
 a. Write a vector equation for the line through  $(2, 1, 7)$  parallel to  $L$ .  
 b. Where does line  $L$  intersect the  $xz$ -plane?
- Find an equation of the plane that is tangent to the sphere with equation  $x^2 + (y + 1)^2 + (z - 3)^2 = 33$  at the point  $(2, -3, -2)$ . 12-6
- Evaluate:   
 a.  $\begin{vmatrix} 2 & 0 & 3 \\ 4 & 2 & -2 \\ -3 & 0 & 1 \end{vmatrix}$       b.  $\begin{vmatrix} 0 & 0 & 0 & 5 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ -4 & 0 & 0 & -1 \end{vmatrix}$  12-7
- Use Cramer's Rule to solve:  $7x + 4y = 19$  12-8  
 $3x - 10y = 14$
- $A(0, 1, -3)$ ,  $B(-6, 1, 1)$ , and  $C(4, 5, 2)$  determine a plane. Find:  
 a. A vector perpendicular to the plane      b. The area of  $\triangle ABC$  12-9



## Uniform Circular Motion

Did you ever wonder why clothes end up pressed against the outside of the washer when the cycle is completed? Did you ever watch a line of skaters make a whip by whirling in a circle? Why does the person on the outside move so much faster than the others? The answer lies in the concepts of force and uniform circular motion. Most likely, you're already familiar with force and motion along a straight line. Here are a few experiments to get a better sense of force and motion in a circle.

### Materials:

- |                                 |                          |
|---------------------------------|--------------------------|
| a piece of string 1 m in length | two pairs of heavy socks |
| a circular paper plate          | a marble or a small ball |

### Make Investigations.

- First, tie one pair of socks securely to one end of the string. Hold the other end of the string and *carefully* swing the socks slowly around your head in a circle. Be sure to note the pull of the weight on your arm.
- Now, roll up the string a little and try to swing the socks so that they stay up in a circle again. Does using a shorter length of string seem to increase or decrease the speed needed?
- Repeat the experiment using both pairs of socks at once. Why do you think the pull on your arm is greater?
- What do you think would happen if you let go of the string? *Don't try it!* To get the same effect, try this experiment instead. Take a paper plate and roll a marble or a small ball around the inside of the rim. Next, cut out a circular sector equal to one-fourth of the circle, and roll the ball again. In which direction does the ball roll when it reaches the cut edge?

### Do a Vector Analysis.

As you know from the examples and exercises concerning force and velocity in Chapter 12, you can use vectors to help explain the results of your investigations.

- Begin by drawing a circle with radius  $r$  to depict the path of your object. Think of the radius as the string that holds the object in the path. Let's use polar coordinates  $(r, \theta)$  with  $\theta = \beta t$ , where  $t$  = elapsed time in seconds and  $\beta$  = the angular speed in radians per second. For simplicity, let  $r = 1$ . Also, choose a smaller value for  $\beta$ , such as 0.5, and a value for  $t$ . Show the position of your object in a diagram by drawing the vector

$$\mathbf{r} = (r \cos \beta t, r \sin \beta t).$$

- Motion, of course, involves a change in position. Suppose that you wanted to find the rate of change of the object's position exactly at time  $t$ . This rate, called the *instantaneous velocity* of the object at time  $t$  (which will be discussed in Chapter 20) can be represented by the vector

$$\mathbf{v} = (-\beta r \sin \beta t, \beta r \cos \beta t).$$

Before locating  $\mathbf{v}$ , recall the result of your paper plate investigation. We hope you found that at the instant the marble or small ball reached the cut edge, it rolled ahead in a direction that was perpendicular to the radius of the plate. Therefore,  $\mathbf{v}$  must be perpendicular to  $\mathbf{r}$ . Why? Draw  $\mathbf{v}$ , placing its initial point at the terminal point of  $\mathbf{r}$ . Use the dot product to show that  $\mathbf{v}$  and  $\mathbf{r}$  are perpendicular.

- The *speed* of an object is defined as the magnitude of the velocity. Show that the magnitude of  $\mathbf{v}$  is a constant. Does this result surprise you?
- In Chapter 20, you'll also learn about acceleration. *Acceleration* is the rate of change of an object's velocity. For example, you can actually feel acceleration when a car in which you're riding suddenly slows down. The instantaneous acceleration of your object can be represented on your diagram by the vector

$$\mathbf{a} = (-\beta^2 r \cos \beta t, -\beta^2 r \sin \beta t) = -\beta^2 \mathbf{r}.$$

Draw  $\mathbf{a}$  and notice its direction. The vector  $\mathbf{a}$  is called *centripetal acceleration* since it always points towards the center of the circular motion.

- One of the most useful formulas in physics is

$$\mathbf{F} = m\mathbf{a},$$

where  $\mathbf{F}$  represents force,  $\mathbf{a}$  represents acceleration, and the scalar  $m$  is the mass of the object. Let  $m = 1.5$ , and locate  $\mathbf{F}$  on your diagram. The vector  $\mathbf{F}$  is called the *centripetal force*. Why? It corresponds, for example, to the force you exert on the object by holding it in place with the string. Now, draw  $-\mathbf{F}$ , which is equal and opposite to  $\mathbf{F}$ . The vector  $-\mathbf{F}$  is called the *centrifugal force*. Why? This force corresponds to the pull that you feel on your arm. Apply the formulas to show that using a heavier weight or a faster angular speed results in a greater pull. Explain why centrifugal force sends the ketchup to the top of the bottle when swung in a big circle.

- Here's an extra challenge: Use concentric circles and the given formulas to illustrate why using a shorter string at the same angular speed decreases the pull on your arm.

## Report Your Results.

In this project there are a lot of little experiments and related analyses to perform, so be sure to try them all and answer each of the questions asked. As a suggestion, you might organize your report so that each investigation is followed by a corresponding vector analysis. Remember to state your choices for  $\beta$ ,  $t$ , and  $m$ , and to clearly label each diagram. Include enough diagrams to provide a careful illustration of each idea. Don't forget to try the extra challenge!



# 13

## *Sequences and Series*





# ■ Finite Sequences and Series

## 13-1 Arithmetic and Geometric Sequences

### **Objective**

To identify an arithmetic or geometric sequence and find a formula for its  $n$ th term.

A sequence is a set of numbers, called terms, arranged in a particular order. We begin with two of the simplest types of sequences: *arithmetic* and *geometric*.

### Arithmetic Sequences

A sequence of numbers is called an **arithmetic sequence** if the difference of any two consecutive terms is constant. This difference is called the *common difference*. The following sequences are all arithmetic.

$$2, 6, 10, 14, 18, \dots \text{ difference} = 4$$

$$17, 10, 3, -4, -11, -18, \dots \text{ difference} = -7$$

$$a, a + d, a + 2d, a + 3d, a + 4d, \dots \text{ difference} = d$$

### Geometric Sequences

A sequence of numbers is called a **geometric sequence** if the ratio of any two consecutive terms is constant. This ratio is called the *common ratio*. The following sequences are all geometric.

$$1, 3, 9, 27, 81, \dots \text{ ratio} = 3$$

$$64, -32, 16, -8, 4, \dots \text{ ratio} = -\frac{1}{2}$$

$$a, ar, ar^2, ar^3, ar^4, \dots \text{ ratio} = r$$

The same notation can be used for all types of sequences. The first term of a sequence is often denoted by  $t_1$ , the second and third terms by  $t_2$  and  $t_3$ , and so on. The  $n$ th term of the sequence is then denoted by  $t_n$ . Some sequences can be defined by particular rules or formulas. If you have a formula for  $t_n$  in terms of  $n$ , you can find the value of any term of the sequence. For example, suppose a sequence has the formula

$$t_n = n^2 + 1.$$

Then:

$$t_1 = 1^2 + 1 = 2$$

$$t_2 = 2^2 + 1 = 5$$

$$t_3 = 3^2 + 1 = 10$$

$$t_4 = 4^2 + 1 = 17, \text{ and so on}$$

Note that this sequence is neither arithmetic nor geometric.

◀ In the building pictured at the left, the architect Harry Wolf was inspired by proportions derived from the Fibonacci sequence, producing a harmonious effect. This sequence runs: 1, 1, 2, 3, 5, 8, 13, ... Can you find how Wolf has used these values?

Formulas for the  $n$ th term of general arithmetic and geometric sequences are given below. Notice how similar these formulas are.

**Arithmetic Sequence:**

$$t_n = t_1 + (n-1)d$$

$\uparrow$                        $\uparrow$                        $\uparrow$   
 To get the      start with      and add the  
 $n$ th term,      the first      difference  
                     term                       $n-1$  times.

**Geometric Sequence:**

$$t_n = t_1 \cdot r^{(n-1)}$$

$\uparrow$                        $\uparrow$                        $\uparrow$   
 To get the      start with      and multiply  
 $n$ th term,      the first      by the ratio  
                     term                       $n-1$  times.

## Definition of a Sequence

By now you probably have a good intuitive idea of what a sequence is, but can you give a precise definition of a sequence? In mathematics, a **sequence** is usually defined to be a function whose domain is the set of positive integers. For example, the sequence with  $n$ th term

$$t_n = 4n - 2$$

can be thought of as the function

$$t(n) = 4n - 2, \text{ where } n \text{ is a positive integer.}$$

Since a sequence is a function, to graph a sequence such as 1, 4, 7, 10, ..., you plot the points (1, 1), (2, 4), (3, 7), (4, 10). You can see how an arithmetic sequence is related to its graph by completing the following activity.

### Activity

For each sequence, (a) graph the sequence and (b) find the slope of the line containing the graph of the sequence.

1. 0, 1, 2, 3, ...

2. 18, 14, 10, ...

3. 5.3, 6, 6.7, ...

The activity above demonstrates that the graph of any arithmetic sequence consists of discrete points on a line whose slope is the same as the common difference of the sequence.

### Example 1

a. Find a formula for the  $n$ th term of the arithmetic sequence

$$3, 5, 7, \dots$$

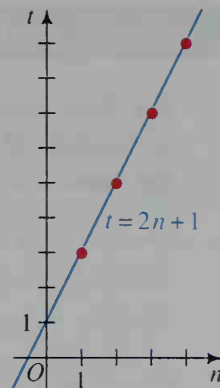
b. Sketch the graph of the sequence.

**Solution**

- a. Use the formula  $t_n = t_1 + (n - 1)d$ . For this sequence,  $t_1$  is 3 and the common difference  $d$  is 2.

$$\begin{aligned} t_n &= 3 + (n - 1) \cdot 2 \\ &= 2n + 1 \end{aligned}$$

- b. The equation  $t_n = 2n + 1$  describes a line, but  $t_n$  is defined only when  $n$  is a positive integer. The graph of the sequence consists of discrete points on the line  $t = 2n + 1$ .

**Example 2**

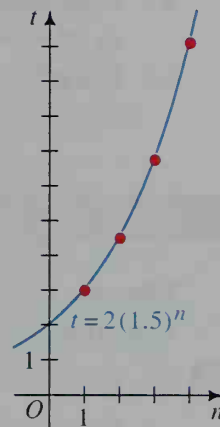
- a. Find a formula for the  $n$ th term of the geometric sequence 3, 4.5, 6.75, ... .
- b. Sketch the graph of the sequence.

**Solution**

- a. Use the formula  $t_n = t_1 \cdot r^{(n-1)}$ . In this case,  $t_1$  is 3 and the common ratio  $r$  is 1.5.

$$\begin{aligned} t_n &= 3 \cdot 1.5^{(n-1)} \\ &= 3 \cdot \frac{1.5^n}{1.5^1} \\ &= \frac{3}{1.5} \cdot (1.5)^n = 2 \cdot (1.5)^n \end{aligned}$$

- b. The equation  $t_n = 2 \cdot (1.5)^n$  is an exponential equation whose graph has an intercept on the  $t$ -axis at 2. The graph of the sequence consists of discrete points on the exponential curve  $t = 2(1.5)^n$ .



The graphs of all geometric sequences are similar to the one in Example 2. The points lie along an exponential curve whose base is the same as the common ratio of the sequence.

The graph of a sequence or the formula for its  $n$ th term can give you information about all of the terms of the sequence. Example 3 shows how you can find any term of a geometric sequence if you know two of its terms.

**Example 3**

- In a geometric sequence,  $t_3 = 12$  and  $t_6 = 96$ . Find  $t_{11}$ .

**Solution**

Substituting in the formula  $t_n = t_1 \cdot r^{(n-1)}$  gives

$$t_3 = t_1 \cdot r^2 = 12 \text{ and } t_6 = t_1 \cdot r^5 = 96.$$

Thus,  $t_1 = \frac{12}{r^2} = \frac{96}{r^5}$ , which gives  $r^3 = 8$  and  $r = 2$ .

Since  $t_1 = 3$ ,  $t_{11} = t_1 \cdot r^{10} = 3 \cdot (2)^{10} = 3072$ .

## CLASS EXERCISES

For Exercises 1–6, state whether the given sequence is arithmetic, geometric, or neither. If arithmetic, give the common difference, and if geometric, give the common ratio.

1. 3, 8, 13, 18, ...

2. 4, 8, 16, 32, ...

3. 2, 5, 10, 17, ...

4. 23, 17, 11, 5, ...

5. 27, -18, 12, -8, ...

6. 1, -3, 5, -7, ...

For Exercises 7–10, state the first four terms of the specified sequence. Then tell whether the sequence is arithmetic, geometric, or neither.

7.  $t_n = 5n + 2$

8.  $t_n = \frac{n+1}{n+2}$

9.  $t_n = 3^n$

10.  $t_n = n^3$

11. **Discussion** Does the sequence 2, 2, 2, ... satisfy the definition of:

a. an arithmetic sequence?

b. a geometric sequence?

12. **Visual Thinking** How does the graph of the sequence with  $n$ th term  $t_n = 3n - 5$  differ from the graph of the line  $y = 3x - 5$ ?

## WRITTEN EXERCISES

Find the first four terms of the given sequence and state whether the sequence is arithmetic, geometric, or neither.

**A**

1.  $t_n = 2n + 3$

2.  $t_n = n^3 + 1$

3.  $t_n = 3 \cdot 2^n$

4.  $t_n = 3 - 7n$

5.  $t_n = n + \frac{1}{n}$

6.  $t_n = (-2)^n$

7.  $t_n = (-1)^n \cdot n$

8.  $t_n = 16 \cdot 2^{2n}$

9.  $t_n = \sin \frac{n\pi}{2}$

10.  $t_n = \cos n\pi$

Find a formula for  $t_n$  and sketch the graph of each arithmetic or geometric sequence.

11. 1, 4, 7, 10, ...

12. 8, 6, 4, 2, ...

13. 8, 4, 2, 1, ...

14. 0.3, 0.9, 2.7, 8.1, ...

15. 30, 26, 22, 18, ...

16. 24, -12, 6, -3, ...



MAY						
		1	2	3	4	
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

The sequence of numbers in each column of a calendar is arithmetic.

State whether the given sequence is arithmetic, geometric, or neither. Find a formula for  $t_n$ , the  $n$ th term of each sequence.

17. 17, 21, 25, 29, ...

18. 15, 7, -1, -9, ...

19. 8, 12, 18, 27, ...

20. 100, -50, 25, -12.5, ...

21. 1, 4, 9, 16, ...

22.  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

23. 11, 101, 1001, 10001, ...

24.  $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \dots$

25.  $2a - 2b, 3a - b, 4a, 5a + b, \dots$

26.  $\frac{a}{9}, \frac{a^2}{18}, \frac{a^3}{36}, \frac{a^4}{72}, \dots$

27.  $2^{2/3}, 2^{5/3}, 2^{8/3}, 2^{11/3}, \dots$

28.  $\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2}, \dots$

Find the indicated term of each arithmetic sequence.

29.  $t_1 = 15, t_2 = 21, t_{20} = ?$

30.  $t_1 = 76, t_3 = 70, t_{101} = ?$

31.  $t_3 = 8, t_5 = 14, t_{50} = ?$

32.  $t_8 = 25, t_{20} = 61, t_1 = ?$

Find the indicated term of each geometric sequence.

33.  $t_1 = 2^{-4}, t_2 = 2^{-3}, t_{12} = ?$

34.  $t_1 = 2, t_2 = 2^{3/2}, t_{13} = ?$

35.  $t_2 = 64, t_5 = -8, t_9 = ?$

36.  $t_1 = 81, t_4 = 24, t_7 = ?$

37. How many terms are in the arithmetic sequence 18, 24, ..., 336?

38. How many terms are in the arithmetic sequence 178, 170, ..., 2?

**B**

39. Find the number of multiples of 7 between 30 and 300.

40. Find the number of multiples of 6 between 28 and 280.

41. How many 3-digit numbers are divisible by 4 and 6?

42. How many 4-digit numbers are *not* divisible by 11?

43. True or false? If the sequence  $a, b, c$  is arithmetic, so is the sequence  $\sin a, \sin b, \sin c$ .

44. True or false? If the sequence  $a, b, c$  is arithmetic, then the sequence  $2^a, 2^b, 2^c$  is geometric.

45. Explain why the logarithms of the terms of the geometric sequence  $a, ar, ar^2, ar^3, \dots$  form an arithmetic sequence.

46. **Writing** One of the principal ideas in the work of the English economist Thomas Malthus (1766–1834) is often expressed as “food supply rises arithmetically, but population increases geometrically.” Write a paragraph or two restating this idea in terms of sequences. If Malthus’s idea is correct, what kind of consequences can be expected? Explain your answer.

47. Find  $x$  if the sequence 2, 8,  $3x + 5$  is (a) arithmetic and (b) geometric.

48. Find  $x$  if the sequence 4,  $x, \frac{3}{2}x$  is (a) arithmetic and (b) geometric.



- B: 2, 9, 16, ...

478 Chapter Thirteen

62. Refer to the figure for Exercise 61. Use similar triangles to prove that  $h$  is the geometric mean of  $x$  and  $y$ .
- C** 63. Prove that the arithmetic mean of two positive numbers is never less than their geometric mean.
64. Consider a circle with radius 6. Find  $A$ , the area of the inscribed equilateral triangle, and  $B$ , the area of the circumscribed equilateral triangle. Show that the geometric mean of  $A$  and  $B$  is the area of the inscribed regular hexagon.

## 13-2 Recursive Definitions

**Objective** To use sequences defined recursively to solve problems.

Sometimes a sequence is defined by giving the value of  $t_n$  in terms of the preceding term,  $t_{n-1}$ . For example, consider the sequence defined by the following formulas:

$$\begin{aligned}t_1 &= 3 \\t_n &= 2t_{n-1} + 1\end{aligned}$$

The second formula above states that the  $n$ th term is one more than twice the  $(n - 1)$ st term. Knowing that the sequence begins with  $t_1 = 3$ , we can determine the first few terms of the sequence as follows:

$$\begin{aligned}t_n = 2t_{n-1} + 1 &\longrightarrow t_2 = 2t_1 + 1 = 2(3) + 1 = 7 \\t_3 &= 2t_2 + 1 = 2(7) + 1 = 15 \\t_4 &= 2t_3 + 1 = 2(15) + 1 = 31\end{aligned}$$

The formulas  $t_1 = 3$  and  $t_n = 2t_{n-1} + 1$  give a *recursive definition* for the sequence 3, 7, 15, 31, . . . . A recursive definition consists of two parts:

1. An *initial condition* that tells where the sequence starts.
2. A *recursion equation* (or *recursion formula*) that tells how any term in the sequence is related to the preceding term.

In Section 13-1, you were given *explicit definitions* for sequences. To see the contrast between a recursive definition and an explicit definition, consider the arithmetic sequence

$$23, 20, 17, 14, \dots$$

**Recursive definition:**  $t_1 = 23$  Initial condition.  
 $t_n = t_{n-1} - 3$  Recursion equation says that each term is 3 less than the preceding term.

**Explicit definition:**  $t_n = 26 - 3n$   $t_n$  is given explicitly in terms of  $n$ .

Sometimes it is possible to find an explicit definition for a sequence when you are given its recursive definition, as the activity on the next page shows.

## Activity

1.  $t_1 = 3$ ;  $t_n = t_{n-1} + 4$ 
  - a. Give the first five terms of this sequence. What kind of sequence is it?
  - b. Find a formula for  $t_n$  in terms of  $n$ .
2.  $t_1 = 64$ ;  $t_n = \frac{1}{2}t_{n-1}$ 
  - a. Give the first five terms of this sequence. What kind of sequence is it?
  - b. Find a formula for  $t_n$  in terms of  $n$ .



Recursive definitions are important tools in modeling, as the example below shows. They are easier to set up than explicit definitions because they reflect what is directly observed. Since many repeated calculations of the same formula can be done quickly using a loop in a computer program, recursion formulas are very useful when using computers to solve problems.

### Example

The population of a certain country grows as a result of two conditions:

1. The annual population growth is 1% of those already in the country. This growth rate equals the birth rate minus the death rate.
2. 20,000 people immigrate into the country each year.

If the population now is 5,000,000 people, what will the population be in 20 years?

### Solution

If you ignore the immigration for a moment and consider only the 1% growth rate, then  $P_n = 1.01P_{n-1}$ . Since immigration adds another 20,000 persons each year, the recursion equation is

$$P_n = 1.01P_{n-1} + 20,000.$$

In many practical situations, it is convenient to use a zero subscript to indicate an initial observation. If we let  $P_0 = 5,000,000$  represent the population now, then  $P_1$  is the population at the end of one year, and  $P_n$  is the population at the end of  $n$  years. To calculate the population for the first few years, you could use a calculator and the recursion equation.

$$P_0 = 5,000,000$$

$$P_1 = 1.01 \cdot 5,000,000 + 20,000 = 5,070,000$$

$$P_2 = 1.01 \cdot 5,070,000 + 20,000 = 5,140,700$$

$$P_3 = 1.01 \cdot 5,140,700 + 20,000 = 5,212,107$$

However, to calculate the population for each of the next 20 years, it is easier to write a simple computer program or use a spreadsheet program. The BASIC program given on the next page uses the recursion equation above to find the population.

```

10 LET P = 5000000 ←———— This is the initial population  $P_0$ .
20 FOR N = 1 TO 20
30 LET P = 1.01*P+20000 ←————  $P_n = 1.01 \cdot P_{n-1} + 20,000$ 
40 NEXT N
50 PRINT "THE POPULATION IN ";N-1;" YEARS IS ";INT(P)
60 END

```

The program produces this output:

```

THE POPULATION IN 20 YEARS IS 6541330

```

## CLASS EXERCISES

Give the first four terms of each sequence.

- $t_1 = 5$ ;  $t_n = t_{n-1} + 3$
- $t_1 = 10$ ;  $t_n = t_{n-1} + n$
- $t_1 = 3$ ;  $t_n = 2t_{n-1}$
- $t_1 = 4$ ;  $t_n = 2t_{n-1} - 1$
- Sometimes a recursive definition will tell how the  $(n + 1)$ st term is related to the  $n$ th term. Give the first four terms of the sequence defined recursively by  $t_1 = 5$  and  $t_{n+1} = 2t_n + n$ .
- Sometimes a recursive definition gives  $t_n$  in terms of more than one preceding term. Give the first four terms of the sequence defined recursively by  $t_1 = 4$ ,  $t_2 = 16$ , and  $t_n = t_{n-1} + t_{n-2}$ .

## WRITTEN EXERCISES

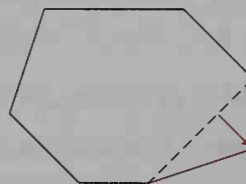
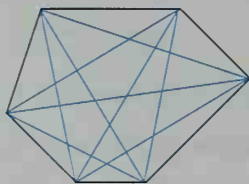
Find the third, fourth, and fifth terms of each sequence.

- A**  $t_1 = 6$ ;  $t_n = t_{n-1} + 4$
- $t_1 = 9$ ;  $t_n = \frac{1}{3}t_{n-1}$
- $t_1 = 1$ ;  $t_n = 3t_{n-1} - 1$
- $t_1 = 4$ ;  $t_n = (t_{n-1})^2 - 10$
- $t_1 = 1$ ;  $t_n = t_{n-1} + 2n - 1$
- $t_1 = \frac{1}{2}$ ;  $t_n = \frac{n}{n+1}(t_{n-1} + 1)$
- $t_1 = 2$ ;  $t_2 = 4$ ;  $t_n = t_{n-1} + t_{n-2}$
- $t_1 = 2$ ;  $t_2 = 4$ ;  $t_n = t_{n-1} \cdot t_{n-2}$
- $t_1 = 5$ ;  $t_2 = 8$ ;  $t_n = (t_{n-1} - t_{n-2})^2$
- $t_1 = 7$ ;  $t_2 = 3$ ;  $t_n = t_{n-1} - 2t_{n-2}$
- Find an explicit definition for the sequence in Exercise 1.
- Find an explicit definition for the sequence in Exercise 2.

Give a recursive definition for each sequence.

- 9, 13, 17, 21, ...
- 81, 27, 9, 3, ...
- 1, 3, 7, 15, 31, 63, ...
- 1, 3, 7, 13, 21, 31, ...
- 1, 3, 6, 10, 15, 21, ...
- 1, 2, 6, 24, 120, 720, ...

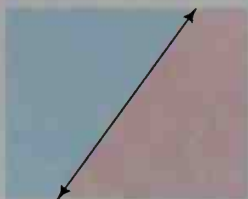
19. a. Give the first eight terms of the sequence defined recursively by  $t_1 = 3$ ,  $t_2 = 5$ , and  $t_n = t_{n-1} - t_{n-2}$ .  
 b. Observing the pattern you get in part (a), tell what the 1000th term of the sequence will be.
20. a. Give the first eight terms of the sequence defined recursively by  $t_1 = 4$ ,  $t_2 = 8$ , and  $t_n = \frac{t_{n-1}}{t_{n-2}}$ .  
 b. Observing the pattern you get in part (a), tell what the 1000th term of the sequence will be.
21. **Geography** Refer to the example on page 480. Suppose the population of those living in the country grows 2% per year, and that an additional 50,000 people immigrate into the country every year.  
 a. Give a recursion equation for  $P_n$ , the population in  $n$  years.  
 b. If the population now is 8,500,000, what will the population be in 5 years?
22. **Chemistry** Each day 8% of a quantity of radioactive iodine will decay.  
 a. Express this fact with a recursion equation.  
 b. Choosing a value for the initial amount of iodine, find the approximate half-life of the iodine.
- B** 23. **Visual Thinking** Let  $S_n$  represent the number of dots in an  $n$  by  $n$  square array. Pretend you have forgotten that  $S_n = n^2$ . Give a recursion equation that tells how  $S_{n+1}$  is related to  $S_n$  by reasoning how many extra dots are needed to form the  $(n + 1)$ st square array from the previous  $n$ th square array. Illustrate your answer with a diagram of dots.
24. Suppose that everyone in a room shakes hands with everyone else exactly once. Let  $H_n$  represent the number of handshakes if there are  $n$  people ( $n \geq 2$ ) in the room. Give a recursion equation that tells how  $H_n$  is related to  $H_{n-1}$ . (*Hint*: Suppose you know  $H_{n-1}$ . If another person enters the room, how many additional handshakes will there be?)
25. **Visual Thinking** Let  $d_n$  represent the number of diagonals that can be drawn in an  $n$ -sided polygon. The diagram at the left below shows that a hexagon has 9 diagonals, and so  $d_6 = 9$ .



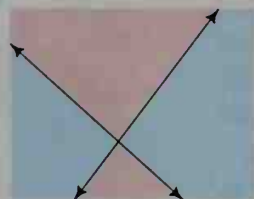
- a. Imagine pushing out one side of the hexagon so that a polygon of 7 sides is formed. (See the diagram at the right above.) How many additional diagonals can be drawn?
- b. Imagine pushing out one side of a polygon with  $n - 1$  sides so that an  $n$ -sided polygon is formed. Tell how many additional diagonals can be drawn. Then write a recursion equation for  $d_n$ .



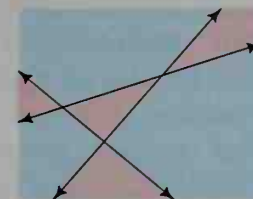
26. **Visual Thinking** Let  $r_n$  represent the number of regions formed when  $n$  lines are drawn in a plane such that no two lines are parallel and no three are concurrent. The diagrams below illustrate  $r_1$ ,  $r_2$ , and  $r_3$ . Give the recursion equation for  $r_n$ .



$r_1 = 2$  regions



$r_2 = 4$  regions



$r_3 = 7$  regions

27. **Visual Thinking** Let  $p_n$  represent the number of intersection points created when  $n$  lines are drawn in a plane such that no two lines are parallel and no three lines are concurrent. The diagrams in Exercise 26 show that  $p_1 = 0$ ,  $p_2 = 1$ , and  $p_3 = 3$ .
- Find  $p_4$  and  $p_5$ .
  - Give a recursion equation for  $p_n$ .
28. **Writing** For what kind of problem would you be more likely to use a recursive definition of a sequence than an explicit definition? Why? For what type of problem would an explicit definition be more useful?

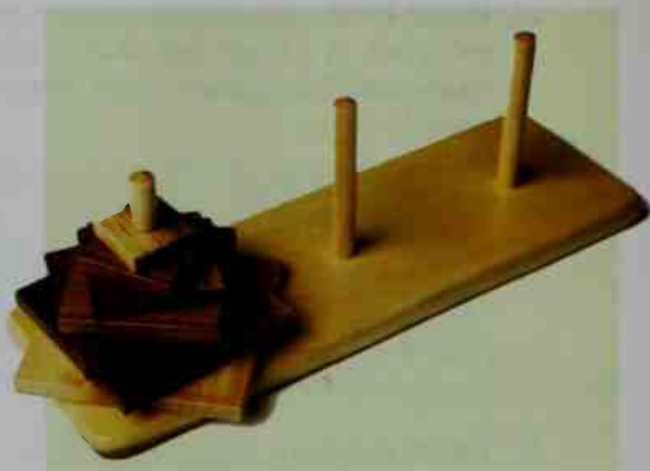


For the parts of the following exercises designated in green, you will need to use a computer or a programmable calculator.

29. **Finance** On the birth of their daughter, Mr. and Mrs. Swift began saving for her college education by investing \$5000 in an annuity account paying 10% interest per year. Each year on their daughter's birthday they invested \$2000 more in the account.
- Let  $A_n$  represent the amount in the account on their daughter's  $n$ th birthday. Give a recursive definition for  $A_n$ .
  - Find the amount that will be in the account on her 18th birthday.
30. **Finance** When Mr. Tallchief reaches his retirement age of 65, he expects to have a retirement account worth about \$400,000. One month after he retires, and every month thereafter, he intends to withdraw \$4000 from the account. The balance will be invested at 9% annual interest compounded monthly.
- Let  $A_n$  represent the amount in the account  $n$  months after Mr. Tallchief's retirement. Give a recursive definition for  $A_n$ .
  - When will there be no money left in the account?



31. The Tower of Hanoi puzzle consists of a block of wood with three posts, A, B, and C. On post A there are eight disks of diminishing size from bottom to top. The task is to transfer all eight disks from post A to one of the other two posts given that:



1. only one disk can be moved at a time;
  2. no disk can be placed on top of a smaller disk.
- a. Let  $M_n$  represent the minimum number of moves needed to move  $n$  disks from post A to one of the other posts. Obviously,  $M_1 = 1$ . What are  $M_2$  and  $M_3$ ? (To try this puzzle, you could use coins of different sizes and three locations A, B, and C on a sheet of paper to represent the three posts.)
  - b. Suppose you know how to move  $(n - 1)$  disks or coins from post A to another post, and that to do so requires  $M_{n-1}$  moves. Explain why  $M_n = 2M_{n-1} + 1$ .
  - c. Use your answer to part (b) to check your values for  $M_2$  and  $M_3$ . Then find  $M_4$ ,  $M_5$ ,  $M_6$ ,  $M_7$ , and  $M_8$ .
  - d. Find an explicit definition for  $M_n$ .
  - e. According to legend, in the great Temple of Benares, there is an altar with three diamond needles. At the beginning of time, 64 gold rings of decreasing radius from bottom to top were placed on one of these needles. Day and night, priests sit before the altar transferring one gold ring per second in accordance with the two rules given above. The legend also says that when all 64 rings have been transferred to one of the other diamond needles, the world will come to an end. How long will it take the priests to transfer all the rings? (You can use your answer to part (d), or you can find a computer solution using the recursion equation given in part (b).)
32. Here is a famous problem posed in the thirteenth century by Leonardo de Pisa, better known as Fibonacci: Suppose we have one pair of newborn rabbits of both genders. We assume that the following conditions are true.
1. It takes a newborn rabbit one month to become an adult.
  2. A pair of adult rabbits of both genders will produce one pair of newborn rabbits of both genders each month, beginning one month after becoming adults.
  3. The rabbits do not die.

How many rabbits will there be one year later? (To help with your solution to this problem, write a recursion equation in which  $r_n$ , the number of pairs of rabbits  $n$  months from now, is expressed in terms of  $r_{n-1}$  and  $r_{n-2}$ . Remember that  $r_0 = 1$ .)

33. **Physics** *Newton's Law of Cooling* states that over equal time periods (for example, one minute) the change in an object's temperature is proportional to the difference between the temperature of the object at the beginning of the time period and the room temperature. Thus, if  $t_0$  is the temperature of the object when it is placed in a room whose temperature is  $R$ , then  $t_n$ , the object's temperature  $n$  minutes later, is given by the recursion equation

$$t_n - t_{n-1} = k(t_{n-1} - R),$$

where  $k$  is a proportionality constant.

- Suppose that a cup of coffee at  $98^\circ\text{C}$  is placed in a room whose temperature is  $18^\circ\text{C}$ , and that 1 min later the coffee has cooled to  $94^\circ\text{C}$ . Find the proportionality constant  $k$ , and explain the significance of its negative sign.
- Show that the recursion equation given above can be rewritten as  $t_n = 0.95t_{n-1} + 0.9$ .
- Use a computer to print out a table of values for  $t_n$ . When will the cup of coffee cool to less than  $75^\circ\text{C}$ ?

34. **Medicine** Suppose that in a closed community with population  $P$ , a flu epidemic begins and that the number of people *newly* exposed to the flu on a given day is proportional to the number not yet exposed on the previous day.

- If  $f_n$  represents the number of people exposed to the flu  $n$  days after it begins, explain how the description above leads to the recursion equation  $f_n - f_{n-1} = k(P - f_{n-1})$ , where  $k$  is a proportionality constant.
- Suppose in a college community of 2500 students, the flu begins with 100 students exposed to the flu; that is,  $f_0 = 100$ . On the next day,  $f_1 = 220$ . Find the value of  $k$ , and then show that  $f_n = 0.95f_{n-1} + 125$ .
- Use a computer to print out a table of values for  $f_n$ . About how long does it take before the flu spreads through the whole student body?

- C** 35. **a. Ecology** Suppose that a certain population grows from one generation to the next in such a way that the population *increase* is proportional to the population of the previous generation. Explain how this growth can be modeled by the equation  $P_n - P_{n-1} = kP_{n-1}$ . In your explanation, tell what  $P_n$  and  $P_{n-1}$  represent.
- Suppose that a population has a carrying capacity  $C$ . This is the maximum size beyond which the population cannot grow. As the population gets closer to  $C$ , overcrowding tends to reduce population increases. Sketch a graph that models this type of population growth with overcrowding.
  - The population growth described in part (b) can be modeled by modifying the equation in part (a) so that the new equation is

$$P_n - P_{n-1} = kP_{n-1}(C - P_{n-1}).$$

Write a sentence or two explaining what this recursion equation says.

- For a population of fish in an aquarium, let  $C = 500$  and  $k = 0.002$ . Choose a value for the initial population, and use a computer to print the values given by the recursion equation in part (c). How well do the computer results agree with your graph in part (b)?

## 13-3 Arithmetic and Geometric Series and Their Sums

### Objective

To find the sum of the first  $n$  terms of arithmetic or geometric series.

The words *sequence* and *series* are often used interchangeably in everyday conversation. For example, a person may refer to a *sequence of events* or to a *series of events*. In mathematics, however, a distinction is made between a sequence of numbers and a series of numbers. This distinction between a sequence and a series can best be made by considering some examples.

Finite sequence: 2, 6, 10, 14

Finite series:  $2 + 6 + 10 + 14$

Infinite sequence:  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

Infinite series:  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

As you can see from the examples above, a **series** is an indicated sum of the terms of a sequence. The *sum* of the finite series  $2 + 6 + 10 + 14$  is 32. In Section 13-5 we will see that the sum of the infinite series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$  is 1. However, not all infinite series have sums. In this section we will consider only finite series and their sums.

If  $S_n$  represents the sum of  $n$  terms of a series, then  $S_n$  can be expressed either explicitly or recursively as follows:

Explicit definition of  $S_n$ :  $S_n = t_1 + t_2 + t_3 + \dots + t_n$

Recursive definition of  $S_n$ :  $S_0 = 0$   
 $S_n = S_{n-1} + t_n$  for  $n \geq 1$



The recursive definition can be used as the basis for a computer program that will find the sum of a series. Example 1 illustrates how this is done.

**Example 1** Find the sum of the cubes of the first twenty positive integers.

### Solution

We want  $S_{20} = 1^3 + 2^3 + 3^3 + \dots + 20^3$ .

First find a formula for  $t_n$  in terms of  $n$ . The  $n$ th term is  $t_n = n^3$ , and is represented by T in line 30 of the computer program given on the next page. The recursive definition of the sum  $S_n$ , which is represented by S, occurs in lines 10 and 40.

```

10 LET S = 0 ←———— Initial condition  $S_0 = 0$ 
20 FOR N = 1 TO 20
30 LET T = N^3
40 LET S = S + T ←———— Recursion equation  $S_n = S_{n-1} + t_n$ 
50 NEXT N
60 PRINT 'THE SUM OF THE FIRST 'N-1;' TERMS IS 'S
70 END

```

When the program is run, the computer gives this output:

```

○ THE SUM OF THE FIRST 20 TERMS IS 44100 ○

```

You can use the method of Example 1 to find the sum of any finite series. However, if the series is arithmetic or geometric, you can also use the formulas given in the following two theorems.

### Sum of a Finite Arithmetic Series

The sum of the first  $n$  terms of an arithmetic series is

$$S_n = \frac{n(t_1 + t_n)}{2}.$$

**Proof:** Write the series for  $S_n$  twice, the second time with the order of the terms reversed. Then add the two equations, term by term:

$$\begin{array}{rcl}
 S_n & = & t_1 + (t_1 + d) + (t_1 + 2d) + \cdots + (t_n - d) + t_n \\
 S_n & = & t_n + (t_n - d) + (t_n - 2d) + \cdots + (t_1 + d) + t_1 \\
 \hline
 2S_n & = & (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \cdots + (t_1 + t_n) + (t_1 + t_n) \\
 \text{Since there are } n \text{ of these } (t_1 + t_n) \text{ terms,} \\
 2S_n & = & n(t_1 + t_n) \\
 S_n & = & \frac{n(t_1 + t_n)}{2}
 \end{array}$$

**Example 2** Find the sum of the first 25 terms of the arithmetic series

$$11 + 14 + 17 + 20 + \cdots$$

**Solution**

**Step 1:** First find the 25th term.

$$t_{25} = t_1 + (n - 1)d = 11 + (25 - 1)3 = 83$$

$$\text{Step 2: } S_{25} = \frac{n(t_1 + t_{25})}{2} = \frac{25(11 + 83)}{2} = 1175$$



## Sum of a Finite Geometric Series

The sum of the first  $n$  terms of a geometric series is

$$S_n = \frac{t_1(1 - r^n)}{1 - r},$$

where  $r$  is the common ratio and  $r \neq 1$ .

**Proof:** Multiply the series for  $S_n$  by the common ratio and then subtract the resulting series from the original one, as shown below.

$$\begin{array}{r} S_n = t_1 + t_1r + t_1r^2 + \cdots + t_1r^{n-2} + t_1r^{n-1} \\ rS_n = \quad \quad t_1r + t_1r^2 + \cdots + t_1r^{n-2} + t_1r^{n-1} + t_1r^n \\ \hline S_n - rS_n = t_1 + 0 + 0 + \cdots + 0 + 0 - t_1r^n \\ S_n - rS_n = t_1 - t_1r^n \\ S_n(1 - r) = t_1(1 - r^n) \\ S_n = \frac{t_1(1 - r^n)}{1 - r}, \quad r \neq 1 \end{array}$$

This formula is not defined for  $r = 1$ . If  $r$  does equal 1, however, the geometric series is simply a series of repeated numbers, such as  $a + a + \cdots + a$ , whose sum is  $na$ .

**Example 3** Find the sum of the first 10 terms of the geometric series

$$2 - 6 + 18 - 54 + \cdots$$

**Solution**

$$\begin{aligned} S_{10} &= \frac{t_1(1 - r^{10})}{1 - r} \\ &= \frac{2(1 - (-3)^{10})}{1 - (-3)} = \frac{2(1 - 3^{10})}{4} = -29,524 \end{aligned}$$

## CLASS EXERCISES

1. Give an example of an arithmetic sequence and of an arithmetic series.
2. Which theorem can be used to find the sum of the finite series  $1 + 2 + 4 + 8 + 16 + \cdots + 1024$ ? What is this sum?
3. a. Is the series  $3 + 5 + 9 + 17 + \cdots + 32,769$  arithmetic, geometric, or neither?  
b. Give a recursive definition of the series.  
c. How would you modify the computer program in Example 1 to find the sum of this series?

## WRITTEN EXERCISES

For each of the arithmetic series in Exercises 1–8, find the specified sum.

- A**
1.  $S_{10}$ :  $t_1 = 3$ ,  $t_{10} = 39$
  2.  $S_{200}$ :  $t_1 = 18$ ,  $t_{200} = 472$
  3.  $S_{50}$ :  $5 + 10 + 15 + \cdots$
  4.  $S_{25}$ :  $17 + 25 + 33 + \cdots$
  5.  $S_{12}$ :  $t_n = 5 + 3n$
  6.  $S_{40}$ :  $t_1 = 5$ ,  $t_3 = 11$
  7.  $1 + 2 + 3 + \cdots + 1000$
  8.  $3 + 7 + 11 + \cdots + 99$
  9. Find  $S_8$  for the geometric series with  $t_1 = 8$  and (a)  $r = \frac{1}{2}$ , (b)  $r = -\frac{1}{2}$ .
  10. Show that the sum of the first 10 terms of the geometric series

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

is twice the sum of the first 10 terms of the series

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

11. Show that  $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$ .
12. Show that  $9 + 90 + 900 + \cdots + 9 \cdot 10^{n-1} = 10^n - 1$ .
13. Show that  $\sqrt{2} + 2 + 2\sqrt{2} + \cdots + 64 = \frac{63\sqrt{2}}{\sqrt{2} - 1}$ .
14. For the series  $1 + \sqrt[5]{3} + \sqrt[5]{3^2} + \sqrt[5]{3^3} + \cdots$ , show that

$$S_{15} = \frac{26}{3^{1/5} - 1}.$$

15. Show that the sum of the first  $n$  positive integers is  $\frac{n(n+1)}{2}$ .
16. Let  $S_n$  be the sum of the first  $n$  positive odd integers.
  - a. Evaluate  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$ .
  - b. Suggest and prove a formula for  $S_n$ .

$$1 + 3 + 5 + 7 = 4^2$$

Ex. 16

Find a formula for  $t_n$ , the  $n$ th term of the series. Then give a recursive definition for  $S_n$ , the sum of  $n$  terms of the series.

17.  $2 + 4 + 6 + 8 + \cdots$
18.  $1 + 3 + 9 + 27 + \cdots$
19.  $8 + 12 + 18 + 27 + \cdots$
20.  $50 + 47 + 44 + 41 + \cdots$



For Exercises 21–26, you will need to use a computer.

Write a computer program, like the one given in Example 1, that will print the sum of each series. Then run the program to find the sum.

21.  $1^4 + 2^4 + 3^4 + \cdots + 10^4$
22.  $1^1 + 2^2 + 3^3 + \cdots + 15^{15}$
23.  $10^2 + 20^2 + 30^2 + \cdots + 1000^2$
24.  $\sqrt{5} + \sqrt{10} + \sqrt{15} + \cdots + \sqrt{200}$

- B** 25. a. Refer to the computer program given in Example 1. Delete line 60 and insert the following line.

45 PRINT N,S

Explain what happens to the computer's output.

- b. Change lines 20 and 30 in the program so that the computer will print the sums  $S_1, S_2, S_3, \dots, S_{25}$  for the series

$$2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-25}.$$

- c. Find  $S_{25}$  using the formula given on page 488. (Your answer should be approximately the same as the last number that the computer prints.)  
d. Guess the sum of the *infinite* series

$$2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n} + \cdots.$$

26. a. Modify the computer program in Exercise 25 and run it for the series

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{25}}.$$

- b. Find  $S_{25}$  using the formula given on page 488.  
c. Guess the sum of the *infinite* series

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} + \cdots.$$

27. The sum of the first  $n$  terms of a series is  $S_n = n^2 + 4n$ . Find  $t_1, t_2$ , and  $t_3$ .

28. The sum of the first  $n$  terms of a series is  $S_n = 2n^2$ .

- a. Find  $t_1, t_2$ , and  $t_3$ .  
b. Find  $S_n - S_{n-1}$ .

29. Find the sum of all multiples of 3 between 1 and 1000.

30. Find the sum of all positive 3-digit numbers divisible by 6.

31. Find the sum of all positive 3-digit numbers whose last digit is 3.

32. Find the sum of all positive odd numbers less than 400 that are divisible by 5.

33. Find the sum of the series  $1 - 3 + 5 - 7 + 9 - 11 + \cdots + 1001$ .

34. Find the sum of the series  $1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \cdots + 299$ , which is the sum of the integers except for multiples of 3.

35. The originator of a chain letter writes 5 letters instructing each recipient to write 5 similar letters to additional people. Then these people each send 5 similar letters to other people. Determine the number of people who should receive letters if the chain continues unbroken for 12 steps. Explain why the process always fails. (There are laws forbidding chain letters that request money.)

36. Value Appliance Store has radios that can be purchased on a daily installment plan. For a particular type of radio, you pay only 1 cent the first day, 2 cents the next day, 4 cents the next day, 8 cents the next day, and so on, for 14 days. How much does the radio cost?

37. a. If you go back through ten generations, how many ancestors do you have? Count your parents as the first generation back, your four grandparents as the second generation, and so on. (Assume there are no duplications.)  
 b. How many generations back must you go in order to have more than one million ancestors?
38. a. **Finance** Suppose a doctor earns \$40,000 during the first year of practice. Suppose also that each succeeding year the salary increases 10%. What is the total of the doctor's salaries over the first 10 years?  
 b. How many years must the doctor work if the salary total is to exceed a million dollars?

39. The number  $T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$  is

sometimes called a **triangular number** because it is possible to represent the number by a triangular array of dots, as shown.

a. Evaluate  $T_1, T_2, T_3, T_4$ , and  $T_5$ .

b. Add any two consecutive triangular numbers. Then make a conjecture and prove it.

40. Consider the series  $1^3 + 2^3 + 3^3 + \cdots + n^3$ . Evaluate  $S_1, S_2, S_3$ , and  $S_4$ . Then suggest a formula for  $S_n$ . (*Hint*: See Exercise 39.)

41. The sum of every row, column, and diagonal of a magic square equals the magic number  $M$ . For the 3-by-3 and 4-by-4 magic squares shown below,  $M$  is 15 and 34, respectively. Find the value of  $M$  for an  $n$ -by- $n$  magic square that contains consecutive integers starting with 1. (*Hint*: How many numbers are in an  $n$ -by- $n$  square? What is their sum? Deduce the row sum  $M$ .) Check that your formula gives  $M(3) = 15$  and  $M(4) = 34$ .

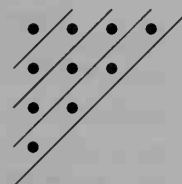
$M = 15$

8	1	6
3	5	7
4	9	2

$M = 34$

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

$$T_4 = 1 + 2 + 3 + 4 = 10$$



42. **Finance** If you invest \$1000 per year every year for 10 years and if your money is compounded annually at 12%, how much money will you have at the time you make your tenth investment?
43. **Finance** If you invest  $P$  dollars at an interest rate  $r$  compounded annually, after  $n$  years you will have  $P(1+r)^n$  dollars. Suppose you invest  $P$  dollars every year for  $n$  years. Show that at the time you make your  $n$ th investment, you will have  $\frac{P[(1+r)^n - 1]}{r}$  dollars.

44. **Finance** The purpose of this exercise is to develop a formula for the monthly payment  $P$  that is required to repay a loan of  $A$  dollars in  $n$  monthly installments, with the interest on the unpaid balance equal to a monthly rate  $r$ . Let  $A_k$  = amount still owed after paying  $k$  installments. Then interest for the  $(k + 1)$ st month is  $rA_k$ , and the principal paid off in the  $(k + 1)$ st payment is  $P - rA_k$ . Thus,

$$A_{k+1} = A_k - (P - rA_k) = (1 + r)A_k - P.$$

- a. Use the equation above and the fact that  $A_0 = A$  to find  $A_1$ ,  $A_2$ , and  $A_3$  in terms of  $A$ ,  $r$ , and  $P$ .
- b. Generalizing from  $A_1$ ,  $A_2$ , and  $A_3$ , you get
- $$A_n = (1 + r)^n A - [(1 + r)^{n-1} + (1 + r)^{n-2} + \cdots + (1 + r) + 1]P.$$

Since the bracketed quantity above is a geometric series, show that the equation can be rewritten

$$A_n = (1 + r)^n A - \left[ \frac{(1 + r)^n - 1}{r} \right] P.$$

- c. Since  $A_n = 0$  (why?), use the last equation in part (b) to show that

$$P = \frac{Ar(1 + r)^n}{(1 + r)^n - 1}.$$

- C** 45. **Finance** A *direct reduction loan* is often used for buying cars and for mortgages on homes. You pay interest only on that portion of the loan that you have not repaid (that is, the unpaid balance). Therefore the monthly payment you make has two parts:  $I$ , which is the interest on the unpaid balance, and  $R$ , which is used to reduce the balance.

Suppose you take out a car loan for \$10,000 at 12% annual interest, calculated monthly, to be repaid in 5 years. Your monthly payment will be  $P$ .

- a. Use the formula in Exercise 44(c) to show that  $P = \$222.44$ . Remember that  $r = \text{monthly rate} = \frac{12\%}{12} = 0.01$ .
- b. Study the table below. Then calculate the entries in the last row of the table.

$n$ = payment number	$A_n$ = amount outstanding after payment	$I_n$ = interest on $A_n$	$R_n$ = loan reduction = $\$222.44 - I_n$
0	10,000.00	100.00	122.44
1	9,877.56	98.78	123.66
2	?	?	?

- c. Express  $A_n$  in terms of  $A_{n-1}$ .
- d. Show that the values of  $R_n$  form a geometric sequence with ratio 1.01, which is  $(1 + \text{monthly rate})$ .
- e. (Optional) Write a computer program that will print out the table above for all 60 payments. Label the column headings N, A, I, and R.





You can often estimate the limit of an infinite sequence by substituting large values of  $n$  in the formula for the  $n$ th term, as illustrated in the next example.

**Example 1** Find: a.  $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right)$       b.  $\lim_{n \rightarrow \infty} (0.99)^n$

**Solution**

a. When  $n = 100$ ,  $\sin\left(\frac{1}{n}\right) = \sin\left(\frac{1}{100}\right) \approx 0.01$ . As  $n$  gets larger,  $\frac{1}{n}$  approaches 0 and so does  $\sin\left(\frac{1}{n}\right)$ . Thus, it seems that  $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n}\right) = 0$ .

b. We can evaluate  $(0.99)^n$  for large  $n$  with logarithms or a calculator. For example,

$$(0.99)^{1000} \approx 4.3 \times 10^{-5} \text{ and } (0.99)^{10,000} \approx 2.2 \times 10^{-44}.$$

Thus, it seems that  $\lim_{n \rightarrow \infty} (0.99)^n = 0$ .

We have just estimated that  $\lim_{n \rightarrow \infty} (0.99)^n$  is 0, and earlier we saw that  $\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$ . These two examples are special cases of the following theorem.

**Theorem**

If  $|r| < 1$ , then  $\lim_{n \rightarrow \infty} r^n = 0$ .

In Example 1 we estimated limits by considering  $t_n$  for large values of  $n$ . In the next example, we show how to change the form of  $t_n$  to assist in finding a limit.

**Example 2** Find: a.  $\lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 - 3n}$       b.  $\lim_{n \rightarrow \infty} \frac{5n^2 + \sqrt{n}}{3n^3 + 7}$

**Solution**

In both parts (a) and (b), we divide numerator and denominator by the highest power of  $n$  that occurs in the denominator.

a. Dividing numerator and denominator by  $n^2$ , we have:

$$\frac{n^2 + 1}{2n^2 - 3n} = \frac{1 + \frac{1}{n^2}}{2 - \frac{3}{n}}$$

Notice that when  $n$  is very large,  $\frac{1}{n^2}$  and  $\frac{3}{n}$  are very near 0. Therefore,

$$\frac{n^2 + 1}{2n^2 - 3n} \approx \frac{1}{2}$$

when  $n$  is very large. For this reason, we say:

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{2n^2 - 3n} = \frac{1}{2}$$

b. Dividing numerator and denominator by  $n^3$ , we have:

$$\frac{5n^2 + \sqrt{n}}{3n^3 + 7} = \frac{\frac{5}{n} + \frac{1}{n^{5/2}}}{3 + \frac{7}{n^3}}$$

Notice that when  $n$  is very large,  $\frac{5}{n}$ ,  $\frac{1}{n^{5/2}}$ , and  $\frac{7}{n^3}$  are very near 0.

Therefore, 
$$\frac{5n^2 + \sqrt{n}}{3n^3 + 7} \approx \frac{0 + 0}{3 + 0} = 0$$

when  $n$  is very large. For this reason, we say:

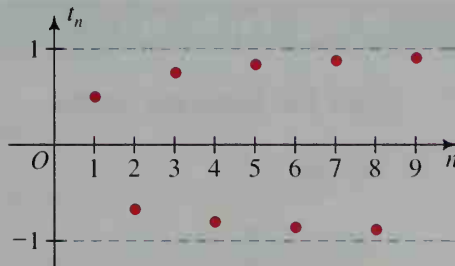
$$\lim_{n \rightarrow \infty} \frac{5n^2 + \sqrt{n}}{3n^3 + 7} = 0$$

## Situations in Which a Sequence Has No Limit

Not all sequences have limits. If the terms of a sequence do not “home in” on a single value, we say that the sequence has no limit or that the limit of the sequence does not exist. For example, the following sequence has no limit:

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots, \frac{(-1)^{n+1} \cdot n}{n+1}, \dots$$

The diagram shows the graph of this sequence. Notice that the odd-numbered terms form a sequence with limit 1. Similarly, the even-numbered terms form a sequence with limit  $-1$ . Nevertheless, there is no single limiting number for all the terms of the sequence. Thus, we say that this sequence has no limit.



## Infinite Limits

Sometimes the terms of a sequence increase or decrease without bound. Here are two examples:

(a)  $3, 7, 11, 15, \dots, 4n - 1, \dots$  (b)  $-10, -100, -1000, \dots, -10^n, \dots$

The terms in sequence (a) continue to increase as  $n$  becomes larger, but they do not approach a fixed number as a target. No matter what fixed number is selected, the terms of the sequence will eventually exceed it. For this reason, we say that the limit of  $4n - 1$  as  $n$  increases without bound is infinity. We write this as

$$\lim_{n \rightarrow \infty} (4n - 1) = \infty.$$

Similarly, the terms in sequence (b) decrease without bound; that is, they are farther and farther to the left of zero on the number line. We say that the limit of  $-10^n$  as  $n$  increases without bound is negative infinity and write

$$\lim_{n \rightarrow \infty} (-10^n) = -\infty.$$

It is important to realize that infinity is neither a number nor a place. When we say that “ $n$  goes to infinity,” we mean that  $n$  increases without bound. When we say that “the limit is infinity,” we mean that the terms are increasing without bound, not that they are approaching some number.

**Example 3** Show that  $\lim_{n \rightarrow \infty} \frac{7n^3}{4n^2 - 5} = \infty$ .

**Solution** Dividing numerator and denominator by  $n^2$ , the highest power of  $n$  in the denominator, we get:

$$\frac{7n^3}{4n^2 - 5} = \frac{7n}{4 - \frac{5}{n^2}}$$

When  $n$  is very large,  $\frac{5}{n^2}$  is very near 0. Therefore,  $\frac{7n^3}{4n^2 - 5} \approx \frac{7n}{4}$  when  $n$

is very large. Thus, since  $\frac{7n}{4}$  increases without bound as  $n$  does,

$$\lim_{n \rightarrow \infty} \frac{7n}{4} = \infty \text{ and } \lim_{n \rightarrow \infty} \frac{7n^3}{4n^2 - 5} = \infty.$$

## CLASS EXERCISES

Find the following limits.

- $\lim_{n \rightarrow \infty} \frac{n}{n+1}$
- $\lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2}$
- $\lim_{n \rightarrow \infty} \frac{2n+1}{3n+1}$
- $\lim_{n \rightarrow \infty} \frac{8n^2 - 3n}{5n^2 + 7}$
- $\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right)$
- $\lim_{n \rightarrow \infty} \log\left[\cos\left(\frac{1}{n}\right)\right]$
- $\lim_{n \rightarrow \infty} (0.999)^n$
- $\lim_{n \rightarrow \infty} (1.001)^n$
- $\lim_{n \rightarrow \infty} \frac{n^4}{2n+1}$
- $\lim_{n \rightarrow \infty} \frac{n^2 + 9,999,999}{n^3}$
- Discussion** Do you think the sequence 1, 0, 1, 0, 1, 0, . . . has a limit? Why?

## WRITTEN EXERCISES

In Exercises 1–12, find the given limit.

- A**  $\lim_{n \rightarrow \infty} \frac{n+5}{n}$
- $\lim_{n \rightarrow \infty} \frac{n^2+1}{n^2}$
- $\lim_{n \rightarrow \infty} \left[1 + \frac{(-1)^n}{n}\right]$
- $\lim_{n \rightarrow \infty} \frac{4n-3}{2n+1}$
- $\lim_{n \rightarrow \infty} \frac{3n^2+5n}{8n^2}$
- $\lim_{n \rightarrow \infty} \frac{2n^4}{6n^5+7}$

7.  $\lim_{n \rightarrow \infty} \tan\left(\frac{1}{n}\right)$       8.  $\lim_{n \rightarrow \infty} \sec\left(\frac{1}{n}\right)$       9.  $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1}$   
 10.  $\lim_{n \rightarrow \infty} \frac{5n^{2/3} - 8n}{6n - 1}$       11.  $\lim_{n \rightarrow \infty} \log\left(\frac{n+1}{n}\right)$       12.  $\lim_{n \rightarrow \infty} \log \sqrt[n]{10}$

In Exercises 13–18, find the limit of the specified sequence or state that the limit does not exist.

13.  $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \frac{1}{243}, \dots$       14.  $1, -4, 9, -16, 25, -36, \dots$   
 15.  $\frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \frac{7}{6}, -\frac{8}{7}, \dots$       16.  $\frac{1}{10}, -\frac{2}{10^2}, \frac{3}{10^3}, -\frac{4}{10^4}, \frac{5}{10^5}, \dots$   
 17.  $t_n = \cos\left(\frac{n\pi}{2}\right)$       18.  $t_n = \sin(n\pi)$

In Exercises 19–30, evaluate the given limit or state that the limit does not exist. If the sequence approaches  $\infty$  or  $-\infty$ , so state.

19.  $\lim_{n \rightarrow \infty} \frac{5n^{5/2} - 7n}{n^2 + 10n}$       20.  $\lim_{n \rightarrow \infty} \frac{5n}{n^{1/2} - 3}$   
 21.  $\lim_{n \rightarrow \infty} \log\left(\frac{1}{n}\right)$       22.  $\lim_{n \rightarrow \infty} \frac{\sin n}{n}$   
 23.  $\lim_{n \rightarrow \infty} \frac{\cos(n\pi)}{n}$       24.  $\lim_{n \rightarrow \infty} e^{-n}$   
 25.  $\lim_{n \rightarrow \infty} \tan\left(\frac{\pi}{4} + n\pi\right)$       26.  $\lim_{n \rightarrow \infty} 2^n$   
**B** 27.  $\lim_{n \rightarrow \infty} \log\left[\sec\left(\frac{1}{n}\right)\right]$       28.  $\lim_{n \rightarrow \infty} [(-1)^n - 1]$   
 29.  $\lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{\sqrt{n-1}}$       30.  $\lim_{n \rightarrow \infty} \frac{\sqrt[3]{8n^2 - 5n + 1}}{\sqrt[3]{n^2 + 7n - 3}}$

31. a. If the  $n$ th term of a geometric series is  $t_n = \left(\frac{1}{3}\right)^{n-1}$ , show that

$$S_n = \frac{3}{2} \left[ 1 - \left(\frac{1}{3}\right)^n \right].$$

b. Find  $\lim_{n \rightarrow \infty} S_n$ .

32. Consider the formula for the sum of the series  $t_1 + t_1 r + \dots + t_1 r^{n-1}$  and then suggest a formula for the sum of the following infinite series:

$$t_1 + t_1 r + t_1 r^2 + \dots, \text{ where } |r| < 1$$

33. Recall that  $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ . Find  $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n$  by noting that

$$\left(1 + \frac{2}{n}\right)^n = \left(1 + \frac{1}{\frac{n}{2}}\right)^n = \left[\left(1 + \frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}}\right]^2.$$

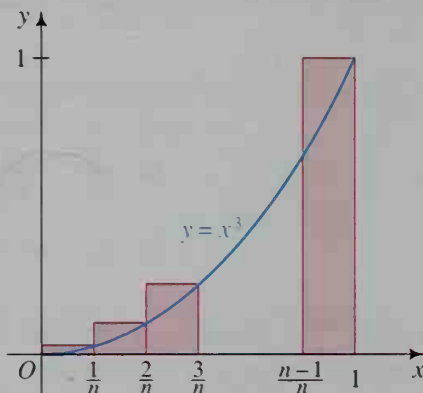


34. a. Show that  $\lim_{n \rightarrow \infty} \left(1 + \frac{3}{n}\right)^n = e^3$ .

b. Evaluate  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{2n}\right)^n$ .

(Hint: See Exercise 33.)

35. The area  $A$  under the curve  $y = x^3$  between  $x = 0$  and  $x = 1$  can be approximated by adding the areas of  $n$  rectangles as shown.



$$\begin{aligned} A \approx A_n &= \left(\frac{1}{n}\right)^3 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^3 \cdot \frac{1}{n} + \left(\frac{3}{n}\right)^3 \cdot \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^3 \cdot \frac{1}{n} \\ &= \frac{1}{n^4} (1^3 + 2^3 + 3^3 + \cdots + n^3) \end{aligned}$$

- a. According to the result of Exercise 40 in Section 13-3,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2.$$

Use this formula to show that  $A_n = \frac{n^2 + 2n + 1}{4n^2}$ .

- b. As  $n$  becomes very large, what value is  $A_n$  approaching?

- c. Find  $A$  by evaluating  $\lim_{n \rightarrow \infty} A_n$ .

- C** 36. Use the procedure in Exercise 35 to find the area under the curve  $y = x^2$  between 0 and 1. You will need to know that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

## ■■■■ CALCULATOR EXERCISES

1. a. Guess the value of  $\lim_{n \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$  by evaluating  $\sqrt{n+1} - \sqrt{n}$  for several large values of  $n$ .
- b. Multiply  $\sqrt{n+1} - \sqrt{n}$  by  $\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$ .
- c. Determine what happens to the expression in part (b) when  $n$  becomes very large. Does your answer agree with your answer to part (a)?

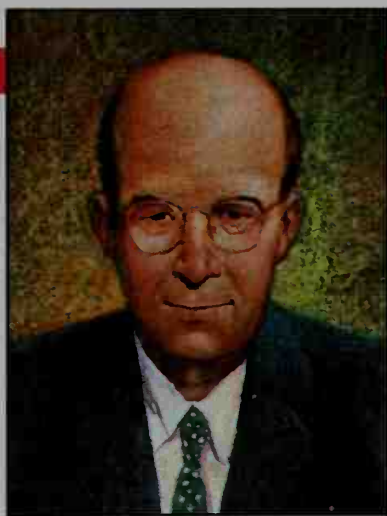
2. A sequence is defined recursively by the equations

$$t_1 = 1, \text{ and } t_n = \frac{t_{n-1}}{2} + \frac{1}{t_{n-1}}.$$

- a. Find decimal approximations for the first five terms of the sequence.  
b. Suggest a limit for this sequence.
3. a. Evaluate  $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$  by considering its value to be the limit of the following sequence as  $n$  approaches  $\infty$ :  
 $\sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}$ , and so on.
- b. Compare your answer with the golden ratio,  $R = \frac{1 + \sqrt{5}}{2}$ .

### ////COMPUTER EXERCISE

A student leaves home to go to the movies. Halfway there, the student remembers some uncompleted homework and heads back home. Halfway back home the student has a change of mind and heads back to the movies. You guessed it! Halfway back to the movies, the student, overcome by an attack of conscience, heads back to complete the homework. Suppose the student continues in this fashion. Write a computer program to calculate how far from home the student is after each of the first twenty changes of mind. (If you think of the student as moving along a number line with home at 0 and the movies at 1, does the student appear to be approaching a limiting point on the number line?)



*Manuel Sandoval Vallarta (1899–1977)*

Manuel Sandoval Vallarta, a mathematician and physicist, was a professor at the Massachusetts Institute of Technology. He studied relativity with Albert Einstein and electromagnetic theory with Max Planck. In 1946 he returned to his native Mexico, where he founded the Instituto Nacional de Energía Nuclear.

A pioneer in the field of cosmic-ray physics, and an advocate for the peaceful uses of atomic energy, Sandoval Vallarta was president of the Latin American Council on Cosmic Rays, and represented Mexico in conferences of the United Nations Atomic Energy Agency.

## 13-5 Sums of Infinite Series

### Objective

To find the sum of an infinite geometric series.

The sum of an infinite series is very closely connected to the limit of an infinite sequence. To see this, consider the infinite geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \left(\frac{1}{2}\right)^n + \cdots$$

Associated with this series is the following *sequence of partial sums*:

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$\vdots$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \left(\frac{1}{2}\right)^n$$

$$= \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \quad \leftarrow \text{using the formula } S_n = \frac{t_1(1 - r^n)}{1 - r}$$

$$= 1 - \left(\frac{1}{2}\right)^n$$

Since the sequence of partial sums  $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots, 1 - \left(\frac{1}{2}\right)^n$  has limit 1, we say that the infinite series has **limit** 1 or has the **sum** 1.

In general, for any infinite series  $t_1 + t_2 + \cdots + t_n + \cdots$ ,

$$S_n = t_1 + t_2 + \cdots + t_n$$

is called the  **$n$ th partial sum**. If the **sequence of partial sums**  $S_1, S_2, \dots, S_n, \dots$  has a finite limit  $S$ , then the infinite series is said to **converge** to the sum  $S$ . If the sequence of partial sums approaches infinity or has no finite limit, the infinite series is said to **diverge**.

Since we already have a formula for the  $n$ th partial sum of a geometric series, we can prove a theorem that tells when such series converge.

### Sum of an Infinite Geometric Series

If  $|r| < 1$ , the infinite geometric series

$$t_1 + t_1 r + t_1 r^2 + \cdots + t_1 r^n + \cdots$$

converges to the sum  $S = \frac{t_1}{1 - r}$ .

If  $|r| \geq 1$  and  $t_1 \neq 0$ , then the series diverges.

**Proof:** The  $n$ th partial sum of the geometric series is

$$S_n = \frac{t_1(1 - r^n)}{1 - r}.$$

1. Therefore, if  $|r| < 1$ :

$$\begin{aligned}\lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{t_1(1 - r^n)}{1 - r} \\ &= \frac{t_1(1 - 0)}{1 - r} \text{ since } \lim_{n \rightarrow \infty} r^n = 0 \text{ when } |r| < 1 \\ &= \frac{t_1}{1 - r}\end{aligned}$$

2. However, if  $|r| > 1$ ,  $r^n$  increases or decreases without bound as  $n$  approaches infinity. Thus,  $S_n$  becomes infinite and the series diverges.

3. If  $r = 1$ , the series becomes the divergent series

$$t_1 + t_1 + t_1 + \cdots$$

4. If  $r = -1$ , the series becomes the divergent series

$$t_1 - t_1 + t_1 - t_1 + \cdots$$

**Example 1** Find the sum of the infinite geometric series

$$9 - 6 + 4 - \cdots$$

**Solution** Since  $t_1 = 9$  and  $r = -\frac{2}{3}$ ,

$$S = \frac{t_1}{1 - r} = \frac{9}{1 - \left(-\frac{2}{3}\right)} = \frac{27}{5}.$$

**Example 2** For what values of  $x$  does the following infinite series converge?

$$1 + (x - 2) + (x - 2)^2 + (x - 2)^3 + \cdots$$

**Solution** This is an infinite geometric series with  $r = x - 2$ . By the theorem on page 500, the series converges when  $|r| < 1$ ; that is, when  $|x - 2| < 1$ , or

$$1 < x < 3.$$

This interval  $1 < x < 3$  for which the series converges is called the *interval of convergence* for the series.

Our final example on the next page illustrates two important facts about repeating decimals. First, they can be written as infinite geometric series, and second, they represent rational numbers.

**Example 3**

The infinite, repeating decimal  $0.454545 \dots$  can be written as the infinite series

$$0.45 + 0.0045 + 0.000045 + \dots$$

What is the sum of this series?

**Solution**

This is a geometric series with  $t_1 = 0.45$  and  $r = 0.01$ . Therefore,

$$S = \frac{t_1}{1 - r} = \frac{0.45}{1 - 0.01} = \frac{0.45}{0.99} = \frac{5}{11}.$$

**CLASS EXERCISES**

For each infinite geometric series, find  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$ . Also, find the sum of the series if it converges.

- $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$
- $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$
- $1 + 3 + 9 + 27 + \dots$
- $1 + 0.1 + 0.01 + 0.001 + \dots$
- Find the interval of convergence for each series.
  - $1 + x + x^2 + x^3 + \dots$
  - $1 + 2x + 4x^2 + 8x^3 + \dots$
- Express  $0.3333 \dots$  as an infinite geometric series. For this series, determine (a)  $t_1$ , (b)  $r$ , and (c) the sum.
- Discussion** Consider any infinite arithmetic series for which  $t_1 \neq 0$  and  $d \neq 0$ . Explain why this series diverges.
- Discussion** Do you think the series  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots + \frac{n}{n+1} + \dots$  converges or diverges? Tell why.

**WRITTEN EXERCISES**

In Exercises 1–8, find the sum of the given infinite geometric series.

- A**  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$
- $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$
- $24 - 12 + 6 - 3 + \dots$
- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$
- $5 + 5^{-1} + 5^{-3} + \dots$
- $\sqrt{27} + \sqrt{9} + \sqrt{3} + \dots$
- $t_1 + t_2 + \dots + t_n + \dots$ ,  
where  $t_n = 8(5)^{-n}$
- $t_1 + t_2 + \dots + t_n + \dots$ ,  
where  $t_n = (-2)^{1-n}$
- Find the common ratio of an infinite geometric series with sum 8 and first term 4.
- Find the first three terms of an infinite geometric series with sum 81 and common ratio  $\frac{1}{3}$ .



11. What is the value of  $x$  if the series  $1 + 2x + 4x^2 + \cdots$  converges to  $\frac{3}{5}$ ?
12. What is the value of  $x$  if the series  $x^2 - x^3 + x^4 - \cdots$  converges to  $\frac{x}{5}$ ?

For each infinite geometric series, find (a) the interval of convergence and (b) the sum, expressed in terms of  $x$ .

13.  $1 + x^2 + x^4 + x^6 + \cdots$
14.  $1 + 3x + 9x^2 + \cdots$
15.  $1 + (x - 3) + (x - 3)^2 + \cdots$
16.  $1 - (x - 1) + (x - 1)^2 - \cdots$
17.  $1 - \frac{2}{x} + \frac{4}{x^2} - \frac{8}{x^3} + \cdots$
18.  $\frac{x^2}{3} - \frac{x^4}{6} + \frac{x^6}{12} - \cdots$
19. Show that the series  $\sin^2 x + \sin^4 x + \sin^6 x + \cdots$  converges to  $\tan^2 x$  if  $x \neq \frac{\pi}{2} + n\pi$ .
20. a. Show that the series  $\tan^2 x - \tan^4 x + \tan^6 x - \cdots$  converges to  $\sin^2 x$  if  $-\frac{\pi}{4} < x < \frac{\pi}{4}$ .
- b. Are there other values of  $x$  for which the series converges?
21. **Writing** Explain why there is no infinite geometric series with first term 10 and sum 4.
22. **Writing** Explain why the sum of an infinite geometric series is positive if and only if the first term is positive.

In Exercises 23–28, use the method of Example 3 to express the given repeating decimal as a rational number.

23. 0.777 ...
24. 0.636363 ...
25. 44.444 ...
26. 5.363636 ...
27. 0.142857142857 ...
28. 0.0123123123 ...

For the series in Exercises 29–32, find the first four partial sums,  $S_1, S_2, S_3, S_4$ , and suggest a formula for  $S_n$ . Then find the sum of the infinite series by evaluating  $\lim_{n \rightarrow \infty} S_n$ . Note that since these series are not geometric, you cannot use the sum formula given on page 500.

- B** 29.  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$
30.  $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} + \cdots$
31.  $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2)(3n+1)} + \cdots$
32.  $\frac{3}{1 \cdot 4} + \frac{5}{4 \cdot 9} + \frac{7}{9 \cdot 16} + \cdots + \frac{2n+1}{n^2(n+1)^2} + \cdots$

33. **Physics** A ball is dropped from a height of 8 m. Each time it hits the ground, it rebounds  $\frac{3}{4}$  of the distance it has fallen. In theory, how far will the ball travel before coming to rest?
34. Repeat Exercise 33 for a ball dropped from a height of 10 m if it rebounds 95% of the distance it falls each time.
35. Each side of a square has length 12. The midpoints of the sides of the square are joined to form another square, and the midpoints of this square are joined to form still another square. If this process is continued indefinitely, find (a) the sum of the areas of all the squares and (b) the sum of the perimeters.
36. Each side of an equilateral triangle has length 12. The midpoints of the sides of the triangle are joined to form another equilateral triangle, and the midpoints of this triangle are joined to form still another triangle. If this process is continued indefinitely, find (a) the sum of the areas of all the triangles and (b) the sum of the perimeters.
37.  $S_n$  is the  $n$ th partial sum and  $S$  is the limit of the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

What is the smallest value of  $n$  for which  $S - S_n < 0.0001$ ?

38. Repeat Exercise 37 for the geometric series  $1 + \frac{2}{3} + \frac{4}{9} + \cdots$ .
39. Here is an old paradox: Achilles races a turtle who has a 100-meter head start. If Achilles runs 10 m/s and the turtle only 1 m/s, when will Achilles overtake the turtle?
- Erroneous Solution* When Achilles covers the 100-meter head start, the turtle has moved 10 m ahead. And when Achilles covers this 10 m, the turtle has moved 1 m ahead. Every time Achilles runs to where the turtle was, the turtle has moved ahead. Thus, Achilles can *never* catch the turtle.
- Correct Solution:* Let  $t_1$  = the time for Achilles to cover the 100-meter head start; let  $t_2$  = the time to cover the next 10 m; let  $t_3$  = the time for the next 1 m, and so on. Find the first three terms and the sum of the infinite series  $t_1 + t_2 + t_3 + \cdots$ .
40. **Writing** Comment on the following paradox: You can *never* leave the room in which you are sitting because in order to do so, you must first walk halfway to the door. Then you must walk half the remaining distance to the door, and then half the next remaining distance. Since you must continue to cover the halves of these remaining distances an infinite number of times, you can never leave the room.

**Exercises 41–44 deal with sequences and series of complex numbers and should be done sequentially.**

41. In an arithmetic sequence of complex numbers,  $t_1 = 1 + 3i$  and  $t_2 = 3 + 4i$ . Find the next three terms. Also find  $t_{25}$  and  $S_{25}$ .

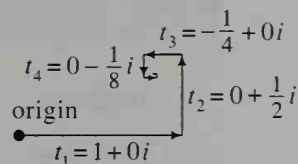
42. In a geometric sequence of complex numbers,  $t_1 = i$  and the common ratio is  $r = 2i$ . Find the next four terms of the sequence and represent all five terms graphically in an Argand diagram. Find the sum of these terms.
43. The formula for the sum of an infinite geometric series given on page 500 holds for complex numbers as well as for real numbers. Use this formula to find the sum of the following series.

a.  $1 + \frac{i}{2} + \frac{i^2}{4} + \frac{i^3}{8} + \frac{i^4}{16} + \cdots$

b.  $27 - 9i + 3i^2 - i^3 + \cdots$

44. A bug leaves the origin and crawls 1 unit east,  $\frac{1}{2}$  unit north,  $\frac{1}{4}$  unit west,  $\frac{1}{8}$  unit south, and so forth, as shown. Each segment of its journey can be considered as a complex number or vector. Hence, the bug's ultimate destination can be considered as the sum

$$S = t_1 + t_2 + t_3 + t_4 + \cdots$$



of an infinite geometric series.

- a. Find  $r$  for this series.

- b. Use the formula  $S = \frac{t_1}{1 - r}$  to show that  $S = \frac{4}{5} + \frac{2}{5}i$ . Thus, the ultimate destination point is  $\left(\frac{4}{5}, \frac{2}{5}\right)$ .

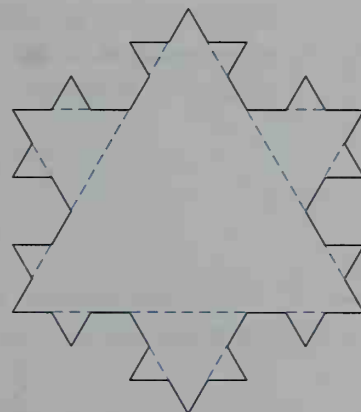
- c. How far must the bug crawl to reach its ultimate destination?

- C** 45. Consider an infinite geometric series of positive terms that converges to  $S$ . What fractional part of  $S$  is the sum of the odd-numbered terms? the sum of the even-numbered terms? (Give answers in terms of  $r$ , the common ratio of the series.)

46. a. Prove that every rational number  $\frac{a}{b}$  can be expressed as a repeating decimal. (For example,  $\frac{2}{11} = 0.181818 \dots$  and  $\frac{1}{4} = 0.25000 \dots$  or  $0.24999 \dots$ )
- b. Prove that every repeating decimal represents a rational number.

**Exercises 47 and 48 refer to the “snowflake curve” defined as follows: The sides of an equilateral triangle are trisected. A new equilateral triangle is placed on the middle third of each trisection. The sides common to the previous figure and the new triangles are then removed. This process continues indefinitely using the sides of the last figure obtained.**

47. Find the area enclosed by the snowflake curve if each side of the initial equilateral triangle is one unit in length.
48. Show that the limit of the sequence of perimeters of the snowflake curve is infinite.



## ■■■■ CALCULATOR EXERCISE

The infinite series  $\frac{1}{5} + \frac{2}{25} + \frac{4}{125} + \frac{8}{625} + \cdots$  converges to  $\frac{1}{3} = 0.3333 \dots$ .

Determine how many terms of the series must be added to make the sum:

- correct in at least the first 2 decimal places, that is 0.33.
- correct in at least the first 3 decimal places, that is 0.333.
- correct in at least the first 4 decimal places, that is 0.3333.

## 13-6 Sigma Notation

**Objective** To represent series using sigma notation.

The Greek letter  $\Sigma$  (**sigma**) is often used in mathematics to express a series or its sum in abbreviated form. For example,  $\sum_{k=1}^{100} k^2$  represents the series whose terms are

obtained by evaluating  $k^2$  first for  $k = 1$ , then for  $k = 2$ , then for  $k = 3$ , and so on until  $k = 100$ . That is,

$$\sum_{k=1}^{100} k^2 = 1^2 + 2^2 + 3^2 + \cdots + 100^2.$$

The symbol on the left above may be read as “the sum of  $k^2$  for values of  $k$  from 1 to 100.” Similarly, the symbol  $\sum_{k=5}^{10} 3k$  may read as “the sum of  $3k$  for values of  $k$

from 5 to 10.” This symbol represents the series whose terms are obtained by evaluating  $3k$  first for  $k = 5$ , then for  $k = 6$ , and so on, to  $k = 10$ . That is,

$$\sum_{k=5}^{10} 3k = 3 \cdot 5 + 3 \cdot 6 + 3 \cdot 7 + 3 \cdot 8 + 3 \cdot 9 + 3 \cdot 10 = 135.$$

The symbol  $3k$  is called the **summand**, the numbers 5 and 10 are called the **limits of summation**, and the symbol  $k$  is called the **index**. Any letter can be used for the index. For example:

$$\begin{aligned} \sum_{n=1}^5 \left(-\frac{1}{2}\right)^n &= \left(-\frac{1}{2}\right)^1 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^4 + \left(-\frac{1}{2}\right)^5 \\ &= -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} = -\frac{11}{32} \end{aligned}$$

Sigma notation can also be used to represent an infinite series and its sum. For example:

$$\begin{aligned} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j &= \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 \end{aligned}$$

The symbol  $\sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j$  is read “the sum of  $\left(\frac{1}{2}\right)^j$  for values of  $j$  from 0 to infinity”; it represents both the infinite geometric series at the bottom of page 506 and its sum, 2.

When you get used to using sigma notation, you will find it much easier than manipulating series that are written in expanded form. The following properties are consequences of the commutative, associative, and distributive properties of the real and imaginary numbers. These properties are proved in Exercises 29 and 30.

### Properties of Finite Sums

$$1. \sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$2. \sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

The properties above can be used together with previously derived sums to derive the sums of many other series. Example 1 illustrates how these properties may be used with the following known sums.

sum of integers:  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$  (Proved in Section 13-3)

sum of squares:  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$  (To be proved in Section 13-7)

sum of cubes:  $\sum_{k=1}^n k^3 = \left[ \frac{n(n+1)}{2} \right]^2$  (To be proved in Section 13-7)

#### Example

Express  $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + 199 \cdot 200$  in sigma notation and evaluate.

#### Solution

$$\begin{aligned} 1 \cdot 2 + 3 \cdot 4 + \cdots + 199 \cdot 200 &= \sum_{k=1}^{100} (2k-1)(2k) \\ &= \sum_{k=1}^{100} (4k^2 - 2k) \\ &= 4 \sum_{k=1}^{100} k^2 - 2 \sum_{k=1}^{100} k \\ &= 4 \left[ \frac{100 \cdot 101 \cdot 201}{6} \right] - 2 \left[ \frac{100 \cdot 101}{2} \right] \\ &= 1,353,400 - 10,100 = 1,343,300 \end{aligned}$$



## CLASS EXERCISES

Give each series in expanded form.

1.  $\sum_{k=1}^4 5k$

2.  $\sum_{k=3}^6 k^2$

3.  $\sum_{j=2}^8 (-1)^j$

4.  $\sum_{n=1}^{\infty} \frac{1}{n}$

Express each of the following series using sigma notation.

5.  $4 + 9 + 16 + 25 + 36$

6.  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

7. The arithmetic series:  $3 + 6 + 9 + \cdots + 300$

8. The infinite geometric series:  $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

## WRITTEN EXERCISES

In Exercises 1–8, write the given series in expanded form.

**A** 1.  $\sum_{k=2}^6 k$

2.  $\sum_{k=1}^{10} 5k$

3.  $\sum_{k=1}^5 \frac{1}{k}$

4.  $\sum_{n=3}^7 (4n - 7)$

5.  $\sum_{n=0}^{\infty} 3^{1-n}$

6.  $\sum_{j=2}^{\infty} j(-1)^j$

7.  $\sum_{t=-2}^2 4^t$

8.  $\sum_{s=0}^6 |13 - 3s|$

In Exercises 9–16, express the given series using sigma notation.

9.  $4 + 8 + 12 + 16 + 20$

10.  $1 + 2 + 4 + 8 + 16 + 32$

11.  $5 + 9 + 13 + \cdots + 101$

12.  $2 + 4 + 6 + 8 + \cdots + 200$

13.  $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

14.  $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$

15.  $\sin x + \sin 2x + \sin 3x + \cdots$

16.  $48 + 24 + 12 + 6 + \cdots$

17. Show that  $\sum_{t=1}^4 \log t = \log 24$ .

18. Show that  $\sum_{k=1}^4 k \log 2 = \log 2^{10}$ .

19. Evaluate:  $\sum_{k=1}^{100} \cos k\pi$

20. Evaluate:  $\sum_{k=1}^{50} \sin \left( k \cdot \frac{\pi}{2} \right)$

**B** 21. Evaluate: a.  $\sum_{n=1}^8 \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right)^n$

b.  $\sum_{n=1}^8 \left| \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right)^n \right|$

22. Evaluate:  $\sum_{n=1}^{\infty} \left( \frac{1}{2} + \frac{1}{2}i \right)^n$  (Hint: This is a geometric series with  $|r| < 1$ .)

23. Show that  $\sum_{k=1}^8 (-1)^k \log k = \log \left( \frac{2 \cdot 4 \cdot 6 \cdot 8}{1 \cdot 3 \cdot 5 \cdot 7} \right)$  24. Evaluate  $\sum_{k=1}^{100} \lfloor \sqrt{k} \rfloor$ , where  $\lfloor k \rfloor$  is the greatest integer of  $k$ .

In Exercises 25–28, express the given series using sigma notation.

25. a.  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$  b.  $-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}$   
 26. a.  $27 - 9 + 3 - 1 + \frac{1}{3} - \frac{1}{9}$  b.  $-27 + 9 - 3 + 1 - \frac{1}{3} + \frac{1}{9}$   
 27.  $1 - 3 + 5 - 7 + \cdots - 99$  28.  $-2 + 4 - 6 + \cdots + 100$

Show how the commutative, associative, and distributive properties are used to prove each of the following.

29.  $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$  30.  $\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$

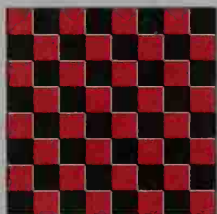
Use the properties of finite sums to prove each of the following.

31.  $\sum_{i=1}^n (a_i + b_i)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i=1}^n a_i b_i + \sum_{i=1}^n b_i^2$   
 32.  $\sum_{x=1}^n (ax^2 + bx + c) = a \sum_{x=1}^n x^2 + b \sum_{x=1}^n x + cn$

Evaluate the following series using the method shown in the example.

33.  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 100 \cdot 101$   
 34.  $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + 99 \cdot 101$   
 35.  $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + 20 \cdot 21 \cdot 22$   
 36.  $2 \cdot 4 + 6 \cdot 8 + 10 \cdot 12 + \cdots + 398 \cdot 400$

- C** 37. a. An 8-by-8 checkerboard has 64 little squares and many other squares of various sizes. How many squares does it have in all?  
 b. Find the total number of squares in an  $n$ -by- $n$  checkerboard.

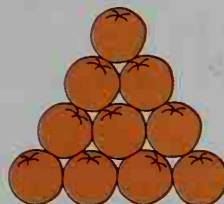


38. a. The 3-by-3-by-3 cube shown has 27 little cubes but also many other cubes of various sizes. How many cubes does it have in all?



- b. Find the total number of cubes in an  $n$ -by- $n$ -by- $n$  cube.

39. A stack of oranges is compactly arranged so the bottom layer consists of oranges in an equilateral triangle with  $n$  oranges on a side. The layer next to the bottom consists of an equilateral triangle of oranges with  $n - 1$  oranges on a side. This pattern continues upward with one orange on the top. How many oranges are there?



Fourth layer

### //// COMPUTER EXERCISE

The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges, but very slowly. Find how many terms are needed to make the partial sum  $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$  greater than: (a) 3; (b) 4; (c) 10. (The answer is surprising!)

## 13-7 Mathematical Induction

### Objective

To use mathematical induction to prove that a statement is true.

Observe the pattern in the following statements:

$$\frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$

It appears that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Nevertheless, this appearance does not constitute a *proof* that the statement is true for all positive integers  $n$ . One way to prove the statement true is to use a method called **mathematical induction**.

## Proof by Mathematical Induction

Let  $S$  be a statement in terms of a positive integer  $n$ .

Step 1. Show that  $S$  is true for  $n = 1$ .

Step 2. Assume that  $S$  is true for  $n = k$ , where  $k$  is a positive integer, and then prove that  $S$  must be true for  $n = k + 1$ .

If you can do both steps (1) and (2) above, then you can conclude that  $S$  is true for all positive integers. The reason is that once you know that  $S$  is true for  $n = 1$ , step (2) tells you it is true for  $n = 1 + 1 = 2$ . Applying step (2) again,  $S$  must be true for  $n = 2 + 1 = 3$ , then for  $n = 3 + 1 = 4$ , and so on.

You might think of mathematical induction as something like setting up dominoes so they will all fall down. Set up the dominoes so that if any one domino falls, the next domino will fall. Then knock over the first domino, which will knock over the second, which will knock over the third, and so on. Thus, you can conclude that all the dominoes will fall down.



**Example 1** Prove that  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$  for all positive integers  $n$ .

### Solution

**Step 1** Show that the statement is true for  $n = 1$ .

$$\frac{1}{1(1+1)} = \frac{1}{1+1}. \text{ Yes, it is true.}$$

**Step 2** Assume that the statement is true for  $n = k$ , and then prove that it must be true for  $n = k + 1$ .

$$\text{Assume: } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

*Prove:*

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)[(k+1)+1]} &= \frac{k+1}{(k+1)+1}, \text{ or} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k+1}{k+2} \end{aligned}$$

*Basic strategy* at this stage of proof: Show that the left side of the “Prove” statement is equal to the right side. To do this, take the left side and try to simplify it by using the commutative, associative, and distributive properties and the “Assume” statement. This is shown in the proof on the next page.



**Proof:**

$$\begin{aligned}
 & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
 & \quad \text{Use the "Assume" statement.} \\
 &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
 &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
 &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\
 &= \frac{(k+1)^2}{(k+1)(k+2)} \\
 &= \frac{k+1}{k+2}
 \end{aligned}$$

## Example 2

Prove that  $n^3 + 2n$  is a multiple of 3 for all positive integers  $n$ .

## Solution

**Step 1** Show that the statement is true for  $n = 1$ .

$$1^3 + 2 \cdot 1 = 3 \text{ is a multiple of 3. Yes, it is true.}$$

**Step 2** Assume that the statement is true for  $n = k$ , and then prove that it must be true for  $n = k + 1$ .

*Assume:*  $k^3 + 2k$  is a multiple of 3.

*Prove:*  $(k+1)^3 + 2(k+1)$  is a multiple of 3.

**Proof:** Use the same basic strategy as in Example 1.

$$\begin{aligned}
 (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
 &= \underbrace{(k^3 + 2k)}_{\text{Use the assumed statement.}} + \underbrace{3(k^2 + k + 1)}_{\text{a multiple of 3}} \\
 &= \text{a multiple of 3} + \text{a multiple of 3} \\
 &= \text{a multiple of 3}
 \end{aligned}$$

In the last line of the proof, we have used the fact that the sum of two multiples of 3, say  $3i$  and  $3j$ , is another multiple of 3, namely  $3(i+j)$ .



## WRITTEN EXERCISES

Use mathematical induction to prove the statements in these exercises. *Note that some of these statements can also be proved by other methods.*

**A** 1.  $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

2.  $1 + 3 + \cdots + (2n-1) = n^2$

3.  $\sum_{i=1}^n 2i = n^2 + n$

4.  $\sum_{i=1}^n 2^{i-1} = 2^n - 1$

5.  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

6.  $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$

7.  $\sum_{i=1}^n (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

8.  $(1+x)^n \geq 1 + nx$ , where  $x > -1$

9.  $\sum_{i=1}^n (i \cdot 2^{i-1}) = 1 + (n-1) \cdot 2^n$

10.  $18^n - 1$  is a multiple of 17.

11.  $11^n - 4^n$  is a multiple of 7.

12.  $n(n^2 + 5)$  is a multiple of 6.

For Exercises 13 and 14, decide which values of  $n$  will make the inequality true and then prove your answer is correct.

**B** 13.  $2^n > 2n$

14.  $n! > 2^n$

15. If, in a room with  $n$  people ( $n \geq 2$ ), every person shakes hands once with every other person, prove that there are  $\frac{n^2 - n}{2}$  handshakes.

16. Prove that the sum of the cubes of any three consecutive positive integers is a multiple of 9.

17. Use the triangle inequality (see Exercise 36 on page 99) to prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

18. Prove De Moivre's theorem (page 408) for every positive integer  $n$ .

In Exercises 19–21, write out the first few terms of each sequence. Then suggest and prove a formula in terms of  $n$  for the  $n$ th term,  $a_n$ .

19.  $a_1 = \frac{1}{2}$ ,  $a_n = a_{n-1} - \frac{a_{n-1}}{n+1}$

20.  $a_1 = 1$ ,  $a_n = 2a_{n-1} + 1$

21.  $a_1 = \frac{1}{4}$ ,  $a_n = a_{n-1} + \frac{1}{(3n-2)(3n+1)}$



In Exercises 22–24, write out the first few terms of each sequence. Then suggest and prove a formula in terms of  $n$  for the  $n$ th partial sum,  $S_n$ .

**C** 22.  $S_n = \sum_{i=1}^n i^3$

23.  $S_n = \sum_{i=1}^n (i \cdot i!)$  (Hint: Look at the sequence of factorials of integers.)

24.  $S_n = \sum_{i=1}^n i(i+1)(i+2)$  (Hint: See Exercises 1 and 6.)

25. Prove that a convex polygon with  $n$  sides ( $n \geq 3$ ) has  $\frac{n^2 - 3n}{2}$  diagonals.

26. Prove that it is possible to pay any debt of \$4, \$5, \$6, \$7, ..., \$ $n$ , and so on, by using only \$2 bills and \$5 bills. For example, a debt of \$11 can be paid with three \$2 bills and one \$5 bill, or  $11 = 3 \cdot 2 + 1 \cdot 5$ .

## Chapter Summary

1. A *sequence* is a function whose domain is the set of positive integers. A sequence can be defined explicitly by a formula such as  $t_n = 3n + 6$  or recursively by a pair of formulas, such as  $t_1 = 6$  and  $t_n = 7 - t_{n-1}$ .
2. A sequence is *arithmetic* if the difference  $d$  of any two consecutive terms is constant. A sequence is *geometric* if the ratio  $r$  of any two consecutive terms is constant.
3. A series is an indicated sum of the terms of a sequence. The formulas on pages 487 and 488 can be used to find the sums of finite arithmetic series and finite geometric series, respectively.
4. If the  $n$ th term of a sequence becomes arbitrarily close to a number  $L$  as  $n$  gets larger and larger, we write  $\lim_{n \rightarrow \infty} t_n = L$ . If there is no single limiting number  $L$  for all terms of the sequence, we say that the sequence has no limit. A sequence whose terms increase (or decrease) without bound is said to have a limit of  $\infty$  (or  $-\infty$ ).
5. Associated with any infinite series  $t_1 + t_2 + t_3 + \cdots$ , there is a *sequence of partial sums*:

$$S_1 = t_1, S_2 = t_1 + t_2, S_3 = t_1 + t_2 + t_3, \text{ and so on.}$$

If this sequence has a limit  $L$ , we say that the series *converges* to the sum  $L$ .

If the sequence of partial sums has no limit, the series *diverges*.

6. The infinite geometric series  $t_1 + t_1 r + t_1 r^2 + \cdots$  converges to the sum

$$S = \frac{t_1}{1 - r} \text{ if } |r| < 1. \text{ The series diverges if } |r| \geq 1 \text{ and } t_1 \neq 0.$$

7. Any infinite repeating decimal can be written as an infinite geometric series whose sum is a rational number.

8. The Greek letter  $\Sigma$  (sigma) is often used to express a series or its sum. Some properties of finite sums are given on page 507.
9. *Mathematical induction*, used to prove that a statement  $S$  is true for all positive integers, involves two steps: (1) Show that  $S$  is true for  $n = 1$ , and (2) prove that  $S$  is true for  $n = k + 1$  whenever it is true for  $n = k$ .

### Key vocabulary and ideas

sequence (p. 473, p. 474)	sequence of partial sums (p. 500)
arithmetic and geometric sequences (p. 473)	sum of an infinite series (p. 500)
recursive and explicit definitions (p. 479)	converge, diverge (p. 500)
series, sum of a series (p. 486)	sigma notation (p. 506)
limit of a sequence (p. 493)	mathematical induction (p. 510)

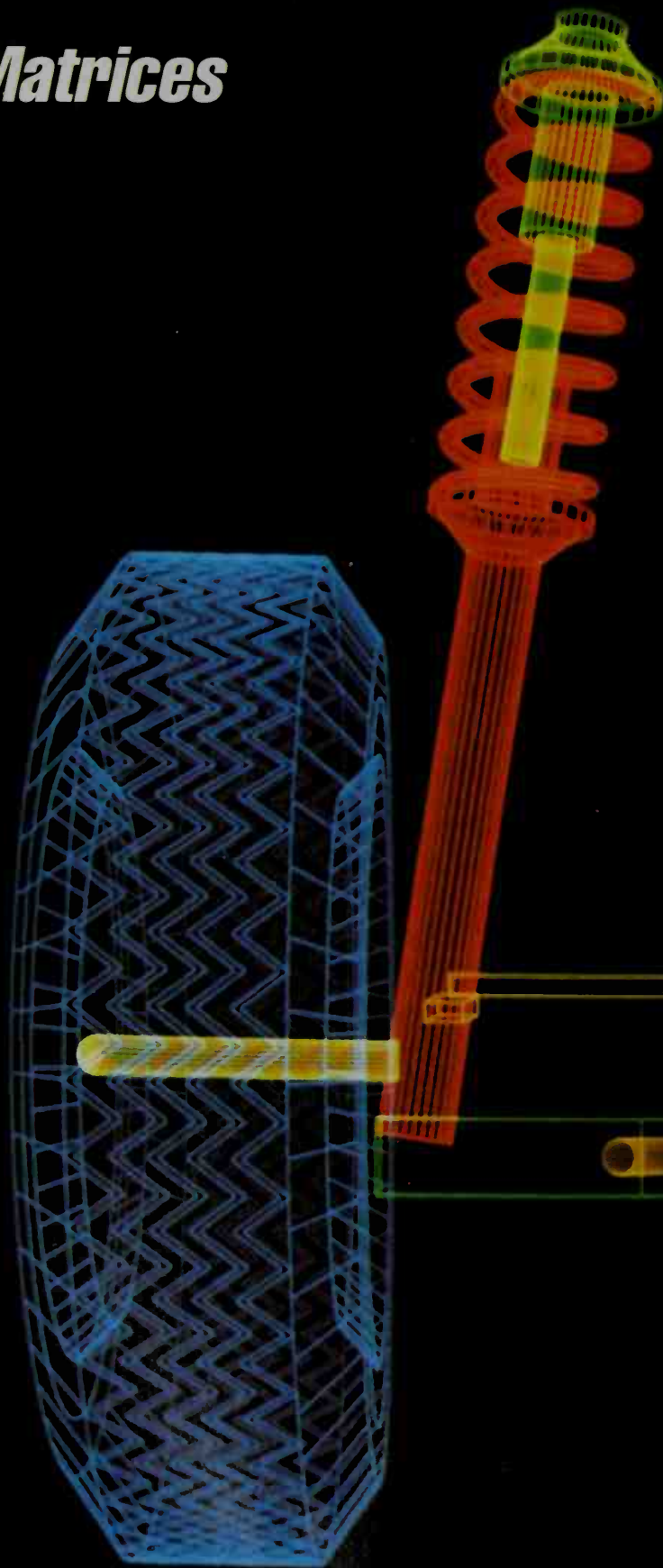
## Chapter Test

1. State whether each sequence is arithmetic, geometric, or neither, and find a formula for  $t_n$  in terms of  $n$ . 13-1  
 a. 10, 6, 2, -2, ...    b. 3, 6, 11, 18, 27, ...    c. -2, 6, -18, 54, ...
2. In an arithmetic sequence,  $t_2 = 6$  and  $t_6 = 16$ . Find  $t_{21}$ .
3. In a geometric sequence,  $t_3 = 9$  and  $t_6 = \frac{9}{8}$ . Find  $t_{12}$ .
4. In a sequence,  $t_1 = 1$ ,  $t_2 = 4$ , and  $t_n = 2t_{n-1} + t_{n-2}$ . Find  $t_6$ . 13-2
5. Give a recursive definition for the sequence 1, 5, 21, 85, ...
6. For the sequence of positive 3-digit integers ending in 4, find an explicit formula for (a)  $t_n$  and (b)  $S_n$ . 13-3
7. Find the sum of the first 6 terms of the series  $27 - 9 + 3 - 1 + \cdots$ .
8. *Writing* Briefly tell what the word “limit” means in mathematics. 13-4
9. Find: a.  $\lim_{n \rightarrow \infty} \frac{3n^2 + 1}{4n^2 - 2n + 5}$     b.  $\lim_{n \rightarrow \infty} n \cos n\pi$     c.  $\lim_{n \rightarrow \infty} (0.59)^n$
10. Express 0.131313... as an infinite series and as a rational number. 13-5
11. Find (a) the interval of convergence and (b) the sum, expressed in terms of  $x$ , for the series  $1 + \frac{3x}{2} + \frac{9x^2}{4} + \cdots$ .
12. Express each of the following series using sigma notation. 13-6  
 a.  $1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81}$     b.  $7 + 3 - 1 - 5 - 9 - 13 - 17$
13. Evaluate  $\sum_{k=1}^{20} 3k(k+2)$ .
14. Prove by mathematical induction: 13-7  

$$2 + 7 + 12 + \cdots + (5n - 3) = \frac{n(5n - 1)}{2}$$

14

# *Matrices*



# 14-1 Matrix Addition and Scalar Multiplication

**Objective** To find the sum, difference, or scalar multiples of matrices.

An automobile dealer sells four different models whose fuel economy is shown below.

	Sports car	Sedan	Station wagon	Van
Miles per gallon for city driving	17	22	17	16
Miles per gallon for highway driving	23	30	24	19

The information in this table can be displayed as a rectangular array of numbers enclosed by brackets, as shown below. Such an array is called a **matrix** (plural, *matrices*). A matrix is usually named by a capital letter. In this example,  $E$  is used to name the matrix.

$$E = \begin{bmatrix} 17 & 22 & 17 & 16 \\ 23 & 30 & 24 & 19 \end{bmatrix}$$

Each number in a matrix is an **element** (or **entry**) of the matrix. The **dimensions** of a matrix are the number of rows and columns. Matrix  $E$  has two rows and four columns, so we say that  $E$  is a  $2 \times 4$  (read “two by four”) matrix, denoted by  $E_{2 \times 4}$ . (The number of rows is first and the number of columns is second.)

Sometimes it is helpful to label the rows and columns to remind us of what each represents. In the matrix below, the row labels remind us of the city and highway driving conditions and the column labels remind us of the model. We say that the fuel economy matrix,  $E$ , is a “driving-condition by model” matrix.

$$\begin{matrix} & \text{sp} & \text{se} & \text{sw} & \text{v} \\ \begin{matrix} \text{c} \\ \text{h} \end{matrix} & \begin{bmatrix} 17 & 22 & 17 & 16 \\ 23 & 30 & 24 & 19 \end{bmatrix} & = & E \end{matrix}$$

Of course, the same information could be given by interchanging the rows and columns of  $E$ . Matrix  $F$ , shown on the next page, is called the **transpose** of matrix  $E$ , and is denoted by  $E^t$ .



Computer-aided design allows us to model three-dimensional objects on a two-dimensional screen. We can adjust the size, shape, and orientation of parts without the time and expense of constructing physical prototypes.



$$\begin{matrix} & \text{c} & \text{h} \\ \text{sp} & 17 & 23 \\ \text{se} & 22 & 30 \\ \text{sw} & 17 & 24 \\ \text{v} & 16 & 19 \end{matrix} = F = E^t$$

Notice that the rows of  $F$  are the same as the columns of  $E$ . The dimensions of  $F$  are  $4 \times 2$  (a “model by driving-condition” matrix).

Suppose that the Environmental Protection Agency (EPA) mandates that all of these fuel performance figures must increase 10% by the year 1998. This means that every element of matrix  $E$  must be multiplied by 1.10. The resulting matrix is denoted by  $1.1E$ .

$$\begin{aligned} 1.1E &= 1.1 \begin{bmatrix} 17 & 22 & 17 & 16 \\ 23 & 30 & 24 & 19 \end{bmatrix} = \begin{bmatrix} (1.1)17 & (1.1)22 & (1.1)17 & (1.1)16 \\ (1.1)23 & (1.1)30 & (1.1)24 & (1.1)19 \end{bmatrix} \\ &= \begin{bmatrix} 18.7 & 24.2 & 18.7 & 17.6 \\ 25.3 & 33 & 26.4 & 20.9 \end{bmatrix} \end{aligned}$$

The operation of multiplying a matrix  $A$  by a real number  $c$  is called **scalar multiplication**. (In matrix algebra any real number is called a **scalar**.) The new matrix,  $cA$ , is the result of multiplying each element in  $A$  by  $c$ .

**Example 1** If  $M = \begin{bmatrix} 3 & 1 & 5 \\ 4 & 0 & -2 \end{bmatrix}$ , find  $M^t$ ,  $2M$ , and  $-3M$ .

**Solution**  $M^t = \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 5 & -2 \end{bmatrix}$  ← Interchange the rows and columns of  $M$  to get the transpose of  $M$ .

$$2M = \begin{bmatrix} 2(3) & 2(1) & 2(5) \\ 2(4) & 2(0) & 2(-2) \end{bmatrix} = \begin{bmatrix} 6 & 2 & 10 \\ 8 & 0 & -4 \end{bmatrix}$$

$$-3M = \begin{bmatrix} -3(3) & -3(1) & -3(5) \\ -3(4) & -3(0) & -3(-2) \end{bmatrix} = \begin{bmatrix} -9 & -3 & -15 \\ -12 & 0 & 6 \end{bmatrix}$$

Two matrices are *equal* if and only if they have the same dimensions, and the elements in all corresponding positions (same row, same column) are equal. Matrices of different dimensions can never be equal.

Two matrices having the same dimensions can be added to produce a new matrix by finding the sums of the corresponding elements of the matrices. This operation is called **matrix addition**. Since corresponding elements are added, two matrices cannot be added if they have different dimensions.

**Matrix subtraction** is similar to real number subtraction: to subtract a matrix, we add the *additive inverse* of the matrix. (The additive inverse of a matrix is defined in Section 14-3.) More simply, two matrices can be subtracted by finding the differences of the corresponding elements of the matrices.

**Example 2**

Let  $A = \begin{bmatrix} 3 & 8 & 1 \\ 4 & 0 & -3 \\ -2 & 1 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 0 & 9 \\ 4 & -6 & -5 \\ 0 & 7 & 2 \end{bmatrix}$ .

Find  $A + B$  and  $A - B$ .

**Solution**

$$A + B = \begin{bmatrix} 3+2 & 8+0 & 1+9 \\ 4+4 & 0+(-6) & -3+(-5) \\ -2+0 & 1+7 & 5+2 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 10 \\ 8 & -6 & -8 \\ -2 & 8 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 3-2 & 8-0 & 1-9 \\ 4-4 & 0-(-6) & -3-(-5) \\ -2-0 & 1-7 & 5-2 \end{bmatrix} = \begin{bmatrix} 1 & 8 & -8 \\ 0 & 6 & 2 \\ -2 & -6 & 3 \end{bmatrix}$$

If two matrices are used to model a real-world application, their sum or difference may not always give a meaningful matrix. For example, the following matrices give baseball statistics for 1981, a year in which the season was divided in half because of a players' strike. Matrices  $A$  and  $B$  give the games won and lost and the winning percentage, expressed as a decimal, for the top three teams in the National League East Division. The teams were the Philadelphia Phillies, the St. Louis Cardinals, and the Montreal Expos.

First half of the season

	W	L	PCT
Phillies	34	21	.618
Cardinals	30	20	.600
Expos	30	25	.545

$$= A$$

Second half of the season

	W	L	PCT
Phillies	25	27	.481
Cardinals	29	23	.558
Expos	30	23	.566

$$= B$$

Notice what happens when  $A$  and  $B$  are added:

	W	L	PCT
Phillies	59	48	1.099
Cardinals	59	43	1.158
Expos	60	48	1.111

$$= A + B$$

Elements in the first two columns make sense because they give the number of wins and losses for the whole season.

The elements in the third column do not make sense because the winning percentage for the whole season is *not* the sum of the percentages for each half of the season. Clearly, no team can win more than 100% of its games.

## CLASS EXERCISES

For Exercises 1–3, use matrices  $A$ ,  $B$ ,  $C$ , and  $D$ .

$$A = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad B = [2 \quad 5] \quad C = [5 \quad 2] \quad D = \left[ 2 \quad \frac{10}{2} \right]$$

1. Give the dimensions of each matrix.
2. Which matrices are transposes of each other?
3. Which matrices are equal to each other?

For Exercises 4–6, use matrices  $A$ ,  $B$ ,  $C$ , and  $D$ .

$$A = \begin{bmatrix} 5 & -1 \\ 3 & -4 \\ 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 7 \\ 2 & 9 \\ -3 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 3 & 1 \\ 4 & 0 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 9 \\ -2 & 3 & 4 \end{bmatrix}$$

4. Give the dimensions of each matrix.
5. Which matrices can be added? Find their sum.
6. a. Give the dimensions of  $A'$  and  $B'$ .  
b. Find  $A'$ .
7. **Discussion** The matrix below gives nutritional information for single servings of tomato juice and orange juice. All elements are given in grams.

	Protein	Carbohydrate	Fat	Sodium	
Tomato juice	2.2	10.4	0.2	0.486	] = $N$
Orange juice	1.5	24.7	0.5	0.002	

- a. Explain the meaning of the number 10.4 in matrix  $N$ .
  - b. Explain the meaning of the number 0.002 in matrix  $N$ .
  - c. Give a real-world meaning for the matrix  $2N$ .
  - d. Matrix  $N$  could be called a “juice-type by nutrient” matrix. How could you form a “nutrient by juice-type” matrix?
8. **Discussion** Explain how a teacher’s grade list could be considered a matrix.

## WRITTEN EXERCISES

Simplify.

**A** 1.  $\begin{bmatrix} -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2.  $[12 \quad 7] + [3 \quad -4]$

3.  $\begin{bmatrix} 8 & 1 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$

4.  $[1 \quad -1 \quad 0] - [6 \quad -9 \quad -1]$

$$5. \begin{bmatrix} 8 & 2 & -2 \\ -3 & 1 & 14 \end{bmatrix} + \begin{bmatrix} 12 & 3 & 10 \\ 0 & 0 & -6 \end{bmatrix}$$

$$6. \begin{bmatrix} 1 & -5 \\ -3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 0 & 7 \\ -8 & 3 \end{bmatrix}$$

$$7. 8 \begin{bmatrix} 5 & -2 \\ 4 & 0 \end{bmatrix}$$

$$8. -4 \begin{bmatrix} 0 & -1 \\ -6 & 5 \end{bmatrix}$$

$$9. 2 \begin{bmatrix} 3 & 0 \\ -4 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 3 & 0 \\ 6 & 11 \end{bmatrix}$$

$$10. \begin{bmatrix} 18 & 12 & 5 \\ 2 & 0 & -3 \end{bmatrix} - 5 \begin{bmatrix} 4 & -1 & 3 \\ -3 & 2 & 5 \end{bmatrix}$$

For Exercises 11–14, use matrices  $A$ ,  $B$ , and  $C$ .

$$A = \begin{bmatrix} 3 & 2 & 4 & 8 \\ 0 & 1 & 8 & -2 \\ 3 & 7 & 9 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 8 & 1 & 4 \\ 3 & 0 & 7 \\ -2 & 5 & 11 \\ 6 & 4 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 9 & 0 & 5 \\ 4 & 3 & 1 \\ 7 & 12 & 2 \\ 6 & 0 & 3 \end{bmatrix}$$

11. a. Give the dimensions of each matrix.  
b. Which sum is defined,  $A + B$  or  $B + C$ ? Find this sum.

12. a. Give the dimensions of  $A'$ .  
b. Find  $A'$ ,  $A' + B$ , and  $C - A'$ .

13. Find  $2B$  and  $2B + C$ .

14. Find  $3C$  and  $3C - B$ .

- B** 15. **Business** The Handy Hardware Company has two locations, one downtown and one at the mall. During April, the downtown store sold 31 of the lowest-priced lawn mowers, 42 of the medium-priced mowers, and 18 of the highest-priced mowers. Also during April, the mall store sold 22 of the lowest-priced mowers, 25 of the medium-priced mowers, and 11 of the highest-priced mowers.
- Represent this information in an April sales matrix  $A$ .
  - Do the dimensions of your matrix  $A$  represent a “mower-type by location” matrix or a “location by mower-type” matrix?
  - Suppose that during May, the Handy downtown store sold 28 of the lowest-priced mowers, 29 of the medium-priced ones and 20 of the highest-priced ones, and that the mall store sold 20 lowest-priced ones, 18 medium-priced ones, and 9 highest-priced ones. Represent this information in a May matrix  $M$  that has the same dimensions as  $A$ .
  - Writing** Find  $A + M$  and describe what this matrix sum tells you.
  - If the manager of the Handy stores expects next year’s lawn mower sales to rise about 8%, about how many highest-priced lawn mowers does the manager expect to sell at the downtown store next April? If you were the manager, would you round your calculations up or down? What scalar multiple of matrix  $A$  would assist you in planning next April’s sales?
  - Suppose the manager estimates that the total lawn mower sales for next April and May will be given by the matrix  $1.09A + 1.15M$ . What sales increases is the manager expecting for next April? for next May?

16. **Manufacturing** The matrix  $S$  gives this week's shipping order for various sofas, chairs, coffee tables, and end tables to be delivered from a warehouse to its store location. Each furniture piece is available in three styles: colonial, traditional, and modern. Matrix  $T$  gives the total living room furniture inventory of the warehouse.



$$\begin{array}{c}
 \text{Styles} \\
 \begin{array}{c} \text{C} \quad \text{T} \quad \text{M} \\ \text{s} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \\ \text{c} \begin{bmatrix} 10 & 2 & 3 \end{bmatrix} \\ \text{ct} \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \\ \text{et} \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \end{array} = S
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Styles} \\
 \begin{array}{c} \text{C} \quad \text{T} \quad \text{M} \\ \text{s} \begin{bmatrix} 12 & 10 & 15 \end{bmatrix} \\ \text{c} \begin{bmatrix} 40 & 15 & 17 \end{bmatrix} \\ \text{ct} \begin{bmatrix} 17 & 42 & 18 \end{bmatrix} \\ \text{et} \begin{bmatrix} 24 & 24 & 24 \end{bmatrix} \end{array} = T
 \end{array}$$

- What is the meaning of the 10 in matrix  $S$ ? of the 40 in matrix  $T$ ?
- Give the dimensions of  $S$  and  $T$ .
- Calculate  $T - S$  and give its real-world significance.
- Because of an upcoming anniversary sale, the number of furniture pieces sold for each model type is expected to be 50% higher next week than this week. Express in terms of  $S$  and  $T$  a matrix that will give the warehouse inventory after next week's order is shipped.

Find the values of the variable for which the given statement is true.

$$17. \begin{bmatrix} x & 3 \\ -7 & 0 \end{bmatrix} = \begin{bmatrix} 2 & y \\ -7 & z + 1 \end{bmatrix} \qquad 18. \begin{bmatrix} a - 3 & 5 \\ 2 - c & 4b \end{bmatrix} = \begin{bmatrix} -1 & 2x + 1 \\ -8 & 12 \end{bmatrix}$$

- For any matrix  $A$ , give a simpler name for the matrix  $(A^t)^t$ .
- Is it true or false that  $(2A)^t = 2A^t$  for any matrix  $A$ ? Give an example to support your answer.
- Is it true that  $(A + B)^t = A^t + B^t$  for any matrices  $A$  and  $B$  that have the same dimensions? Give an example to support your answer.
- Writing** Give a real-world example of two matrices having the same dimensions and whose sum makes no sense. Write a sentence explaining why the resulting matrix is not meaningful.
- Research** Refer to a world almanac. Write a matrix that gives the number of stories and the heights in meters of the five tallest buildings in the United States.
- Research** In a newspaper or magazine, find a table of numbers that can be expressed as a matrix.



## 14-2 Matrix Multiplication

### Objective

To find the product of two matrices.

Suppose a teacher calculates your test average for the term by using a formula that counts, or weighs, each of your five tests a certain percentage of your grade, as shown in matrix  $W$  below. Notice that the weights must add up to 100%.

$$\begin{array}{c} \text{weight} \end{array} \begin{array}{c} \text{Test 1} \quad \text{Test 2} \quad \text{Test 3} \quad \text{Test 4} \quad \text{Test 5} \\ \left[ \begin{array}{ccccc} 15\% & 15\% & 25\% & 15\% & 30\% \end{array} \right] = W \end{array}$$

How would you use these weights to find the weighted score, or *weighted average*, of students A, B, and C, whose test scores are given in matrix  $T$  at the right.

$$\begin{array}{c} \text{Test 1} \\ \text{Test 2} \\ \text{Test 3} \\ \text{Test 4} \\ \text{Test 5} \end{array} \begin{array}{c} \text{A} \quad \text{B} \quad \text{C} \\ \left[ \begin{array}{ccc} 82 & 92 & 74 \\ 85 & 88 & 68 \\ 78 & 95 & 73 \\ 75 & 85 & 82 \\ 84 & 94 & 81 \end{array} \right] = T \end{array}$$

To calculate a student's weighted score (or weighted average), you must multiply each test score by its weight and then add the products.

$$\begin{aligned} \text{A's weighted score} &= 0.15(82) + 0.15(85) + 0.25(78) + 0.15(75) + 0.30(84) \\ &= 81 \end{aligned}$$

$$\begin{aligned} \text{B's weighted score} &= 0.15(92) + 0.15(88) + 0.25(95) + 0.15(85) + 0.30(94) \\ &= 91.70 \approx 92 \end{aligned}$$

$$\begin{aligned} \text{C's weighted score} &= 0.15(74) + 0.15(68) + 0.25(73) + 0.15(82) + 0.30(81) \\ &= 76.15 \approx 76 \end{aligned}$$

The students' weighted scores for the term are shown in matrix  $S$ .

$$\begin{array}{c} \text{A} \quad \text{B} \quad \text{C} \\ \text{weighted score} \left[ \begin{array}{ccc} 81 & 92 & 76 \end{array} \right] = S \end{array}$$

Matrix  $S$  can be thought of as the *matrix product* of matrices  $W$  and  $T$ .

$$\begin{array}{c} W \quad \cdot \quad T \quad = S \\ \swarrow \quad \downarrow \quad \searrow \\ \left[ \begin{array}{ccccc} 0.15 & 0.15 & 0.25 & 0.15 & 0.30 \end{array} \right] \cdot \left[ \begin{array}{ccc} 82 & 92 & 74 \\ 85 & 88 & 68 \\ 78 & 95 & 73 \\ 75 & 85 & 82 \\ 84 & 94 & 81 \end{array} \right] = \left[ \begin{array}{ccc} 81 & 92 & 76 \end{array} \right] \end{array}$$

In the example on the previous page, notice how the dimensions of the matrices agree:  $W$  has five *columns* and  $T$  has five *rows*. Also, notice that the column labels of  $W$  match the row labels of  $T$ .

$$W_{1 \times 5} \cdot T_{5 \times 3} = S_{1 \times 3}$$

weight of **test** • **test score** by student = weighted score by student

In general, the **matrix product** of  $A_{m \times n}$  and  $B_{n \times p}$  is the matrix whose dimensions are  $m \times p$  and whose element in the  $i$ th row and  $j$ th column is the sum of the products of the corresponding elements of the  $i$ th row of  $A$  and the  $j$ th column of  $B$ . From this definition we can see that matrices  $A$  and  $B$  can be multiplied only if the number of columns of  $A$  equals the number of rows of  $B$ .

**Example 1** Find the matrix product  $\begin{bmatrix} 1 & 2 & 0 \\ 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 0 & 4 \\ 1 & -1 \end{bmatrix}$ .

**Solution** First determine if the product is defined. The dimensions are:

$$2 \times 3 \text{ and } 3 \times 2.$$

Since the number of columns in the first matrix equals the number of rows in the second matrix, the two matrices can be multiplied. The matrix product has dimensions  $2 \times 2$ .

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 0 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1(-3) + 2(0) + 0(1) & 1(2) + 2(4) + 0(-1) \\ 3(-3) + (-5)(0) + 2(1) & 3(2) + (-5)(4) + 2(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & -5 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 0 & 4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ -7 & -16 \end{bmatrix}$$

**Example 2** Machine I and Machine II produce items X, Y, and Z at the hourly rate given in matrix  $H$ . Matrix  $D$  gives the number of hours each machine runs during the week.

$$\begin{array}{c} \text{I} \quad \text{II} \\ \text{X} \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 1 & 2 \end{bmatrix} = H \end{array} \quad \begin{array}{c} \text{M} \quad \text{T} \quad \text{W} \quad \text{Th} \quad \text{F} \\ \text{I} \begin{bmatrix} 8 & 8 & 8 & 7 & 7 \\ 6 & 10 & 12 & 11 & 9 \end{bmatrix} = D \end{array}$$

- Give the dimensions of  $H$ ,  $D$ , and  $HD$ .
- Find  $HD$ . What information does  $HD$  give?
- How many Y items are produced on Monday? How many Z items are produced on Thursday?

### Solution

- $H$  is a  $3 \times 2$  matrix and  $D$  is a  $2 \times 5$  matrix. So  $HD$  is a  $3 \times 5$  matrix.

b.

$$HD = \begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 8 & 8 & 8 & 7 & 7 \\ 6 & 10 & 12 & 11 & 9 \end{bmatrix} = \begin{matrix} \text{X} \\ \text{Y} \\ \text{Z} \end{matrix} \begin{matrix} \text{M} & \text{T} & \text{W} & \text{Th} & \text{F} \\ \begin{bmatrix} 36 & 44 & 48 & 43 & 39 \\ 64 & 80 & 88 & 79 & 71 \\ 20 & 28 & 32 & 29 & 25 \end{bmatrix} \end{matrix}$$

$HD$  gives a meaningful result since the column labels of matrix  $H$  match the row labels of matrix  $D$ .  $HD$  is an “item by day” matrix, and so each element gives the quantity of a particular item that is produced on a particular day.

- From the matrix product  $HD$  above, you can see that 64 Y items are produced on Monday and 29 Z items are produced on Thursday.

Notice that in Example 2 the product  $H_{3 \times 2} \cdot D_{2 \times 5}$  is a  $3 \times 5$  matrix. However, the product  $D_{2 \times 5} \cdot H_{3 \times 2}$  is not defined because there are more elements in each row of  $D$  than there are in each column of  $H$ .

## CLASS EXERCISES

In Exercises 1–6, you are given the dimensions of matrices  $A$  and  $B$ . Is  $AB$  defined? Is  $BA$  defined? Give the dimensions of each possible product.

1.  $A_{3 \times 5}$

$B_{5 \times 2}$

2.  $A_{7 \times 6}$

$B_{2 \times 7}$

3.  $A_{1 \times 4}$

$B_{3 \times 1}$

4.  $A_{5 \times 7}$

$B_{7 \times 3}$

5.  $A_{2 \times 4}$

$B_{3 \times 4}$

6.  $A_{3 \times 4}$

$B_{4 \times 3}$

Find each matrix product.

7.  $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 5 & 6 \end{bmatrix}$

8.  $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ -1 & 0 \end{bmatrix}$

9. Matrix  $M$  gives the number of different models sold per month by a certain car dealership for a three month period. Matrix  $C$  gives the costs per model. Which matrix is defined,  $MC$  or  $CM$ ?

10. **Discussion**  $A$  and  $B$  are both  $2 \times 3$  matrices. Can you multiply them? Can you multiply  $A'$  and  $B$ ? How might this be useful?

## WRITTEN EXERCISES

Find each matrix product, if it is defined.

**A** 1.  $\begin{bmatrix} 4 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2.  $\begin{bmatrix} -6 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 12 \\ 0 & -4 \end{bmatrix}$

3.  $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ 0 & 1 \end{bmatrix}$

4.  $\begin{bmatrix} -2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -6 & 5 \end{bmatrix}$

5.  $\begin{bmatrix} 8 & -10 \\ 0 & 3 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -9 \\ 1 \end{bmatrix}$

6.  $\begin{bmatrix} 7 & 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -3 & 8 \\ 9 & 5 \\ -2 & 6 \end{bmatrix}$

7.  $\begin{bmatrix} 9 & -4 & 4 \\ 2 & -1 & -6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -3 \\ 3 & 5 & 2 \end{bmatrix}$

8.  $\begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

9.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

10.  $\begin{bmatrix} 9 & 4 \\ 3 & 1 \\ 2 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$

- B** 11. a. Multiplication of real numbers is commutative. That is,  $ab = ba$  for all real numbers  $a$  and  $b$ . However, multiplication of matrices is *not* commutative. Give an example of two  $2 \times 2$  matrices  $A$  and  $B$  for which  $AB \neq BA$ .  
 b. **Investigation** Is matrix addition commutative? Give examples and write a convincing argument to explain your answer.
12. In Exercise 11, you found matrices  $A$  and  $B$  such that  $AB \neq BA$ . However, it is possible to find other matrices  $C$  and  $D$  such that  $CD = DC$ . Verify this for

$$C = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}.$$

13. a. Multiplication of real numbers is associative. That is,  $a(bc) = (ab)c$  for all real numbers  $a$ ,  $b$ , and  $c$ . Choose three  $2 \times 2$  matrices  $A$ ,  $B$ ,  $C$  and check to see whether  $A(BC) = (AB)C$ .  
 b. Compare your results in part(a) with those of your classmates. Does it appear that matrix multiplication is associative?  
 c. **Investigation** Decide whether matrix addition is associative. Give examples to support your answer.
14. a. Multiplication of real numbers is distributive over addition. That is,  $a(b + c) = ab + ac$  for all real numbers  $a$ ,  $b$ , and  $c$ . Choose three  $2 \times 2$  matrices  $A$ ,  $B$ ,  $C$  and check to see whether  $A(B + C) = AB + AC$ .  
 b. Does  $(B + C)A = BA + CA$ ?

15. **Business** Matrix  $S$  gives the number of three types of cars sold in March by two car dealers, and matrix  $P$  gives the profit for each type of car sold.

$$\begin{array}{l} \text{compact} \\ \text{mid-size} \\ \text{full-size} \end{array} \begin{array}{c} \text{dealer} \\ \begin{matrix} 1 & 2 \end{matrix} \\ \left[ \begin{array}{cc} 18 & 15 \\ 24 & 17 \\ 16 & 20 \end{array} \right] = S \end{array} \quad \begin{array}{c} \text{compact} \quad \text{mid-size} \quad \text{full-size} \\ \text{profit} \left[ \begin{array}{ccc} \$400 & \$650 & \$900 \end{array} \right] = P \end{array}$$

Which matrix is defined,  $SP$  or  $PS$ ? Find this matrix and interpret its elements.

16. **Animal Science** A dog breeder finds that certain brands of dog food contain different amounts of three main nutrients, measured in milligrams per serving, as shown in the matrix  $N$ . The dog breeder decides to mix the brands in order to give the healthiest feeding mixture possible. Matrix  $P$  gives the portion of the mixture for each brand.

$$\begin{array}{l} \text{nutrient 1} \\ \text{nutrient 2} \\ \text{nutrient 3} \end{array} \begin{array}{c} \text{brands} \\ \begin{matrix} W & X & Y & Z \end{matrix} \\ \left[ \begin{array}{cccc} 250 & 480 & 360 & 200 \\ 320 & 510 & 475 & 315 \\ 180 & 200 & 230 & 155 \end{array} \right] = N \end{array} \quad \begin{array}{c} \text{part of mixture} \\ \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} \left[ \begin{array}{c} 40\% \\ 10\% \\ 15\% \\ 35\% \end{array} \right] = P \end{array}$$

- a. Which matrix is defined,  $NP$  or  $PN$ ? Find this matrix.  
b. How many milligrams of nutrient 2 are in a serving of the mixture?
17. **Business** Matrix  $P$  gives the monthly production schedule for three models of calculators. Matrix  $M$  gives the number of components needed to construct each model. Matrix  $R$  gives the number of relays needed for each component.

$$\begin{array}{c} \text{scientific} \\ \text{business} \\ \text{graphing} \end{array} \begin{array}{c} \text{Jan.} \quad \text{Feb.} \quad \text{Mar.} \\ \left[ \begin{array}{ccc} 500 & 600 & 600 \\ 200 & 200 & 200 \\ 100 & 300 & 400 \end{array} \right] = P \end{array}$$

$$\begin{array}{c} \text{scientific} \\ \text{business} \\ \text{graphing} \end{array} \begin{array}{c} \text{components} \\ \begin{matrix} A & B & C \end{matrix} \\ \left[ \begin{array}{ccc} 2 & 1 & 2 \\ 2 & 1 & 4 \\ 2 & 5 & 6 \end{array} \right] = M \end{array}$$

$$\begin{array}{c} \text{relay x} \\ \text{relay y} \end{array} \begin{array}{c} \text{components} \\ \begin{matrix} A & B & C \end{matrix} \\ \left[ \begin{array}{ccc} 5 & 3 & 2 \\ 6 & 4 & 3 \end{array} \right] = R \end{array}$$

- a. Explain why the product  $PM$  is defined, but not meaningful.  
b. Find  $M'P$ . Use your answer to tell how many A components will be needed in March.  
c. Explain why  $MR$  is not defined.  
d. Find  $RM'$ . What information does this product give?  
e. Find  $RM'P$ . What information does this product give?



18. **Nutrition** Patients in a large hospital have a choice of four menus, each with meat, potato, vegetable, salad, and dessert. For example, Menu 1 has baked ham, sweet potatoes, peas, green salad, and baked apples. Matrix  $C$  gives the costs per serving (in cents) for each item in these menus. Matrix  $D$  gives the number of menus ordered each day last week.

$$\begin{array}{l} \text{Menu 1} \\ \text{Menu 2} \\ \text{Menu 3} \\ \text{Menu 4} \end{array} \begin{bmatrix} \text{meat} & \text{potato} & \text{vegetable} & \text{salad} & \text{dessert} \\ 74 & 8 & 15 & 11 & 12 \\ 48 & 10 & 16 & 12 & 14 \\ 87 & 9 & 12 & 16 & 20 \\ 62 & 7 & 18 & 15 & 22 \end{bmatrix} = C$$

$$\begin{array}{l} \text{Menu 1} \\ \text{Menu 2} \\ \text{Menu 3} \\ \text{Menu 4} \end{array} \begin{bmatrix} \text{M} & \text{Tu} & \text{W} & \text{Th} & \text{F} \\ 50 & 48 & 44 & 52 & 38 \\ 63 & 60 & 55 & 55 & 53 \\ 71 & 79 & 83 & 87 & 78 \\ 38 & 42 & 47 & 41 & 44 \end{bmatrix} = D$$

- Explain the meaning of  $C^tD$  and  $D^tC$ .
- Let  $A$  be the  $5 \times 1$  matrix whose elements are all 1. Find  $CA$ . What information does  $CA$  give?
- Let  $B$  be the  $1 \times 4$  matrix whose elements are all 1. Find  $BD$ . What information does  $BD$  give?

19. **Industrial Design** At a regional competition for industrial design students, each project submitted is given a score from 1 to 10 in each of three categories: precision, weight = 30%; surface finish, weight = 20%; artistic design, weight = 50%. The total score for a project is found by adding the products of each category score and its weight.



- Stephanie has a precision score of 8, a surface finish score of 7, and an artistic design score of 9. What is her total project score?
- The six finalists have the scores given in the table below. Use matrix multiplication to determine the first, second, and third place winners.

	precision	finish	design
Stephanie	8	7	9
Joyce	8	6	10
Eduardo	6	8	10
Frank	9	10	7
Asabi	10	10	6
Dara	8	7	8

20. Tickets to the Senior Class Play cost \$2.00 for students, \$5.00 for adults, and \$4.00 for senior citizens. At Friday night's performance, there were 121 students, 164 adults, and 32 senior citizens. At Saturday night's performance, there were 183 students, 140 adults, and 25 senior citizens. Display this information in matrix form. Then use matrix multiplication to find the ticket sales income for Friday and Saturday nights' performances.



21. a. **Business** A Chicago company wants to send some of its key personnel to a convention in London. In the company's Research and Development Division, five people plan to fly first class, three people plan to fly business class, and two people plan to fly coach class. In the Sales Division, four people plan to fly business class, and eight people coach class. Display this information in a  $2 \times 3$  travel matrix  $T$ .
- b. Round-trip prices for four different airlines are as follows: Airline A charges \$1,280 for first class, \$922 for business class, and \$676 for coach. Airline B charges \$1,400 for first class, \$1,024 for business class, and \$728 for coach. Airline C charges \$1,320 for first class, \$905 for business class, and \$654 for coach. Finally, Airline D charges \$1,450 for first class, \$1,050 for business class, and \$734 for coach. Display this information in a price matrix  $P$  that can be multiplied with matrix  $T$  to give the travel costs for each company division per airline. Then find this product.
- c. How much will it cost to fly the Sales Division on Airline D?
- d. Which airline will cost the Research and Development Division the least?

- C** 22. **Business** Consider the price matrix  $P$ , shown below, which gives the costs for flying first class, business class, and coach class on four different airlines, A, B, C, and D.

$$\begin{array}{l} \text{first} \\ \text{business} \\ \text{coach} \end{array} \begin{bmatrix} A & B & C & D \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} = P$$

Suppose that the prices in matrix  $P$  are all going to increase as follows:

- Airline A: increase all prices by 5%;
- Airline B: increase all prices by 4%;
- Airline C: increase all prices by 6%;
- Airline D: increase all prices by 2%.

Write a  $4 \times 4$  increase matrix  $N$  such that the product  $PN$  is the new price matrix; that is, each element of  $PN$  gives the new price for each class on a particular airline. Show the matrix  $PN$ .

## 14-3 Applying Matrices to Linear Systems

### Objective

To find the inverse of a  $2 \times 2$  matrix and to solve linear systems using matrices.

In the previous sections, you may have discovered that matrix addition and matrix multiplication have many of the properties of ordinary addition and multiplication.

	Properties of Real Numbers	Properties of Matrices
Property	Let $a$ , $b$ , and $c$ be real numbers.	Let $A$ , $B$ , and $C$ be $m \times n$ matrices.
Commutative	$a + b = b + a$ $ab = ba$	$A + B = B + A$ $AB \neq BA$
Associative	$(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$	$(A + B) + C = A + (B + C)$ $(AB)C = A(BC)$
Distributive	$a(b + c) = ab + ac$ $= ba + ca$ $= (b + c)a$	$A(B + C) = AB + AC$ $(B + C)A = BA + CA$

An important exception to the similarity of these properties is that matrix multiplication is *not* commutative: in general,  $AB \neq BA$ . Since products do not commute, *left-multiplication* can give a different product from *right-multiplication*.

Before discussing other properties, we first need to identify some important matrices. Any  $m \times n$  matrix whose elements are all zero is called a **zero matrix**, denoted by  $O_{m \times n}$ . The following matrices are zero matrices.

$$O_{1 \times 3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad O_{2 \times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad O_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad O_{3 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A **square matrix** is any matrix having the same number of rows as columns. The *main diagonal* of a square matrix is the diagonal that extends from upper left to lower right. Any  $n \times n$  matrix whose main diagonal elements are 1 and whose other elements are 0 is called an **identity matrix**, denoted by  $I_{n \times n}$ . The following matrices are identity matrices.

$$I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Powers of square matrices are defined just like powers of real numbers:

$$A^n = \underbrace{A \cdot A \cdot A \cdot \cdots \cdot A}_{n \text{ factors}}$$

As with real numbers, matrix addition and multiplication both have identity properties. It is also true that matrix addition and multiplication both have inverse properties. The real numbers 2 and  $-2$  are called additive inverses since  $2 + (-2) = 0$ . Similarly, a matrix can have an additive inverse. The **additive inverse** of a matrix  $A$ , denoted by  $-A$ , is the matrix in which each element is the opposite of its corresponding element in  $A$ . For example,

$$\text{if } A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -8 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -2 & 3 & -5 \\ 0 & -1 & 8 \end{bmatrix}.$$

	Properties of Real Numbers	Properties of Matrices
Properties for Addition	Let $a$ be any real number.	Let $A$ be any $m \times n$ matrix. Let $O$ be the $m \times n$ zero matrix.
Identity	$a + 0 = 0 + a = a$	$A + O = O + A = A$
Inverse	$a + (-a) = -a + a = 0$	$A + (-A) = -A + A = O$

You should verify that the matrices  $A$  and  $-A$  given above satisfy the inverse property for addition. The real numbers 2 and  $2^{-1}$  are called multiplicative inverses since  $2 \cdot 2^{-1} = 1$ . Similarly, a matrix can have a *multiplicative inverse*. The examples below illustrate the identity and inverse properties for multiplication.

$$\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the multiplicative inverse of a square matrix can involve a lot of computations. The easiest inverse to find is that of a  $2 \times 2$  matrix. In Written Exercise 29, you are asked to derive the following formula.

### Inverse of a $2 \times 2$ Matrix

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } |A| = ad - bc.$$

Recall that  $|A|$  is called the *determinant* of  $A$  (see page 458). The formula above tells you that if  $|A| \neq 0$  then  $A^{-1}$  exists. Thus, the multiplicative inverse property of matrices is not quite the same as the corresponding property of real numbers. Whereas 0 is the only real number that has no inverse, there are infinitely many matrices that have no inverse. The properties for multiplication are summarized on the next page.

	Properties of Real Numbers	Properties of Matrices
Properties for Multiplication	Let $a$ be any real number.	Let $A$ be any $n \times n$ matrix. Let $I$ be the $n \times n$ identity matrix and $O$ be the $n \times n$ zero matrix.
Identity	$a \cdot 1 = 1 \cdot a = a$	$A \cdot I = I \cdot A = A$
Inverse	$a \cdot a^{-1} = a^{-1} \cdot a = 1$ ( $a \neq 0$ )	$A \cdot A^{-1} = A^{-1} \cdot A = I$ ( $ A  \neq 0$ )
Multiplicative Property of Zero	$a \cdot 0 = 0 \cdot a = 0$	$A \cdot O = O \cdot A = O$

**Example 1** If  $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ , find  $A^{-1}$ . Check your answer by finding  $A \cdot A^{-1}$ .

**Solution**  $|A| = 5 \cdot 3 - 4 \cdot 2 = 7$ . Since  $|A| \neq 0$ ,  $A^{-1}$  exists.

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix}$$

$$\text{Check: } A \cdot A^{-1} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{5}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

When solving the linear equation  $3x + 7 = 20$ , you add  $-7$  to both sides of the equation and then multiply both sides by  $3^{-1}$ , or  $\frac{1}{3}$ . Any matrix equation in the form  $AX + B = C$  can be solved in a similar way.

**Example 2** Let  $A$ ,  $B$ ,  $C$ , and  $X$  be  $n \times n$  matrices, where  $A^{-1}$  exists. Solve  $AX + B = C$  for  $X$ .

**Solution**

$$AX + B = C$$

$$AX + B + (-B) = C + (-B)$$

$$AX + O = C - B$$

$$AX = C - B$$

$$A^{-1}(AX) = A^{-1}(C - B)$$

$$(A^{-1}A)X = A^{-1}(C - B)$$

$$IX = A^{-1}(C - B)$$

$$X = A^{-1}(C - B)$$

Add the additive inverse of  $B$ .

Additive Inverse Property

Additive Identity Property

Left-multiply by  $A^{-1}$ .

Associative Property

Multiplicative Inverse Property

Multiplicative Identity Property



**Example 3**

Use matrices to solve the given system of equations:

$$5x + 4y = -2$$

$$2x + 3y = -5$$

**Solution**

This system of two equations in two unknowns can be replaced by a single matrix equation:

$$A \cdot X = C$$

$$\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

Notice that the *coefficient matrix*  $A$  is the matrix whose inverse was found in Example 1.

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}$$

If you left-multiply both sides of the matrix equation by  $A^{-1}$ , you get:

$$\frac{1}{7} \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$\frac{1}{7} \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 14 \\ -21 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Thus, the solution to the system is  $x = 2$  and  $y = -3$ .

**Example 4**

Rewrite the following system as a single matrix equation.

$$3x_1 - 5x_2 + 4x_3 + 7x_4 = 12$$

$$6x_1 + 7x_2 - 8x_4 = 29$$

$$5x_1 + 9x_2 - 7x_3 - 5x_4 = -52$$

$$x_2 - 5x_3 + 9x_4 = 41$$

**Solution**

$$\begin{bmatrix} 3 & -5 & 4 & 7 \\ 6 & 7 & 0 & -8 \\ 5 & 9 & -7 & -5 \\ 0 & 1 & -5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 29 \\ -52 \\ 41 \end{bmatrix}$$

The matrix equation in Example 4 has the form  $AX = C$ , whose solution is  $X = A^{-1}C$ , provided that  $A^{-1}$  exists. This textbook will not show how to find the inverse of matrices larger than  $2 \times 2$  because the calculations can be tedious. However, many computer software programs and graphing calculators can calculate  $A^{-1}C$  efficiently when given specific matrices for  $A$  and  $C$ . If you have access to a graphing calculator or computer program, you should find the solution to the system in Example 4.

## CLASS EXERCISES

For Exercises 1–4, use matrices  $A$ ,  $B$ ,  $C$ , and  $D$ .

$$A = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 7 & -1 \\ 6 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 3 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

- Find the additive inverse of  $B$  and of  $C$ .
- Find the multiplicative inverse of  $A$  and of  $B$ .
- Explain why  $C$  and  $D$  do not have multiplicative inverses.
- Discussion** Find  $A^2$ . To find  $A^3$ , do you calculate  $A^2 \cdot A$  or  $A \cdot A^2$ ? Explain.
- Discussion** Refer to the solution to Example 2 on page 532. If the equation  $AX = C - B$  is right-multiplied by  $A^{-1}$  instead of left-multiplied, then you would get  $(AX)A^{-1}$  on the left side of the equation. Explain why  $(AX)A^{-1}$  cannot be simplified.
- The matrix equation below contains the unknown matrix  $X$ . The first two steps of the solution are shown.

$$2 \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} + 3X = \begin{bmatrix} 1 & -4 \\ 3 & -7 \end{bmatrix}$$

Simplify by using  
scalar multiplication.

$$\begin{bmatrix} 10 & 2 \\ 6 & 8 \end{bmatrix} + 3X = \begin{bmatrix} 1 & -4 \\ 3 & -7 \end{bmatrix}$$

Subtract the first matrix  
from both sides.

$$3X = \begin{bmatrix} 1 & -4 \\ 3 & -7 \end{bmatrix} - \begin{bmatrix} 10 & 2 \\ 6 & 8 \end{bmatrix}$$

Complete the solution, giving reasons for each step.

## WRITTEN EXERCISES

For Exercises 1–6, use the following matrices to find the specified matrices. If a matrix does not exist, write *not defined*.

$$M = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix} \quad N = \begin{bmatrix} 5 & 11 \\ -3 & -7 \end{bmatrix} \quad C = \begin{bmatrix} 9 & 3 \\ 12 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & -8 & 1 \\ 5 & 7 & 3 & 9 \end{bmatrix}$$

**A**

- |                       |              |                |              |
|-----------------------|--------------|----------------|--------------|
| 1. a. $M^{-1}$        | b. $MM^{-1}$ | 2. a. $N^{-1}$ | b. $N^{-1}N$ |
| 3. a. $-C$            | b. $C^{-1}$  | 4. a. $D^{-1}$ | b. $-D$      |
| 5. a. $-\frac{1}{3}C$ | b. $-2D + C$ | 6. a. $D^2$    | b. $N^3$     |

Solve each matrix equation for  $X$ . Assume any required inverse matrix exists.

- |                |                    |
|----------------|--------------------|
| 7. a. $DX = E$ | 8. a. $TX - S = R$ |
| b. $XD = E$    | b. $XT - S = R$    |

Solve each matrix equation for  $X$ .

$$9. X + 3 \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 0 & -8 \end{bmatrix}$$

$$10. 2X - \begin{bmatrix} 1 & 5 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -1 & 12 \end{bmatrix}$$

$$11. \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} X = \begin{bmatrix} 9 & 3 \\ 9 & 1 \end{bmatrix}$$

$$12. \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix} X = \begin{bmatrix} 3 & 0 & 5 \\ 1 & 4 & 2 \end{bmatrix}$$

$$13. \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & 7 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$14. \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} X - \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Write each system of equations as a single matrix equation. Solve the system using the method shown in Example 3.

$$15. \text{ a. } \begin{aligned} 8x - 2y &= -14 \\ 12x + 3y &= 9 \end{aligned}$$

$$16. \text{ a. } \begin{aligned} 12x - 4y &= 3 \\ 17x - 5y &= 8 \end{aligned}$$

$$\text{ b. } \begin{aligned} 8x - 2y &= 38 \\ 12x + 3y &= 39 \end{aligned}$$

$$\text{ b. } \begin{aligned} 12x - 4y &= 0 \\ 17x - 5y &= -16 \end{aligned}$$

**B** 17. a. Solve the system using matrices.

$$\begin{aligned} 12x + 20y &= 40 \\ 9x + 15y &= 45 \end{aligned}$$

b. Sketch the graphs of the two equations on the same set of axes. What do the graphs tell you about the solution of this system?

18. a. Solve the system using matrices.

$$\begin{aligned} 14x - 21y &= 56 \\ x - \frac{3}{2}y &= 4 \end{aligned}$$

b. Sketch the graphs of the two equations on the same set of axes. What do the graphs tell you about the solution of this system?

19. **Writing** Write a paragraph explaining why the identity and the inverse properties for multiplication apply only to square matrices.

20. a. Using the distributive property, factor the following expression. Then simplify.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & -5 \\ 7 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & -3 \end{bmatrix}$$

b. Solve the following equation:

$$X \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix} - X \begin{bmatrix} 4 & -7 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -5 & 7 \end{bmatrix}$$

21. Solve the matrix equation  $X - AX = D$  for  $X$ . (*Hint*: Rewrite  $X$  either as  $XI$  or as  $IX$ . Which way is helpful?)

22. Solve the matrix equation  $X - XA = D$  for  $X$ .



For Exercises 23–27, first write a matrix equation, and then solve the equation. You will need to use computer software or a calculator that computes matrix operations in order to solve the equation.

23.  $3x + 5y - 9z = 26$   
 $4x + 7y + 2z = -7$   
 $6x - 9y - 8z = 3$

24.  $3a + 4b - 5c - 11d = 5$   
 $5a - 2b - 7c + 8d = 6$   
 $2a - 4b - 8c + 16d = -3$   
 $3b + 2c + 8d = 4$

25. **Nutrition** A dietician wants to combine four foods (I, II, III, and IV) to make a meal having 78 units of vitamin A, 67 units of vitamin B, 146 units of vitamin C, and 153 units of vitamin D. The matrix below gives the vitamin content per ounce of each food. How many ounces of each food should be included in the meal? Begin by writing the unknowns in a  $4 \times 1$  matrix. Set up a matrix equation and then solve, giving answers to the nearest ounce.

	I	II	III	IV
A	3	2	2	6
B	2	3	5	0
C	8	6	4	7
D	5	5	8	6



26. a. **Manufacturing** A company makes oak tables, chairs, and desks. Each item requires labor time in minutes, as given in the matrix below. The amount of time available for labor each week is 20,250 min for carpentry, 12,070 min for assembly, and 17,000 min for finishing. If the production manager wants to use all of the available labor, how many tables, chairs, and desks should the manager schedule for production each week?

	tables	chairs	desks
carpentry	120	105	125
assembly	40	65	110
finishing	80	90	125

- b. Suppose that because of vacation schedules the amount of labor available is less for the coming week. The amount of labor available is 14,960 min for carpentry, 8,970 min for assembly, and 12,590 min for finishing. How many tables, chairs, and desks should the manager schedule for this week?
27. Suppose the points  $(4, 11)$ ,  $(-6, -9)$ , and  $(8, 61)$  are known to lie on the parabola  $y = ax^2 + bx + c$ . Write three equations in terms of  $a$ ,  $b$ , and  $c$ . Express this system as a single matrix equation, and then solve to find an equation of the parabola.

- C** 28. **Investigation** If  $a$  and  $b$  are real numbers and  $ab = 0$ , then either  $a = 0$  or  $b = 0$ . If  $A$  and  $B$  are  $n \times n$  matrices, does the fact that  $AB = O_{n \times n}$  imply that either  $A$  or  $B$  must be a zero matrix? Explain, giving examples for  $A$  and  $B$ .
29. The purpose of this exercise is to derive the formula for the inverse of a  $2 \times 2$  matrix. Let the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Suppose that the inverse of  $A$  is represented by  $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ . Use the fact that  $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  to solve for  $w$ ,  $x$ ,  $y$ , and  $z$ .



**Olga Taussky-Todd (1906– )**

Born in what is now the Czech Republic, Olga Taussky-Todd studied number theory in Vienna. During World War II, she worked outside London studying the stability of matrices and applying her results to the oscillations of airplane wings at supersonic speeds. Later in her career, she said that matrix theory emerged as her major subject.

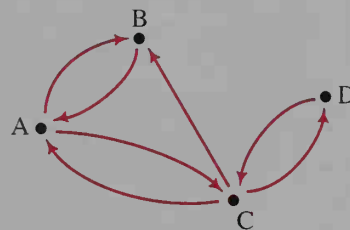
After working ten years in the United States as a consultant for the National Bureau of Standards, Taussky-Todd began teaching at the California Institute of Technology. In 1970 she received a Ford Prize for a paper on the sums of squares.

## Applications of Matrices

### 14-4 Communication Matrices

**Objective** To solve communication network problems using matrices.

The points A, B, C, and D in the diagram at the right represent four ships at sea that are within communication range. The arrows indicate the direction of radio transmissions. For example, Ship B can send a message to Ship A, but it cannot send a message to Ship C. Although Ship D cannot communicate directly with Ship A, it can communicate indirectly by using Ship C as a relay.





Although the diagram for the communication network of just four ships is easy to understand, a communication diagram can become very complicated when many ships are involved. Using matrices gives a way to keep track of who can communicate with whom. For example, the *communication matrix*  $M$  shown below models the communication between the four ships shown in the diagram on page 537. In such a matrix, a “1” in any row indicates that direct communication from one ship to another is possible. A “0” indicates that direct communication is not possible.



$$\begin{array}{c} \text{to ship} \\ \text{from ship} \end{array} \begin{array}{c} A \quad B \quad C \quad D \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = M$$

Notice that in matrix  $M$ , the main diagonal elements are zeros because we are assuming that no ship will send a message directly to itself. Make sure that you understand what the elements in  $M$  represent.

While  $M$  gives the direct communication links between ships,  $M^2$  gives the indirect two-step paths that use one ship as a relay. For example, A can communicate with D using C as a relay. This gives a two-step path from A to D, namely, A–C–D.

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

This means there are 2 two-step paths from A to itself. (A–B–A and A–C–A)

This means there is exactly 1 two-step path from D to B. (D–C–B)

If  $M^2$  gives the number of two-step paths and  $M$  gives the number of one-step paths, then  $M + M^2$  gives the number of paths that take either one or two steps.

$$M + M^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

This means there are two paths from C to B that take no more than 2 steps.

Similarly, it can be proved that the matrix  $M^3$  gives the number of three-step paths between any two ships. The matrix  $M + M^2 + M^3$  gives the number of communication paths that take no more than three steps.

### Example

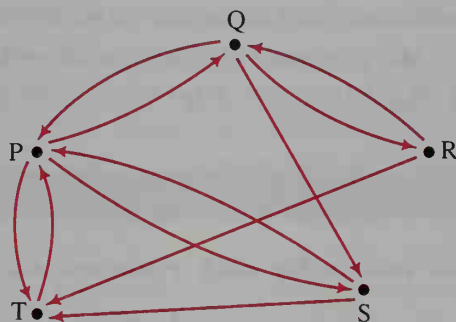
The matrix  $M$  below models the communication between Computers P, Q, R, S, and T.

$$\begin{array}{c} \text{to} \\ \text{from} \end{array} \begin{array}{c} P \\ Q \\ R \\ S \\ T \end{array} \begin{bmatrix} P & Q & R & S & T \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = M$$

- Draw a diagram that illustrates this communication network.
- Name the computer(s) that can send data along the greatest number of routes via one relay.

### Solution

a.



- $M^2$  shows the number of ways data can be sent from one computer to another using one relay.

$$\begin{array}{c} P \\ Q \\ R \\ S \\ T \end{array} \begin{bmatrix} P & Q & R & S & T \\ 3 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 3 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} = M^2$$

The row sums of  $M^2$  for Computers P, Q, R, S, and T are 6, 7, 4, 4, and 3, respectively. Thus, Computer Q, with a total of 7, sends data along the greatest number of two-step paths. You can verify this by checking the diagram in part (a).



Using a computer network, students share files.

## CLASS EXERCISES

Matrix  $M$  below describes a communication network for weather stations.

$$\begin{array}{c} \text{from station} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array} \begin{array}{c} \text{to station} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = M$$

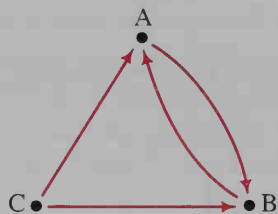
1. To which stations can A send messages?
2. From which stations can B receive messages?
3. Draw a diagram that illustrates this communication network.
4. What information does  $M^2$  give you? (Do not calculate  $M^2$ .)
5. *Discussion* Will the main diagonal of  $M^2$  necessarily be all zeros?

## WRITTEN EXERCISES

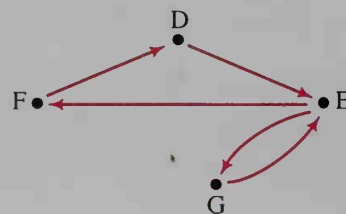
Write the communication matrix for each communication network given.

**A**

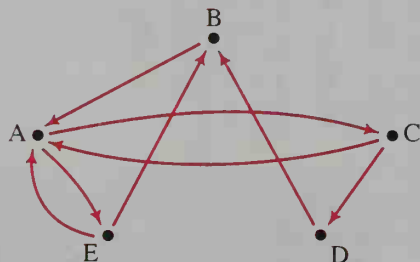
1.



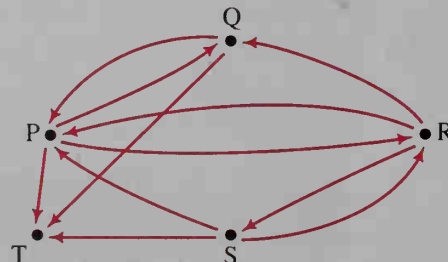
2.



3.



4.



Draw a communication network that is described by the given matrix.

5.

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

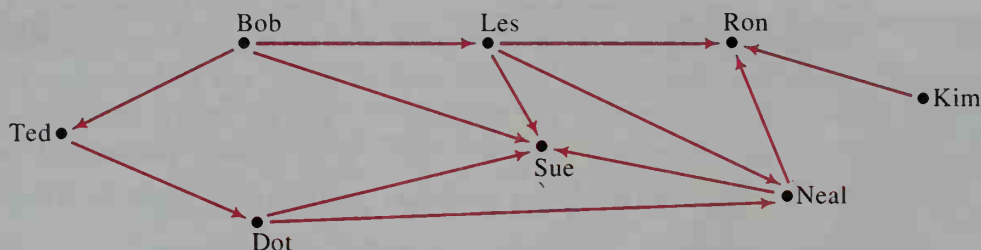
6.

$$\begin{array}{c} R \\ S \\ T \\ U \\ V \end{array} \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



For some parts of the following exercises, you may wish to use computer software or a calculator that performs matrix operations.

7. a. Ship A sends messages to Ship B. Ship B sends and receives messages from Ships C and E. Ship D sends and receives messages from Ship C. Ship E receives messages from Ships A and C. Draw this network.  
 b. Write the matrix  $M$  that models this communication network, labeling rows and columns alphabetically.  
 c. Find  $M^2$ . Explain what the element in the fifth row, second column means.  
 d. Find the matrix that represents the number of ways messages can be sent from one ship to another using at most one relay.  
 e. Reason from the network diagram what the last element in row 4 of  $M^3$  is. (You do not need to calculate  $M^3$ .)
  8. a. Suppose radios of varying quality are used by forest rangers to communicate with each other in a large national park. Make a diagram showing five rangers, some pairs having two-way communication, some pairs having one-way communication, and some having no communication.  
 b. Write a matrix  $M$  to model the communication between the forest rangers. Find  $M + M^2$ . What information does this matrix give you?
- B** 9. The diagram below models a rumor network.

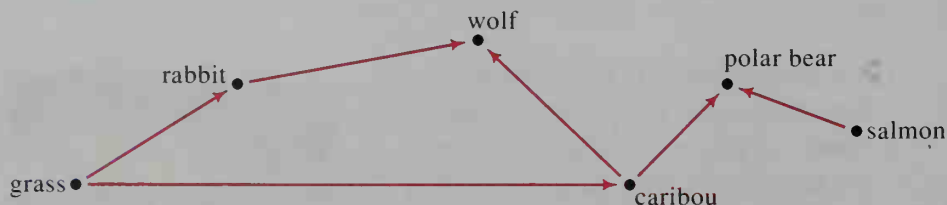


If a person has only outgoing arrows, then that person is a transmitter of the rumor. A person having only incoming arrows is a receiver of the rumor. A person having both outgoing and incoming arrows is a relay point for the spreading of the rumor.

- a. Identify the transmitters, receivers, and relays for this network.
  - b. How can you identify a transmitter by looking at the corresponding rumor matrix? How can you identify a receiver?
10. Let  $M$  be a matrix that illustrates communication among several Ham radio operators. If  $M^2$  contains no zeros, what can you conclude?



11. There are  $n$  ships at sea. Each ship in the network can communicate directly by radio with every ship (except itself). How many zeros are in the communications matrix? How many ones?
12. The *dominance matrix* shown below is a special kind of communication matrix. A **dominance relation** exists between members when, between any two members, one dominates the other. For example, the winner in the pair “dominates” the loser. In terms of a communication network, this means that, between any two members, only a one-way communication is possible.
- a. Matrix  $M$  at the right represents the result of a chess tournament played by Ajani, Bella, Carmen, Denise, and Enrico. Make a diagram to illustrate this tournament.
- b. In the matrix above, the number of games won is indicated in the rows; the number of games lost is indicated in the columns. Which two players are tied with the best record for the number of wins and losses?
- c. One method used to decide who comes in first, second, and third in a tournament is to look at the *second-stage dominances*. For example, Carmen has a second-stage dominance over Denise since Carmen beat Ajani and Ajani beat Denise. Find  $M^2$ , which gives the second-stage dominances.
- d. The person who has the greatest total of direct and second-stage dominances is the winner of the tournament. Find the matrix that gives this information. Who won the tournament? Who finished second and third?
13. a. **Ecology** The diagram below models a possible food chain in Alaska.



The arrow from the rabbit to the wolf indicates the rabbit is a source of food for the wolf. Any contamination of the rabbit's food, such as DDT poisoning, will also be harmful to the wolf. Write a matrix  $M$  that represents this food chain. Label the rows and columns alphabetically.

- b. Find  $M^2$ . Tell the meaning of the element in the grass row, wolf column.
- c. Although grass is not food for salmon, any insecticides on grasslands can wash into rivers, possibly killing salmon or affecting their reproduction. How would you change matrix  $M$  to reflect this fact?





**14. Research** Draw a food chain diagram for an ecological area of interest to you (for example, a swamp, a field, a pond, or a tropical region). Model this food chain with a matrix  $M$  and calculate  $M^2$ ,  $M^3$ , and  $M + M^2 + M^3$ . Write a paragraph interpreting what the matrix  $M + M^2 + M^3$  tells you.

**C 15. Writing** Write a paragraph explaining why  $M^3$  gives the number of three-step paths between points in a communication network.

## 14-5 Transition Matrices

**Objective** To make predictions using powers of matrices.

The manufacturers of Highlight shampoo, the best-selling shampoo in the nation, have done a market analysis to predict the buying trends of customers in the marketplace over the long run. Their research tells them two things:

- (1) the current market shares
- (2) the percentages of people who change from brand to brand each month

The percentages of people who change brands each month can be written as a *transition matrix*, as illustrated in Example 1.

### Example 1

Of all people who buy Highlight one month, 90% will buy Highlight the next month, 5% will change to Silky Shine the next month, and 5% will change to another brand. Of those people who buy Silky Shine one month, 10% will change to Highlight the next month, 70% will stay with Silky Shine, and 20% will change to another brand. Of the users of other brands, 20% will change to Highlight, and 20% will change to Silky Shine. Write this information as a transition matrix  $T$ .



### Solution

$$\begin{array}{l}
 \text{Highlight} \\
 \text{Silky Shine} \\
 \text{Other}
 \end{array}
 \begin{bmatrix}
 \text{Highlight} & \text{Silky Shine} & \text{Other} \\
 0.9 & 0.05 & 0.05 \\
 0.1 & 0.7 & 0.2 \\
 0.2 & 0.2 & 0.6
 \end{bmatrix}
 = T$$

The rows in matrix  $T$  above tell what the users of each product will use next month. Notice that the percentages have been expressed as decimals and that the elements in each row add up to 1.

Suppose the market survey also indicates that the current market shares are Highlight users, 40%; Silky Shine users, 30%; and other brands, 30%. We can write this information as a current market share matrix  $M_0$ .

$$\begin{array}{c} \text{Highlight} \quad \text{Silky Shine} \quad \text{Other} \\ \text{market share } [ \quad 0.4 \quad \quad 0.3 \quad \quad 0.3 \quad ] = M_0 \end{array}$$

**Example 2** Use the transition matrix  $T$  and the current market share matrix  $M_0$  to calculate the market shares for the next month.

**Solution** The matrix  $M_1$ , which gives the market shares one month from now, is the product  $M_0T$ .

$$\begin{aligned} M_1 &= M_0T \\ &= [0.4 \quad 0.3 \quad 0.3] \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} \\ &= [0.45 \quad 0.29 \quad 0.26] \end{aligned}$$

To see why  $M_0T$  gives the next month's market share, let us consider how the first entry in the product  $M_0T$  is formed.

$$\underbrace{(0.4)(0.9)}_{\substack{\text{40\% use Highlight,} \\ \text{and 90\% of these} \\ \text{will buy it next} \\ \text{month.}}} + \underbrace{(0.3)(0.1)}_{\substack{\text{30\% use Silky} \\ \text{Shine, and 10\% of} \\ \text{these will change} \\ \text{to Highlight.}}} + \underbrace{(0.3)(0.2)}_{\substack{\text{30\% use another} \\ \text{brand, and 20\% of} \\ \text{these will change} \\ \text{to Highlight.}}} = \underbrace{0.45}_{\substack{\text{45\% will use} \\ \text{Highlight next} \\ \text{month.}}}$$

To find a matrix  $M_2$  that gives the market shares two months from now, multiply the next month's market share matrix  $M_1$ , which you found in Example 1, by the transition matrix  $T$ .

$$\begin{aligned} M_2 &= M_1T \\ &= (M_0T)T \leftarrow \text{substituting } M_1 = M_0T \\ &= M_0T^2 \end{aligned}$$

Likewise, the market share matrix three months from now is:

$$\begin{aligned} M_3 &= M_2T \\ &= (M_0T^2)T \\ &= M_0T^3 \end{aligned}$$

Generalizing, we see that  $M_0T^n$  gives the market share  $n$  months from now. Using computer software or a calculator that performs computations of matrices makes finding the calculations of  $M_0T^n$  easy. Some of these calculations are shown on the next page.

$$M_1 = [0.45 \quad 0.29 \quad 0.26]$$

$$M_2 = [0.4860 \quad 0.2775 \quad 0.2365]$$

$$M_3 = [0.51245 \quad 0.26585 \quad 0.22170]$$

$$M_4 = [0.532130 \quad 0.256058 \quad 0.211813]$$

$$M_5 = [0.546885 \quad 0.248209 \quad 0.204906]$$

.

.

.

$$M_{10} = [0.581189 \quad 0.228840 \quad 0.189971]$$

.

.

.

$$M_{20} = [0.591878 \quad 0.222638 \quad 0.185485]$$

.

.

.

$$M_{30} = [0.592548 \quad 0.222248 \quad 0.185204]$$

The situation we have been discussing is an example of a **Markov chain**, a sequence of observations or predictions  $M_0, M_1, M_2, \dots$ , each of which depends on the immediately preceding one, and only on it. The matrices  $M_n$  give the predicted market shares  $n$  months from now. Notice that in Example 2, there is little difference between  $M_{20}$  and  $M_{30}$ . If you were to continue calculating  $M_n$  for larger values of  $n$ , you would see that the market share values eventually *stabilize* at about 59%, 22%, and 19%. We say that the market shares have reached a *steady state*. Of course, achieving a steady state assumes that the transition matrix  $T$  remains the same. However,  $T$  *could* change due to an advertising campaign or a price change for one of the shampoos. If this happens, the Markov chain based on  $T$  terminates and a new chain based on a new transition matrix could begin.

The *steady-state* matrix  $[0.59 \quad 0.22 \quad 0.19]$  was calculated by assuming not only a constant transition matrix  $T$  but also by assuming a current market share  $M_0 = [0.4 \quad 0.3 \quad 0.3]$ . If the current market share were different, how would the steady-state matrix be affected, if at all? The answers to this and other interesting prediction problems can be found in the exercises.

## CLASS EXERCISES

1. **Discussion** For the product  $M_0T = M_1 = [0.45 \quad 0.29 \quad 0.26]$ , the calculation of the number 0.45 is explained in the text. Give similar explanations for the numbers 0.29 and 0.26.
2. In the shampoo example, we saw that  $M_2 = M_1T = (M_0T)T$ . We then said that  $(M_0T)T = M_0T^2$ . What property of matrix multiplication justifies this last equation?

3. a. **Geography** Suppose that census takers have found that from one year to the next, 10% of those living in a certain city will move to its suburbs while 3% of those living in the suburbs will move into the city. Write a transition matrix  $T$  that describes this situation.
- b. If 60% of the metropolitan population is now in the city and 40% is in the suburbs, give the current population as a  $1 \times 2$  matrix  $M_0$ .
- c. Calculate  $M_0T$  and interpret its elements.
- d. What do the elements of  $M_0T^2$  tell you?
- e. How could one use a computer to determine if the population distribution reaches a steady state?


## WRITTEN EXERCISES

- A**
1. **Transportation** Commuters to a city commute either by car or by public transportation. Suppose each year 18% of the car commuters change to public transportation and 5% of those using public transportation change to cars.
    - a. Give the transition matrix  $T$ .
    - b. If 60% of the commuters now use public transportation and 40% use cars, write a  $1 \times 2$  matrix  $M_0$  to represent this information.
    - c. Calculate  $M_1 = M_0T$  and explain the meaning of each element.
    - d. Calculate  $M_2 = M_1T$  and explain the meaning of each element.
  2. **Geography** Over the years, it has been observed that each year, 3% of those living in a certain city's urban area move to the suburbs and 2% of those in the suburbs move into the urban area.
    - a. Give the transition matrix  $T$ .
    - b. Suppose one third of the city's population now lives in the suburbs. Give a  $1 \times 2$  matrix  $M_0$  that describes the current population.
    - c. Find the percentages of the population living in the urban area and in the suburbs one year from now and also two years from now.
    - d. With a computer one could determine that  $M_0T^{80} = [0.4 \ 0.6]$ , so in 80 years, 40% will live in the urban area and 60% in the suburbs. Find the population percentages for 81 years from now without using a computer.
    - e. Use your answer to part (d) to give the steady-state matrix.
  3. **Business** Dixie's and Sargent's are two competing restaurants located next door to each other. A survey of those who eat regularly at one place or the other shows that customer loyalty is quite high. Only 10% of those who eat at Dixie's on one day will switch to Sargent's the next day and only 20% of those who eat at Sargent's will switch to Dixie's the next day. Suppose that of those who always eat lunch at one place or the other, 60% ate at Dixie's on Monday.
    - a. What percentage ate at Dixie's on Tuesday? on Wednesday?
    - b. Suppose that two months from now, two thirds of the regular lunch crowd ate at Dixie's and one third ate at Sargent's. What fractional amount of the lunch crowd will be at each place on the next day?
    - c. Use your answer to part (b) to give the steady-state matrix.

4. **Psychology** In an experiment to see how well mice can learn, mice are placed in the maze shown. A mouse that enters room 1 is rewarded with cheese, but there is no reward in room 2. Studies show that 70% of mice who go to room 1 will go to room 1 to be rewarded the next time. Those who go to room 2 one time are as likely to go to room 1 as to room 2 the next time.



- Write a transition matrix.
- If on the twentieth trial, 62.5% of the mice go to room 1 to get the cheese, what percentage will do this on the twenty-first trial?
- What appears to be the steady-state matrix?

 For the parts of the exercises designated in green, you will need to use computer software or a calculator that performs matrix operations.

5. **Statistics** In the land of Oz, the weather is a little unpredictable. However, the Wizard of Oz, who is an amateur weatherman, has made the following observations through the centuries: 80% of all sunny days are followed by another sunny day, 10% of sunny days are followed by cloudy ones, and 10% are followed by rainy ones. A cloudy day has a 40% chance of being followed by another cloudy day and a 60% chance of being followed by a rainy day. A rainy day is always followed by a sunny one.
- Write the matrix  $T$  that represents the day-to-day weather transitions.
  - Calculate  $T^2$ . Give the chance that if Monday is sunny, then Wednesday will be rainy.
  - Calculate  $T^3$ . Give the chance that Thursday will be cloudy if Monday is sunny.
  - Calculate  $T^{30}$ . What is the chance that 30 days from now will be sunny if today is sunny? if today is cloudy? if today is rainy?
  - Calculate  $T^{40}$ . Then answer the three questions in part (d).
  - Over a long period, what percentage of the days are sunny? cloudy? rainy?
6. **Business** Every year, each of several rent-a-car companies will buy all new cars from the “big three” automobile manufacturing companies (A, B, and C). The three companies keep track of how these orders change from year to year. They find that 50% of A cars will be replaced by new A cars the next year, while the other 50% of A cars will be replaced by equal numbers of B and C cars. Sixty percent of B cars will be replaced by new B cars while 30% will be replaced by A cars and 10% by C cars. Thirty percent of C cars will be replaced by B cars and 25% will be replaced by A cars.
- Write a transition matrix. Label the rows and columns.
  - The 1990 fleet of rental cars was distributed as follows: 42% A cars, 37% B cars, and 21% C cars. Find the following:
    - the 1991 percentages for the three auto manufacturers.
    - the percentage of B cars in the year 2000.
    - the steady-state matrix.



7. **Statistics** Last winter a certain strain of flu was studied by medical researchers. They determined that from one five-day period to the next, a well person had a 20% chance of getting the flu. Of those persons who had the flu during a five-day period, 50% still had it during the next five-day period, while the other 50% recovered and became immune.
- Write a transition matrix  $T$  for the three conditions: well, flu, and immune.
  - Suppose that after Christmas vacation, 5% of the students at a college returned with the flu, 10% were immune, and 85% were well but not immune. After 5 days, what percentage of the students had the flu? What percentage were immune?
  - Find the percentages of students who were well, who had the flu, and who were immune after 10 days, after 30 days, and after 50 days.

8. Consider the transition matrix  $T = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$  and the matrix  $S = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ .

- Show that  $ST = S$ .
- Writing** Write a sentence or two explaining how part (a) shows that matrix  $S$  is the steady-state matrix.

- B** 9. The purpose of this exercise is to find without a computer the steady-state matrix  $S$  for a given transition matrix. Let  $T = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix}$ . By definition,  $S$  has the form  $S = [x \quad 1 - x]$ . Since  $ST$  must equal  $S$ , you can solve for  $x$ . Do so, and then write  $S$ .
10. Use the technique of Exercise 9 to find the steady-state matrix  $S$  for the given transition matrix  $T$ .

a.  $T = \begin{bmatrix} 0.25 & 0.75 \\ 0.45 & 0.55 \end{bmatrix}$

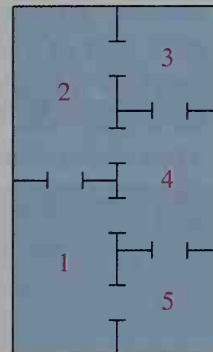
b.  $T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{4}{5} & \frac{1}{5} \end{bmatrix}$

11. If  $T$  is a  $3 \times 3$  transition matrix, then the steady-state matrix must have the form  $S = [x \quad y \quad 1 - x - y]$ . Find  $S$  if:

$$T = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

12. **Business** Refer to the shampoo problem on page 543. Recall that when the initial market share matrix was  $M_0 = [0.4 \ 0.3 \ 0.3]$ , the market share matrix for  $n$  months in the future stabilized at  $M_0 T^n = [0.59 \ 0.22 \ 0.19]$  when  $n$  was large. This matrix described the steady state of the market shares. What would this matrix be if the initial market share had been different? Consider the following two possibilities for the initial market share.
- $M_0 = [0.6 \ 0.3 \ 0.1]$
  - $M_0 = [0.8 \ 0.1 \ 0.1]$

13. The diagram at the right shows a five-room maze for mice. During a one-minute time period each mouse moves to an adjoining room. Thus from room 1, a mouse moves to either room 2, room 4, or room 5. If we assume that any of these possibilities is as likely as another, then the chances that a mouse moves from room 1 to room 2, from room 1 to room 4, or from room 1 to room 5 are all  $\frac{1}{3}$ . This means that the first



row of the transition matrix is  $\begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ .

- Complete the transition matrix.
  - If a mouse is placed in room 1, what are the chances that after one hour it will be in room 1? room 2? room 3? room 4? room 5?
  - Answer part (b) if the mouse begins in room 4.
  - Does the answer to part (b) seem to depend on where the mouse begins?
14. **Biology** Consider a species of insects that lives for four years subject to the following conditions:
- 40% of those in their first year survive to the second year.
  - 50% of those in their second year survive to the third year.
  - 30% of those in their third year survive to the fourth year.
  - A female alive in the fourth year lays many eggs of which only an average of 40 will hatch the next spring. Assume that each male mates with exactly one female and that there are equal numbers of males and females. Thus, for each insect, male or female, alive in the fourth year, 20 new insects are produced the next year.
- The above information has been incorporated into the matrix  $T$ . The 0.4 in the first row means that 40% of those in their first year survive to the second. Since it is impossible to go directly from year 1 to year 3, a zero appears in the third column of row 1. Explain the other elements in  $T$ .

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.3 \\ 20 & 0 & 0 & 0 \end{bmatrix} \end{matrix} = T$$



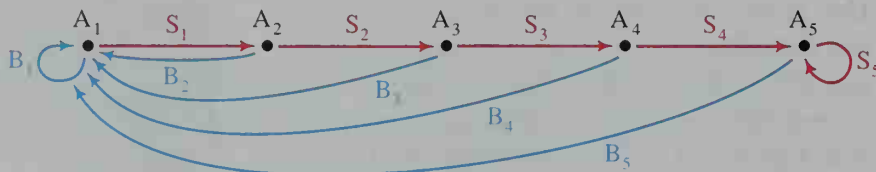
- Suppose that in a garden area the number of insects alive in their first, second, third, and fourth years is given by the population matrix  $M_0 = [800 \ 300 \ 200 \ 100]$ . The population one year later is  $M_1 = M_0 T$ . Calculate this product and explain the meaning of each element. Assume that the insects and their offspring continue to live in the same garden area.
- The population two years later is  $M_2 = M_1 T = M_0 T^2$ . Calculate  $M_2$ .
- Find the insect population ten years later.

15. **Biology** Suppose that the insects in Exercise 14 are able to reproduce in their third year as well as in their fourth year. Suppose also that the birthrate is the same for the three-year-olds as it is for the four-year-olds: 20 new insects for each older one. Change one element of  $T$  to account for this new information. Then answer parts (b), (c), and (d) of Exercise 14.

- C** 16. **Biology** Consider the following population cycle for beavers that are subdivided into five age categories:

$A_1$  (0–2 years),  $A_2$  (2–4 years),  $A_3$  (4–6 years),  $A_4$  (6–8 years), and  $A_5$  (8–10 years)

The numbers  $S_1, S_2, S_3, S_4$ , and  $S_5$  are the survival rates from one age group to the next; for example,  $S_2$  gives the percentage that survives from age group  $A_2$  to  $A_3$ . Since beavers do not live more than 10 years,  $S_5 = 0$ . The numbers  $B_1, B_2, B_3, B_4$ , and  $B_5$  are the birthrates for each age group; for example, if  $B_3 = 1.6$ , then for each beaver in age group  $A_3$ , an average of 1.6 beavers will be born into age group  $A_1$  during the next two-year period. Since very young beavers cannot reproduce,  $B_1 = 0$ . The following diagram illustrates this population cycle.



Write a  $5 \times 5$  matrix  $T$  that gives the population rates for moving from one age group to another over a two-year period. (*Hint:* The first row of  $T$  is  $[B_1 \ S_1 \ 0 \ 0 \ 0]$ .)

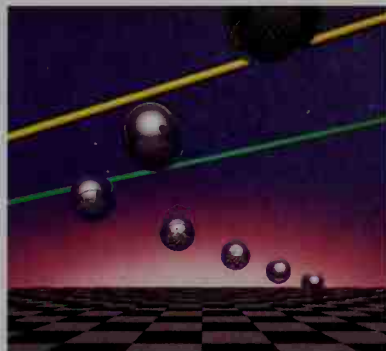
17. **Biology** Use the matrix found in Exercise 16 to solve the following problem: Let the birthrates  $B_1 = 0$ ,  $B_2 = 1.2$ ,  $B_3 = 1.5$ ,  $B_4 = 2.0$ ,  $B_5 = 0.2$  and the survival rates  $S_1 = 0.4$ ,  $S_2 = 0.6$ ,  $S_3 = 0.7$ ,  $S_4 = 0.3$ ,  $S_5 = 0$ . Suppose that in a certain locale, the population for the five age groups is  $M_0 = [58 \ 68 \ 70 \ 48 \ 30]$ .
- Find the population for the various age groups 20 years from now. Note that 20 years represents 10 two-year periods.
  - When will the population reach 1000 or more? (To find the total of the elements in a  $1 \times 5$  population matrix, multiply it on the right by a  $5 \times 1$  matrix whose elements are all 1.)

# 14-6 Transformation Matrices

## Objective

To find the images of points under different types of transformations using matrices.

Many movies and TV programs have introductions in which computer graphics are used to enlarge or rotate an image or printed message. Often the image or message seems to be transformed into three dimensions even though you are looking at a two-dimensional screen. Geometric transformations in two dimensions can be represented by  $2 \times 2$  matrices and three-dimensional transformations can be represented by  $3 \times 3$  matrices.



In Chapter 4 you learned that a function is a correspondence or rule that assigns to every element in one set of objects (the domain) exactly one element in a second set of objects (the range). For most of the functions studied in this book, the domain and range have been sets of numbers. If the domain and range are sets of points, then the function is usually called a **transformation of the plane**.

Consider the two-dimensional transformation  $T$  that maps each point  $P(x, y)$  to its image point  $P'(x + 2y, 3x + y)$ . Under this *linear transformation*, line segments are mapped onto line segments, so the effect of  $T$  on an  $n$ -sided figure is easily found by finding just the images of the vertices of the figure. For example, consider  $\triangle ABC$  shown below. To find its image,  $\triangle A'B'C'$ , we find the images of  $A(0, 2)$ ,  $B(3, 1)$ , and  $C(2, -3)$  using  $T: (x, y) \rightarrow (x + 2y, 3x + y)$  as follows:

$$\begin{array}{lll} \text{Since } A = (0, 2), & T: (0, 2) \rightarrow (0 + 2(2), 3(0) + 2), & \text{so } A' = (4, 2). \\ \text{Since } B = (3, 1), & T: (3, 1) \rightarrow (3 + 2(1), 3(3) + 1), & \text{so } B' = (5, 10). \\ \text{Since } C = (2, -3), & T: (2, -3) \rightarrow (2 + 2(-3), 3(2) + (-3)), & \text{so } C' = (-4, 3). \end{array}$$

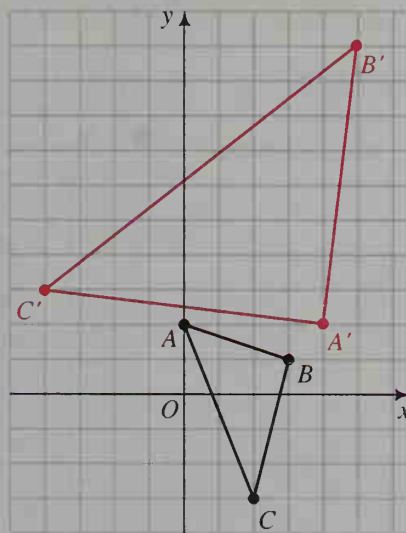
Plot  $A'(4, 2)$ ,  $B'(5, 10)$ , and  $C'(-4, 3)$  and draw  $\triangle A'B'C'$ . Notice that  $\triangle A'B'C'$  has the reverse orientation to that of  $\triangle ABC$ : reading the vertices in alphabetical order, one must move clockwise around  $\triangle ABC$  and counterclockwise around  $\triangle A'B'C'$ .

In the transformation  $T$  above, if the image of  $P(x, y)$  is  $P'(x', y')$ , then:

$$\begin{aligned} x + 2y &= x' \\ 3x + y &= y' \end{aligned}$$

These equations can be simplified by writing a single matrix equation.

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$



It is customary to use the letter  $T$  to represent both the transformation and its corresponding matrix.

$$T = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \text{ represents } T: (x, y) \rightarrow (x + 2y, 3x + y).$$

Although we have already found the images of  $A$ ,  $B$ , and  $C$  using the transformation  $T: (x, y) \rightarrow (x + 2y, 3x + y)$ , a more efficient method is to multiply the transformation matrix  $T$  by a matrix whose columns give the  $x$ - and  $y$ -coordinates of  $A$ ,  $B$ , and  $C$ .

$$\begin{array}{c} T \quad A \quad B \quad C \end{array} \quad \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix} = \begin{array}{c} A' \quad B' \quad C' \\ \begin{bmatrix} 4 & 5 & -4 \\ 2 & 10 & 3 \end{bmatrix} \end{array}$$

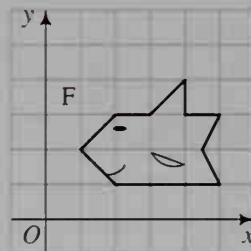
### Example 1

Consider the transformation

$$T: (x, y) \rightarrow (2x, -y),$$

and the “fish” figure  $F$  shown at the right.

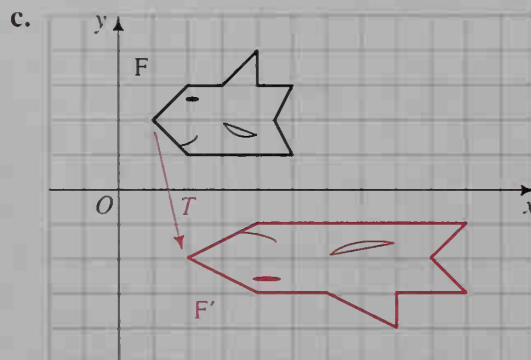
- Find the transformation matrix  $T$ .
- Using matrices, find the images of the nine points determining  $F$ .
- Plot the image points and draw  $F'$ . Compare the orientations of  $F$  and  $F'$ .



### Solution

a.  $T = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

b.  $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 4 & \frac{9}{2} & 5 & 5 \\ 2 & 1 & 3 & 3 & 3 & 4 & 2 & 1 & 3 \end{bmatrix}$   
 $= \begin{bmatrix} 2 & 4 & 4 & 6 & 8 & 8 & 9 & 10 & 10 \\ -2 & -1 & -3 & -3 & -3 & -4 & -2 & -1 & -3 \end{bmatrix}$



$F$  and  $F'$  have opposite orientations.



One advantage of using matrices is that the determinant of the transformation matrix gives two useful pieces of information, as you will discover by completing the following activity.

### Activity

- By counting squares, approximate the areas of  $F$  and  $F'$  in the diagram given in the solution to Example 1(c).
- Find the ratio of the area of  $F'$  to the area of  $F$ .
- Find the determinant of  $T$ .
- Compare your answers to parts (b) and (c).

Depending on how good your approximations are in the previous activity, you can see that the following theorem is true for the transformation in Example 1.

### Theorem 1

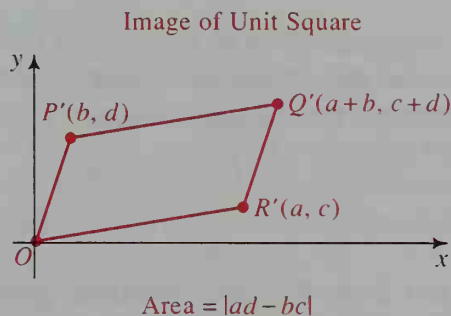
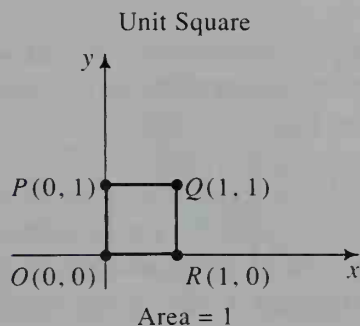
Let  $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  represent a linear transformation that maps a region  $R$  to an image region  $R'$ . Then:

- $\frac{\text{area of } R'}{\text{area of } R} = \text{absolute value of the determinant of } T$ .
- If the determinant of  $T$  is positive, then  $R'$  and  $R$  have the same orientation.

If the determinant of  $T$  is negative, then  $R'$  and  $R$  have opposite orientations.

If the determinant of  $T$  is zero, then  $R'$  lies on a line through the origin.

The reason behind the first statement of Theorem 1 can be seen by examining the unit square under the transformation  $T: (x, y) \rightarrow (ax + by, cx + dy)$ . Its image is a parallelogram, whose area is  $|ad - bc|$  times the area of the original figure. You are asked to prove this fact in Written Exercise 30.



## Theorem 2

The columns of a transformation matrix are the images of the points  $(1, 0)$  and  $(0, 1)$ , respectively.

**Proof:** Let  $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and}$$

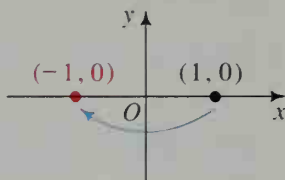
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Therefore  $T: (1, 0) \rightarrow (a, c)$  and  $T: (0, 1) \rightarrow (b, d)$ .

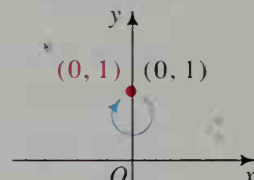
Theorem 2 gives a method for finding the corresponding matrix of a transformation described in words. Example 2 illustrates this method.

**Example 2** Let  $S$  be a reflection in the  $y$ -axis. Find the matrix  $S$ .

**Solution** Visualize what happens to the points  $(1, 0)$  and  $(0, 1)$  when each is reflected in the  $y$ -axis. Then use Theorem 2 to write the matrix  $S$ .



$$\text{image of } \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



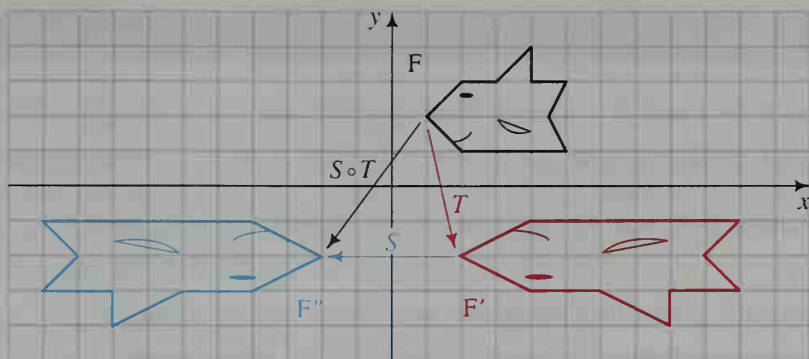
$$\text{image of } \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\therefore S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ represents the reflection in the } y\text{-axis.}$$

Suppose  $T: F \rightarrow F'$  and  $S: F' \rightarrow F''$ . The single transformation that maps  $F$  to  $F''$  is called the composite of  $S$  and  $T$ , written as  $S \circ T$ . Recall that with composition of functions, the rightmost function is performed first.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) & (S \circ T)(F) &= S(T(F)) \\ & & &= S(F') \\ & & &= F'' \end{aligned}$$

If  $T$  and  $S$  are the transformations given in Examples 1 and 2, the composite mapping of fish  $F$  is shown on the next page.



The next theorem, illustrated in Written Exercise 5, gives a nice connection between composition of functions and multiplication of matrices.

### Theorem 3

Let  $S$  and  $T$  be transformations with corresponding matrices

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } T = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

Then the matrix representing the transformation  $S \circ T$  is the matrix  $ST$ , where

$$ST = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

### Example 3

Consider the transformations given in Examples 1 and 2. Find the corresponding matrix for the transformation  $S \circ T$ .

### Solution

Refer to  $S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $T = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$  given in Examples 2 and 1.

Then by Theorem 2, the matrix representing the transformation  $S \circ T$  is

$$ST = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

You will see that the transformation matrices given in the exercises correspond to the transformations you studied in Chapter 4: reflections, translations, and expansions (stretching) or contractions (shrinking). In the Written Exercises you will explore another basic transformation that is not as familiar, namely, a rotation. New transformations can be created by the composition of these four basic transformations in any combination. For example,  $T: (x, y) \rightarrow (2x, -y)$ , given in Example 1, is the composite of an expansion and a reflection.

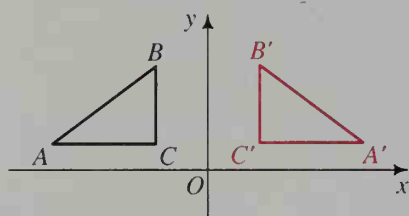
## CLASS EXERCISES

- Let  $g: 5 \rightarrow 3$  and  $f: -1 \rightarrow 5$ . Find  $(g \circ f)(-1)$ .
- Let  $g: x \rightarrow 2x - 4$  and  $f: x \rightarrow -x + 1$ . Find  $(g \circ f)(x)$ .

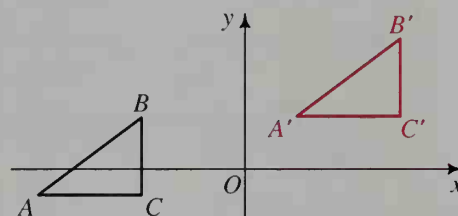
For Exercises 3–6, use  $T: (x, y) \rightarrow (x', y')$ , where  $3x - 2y = x'$   
 $4x + y = y'$ .

- Express the given two equations as a single matrix equation.
- State the transformation matrix  $T$  and find its determinant.
- If  $T$  maps  $\triangle ABC$  to  $\triangle A'B'C'$ , how do the areas compare?
- Do the triangles have the same or opposite orientation?

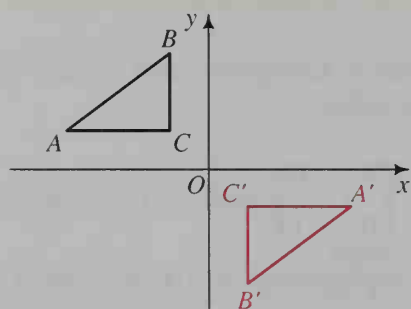
The graphs below illustrate four basic transformations. (An expansion and a contraction are considered the same type of transformation: a *dilation*.) Compare  $\triangle ABC$  and  $\triangle A'B'C'$  in each graph. Then complete the table.



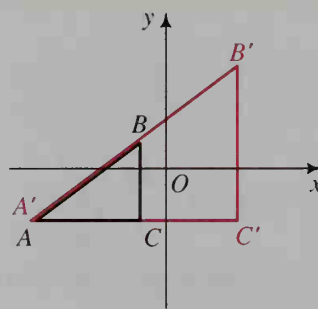
reflection



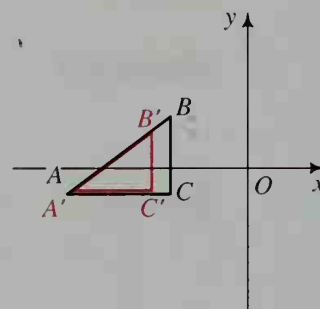
translation



rotation



expansion



contraction

	Transformation	Changes orientation	Changes size	Changes shape
	Reflection	yes	no	no
7.	Translation	?	?	?
8.	Rotation	?	?	?
9.	Dilation	?	?	?

## WRITTEN EXERCISES

**A**

1. Consider the transformation  $T: (x, y) \rightarrow (2x, 3y)$ .
  - a. Plot points  $A(2, 1)$ ,  $B(4, 0)$ , and  $C(1, -3)$  and their images  $A'$ ,  $B'$ , and  $C'$ .
  - b. Write the transformation matrix  $T$ . Find the determinant of this matrix.
  - c. Find the ratio of the area of  $\triangle A'B'C'$  to the area of  $\triangle ABC$ .
  - d. Compare the orientations of  $\triangle ABC$  and  $\triangle A'B'C'$ .
2. Consider the transformation matrix  $T = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ .
  - a. Draw  $\triangle ABC$  with vertices  $A(0, 3)$ ,  $B(2, 4)$ , and  $C(5, 0)$ .
  - b. Write a matrix  $M$  that gives the coordinates of points  $A$ ,  $B$ , and  $C$ . Calculate the product  $TM$  to find the coordinates of  $A'$ ,  $B'$ , and  $C'$ .
  - c. Draw  $\triangle A'B'C'$  on the same set of axes used in part (a).
  - d. Find the determinant of  $T$ . What two things does this determinant tell you about  $\triangle ABC$  and  $\triangle A'B'C'$ ?
3.
  - a. Draw  $\triangle PQR$  with vertices  $P(-1, 0)$ ,  $Q(0, 3)$ , and  $R(2, -3)$ .
  - b. Draw  $\triangle P'Q'R'$ , the image of  $\triangle PQR$  under the following transformation:  $T: (x, y) \rightarrow (2x + y, 3x + y)$ .
  - c. Find the transformation matrix  $T$  and its determinant. Then compare the areas and the orientations of the two triangles.
4.
  - a. Draw  $\triangle XYZ$  with vertices  $X(1, 0)$ ,  $Y(-1, 2)$ , and  $Z(0, 2)$ .
  - b. Draw the image of  $\triangle XYZ$  under the following transformation:  $T: (x, y) \rightarrow (4x - 6y, 2x - 3y)$ .
  - c. Find the transformation matrix  $T$  and its determinant. Explain what this tells you about the images of  $X$ ,  $Y$ , and  $Z$ .

5. Consider the point  $P(1, 2)$  and the transformation matrices

$$S = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} \text{ and } T = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}.$$

- a. Find  $(S \circ T)(P)$  by first finding  $T(P) = P'$  and then  $S(P') = P''$ .
  - b. Find  $(S \circ T)(P)$  in a different way by first finding the transformation matrix  $ST$  and then multiplying  $ST$  by the column matrix for point  $P$ .
6. Consider the point  $P(x, y)$  and the transformation matrices

$$S = \begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

Find  $(S \circ T)(P)$ .

7. **Investigation** The transformation  $T: (x, y) \rightarrow (4x + 2y, 2x + y)$  maps every point of the plane onto a line  $L$ . By experimenting with different points and their images, find the equation of  $L$ . Also find the matrix of the transformation and its determinant.

8. **Investigation** Repeat Exercise 7 for the following transformation:

$$T: (x, y) \rightarrow (5x - 15y, -x + 3y)$$



The translation  $G: (x, y) \rightarrow (x + h, y + k)$  moves every point  $P$  horizontally  $h$  units and vertically  $k$  units to its image point  $P'$ . This can be expressed by either a system of equations or a single matrix equation:

$$\begin{aligned} x + h &= x' \\ y + k &= y' \end{aligned} \qquad \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Describe the motion of each translation.

9.  $G: (3, -1) \rightarrow (8, -2)$

10.  $x + 6 = x'$   
 $y - 1 = y'$

11.  $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Express the translation  $G: P(x, y) \rightarrow P'(x', y')$  by a matrix equation.

12.  $P(2, -4); P'(3, 1)$

13.  $P(5, 3); P'(5, -1)$

14.  $P(-2, -2); P'(4, -2)$

Use the method shown in Example 2 to find the matrix that represents each transformation described.

**B** 15.  $X$ , a reflection in the  $x$ -axis

16.  $L$ , a reflection in the line  $y = x$

17.  $M$ , a reflection in the line  $y = -x$

18.  $R_{180}$ , a half-turn rotation about the origin  $O$

19.  $R_{90}$ , a  $90^\circ$  counterclockwise rotation about the origin  $O$

20.  $R_{-90}$ , a  $90^\circ$  clockwise rotation about the origin  $O$

Let  $Y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  represent a reflection in the  $y$ -axis. Use this matrix as well as the matrices found in Exercises 15–20 to do Exercises 21–26.

21. Find  $XY$  and show that the result equals  $R_{180}$ .

22. Find  $LX$  and show that the result equals  $R_{90}$ .

23. Find  $R_{90}R_{-90}$ . What special name is given to this matrix?

24. Find  $LM$  and show that the result equals  $R_{180}$ .

25. Find  $(R_{90})^2$ . Which matrix equals this product?

26. Find  $(R_{-90})^{51}$  by considering the effect of performing  $R_{-90}$  fifty-one times.

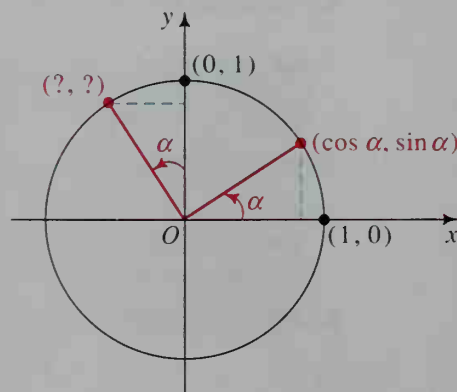
27. **Writing** Find  $X^2$ ,  $L^2$ , and  $M^2$ . Write a paragraph to explain what conclusion you can draw.

28. Consider  $R_\alpha$ , the counterclockwise rotation of  $\alpha$  degrees about the origin. The diagram shows that the image of  $(1, 0)$  under this transformation is  $(\cos \alpha, \sin \alpha)$ .

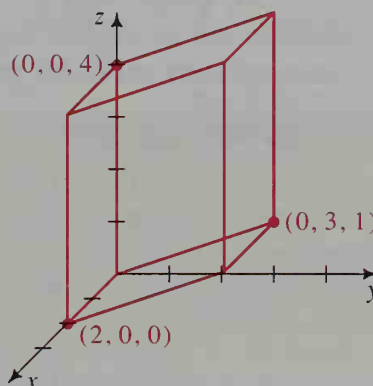
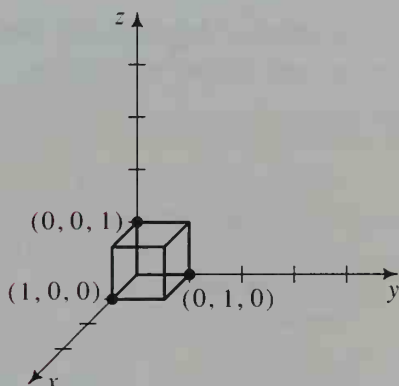
a. Find the image of  $(0, 1)$ .

b. Find the transformation matrix  $R_\alpha$ .

c. Find the determinant of the transformation matrix. What does this tell you about the area of any figure  $F$  and the area of its image  $F'$ ?



- C** 29. Let  $R_\alpha$  be the counterclockwise rotation of  $\alpha$  degrees about the origin, as shown in Exercise 28. Let  $R_\beta$  be the counterclockwise rotation of  $\beta$  degrees about the origin.
- What is the transformation matrix  $R_\beta$ ?
  - Writing** Find  $(R_\alpha)^2$  and the product  $R_\alpha R_\beta$ , and simplify your answers. Write a paragraph explaining why your answers make sense.
30. Consider the transformation  $T: (x, y) \rightarrow (ax + by, cx + dy)$ , which maps a unit square to a parallelogram.
- Refer to the unit square and its image on page 553, and verify that the coordinates of the parallelogram are correct.
  - Show that the area of the image is  $|ad - bc|$ . (*Hint*: Draw line segments from point  $Q'$  perpendicular to both axes.)
31. Consider the transformation  $T$  that maps a point  $P(x, y, z)$  in a three-dimensional coordinate system to the point  $P'(2x, 3y, y + 4z)$ .
- Give a  $3 \times 3$  matrix for  $T$ .
  - The diagrams below show a unit cube and its image, which is a parallelepiped. A parallelepiped is a polyhedron, all of whose faces are parallelograms. Guess what the determinant of  $T$  tells you about the cube and the parallelepiped.



- How are the columns of  $T$  related to the three points labeled in the diagram at the left above?
32. a. Find the transformation  $M$  for a  $90^\circ$  counterclockwise rotation about the  $z$ -axis. (*Hint*: See Exercise 31(c).)
- Find the transformation matrix  $N$  for a  $180^\circ$  rotation about the  $y$ -axis.
  - Find the product  $NM$ . Tell what transformation this product represents.
  - Investigation** Do you think that  $NM = MN$ ? Explain your answer. Then find each product to see if you are correct.
33. Consider the transformations  $Y: (x, y) \rightarrow (-x, y)$  and  $G: (x, y) \rightarrow (x + h, y + k)$ . Suppose that  $G \circ Y: P(x, y) \rightarrow P'(x', y')$ .
- Give transformation matrices  $Y$  and  $G$  and column matrices  $P$  and  $P'$ .
  - Which of the following matrix equations is correct?
    - $P' = GYP$
    - $P' = YP + G$

## Chapter Summary

1. A *matrix* is a rectangular array of numbers. These numbers are the *elements* of the matrix. A matrix with  $m$  rows and  $n$  columns has *dimension*  $m \times n$ . If you interchange the rows and columns of a matrix, you get the *transpose* of the given matrix. The matrix  $cA$  is the result of multiplying each element in a matrix  $A$  by a real number  $c$ . This operation is called *scalar multiplication*.
2. Two matrices having the same dimensions can be *matrix added* (or *subtracted*) by finding the sums (or differences) of the corresponding elements.
3. The *matrix product* of  $A_{m \times n}$  and  $B_{n \times p}$  is an  $m \times p$  matrix whose element in the  $i$ th row and  $j$ th column is the sum of the products of the corresponding elements of the  $i$ th row of  $A$  and the  $j$ th column of  $B$ . Matrices  $A$  and  $B$  can be multiplied only if the number of columns of  $A$  equals the number of rows of  $B$ .
4. Matrix properties (pages 530–532) can be used to solve matrix equations. Matrices can also be used to solve systems of linear equations (see page 533).
5.  $A^{-1}$  is called the *multiplicative inverse* of  $A$ . If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ , where  $|A| = ad - bc$ . If  $|A| \neq 0$ , then  $A^{-1}$  exists.
6. Matrices can be used to model a variety of problems. Applications include communication networks, prediction problems, and motion of graphics.

### Key vocabulary and ideas

matrix, elements of a matrix (p. 517)  
dimension of a matrix (p. 517)  
transpose of a matrix (p. 517)  
scalar multiplication (p. 518)  
matrix addition and subtraction (p. 518)

matrix product (p. 524)  
multiplicative inverse (p. 531)  
communication matrix (p. 538)  
transition matrix (p. 543)  
transformation matrix (p. 552)

## Chapter Test

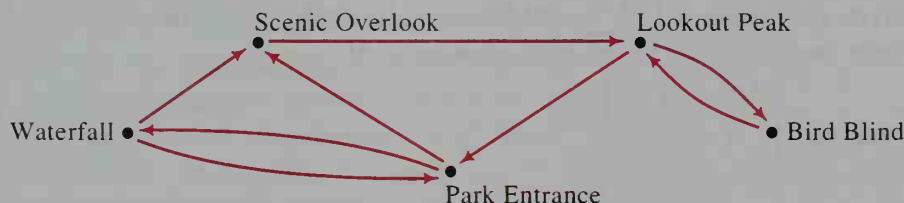
1. A store sells Mama, Papa, and Baby Bears, each in small, medium, and large sizes. One week, the store sold 7 small, 14 medium, and 10 large Mama Bears. It also sold 8 small, 20 medium, and 9 large Papa Bears and 15 small, 20 medium, and 12 large Baby Bears.  
a. Express this information in a sales matrix  $S$ .  
b. Next week, the store hopes to sell 50% more of each type of bear. Write a sales matrix that reflects the projected increase in sales. 14-1
2. If  $A = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 1 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} -3 & 4 & 6 \\ -2 & -1 & 2 \end{bmatrix}$ , find  $2A - B$ .
3. **Writing** Give two  $2 \times 2$  matrices  $A$  and  $B$  such that  $AB \neq BA$ . Use  $A$  and  $B$  to explain why matrix multiplication is not commutative. 14-2

4. Buddy's Nut Shop sells two varieties of mixed nuts. The mixes contain different amounts of peanuts, pecans, cashews, and walnuts. Matrix  $P$  below gives the portion of the total weight of each mix for each type of nut. The owner plans to change the mixes soon, and so decides to combine what is left of the two mixes and sell this combined mixture. Matrix  $N$  gives the current amounts for each mix.

$$\begin{array}{l} \text{Regular Mix} \\ \text{Deluxe Mix} \end{array} \begin{array}{cccc} \text{pt} & \text{pn} & \text{c} & \text{w} \\ \left[ \begin{array}{cccc} 40\% & 25\% & 15\% & 20\% \\ 25\% & 25\% & 25\% & 25\% \end{array} \right] = P \end{array}$$

$$\begin{array}{l} \text{Regular} \\ \text{weight in lbs} \end{array} \begin{array}{cc} \text{Deluxe} \\ \left[ \begin{array}{cc} 12 & 4 \end{array} \right] = N \end{array}$$

- Which matrix is defined,  $PN$  or  $NP$ ? Find this matrix.
  - What is the amount of cashews in the combined mixture?
5. Solve this system using matrices:  $3x + 5y = -10$  14-3  
 $2x - 7y = 45$
6. The diagram below indicates how various sites in a park are connected by roads. 14-4



- Write the matrix  $M$  that models this network. Label the rows and columns in alphabetical order.
  - Find  $M^2$ . Which site has the greatest number of two-step paths to the other sites?
7. a. A survey done by a certain city has found that from one year to the next, 8% of those living in the city will move to its suburbs while 3% of those living in its suburbs will move into the city. Write a transition matrix  $T$  that describes this situation. 14-5
- If 60% of the population is now in the city and 40% is in the suburbs, write the current population matrix  $M_0$ .
  - Find the percentages of the population living in the city and in the suburbs one year from now.
8. Consider the transformation  $T: (x, y) \rightarrow (2x + y, x - y)$ . 14-6
- Draw  $\triangle PQR$  with vertices  $P(0, -1)$ ,  $Q(1, 3)$ , and  $R(-2, 2)$ .
  - Draw  $\triangle P'Q'R'$ , the image of  $\triangle PQR$  under the transformation  $T$ .
  - Find the transformation matrix  $T$  and its determinant. Explain what this value tells you about the relationship between  $\triangle PQR$  and  $\triangle P'Q'R'$ .

1. A ship travels 300 km south from port and then 120 km on a course of  $200^\circ$ .
  - a. Make a diagram that illustrates the result of the ship's sailing.
  - b. Find the ship's current distance and course from port.
2. If  $\mathbf{u} = (2, -1)$  and  $\mathbf{v} = (-3, 4)$ , find (a)  $2\mathbf{v}$ , (b)  $3\mathbf{u} + \mathbf{v}$ , and (c)  $|\mathbf{u} - \mathbf{v}|$ .
3. a. Find a vector equation of the line through the point  $(-2, -4)$  and parallel to the line  $(x, y) = (1, 3) + t(-2, 5)$ .  
 b. What are the corresponding parametric equations for this line?
4. At time  $t$ , the position of an object with constant velocity is given by the parametric equations  $x = 1 + 2t$  and  $y = -1 - 4t$ .
  - a. What are the velocity and speed of the object?
  - b. When and where does it cross the line  $2x - y = 11$ ?
5. Given the points  $A(1, 3)$ ,  $B(-2, -5)$ , and  $C(6, -4)$ , find the measure of  $\angle B$  to the nearest tenth of a degree.
6. Write vector and parametric equations for the line containing the points  $A(2, 0, 5)$  and  $B(-3, -1, 4)$ .
7. Given the points  $A(1, 7, 3)$  and  $B(-3, 5, 5)$ , find a Cartesian equation of the plane perpendicular to  $\overline{AB}$  at its midpoint,  $M$ .
8. Evaluate: 
$$\begin{vmatrix} 10 & 4 & 3 \\ 0 & 2 & 5 \\ -1 & -7 & 0 \end{vmatrix}$$
9. Use Cramer's rule to solve: 
$$\begin{aligned} 4x - 5y &= 2 \\ 5x - 7y &= 1 \end{aligned}$$
10. Find a vector perpendicular to the plane containing the points  $P(1, 0, 5)$ ,  $Q(-4, 3, 2)$ , and  $R(1, -1, -4)$ . Then find an equation of this plane.
11. State whether each sequence is arithmetic, geometric, or neither, and find a formula for  $t_n$ .
  - a.  $\frac{2}{3}, -1, \frac{3}{2}, -\frac{9}{4}, \dots$
  - b.  $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \dots$
  - c.  $\frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \dots$
12. A sequence is defined recursively by  $t_1 = -2$  and  $t_n = 4 + 2t_{n-1}$ .
  - a. Find the first five terms of the sequence.
  - b. Find an explicit definition for the sequence.
  - c. Find  $t_{10}$ .
13. Find  $S_{20}$  for the arithmetic series with  $t_1 = 10$  and  $t_5 = 2$ .
14. Find  $S_6$  for the geometric series with  $r = \frac{1}{3}$  and  $t_1 = 81$ .
15. Find the limit of each sequence. If the limit does not exist, say so.
  - a.  $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$
  - b.  $\frac{1}{2}, -1, 2, -4, 8, -16, \dots$
16. Evaluate the given limit or state that the limit does not exist. If the sequence approaches  $\infty$  or  $-\infty$ , say so.
  - a.  $\lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 7}{n^2 - 9}$
  - b.  $\lim_{n \rightarrow \infty} \frac{5n^3 - 2n}{n^2 + 1}$
  - c.  $\lim_{n \rightarrow \infty} [1 + (-1)^n]$



17. Find the interval of convergence for the geometric series  $2 - \frac{x}{2} + \frac{x^2}{8} - \cdots$ , and express the sum of this series in terms of  $x$ .
18. Express the repeating decimal  $0.345345345 \dots$  as a rational number.
19. Express the series  $7 + 12 + 17 + \cdots + 102$  using sigma notation, and then evaluate.

20. Use mathematical induction to prove that  $\sum_{i=1}^n \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$ .

21. Let  $A = \begin{bmatrix} -1 & 6 & -2 \\ 3 & 4 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -1 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ .

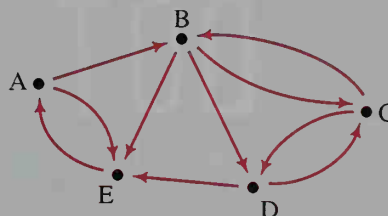
- a. Find  $2A^t - B$ .
- b. Find  $AB$  and  $BA$ . Is the matrix  $A^tB$  defined?

22. Solve this matrix equation for  $X$ :  $\begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix}X - \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -2 & 1 \end{bmatrix}$

23. Write the system of equations shown at the right as a single matrix equation, and then solve the system by solving the matrix equation.

$$\begin{aligned} 3x + 5y &= 4 \\ x + 6y &= 10 \end{aligned}$$

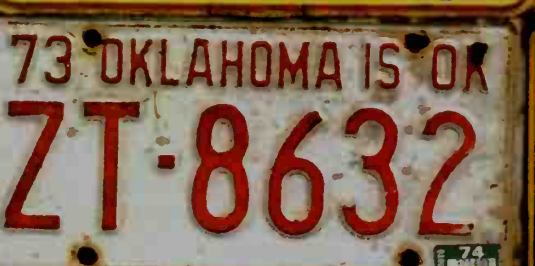
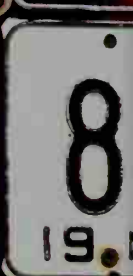
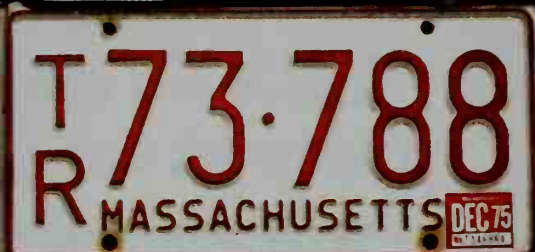
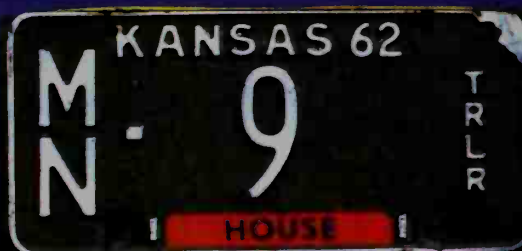
24. a. Write the communication matrix for the communication network shown at the right.
- b. Find the matrix that represents the number of ways messages can be sent from one point to another point using at most one relay.



25. A store sells three different brands of granola cereal. Brand A accounts for 20% of the granola purchases, Brand B for 50%, and Brand C for the remaining 30%. The store plans to include a discount coupon for Brand A in its advertising circular for the next week. In the past, the coupon has caused 35% of Brand B buyers and 45% of Brand C buyers to switch to Brand A, with all other purchasers staying with their original brand.
- a. Write a transition matrix that describes this situation.
- b. What will be the market share for each brand after the coupon is run?
- c. If the coupon were to run for a second week, what would be the market share of each brand after the coupon is run again?
26. a. Draw  $\triangle XYZ$  with vertices  $X(-1, 0)$ ,  $Y(3, 5)$ , and  $Z(9, 1)$ .
- b. Draw  $\triangle X'Y'Z'$ , the image of  $\triangle XYZ$  under the following transformation:  
 $T: (x, y) \rightarrow (2x - 3y, x + y)$
- c. Find the transformation matrix  $T$  and its determinant. Explain what this value tells you about the relationship between  $\triangle XYZ$  and  $\triangle X'Y'Z'$ .

# 15

## *Combinatorics*



## 15-1 Venn Diagrams

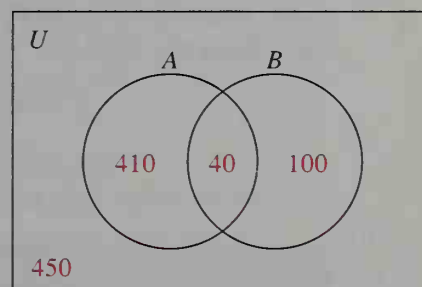
### Objective

To use Venn diagrams to illustrate intersections and unions of sets and to use the inclusion-exclusion principle to solve counting problems involving intersections and unions of sets.

In this chapter we investigate the theory of counting, more formally known as **combinatorics**. Although counting may seem to be a very simple activity—one that merely involves the pairing of the natural numbers  $1, 2, 3, \dots$  with the objects in some set—the task can be challenging when the objects are numerous. Shortcuts in counting are therefore an important part of combinatorics.

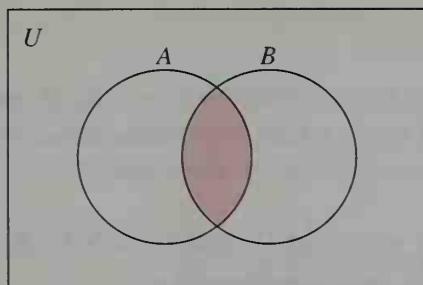
Before we can get started counting, we must know precisely what we are counting. We can use sets to separate the objects to be counted from all others, and we can use a *Venn diagram* to illustrate those sets.

When drawing a Venn diagram, we begin with a rectangle, which represents a *universal set*  $U$ . Inside the rectangle we draw circles to represent *subsets* of the universal set. For example, consider the Venn diagram at the right. Here we have a universal set of 1000 typical Americans. The set  $A$  represents those who have a certain protein called antigen A on the surface of their red blood cells. Likewise, the set  $B$  represents those who have antigen B. Of our 1000 typical Americans, a total of 450 are in set  $A$  and a total of 140 are in set  $B$ .

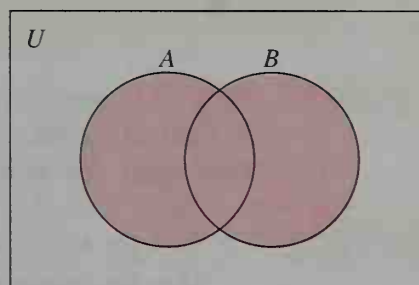


Notice that sets  $A$  and  $B$  overlap. This represents the fact that 40 of our 1000 typical Americans have *both* antigen A and antigen B on the surface of their red blood cells. We call the set of elements that any two sets  $A$  and  $B$  have in common the **intersection** of  $A$  and  $B$ , which we denote by  $A \cap B$ . It is the shaded region in the Venn diagram at the left below.

The set of people who have *either* antigen A or antigen B represents the **union** of sets  $A$  and  $B$ , which we denote by  $A \cup B$ . It is the shaded region in the Venn diagram at the right below.



$A \cap B$



$A \cup B$

◀ How many license plates can be made using three letters followed by three digits? In this chapter you will learn how to find the answer—17,576,000.

The set of all elements *not* in a set  $A$  is called the **complement** of  $A$  and is denoted  $\bar{A}$ . In the union diagram on the right at the bottom of the preceding page, the unshaded region inside the rectangle represents  $\overline{A \cup B}$ , the complement of  $A \cup B$ . This complement consists of people who have *neither* antigen A nor antigen B on the surface of their red blood cells. (Such people are said to have “type O” blood. Moreover, those with antigen A but not antigen B have “type A” blood, those with antigen B but not antigen A have “type B” blood, and those with both antigen A and antigen B have “type AB” blood.)

Look again at the first Venn diagram shown on the preceding page. If we let  $n(A \cup B)$  designate the number of elements in the union of sets  $A$  and  $B$ , we see that  $n(A \cup B) = 550$ . Notice that

$$n(A \cup B) \neq n(A) + n(B)$$

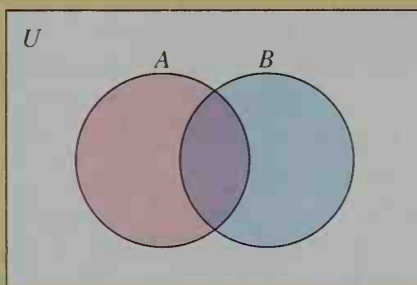
(that is,  $550 \neq 450 + 140$ ), because the number of people in  $A \cap B$  is counted twice when  $n(A)$  and  $n(B)$  are added. To compensate, we must subtract  $n(A \cap B)$  once from the sum:

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ 550 &= 450 + 140 - 40 \end{aligned}$$

This gives us the following counting principle.

### The Inclusion-Exclusion Principle

For any sets  $A$  and  $B$ ,  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ .



#### Example 1

Of the 540 seniors at Central High School, 335 are taking mathematics, 287 are taking science, and 220 are taking both mathematics and science. How many are taking neither mathematics nor science?

#### Solution

Let  $U$  = the universal set of seniors at Central High School,

$M$  = the subset of seniors taking mathematics,

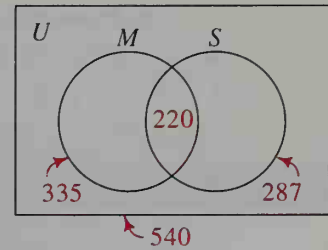
and  $S$  = the subset of seniors taking science.

These sets and the number of people in each are shown in the Venn diagram at the top of the next page.



Using the inclusion-exclusion principle, we have:

$$\begin{aligned} n(M \cup S) &= n(M) + n(S) - n(M \cap S) \\ &= 335 + 287 - 220 \\ &= 402 \end{aligned}$$



Thus, the number of seniors *not* taking mathematics or science is:

$$n(\overline{M \cup S}) = 540 - 402 = 138$$

## Example 2

In a survey, 113 business executives were asked if they regularly read the *Wall Street Journal*, *Business Week* magazine, and *Time* magazine. The results of the survey are as follows:

88 read the *Journal*.

76 read *Business Week*.

85 read *Time*.

42 read all three.

6 read only the *Journal*.

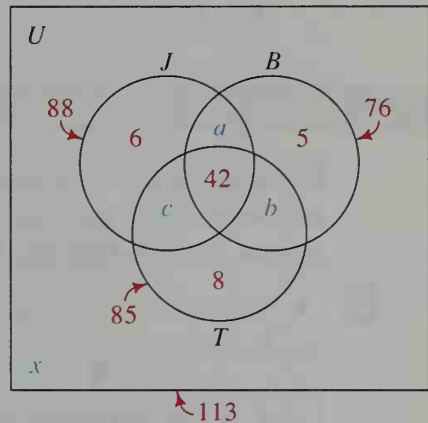
5 read only *Business Week*.

8 read only *Time*.

How many read none of the three publications?

## Solution

Letting  $J$ ,  $B$ , and  $T$  represent the sets of *Journal*, *Business Week*, and *Time* readers, we draw a Venn diagram and place the numbers given above in the diagram, as shown at the right. (Be sure you understand the placement of each number in the diagram.) To find the numbers for the remaining regions of the diagram, we let  $a$  represent the number of executives who read just the *Journal* and *Business Week*,  $b$  represent the number who read just *Business Week* and *Time*, and  $c$  represent the number who read just the *Journal* and *Time*. We can then write the following system of equations:



$$\begin{aligned} 88 &= 6 + 42 + a + c, & \text{or} & \quad 40 = a + c \\ 76 &= 5 + 42 + a + b, & \text{or} & \quad 29 = a + b \\ 85 &= 8 + 42 + b + c, & \text{or} & \quad 35 = b + c \end{aligned}$$

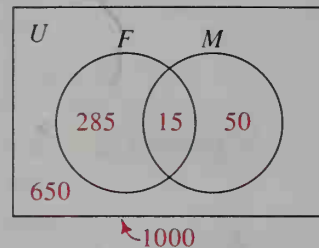
Solving this system gives  $a = 17$ ,  $b = 12$ , and  $c = 23$ . The number  $x$ , representing those who read none of the three publications, can be found by adding the numbers within the three circles and subtracting the sum from 113. This gives  $x = 0$ , which means that every one of the business executives surveyed reads at least one of the three publications.



In Example 2 we found that the set  $\overline{J \cup B \cup T}$  has no elements in it. Such a set is called the **empty set** and is denoted by the Greek letter  $\phi$  (phi). Thus,  $\overline{J \cup B \cup T} = \phi$ .

## CLASS EXERCISES

A certain small college has 1000 students. Let  $F$  = the set of college freshmen, and let  $M$  = the set of music majors. These sets are shown in the Venn diagram at the right. Describe each of the following sets in words and tell how many members it has.



1.  $F \cap M$
2.  $F \cup M$
3.  $\overline{F}$
4.  $\overline{M}$
5.  $\overline{F} \cap M$
6.  $F \cap \overline{M}$
7.  $\overline{F \cup M}$
8.  $F \cup \overline{M}$
9. **Reading** Refer to the first Venn diagram on page 565. People who have "type A" blood belong to the set  $A \cap \overline{B}$ . To what set do people with each of the following blood types belong?
  - a. B
  - b. AB
  - c. O
10. If  $A$  is any subset of a universal set  $U$ , complete the following.
  - a.  $A \cup \phi = \underline{\quad ? \quad}$
  - b.  $A \cap \phi = \underline{\quad ? \quad}$
  - c.  $A \cup \overline{A} = \underline{\quad ? \quad}$
  - d.  $A \cap \overline{A} = \underline{\quad ? \quad}$
  - e.  $A \cup U = \underline{\quad ? \quad}$
  - f.  $A \cap U = \underline{\quad ? \quad}$

## WRITTEN EXERCISES

For Exercises 1–4, draw a Venn diagram and shade the region representing the set given in part (a). Then draw a separate Venn diagram and shade the region representing the set given in part (b).

- A**
- |                  |                             |                                     |                                     |
|------------------|-----------------------------|-------------------------------------|-------------------------------------|
| 1. a. $P \cap Q$ | 2. a. $\overline{P} \cap Q$ | 3. a. $\overline{P \cup Q}$         | 4. a. $\overline{P \cap Q}$         |
| b. $P \cup Q$    | b. $P \cap \overline{Q}$    | b. $\overline{P} \cap \overline{Q}$ | b. $\overline{P} \cup \overline{Q}$ |

Let  $U$  = the universal set of all teachers in your school. Let the subsets of mathematics teachers, biology teachers, physics teachers, and chemistry teachers be represented by  $M$ ,  $B$ ,  $P$ , and  $C$ , respectively. Describe in words each of the following sets, and name a teacher belonging to each set if such a teacher exists in your school.

- |                           |                                 |
|---------------------------|---------------------------------|
| 5. a. $M \cup P$          | b. $M \cap \overline{P}$        |
| 6. a. $P \cap C$          | b. $\overline{P \cup C}$        |
| 7. a. $B \cup (P \cap C)$ | b. $(B \cup P) \cap (B \cap C)$ |
| 8. a. $(B \cup C) \cap M$ | b. $(B \cap M) \cup (C \cap M)$ |

9. a.  $\overline{M \cup P \cup C}$                       b.  $\overline{M \cap \bar{P} \cap \bar{C}}$   
 10. a.  $\overline{M \cap \bar{B} \cap \bar{P} \cap \bar{C}}$                       b.  $\overline{M \cup B \cup P \cup C}$

In Exercises 11–14, draw a Venn diagram to illustrate each situation described. Then use the diagram to answer the question asked.

11. In an election-day survey of 100 voters leaving the polls, 52 said they voted for Proposition 1, and 38 said they voted for Proposition 2. If 18 said they voted for both, how many voted for neither?
12. Although the weather was perfect for the beach party, 17 of the 30 people attending got a sunburn and 25 people were bitten by mosquitoes. If 12 people were both bitten and sunburned, how many had neither affliction?
13. In a survey of 48 high school students, 20 liked classical music and 16 liked bluegrass music. Twenty students said they didn't like either. How many liked classical but not bluegrass?
14. Of the 52 teachers at Roosevelt High School, 27 said they like to go sailing, 25 said they like to go fishing, and 12 said they don't enjoy either recreation. How many enjoy fishing but not sailing?



- B** 15. **Astronomy** Consider the sets defined below. (You may need to consult an encyclopedia to determine the elements of each set.)

Let  $U$  = the universal set of planets in our solar system,  
 $S$  = the subset of planets smaller than Earth,  
 and  $F$  = the subset of planets farther from the Sun than Earth.

- a. Draw a Venn diagram with overlapping circles representing  $S$  and  $F$  inside a rectangular region  $U$ . Inside each of the four regions of your diagram, list the planets described by that region.
- b. The smallest set to which the planet Venus belongs is  $S \cap \bar{F}$ . What is the smallest set to which the planet Uranus belongs?
16. **Geography** Consider the sets defined below.

Let  $U$  = the universal set of states in the United States,  
 $P$  = the subset of states bordering the Pacific Ocean,  
 and  $M$  = the subset of states bordering Mexico.

- a. Name all the states in the set  $\bar{P} \cap M$ .
- b. What is the only state in the set  $\overline{P \cap M}$ ?
- c. How many states are in the set  $\overline{P \cup M}$ ?

17. **Geography** Consider the sets defined below. (You may need to consult an encyclopedia or atlas to determine the elements of each set.)

Let  $U$  = the universal set of 13 countries in South America,

$A$  = the subset of South American countries bordering the Atlantic Ocean or the Caribbean Sea,

$P$  = the subset of South American countries bordering the Pacific Ocean,

and  $R$  = the subset of countries through which the Amazon River runs.

- a. Name two countries that belong to  $\bar{A} \cap \bar{P} \cap \bar{R}$ .
- b. Draw a Venn diagram with overlapping circles representing  $A$ ,  $P$ , and  $R$  all inside a rectangular region  $U$ . Inside each of the eight regions of your diagram, list the South American countries described by that region.
18. **Medical Science** In addition to antigens  $A$  and  $B$ , a red blood cell may have a protein called the Rhesus factor on its surface. If the protein is present, the blood is said to be “Rh positive”; if not, the blood is “Rh negative.” About 85% of the American population is “Rh positive.” Using this information, redraw the first Venn diagram on page 565 and introduce a third set,  $Rh+$ , that overlaps sets  $A$  and  $B$ . Then indicate the approximate number of people who belong to each of the eight regions of the new Venn diagram.
19. In a parking lot containing 85 cars, there are 45 cars with automatic transmissions, 43 cars with rear-wheel drive, and 46 cars with four-cylinder engines. Of the cars with automatic transmissions, 26 also have rear-wheel drive. Of the cars with rear-wheel drive, 29 also have four-cylinder engines. Of the cars with four-cylinder engines, 27 also have automatic transmissions. There are 21 cars with all three features.
- a. How many cars do not have automatic transmissions and rear-wheel drive but do have four-cylinder engines?
- b. How many cars do not have any of the three features?
20. Of the 415 girls at Gorham High School last year, 100 played fall sports, 98 played winter sports, and 96 played spring sports. Twenty-two girls played sports all three seasons while 40 played only in fall, 47 only in winter, and 33 only in spring.
- a. How many girls played fall and winter sports but not a spring sport?
- b. How many girls did not play sports in any of the three seasons?



- C

Similarly, when we say that intersection is distributive over union, we mean that for all sets  $A$ ,  $B$ , and  $C$ ,

Show that this equation is true by drawing two three-circle Venn diagrams. In one diagram shade the region corresponding to  $A \cap (B \cup C)$ , and in the other shade the region corresponding to  $(A \cap B) \cup (A \cap C)$ .

- a.  $A \cap (\bar{A} \cup B)$

**b.**  $A \cup (\overline{A \cup B})$

### Objective

In Section 15-1 we saw that Venn diagrams and the inclusion-exclusion principle are helpful in certain counting problems. In this section we consider three other counting principles.

If an action can be performed in  $n_1$  ways, and for each of these ways another action can be performed in  $n_2$  ways, then the two actions can be performed together in  $n_1 n_2$  ways.

**Example 1**

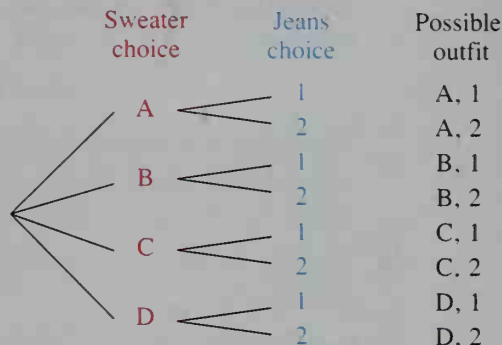
If you have 4 sweaters and 2 pairs of jeans, how many different sweater-and-jeans outfits can you make?

**Solution**

There are two actions to perform:

- (1) choosing a sweater    and    (2) choosing a pair of jeans

Since there are 4 sweaters from which to choose and 2 pairs of jeans from which to choose, there are  $4 \times 2 = 8$  ways to choose a sweater-and-jeans outfit. If we call the sweaters A, B, C, and D, and the pairs of jeans 1 and 2, the so-called *tree diagram* at the right shows the 8 possible outfits.



The multiplication principle can be extended to three or more actions that are performed together, as the next two examples show.

**Example 2**

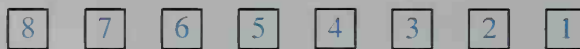
In how many ways can 8 people line up in a cafeteria line?

**Solution**

The diagram below represents the eight places in the cafeteria line.



The first place in line can be filled by any of the 8 people. Then the second place in line can be filled by any of the remaining 7 people, the third place by any of the remaining 6 people, and so on. The diagram below illustrates this reasoning.



By the multiplication principle, the answer is:

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320$$

In Example 2, the product  $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$  can be abbreviated as  $8!$  (read “8 factorial”). The definition of  $n!$ , where  $n$  is a nonnegative integer, is:

$$n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$$

$$0! = 1$$

Notice in Example 2 that no person can occupy more than one space in the cafeteria line. Thus, the spaces of the cafeteria line are filled *without repetition*. This is in contrast with the next example, where repetition does occur.



**Example 3**

How many license plates can be made using 2 letters followed by 3 digits?

**Solution**

The diagram below represents the five spaces of a license plate.



Each of the first two spaces can be filled with any of the 26 letters of the alphabet. Each of the last three spaces can be filled with any of the 10 digits. The diagram below illustrates this reasoning.



Thus, by the multiplication principle, the number of license plates is:

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676,000$$

In Example 3, suppose that each of the five spaces of a license plate can be filled with *either* a letter *or* a digit. For any given space, then, there are 26 ways of choosing a letter, 10 ways of choosing a digit, and  $26 + 10 = 36$  ways of filling the space. Note that we *add* the number of ways of performing the actions of choosing a letter and choosing a digit because the actions are **mutually exclusive**, that is, they cannot be performed together. This leads us to the following principle.

### The Addition Principle

If two actions are mutually exclusive, and the first can be done in  $n_1$  ways and the second in  $n_2$  ways, then one action *or* the other can be done in  $n_1 + n_2$  ways.

**Example 4**

In Morse code, the letters of the alphabet are represented by sequences of dots (•) and dashes (–). For example, • – represents the letter A and – – • represents the letter G. Show that sequences of no more than 4 symbols (dots and dashes) are needed to represent all of the alphabet.

**Solution**

We can think of a 1-symbol sequence as a box to be filled with either a dot or a dash; a 2-symbol sequence as two boxes to be filled, each with a dot or a dash; and so on. Using the multiplication principle and then the addition principle, we have:

$$\begin{aligned} n_1 &= \text{number of 1-symbol sequences} = 2 &&= 2 \\ n_2 &= \text{number of 2-symbol sequences} = 2 \times 2 &&= 4 \\ n_3 &= \text{number of 3-symbol sequences} = 2 \times 2 \times 2 &&= 8 \\ n_4 &= \text{number of 4-symbol sequences} = 2 \times 2 \times 2 \times 2 &&= 16 \\ \text{Total number of sequences with no more than 4 symbols} &= 30 \end{aligned}$$

Thus, the 30 possible sequences are obviously enough to accommodate the 26 letters of the alphabet.

When counting the number of elements in a set, you may find it easier to count the number in the complement of the set and then use the following principle, which is illustrated in Example 5.

### The Complement Principle

If  $A$  is a subset of a universal set  $U$ , then:

$$n(A) = n(U) - n(\bar{A})$$

**Example 5** Find the number of 4-digit numbers containing at least one digit 5.

#### Solution

Let  $U$  = the universal set of all 4-digit numbers,

$A$  = the set of 4-digit numbers containing at least one digit 5,

and  $\bar{A}$  = the set of 4-digit numbers containing no 5's.

$$\begin{aligned} \text{Then: } n(A) &= n(U) - n(\bar{A}) \\ &= 9 \cdot 10 \cdot 10 \cdot 10 - 8 \cdot 9 \cdot 9 \cdot 9 \\ &= 9000 - 5832 = 3168 \end{aligned}$$

### CLASS EXERCISES

- Evaluate: a.  $2!$    b.  $3!$    c.  $4!$
- If  $9! = 362,880$ , what is  $10!$ ?
- Reading** In Example 3, suppose the first of the three digits in a license plate cannot be 0. How many such license plates are possible?
- If a girl has 6 different skirts and 10 different blouses, how many different skirt-and-blouse outfits are possible?
  - If she also has 3 different sweaters, how many skirt-blouse-and-sweater outfits are possible?
- A boy has 2 sports coats and 4 sweaters.
  - How many coat-and-sweater outfits can he wear?
  - Suppose he decides to wear either a sports coat or a sweater, but not both. How many choices does he have?
- If 10 runners compete in a race, in how many different ways can prizes be awarded for first, second, and third places?
- The "home row" of a standard typewriter gives one arrangement of the letters A, S, D, F, G, H, J, K, L.
  - How many *other* arrangements of these letters are possible?
  - If any 9 letters of the alphabet could be placed on the "home row" of a typewriter, how many arrangements of the letters would be possible?
- Reading** Give the conclusion of the complement principle in words rather than symbols. Then use a Venn diagram to illustrate this principle.

## WRITTEN EXERCISES

- A**
1. Evaluate: a.  $5!$                       b.  $6!$                       c.  $7!$                       d.  $0!$
  2. Evaluate: a.  $\frac{10!}{9!}$                       b.  $\frac{20!}{18!}$                       c.  $\frac{n!}{(n-1)!}$                       d.  $\frac{(n+1)!}{(n-1)!}$

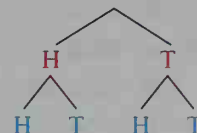
3. In how many different orders can you arrange 5 books on a shelf?
4. In how many different orders can 9 people stand in a line?
5. In how many different ways can you answer 10 true-false questions?
6. In how many different ways can you answer 10 multiple-choice questions if each question has 5 choices?

7. Many radio stations have 4-letter call signs beginning with K. How many such call signs are possible if letters (a) can be repeated? (b) cannot be repeated?



8. How many 3-digit numbers can be formed using the digits 4, 5, 6, 7, 8 if the digits (a) can be repeated? (b) cannot be repeated?
9. In how many ways can 4 people be seated in a row of 12 chairs?
10. In how many ways can 4 different prizes be given to any 4 of 10 people if no person receives more than 1 prize?

11. The tree diagram shows the four possible outcomes when a coin is tossed twice. If H and T represent “heads” and “tails,” respectively, then the four outcomes are HH, HT, TH, and TT.



- a. Make a tree diagram showing the outcomes if a coin is tossed 3 times.
- b. The solid lines of the tree diagram are called *branches* and the elements of the bottom row (H, T, H, and T) are called *leaves*. How many branches and leaves are in your diagram for part (a)?
- c. How many branches and leaves would there be in a tree diagram showing the toss of a coin 10 times?

12. Four cards numbered 1 through 4 are shuffled and 3 different cards are chosen one at a time. Make a tree diagram showing the various possible outcomes.

- B**
13. **Sports** A high school coach must decide on the batting order for a baseball team of 9 players.
    - a. The coach has how many different batting orders from which to choose?
    - b. How many different batting orders are possible if the pitcher bats last?
    - c. How many different batting orders are possible if the pitcher bats last and the team’s best hitter bats third?

14. **Sports** A track coach must choose a 4-person 400 m relay team and a 4-person 800 m relay team from a squad of 7 sprinters, any of whom can run on either team. If the fastest sprinter runs last in both races, in how many ways can the coach form the two teams if each of the 6 remaining sprinters runs only once and each different order is counted as a different team?



15. How many numbers consisting of 1, 2, or 3 digits (without repetitions) can be formed using the digits 1, 2, 3, 4, 5, 6?
16. If you have 5 signal flags and can send messages by hoisting one or more flags on a flagpole, how many messages can you send?
17. In some states license plates consist of 3 letters followed by 2 or 3 digits (for example, RRK-54 or ABC-055). How many such possibilities are there for those plates with 2 digits? for those with 3 digits? In all, how many license plates are possible?
18. How many possibilities are there for a license plate with 2 letters and 3 or 4 nonzero digits?
19. a. How many 3-digit numbers contain no 7's?  
b. How many 3-digit numbers contain at least one 7?
20. a. How many 4-digit numbers contain no 8's or 9's?  
b. How many 4-digit numbers contain at least one 8 or 9?
21. How many numbers from 5000 to 6999 contain at least one 3?
22. Many license plates in the U.S. consist of 3 letters followed by a 3-digit number from 100 to 999. How many of these contain at least one of the vowels A, E, I, O, and U?
23. Telephone numbers in the U.S. and Canada have 10 digits as follows:
- 3-digit area code number: first digit is *not* 0 or 1;  
second digit *must be* 0 or 1
- 3-digit exchange number: first and second digits are *not* 0 or 1
- 4-digit line number: *not* all zeros
- a. How many possible area codes are there?
- b. The area code for Chicago is 312. Within this area code how many exchange numbers are possible?
- c. One of the exchange numbers for Chicago is 472. Within this exchange, how many line numbers are possible?
- d. How many 7-digit phone numbers are possible in the 312 area code?
- e. How many 10-digit phone numbers are possible in the U.S. and Canada?

24. Suppose a state has 5 telephone area codes. Refer to Exercise 23 and tell how many phone numbers there could be in the state without adding any more area codes.
25. a. How many 9-letter “words” can be formed using the letters of the word FISHERMAN? (Note: We allow any arrangement of letters, such as “HAMERSNIF,” to count as a “word.” We also assume each letter is used exactly once.)
- b. How many 9-letter “words” begin and end with a vowel? (Hint: There are 3 choices for the first letter and 2 for the last letter, as shown below.)

3
?
?
?
?
?
?
?
2

26. Suppose the letters of VERMONT are used to form “words.”
- a. How many 7-letter “words” can be formed?
- b. How many 6-letter “words” can be formed? How does your answer here compare with your answer in part (a)?
- c. How many 5-letter “words” begin with a vowel and end with a consonant?
27. A school has 677 students. Explain why at least two students must have the same pair of initials.
28. a. Ohio State University has about 40,000 students. Explain why at least two students must have the same first, middle, and last initials.
- b. What is the minimum number of students needed to be *certain* that at least two students have the same three initials?
29. **Writing** Exercises 27 and 28 involve the *pigeonhole principle*, which states:

If you are putting  $x$  pigeons in  $y$  pigeonholes, and  $x > y$ , then   ?  .

Write a paragraph in which you complete the statement of the pigeonhole principle and explain its use in Exercises 27 and 28. Be sure to tell what corresponds to pigeons and what corresponds to pigeonholes in those exercises.

30. **Research** In a computer, a *bit* stores the smallest unit of information. Bits are usually arranged in groups of 8, called *bytes*, in order to store larger pieces of information. Find out what a bit is and from this determine how many different pieces of information can be stored in a byte. Then find out how “byte-sized” pieces of information are part of the ASCII (pronounced “ask’ ee”) code used by many computers for converting text characters to numbers.
31. Use your calculator to evaluate  $\log_{10} 9!$ . Then, without using your calculator, find the value of  $\log_{10} 10!$ . Check your answer with a calculator.
32. Use the properties of logarithms to evaluate  $\log_{10} 100! - \log_{10} 99!$
33. a. Show that  $10 \cdot 9 \cdot 8 \cdot 7 = \frac{10!}{6!}$ .
- b. For  $1 \leq r \leq n$ , show that:
- $$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$$
- C** 34. In how many zeros does the number  $100!$  end?



## 15-3 Permutations and Combinations

**Objective** To solve problems involving permutations and combinations.

In some situations involving choices, the order in which the choices are made is important, whereas in others it is not. Suppose that a club with 12 members wishes to choose a president, a vice president, and a treasurer. In this case, the order of the choices is important: for instance, the order A, B, and C for president, vice president, and treasurer, respectively, is different from the order B, A, and C for the 3 offices. The number of ways of filling the 3 offices is:

$$12 \cdot 11 \cdot 10 = 1320$$

Suppose, on the other hand, that the club merely wants to choose a governing council of 3. In this case, the order of selection is *not* important, since a selection of A, B, and C is the same as a selection of B, A, and C. Now for each governing council of 3 that can be chosen (for example, A, B, C), there are  $3!$  different slates of officers (ABC, ACB, BAC, BCA, CAB, CBA). Thus:

$$\begin{aligned} n(\text{governing councils}) \times 3! &= n(\text{slates of officers}) \\ n(\text{governing councils}) &= \frac{n(\text{slates of officers})}{3!} \\ &= \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220 \end{aligned}$$

In the first situation, where the order of selection is important, each selection is called a **permutation** of 3 people from a set of 12; the number of such selections is denoted  ${}_{12}P_3$ . In the second situation, where the order of selection is not important, each selection is called a **combination** of 3 people from a set of 12; the number of such selections is denoted  ${}_{12}C_3$ .

In general, for  $0 \leq r \leq n$ , the symbols  ${}_nP_r$  and  ${}_nC_r$  denote the number of permutations and the number of combinations, respectively, of  $r$  things chosen from  $n$  things. (Sometimes  $P(n, r)$  and  $C(n, r)$  are used instead of  ${}_nP_r$  and  ${}_nC_r$  in order to emphasize that permutations and combinations are *functions* of  $n$  and  $r$ .)

	Formula	Example
<b>Permutation</b> (order important)	${}_nP_r = n(n-1)(n-2) \cdots (n-r+1)$ $= \frac{n!}{(n-r)!}$ { Formula derived in Exercise 33 on page 577	${}_{12}P_3 = 12 \cdot 11 \cdot 10$ $= \frac{12!}{(12-3)!}$ $= 1320$
<b>Combination</b> (order not important)	${}_nC_r = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$ $= \frac{n!}{(n-r)!r!}$ { Formula derived in Exercise 28 on page 582	${}_{12}C_3 = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3}$ $= \frac{12!}{(12-3)! 3!}$ $= 220$

**Example 1**

A company advertises two job openings, one for a copywriter and one for an artist. If 10 people who are qualified for either position apply, in how many ways can the openings be filled?

**Solution**

Since the jobs are different, the order of selecting people matters: X as copywriter and Y as artist is different from Y as copywriter and X as artist. Thus, the solution is found by counting permutations:

$${}_{10}P_2 = 10 \cdot 9 = 90$$

**Example 2**

A company advertises two job openings for computer programmers, both with the same salary and job description. In how many ways can the openings be filled if 10 people apply?

**Solution**

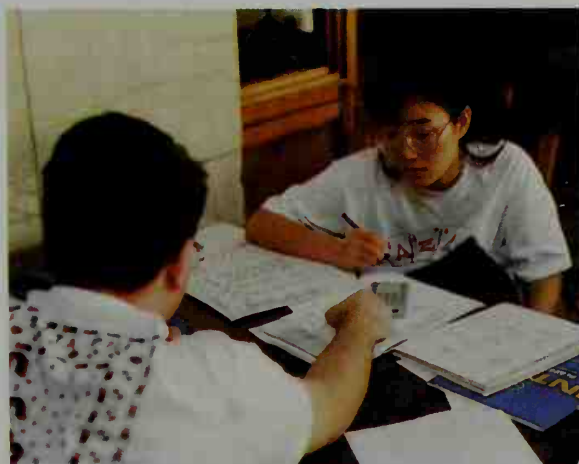
Since the two jobs are identical, the order of selecting people is not important: Choosing X and Y for the positions is the same as choosing Y and X. Thus, the solution is found by counting combinations:

$${}_{10}C_2 = \frac{10 \cdot 9}{1 \cdot 2} = 45$$

If you have a calculator with permutation and combination keys (or just a factorial key), you can use it to help solve counting problems that involve large numbers, as in the next example.

**Example 3**

As shown below, a standard deck of playing cards consists of 52 cards, with 13 cards in each of four *suits* (clubs, spades, diamonds, and hearts). Clubs and spades are black cards, and diamonds and hearts are red cards. Also, jacks, queens, and kings are called *face cards*.



Clubs (♣): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

Spades (♠): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

Diamonds (♦): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

Hearts (♥): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

How many ways are there to deal 13 cards from a standard deck if the order in which the cards are dealt is (a) important? (b) not important?

**Solution**

$$\text{a. } {}_{52}P_{13} = \frac{52!}{39!} \approx 3.95 \times 10^{21} \quad \text{b. } {}_{52}C_{13} = \frac{52!}{39!13!} \approx 6.35 \times 10^{11}$$

## CLASS EXERCISES

In Exercises 1–4, find the value of each expression.

1. a.  ${}_5P_2$                       b.  ${}_5C_2$                       2. a.  ${}_6P_3$                       b.  ${}_6C_3$
3. a.  ${}_{10}P_3$                       b.  ${}_{10}C_3$                       4. a.  ${}_4P_4$                       b.  ${}_4C_4$
5. In how many ways can a club with 10 members choose a president, a vice president, and a treasurer?
6. In how many ways can a club with 10 members choose a 3-person governing council?
7. A lock on a safe has a dial with 50 numbers on it. To open it, you must turn the dial left, then right, and then left to 3 different numbers. Such a lock is usually called a combination lock, but a more accurate name would be permutation lock. Explain.
8. a. Four people (A, B, C, and D) apply for three jobs (clerk, secretary, and receptionist). If each person is qualified for each job, make a list of the ways the jobs can be filled. For example, ABC means that A is clerk, B is secretary, and C is receptionist. This is different from BAC and CBA.  
b. Your list should contain  ${}_4P_3$  entries. How many entries involve persons A, B, and C? How many entries involve persons A, C, and D?  
c. If A, B, C, and D apply for three identical job openings as a clerk, make a list of the number of ways the openings can be filled.

## WRITTEN EXERCISES

- A**
1. a. In how many ways can a club with 20 members choose a president and a vice president?  
b. In how many ways can the club choose a 2-person governing council?
  2. a. In how many ways can a club with 13 members choose 4 different officers?  
b. In how many ways can the club choose a 4-person governing council?
  3. a. In how many ways can a host-couple choose 4 couples to invite for dinner from a group of 10 couples?  
b. Ten students each submit a woodworking project in an industrial arts competition. There are to be first-, second-, and third-place prizes plus an honorable mention. In how many ways can these awards be made?
  4. A teacher has a collection of 20 true-false questions and wishes to choose 5 of them for a quiz. How many quizzes can be made if the order of the questions is considered (a) important? (b) unimportant?
  5. Each of the 200 students attending a school dance has a ticket with a number for a door prize. If 3 different numbers are selected, how many ways are there to award the prizes, given that the 3 prizes are (a) identical? (b) different?
  6. Suppose you bought 4 books and gave one to each of 4 friends. In how many ways can the books be given if they are (a) all different? (b) all identical?

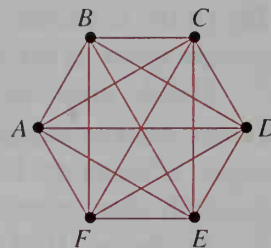
7. Eight people apply for 3 job positions. In how many different ways can the 3 positions be filled if the positions are (a) all different? (b) all the same?
8. How many different ways are there to deal a hand of 5 cards from a standard deck of 52 cards if the order in which the cards are dealt is (a) important? (b) not important?
9. In how many ways can 6 hockey players be chosen from a group of 12 if the playing positions are (a) considered? (b) not considered?
10. Of the 12 players on a school's basketball team, the coach must choose 5 players to be in the starting lineup. In how many ways can this be done if the playing positions are (a) considered? (b) not considered?
11. a. A hiker would like to invite 7 friends to go on a trip but has room for only 4 of them. In how many ways can they be chosen?  
b. If there were room for only 3 friends, in how many ways could they be chosen? How is your answer related to the answer for part (a)? Why?
12. a. In how many ways can you choose 3 letters from the word LOGARITHM if the order of letters is unimportant?  
b. In how many ways can you choose 6 letters from LOGARITHM if the order of letters is unimportant? Compare with part (a).
13. Show that  ${}_{100}C_2 = {}_{100}C_{98}$  in the following ways: (a) by using the formula for  ${}_nC_r$ ; (b) by explaining how choosing 2 out of 100 is related to choosing 98 out of 100.
14. a. Show that  ${}_{11}C_3 = {}_{11}C_8$ .  
b. Study part (a) and Exercise 13. Then make a generalization and prove it.
15. a. Evaluate  ${}_5C_0$  to find how many ways you can select no objects from a group of 5.  
b. Evaluate  ${}_5C_5$  to find how many ways you can select 5 objects from a group of 5.
16. How does the formula for  ${}_nC_n$  suggest the definition  $0! = 1$ ?

- B** 17. A certain chain of ice cream stores sells 28 different flavors, and a customer can order a single-, double-, or triple-scoop cone. Suppose on a multiple-scoop cone that the order of the flavors is important and that the flavors can be repeated. How many possible cones are there?
18. Refer to Exercise 17 and find the number of double-scoop ice cream cones that are possible if repetition of flavors is allowed but the order of flavors is unimportant.





19. Three couples go to the movies and sit together in a row of six seats. In how many ways can these people arrange themselves if each couple sits together?
20. A mathematics teacher uses 4 algebra books, 2 geometry books, and 3 precalculus books for reference. In how many ways can the teacher arrange the books on a shelf if books covering the same subject matter are kept together?
21. From a standard deck of 52 cards, 5 cards are dealt and the order of the cards is unimportant. In how many ways can you receive (a) all face cards? (b) no face cards? (c) at least one face card?
22. Answer part (c) of Exercise 21 if the order of receiving the 5 cards is important.
23. a. How many line segments can be drawn joining the six points A, B, C, D, E, and F shown?  
b. If the six points were positioned differently, the answer to part (a) could be different. Explain.
24. A convex polygon has  $n$  vertices. Find the number of diagonals by calculating the total number of ways of connecting two vertices and then subtracting the number of sides. (You may want to compare your answer with that for Exercise 25 on page 482 where difference equations were used.)
25. Solve for  $n$ :  ${}_nC_2 = 45$
26. Solve for  $n$ :  ${}_nC_2 = {}_{n-1}P_2$
27. **Investigation** Give the values of the combinations shown in the triangular array below. What patterns can you discover in this array?



$$\begin{array}{ccccccc}
 & & {}_1C_0 & {}_1C_1 & & & \\
 & & {}_2C_0 & {}_2C_1 & {}_2C_2 & & \\
 & & {}_3C_0 & {}_3C_1 & {}_3C_2 & {}_3C_3 & \\
 & & {}_4C_0 & {}_4C_1 & {}_4C_2 & {}_4C_3 & {}_4C_4 \\
 & & {}_5C_0 & {}_5C_1 & {}_5C_2 & {}_5C_3 & {}_5C_4 & {}_5C_5
 \end{array}$$

28. For  $1 \leq r \leq n$ , show that  $\frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} = \frac{n!}{(n-r)!r!}$ .
29. If the 52 cards of a standard deck are dealt to four people, 13 cards at a time, the first person can receive  ${}_{52}C_{13}$  possible hands. Then the second person can receive 13 of the remaining 39 cards in  ${}_{39}C_{13}$  possible ways. The third person can receive 13 of the remaining 26 cards in  ${}_{26}C_{13}$  ways, and the fourth person can receive 13 of the remaining 13 cards in  ${}_{13}C_{13}$  ways. Thus, the total number of ways of distributing the 52 cards into four 13-card hands is:

$${}_{52}C_{13} \cdot {}_{39}C_{13} \cdot {}_{26}C_{13} \cdot {}_{13}C_{13}$$

Without using a calculator, show that the above product simplifies to  $\frac{52!}{(13!)^4}$ .

Then use a calculator to find this number.



## ////COMPUTER EXERCISE

Write a program that outputs the values of  ${}_nP_r$  and  ${}_nC_r$  for input values of  $n$  and  $r$  where  $0 \leq r \leq n$ .

### 15-4 Permutations with Repetition; Circular Permutations

#### Objective

To solve counting problems that involve permutations with repetition and circular permutations.

#### Activity

- Write down all possible arrangements of the letters of the word MOP.
- Write down all *distinguishable* arrangements of the letters of the word MOM.
- What accounts for the fact that part (b) gives fewer permutations than part (a)?

As the preceding activity shows, fewer permutations result when some of the objects being rearranged are the same. For example, consider the two 6-letter words MEXICO and CANADA. Since the letters in the word MEXICO are all different, there are  $6! = 720$  possible arrangements of the letters. On the other hand, the word CANADA has fewer distinguishable arrangements of its letters because of the 3 identical A's. To see why, suppose we distinguish the A's by color, as follows:

CANADA

Since the six letters are now all different, there are  $6!$  arrangements of the letters, including:

NADACA	NADACA	NADACA
NADACA	NADACA	NADACA

As soon as the 3 different A's are changed back to the same A, however, the 6 different arrangements above become the *same* arrangement NADACA. Because the 3 different A's have  $3!$  different arrangements, we have:

$$\begin{aligned}n(\text{arrangements of CANADA}) &= \frac{n(\text{arrangements of CANADA})}{3!} \\&= \frac{6!}{3!} = 120\end{aligned}$$

In generalizing this discussion, let us say that CANADA has 6 letters of several different "types": 3 A's, 1 C, 1 N, and 1 D. The number of permutations is:

$$\frac{6!}{3! 1! 1! 1!}$$

The general form of this result is given at the top of the next page.

## The Number of Permutations of Things Not All Different

Let  $S$  be a set of  $n$  elements of  $k$  different types. Let  $n_1$  = the number of elements of type 1,  $n_2$  = the number of elements of type 2, ...,  $n_k$  = the number of elements of type  $k$ . Then the number of distinguishable permutations of the  $n$  elements is:

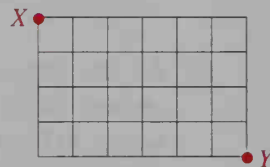
$$\frac{n!}{n_1! n_2! n_3! \cdots n_k!}$$

**Example 1** How many permutations are there of the letters of MASSACHUSETTS?

**Solution** Of the 13 letters of MASSACHUSETTS, there are 4 S's, 2 A's, 2 T's, 1 M, 1 C, 1 H, 1 U, and 1 E. Thus, the number of permutations is:

$$\frac{13!}{4! 2! 2! 1! 1! 1! 1! 1!} = 64,864,800$$

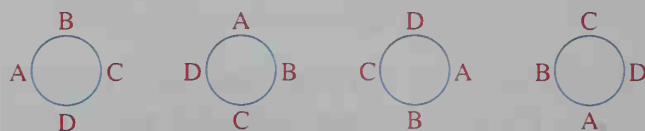
**Example 2** The grid shown at the right represents the streets of a city. A person at point  $X$  is going to walk to point  $Y$  by always traveling south or east. How many routes from  $X$  to  $Y$  are possible?



**Solution** **Method 1** To get from  $X$  to  $Y$ , the person must travel 4 blocks south (S) and 6 blocks east (E). One possible route can be symbolized by the "word" SSEEEEESEE. Other routes can be symbolized by other 10-letter words having 4 S's and 6 E's. The number of these "words" is  $\frac{10!}{4! 6!} = 210$ .

**Method 2** Every route from  $X$  to  $Y$  covers 10 blocks of which 4 must be south. The number of ways to choose which 4 of the 10 blocks are to be south is  ${}_{10}C_4 = \frac{10!}{6! 4!} = 210$ .

Thus far we have considered only *linear* permutations, but permutations may also be *circular* (or *cyclic*). For example, the diagrams below show that seating four people around a circular table is different from seating them in a row.



ABCD

DABC

CDAB

BCDA

These circular permutations are the *same*, because in each one, A is to the right of B, who is to the right of C, who is to the right of D.

These linear permutations are *different*.

**Example 3**

How many circular permutations are possible when seating four people around a table?

**Solution**

The diagrams at the bottom of the preceding page show that 1 circular permutation corresponds to 4 linear permutations. Thus:

$$\begin{aligned}\text{number of circular permutations} &= \frac{1}{4} (\text{number of linear permutations}) \\ &= \frac{1}{4} \cdot 4! \\ &= 3! = 6\end{aligned}$$

**CLASS EXERCISES**

In Exercises 1–4, find the number of permutations of the letters of each word.

1. a. MALE                      b. MALL                      2. a. MOUSE                      b. MOOSE
3. a. VERIFY                      b. VIVIFY                      4. a. SHAKEUPDOWN                      b. SHAKESPEARE
5. Find the number of permutations of the letters of:
  - a. your last name                      b. your full name
6. **Reading** Of the two solutions given for Example 2, which do you prefer? Why?
7. Refer to the diagram for Example 2 and suppose that the street corner just north of Y is Z. How many ways are there to walk from X to Z if you always travel south or east?
8. All but one of the following linear permutations correspond to the same circular permutation. Which one does not?

CDEAB

EABCD

BCDAE

DEABC

**WRITTEN EXERCISES**

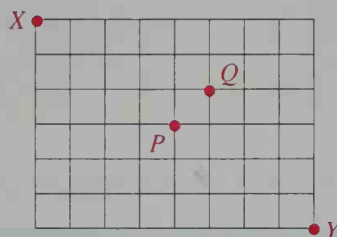
In Exercises 1–4, find the number of permutations of the letters of the given word.

**A**

1. MISSOURI                      2. PENNSYLVANIA
3. MISSISSIPPI                      4. CONNECTICUT
5. Each of the 10 finalists in the state spelling bee contest receives a prize. The prizes are six \$25 bonds, three \$50 bonds, and one \$100 bond. In how many ways can these prizes be given?
6. The 12 workers in a cafeteria crew rotate among three kinds of jobs. In how many ways can the crew be assigned the jobs of 2 cooks, 7 servers, and 3 dishwashers?

7. In the English language, *antidisestablishmentarianism* is one of the longest words. If a computer could print out every permutation of this word at the rate of one word per second, how long would it take?
8. Which of the 50 states has the least number of permutations of the letters of its name?

In Exercises 9 and 10, a person bicycles along the city streets shown at the right by always traveling south and east.



9. Find the number of possible routes from:
  - a.  $X$  to  $Y$
  - b.  $X$  to  $Q$
  - c.  $Q$  to  $Y$
  - d.  $X$  to  $Y$  via  $Q$
10. To the nearest tenth of a percent, what percent of the routes from  $X$  to  $Y$  pass through  $P$ ?
11. How many circular permutations are possible when seating 5 people around a circular table?
12. The photo shows the mythical Round Table of King Arthur. Note that the table indicates where King Arthur sat as well as where 24 knights sat. In how many ways can  $n$  people be seated around a similar table if:
  - a. a head of the table is designated?
  - b. no head of the table is designated?



- B** 13. **Meteorology** A typical January in Boston has 3 days of snow, 12 days of rain, and 16 days without precipitation. In how many ways can such weather be distributed throughout the month? [*Hint*: Think of a weather distribution pattern as a 31-letter word made up of S's (snow), R's (rain), and N's (no precipitation).]
14. In the card game of bridge, 52 cards are dealt to 4 people who are called North, East, South, and West. A 52-letter "word" consisting of 13 N's, 13 E's, 13 S's, and 13 W's will tell you who has what cards. For example, if a word begins NEEWS . . . , then North has the ace of spades, East has the 2 and 3 of spades, West has the 4 of spades, South has the 5 of spades, and so on. How many such arrangements are there? (Compare your answer with the method shown in Exercise 29 of Section 15-3.)
15. A teacher's grade book alphabetically lists 25 students in a class. This teacher, who does not give pluses and minuses with letter grades, announces that on the last test there were 5 A's, 10 B's, 6 C's, 3 D's, and 1 E.
  - a. Explain how a 25-letter "word" could tell who got what grade.
  - b. In how many ways can these grades be distributed to the 25 students?

16. A group of 4 people orders from a restaurant menu that lists 3 main courses: fish, chicken, and steak. How many combinations of main-course orders are possible in this group? (*Hint:* You can form “words” by using 4 x’s to tally the orders and 2 bars to separate the three types of orders. For example, the “word”  $xx|x|x$  represents 2 orders of fish, 1 order of chicken, and 1 order of steak. Similarly, the “word”  $xxx||x$  represents 3 orders of fish, 0 orders of chicken, and 1 order of steak.)
17. A group of 12 friends goes to a cinema complex that is showing 6 different movies. If the group splits up into subgroups based on movie preferences, how many subgroup combinations are possible? (See the hint for Exercise 16.)
- C** 18. The word ABSTEMIOUS contains the 5 vowels A, E, I, O, and U in alphabetical order. How many permutations of this word have the vowels in alphabetical order?

## MIXED COMBINATORICS EXERCISES

The following exercises are counting problems like those from the last four sections. They test your ability to choose the appropriate counting technique.

- A** 1. Three identical door prizes are to be given to three lucky people in a crowd of 100. In how many ways can this be done?
2. The license plates in a certain state consist of 3 letters followed by 3 nonzero digits. How many such license plates are possible?
3. How many 4-digit numbers (a) contain no 0’s? (b) contain no 1’s? (c) begin with an even digit and end with an odd digit?
4. A student must take four final exams, scheduled by computer, during the morning and afternoon testing periods on Monday through Friday of exam week. If the order of the student’s four exams is important, in how many ways can a computer schedule the exams?
5. a. A railway has 30 stations. On each ticket, the departure station and the destination station are printed. How many different tickets are possible?  
 b. If a ticket can be used in either direction between two stations, how many different tickets are needed?
6. In how many ways can the letters of each of the following words be arranged?  
 a. RADISH  
 b. SQUASH  
 c. TOMATO





7. There are 3 roads from town A to town B, 5 roads from town B to town C, and 4 roads from town C to town D. How many ways are there to go from A to D via B and C? How many different round trips are possible?
8. A teacher must pick 3 high school students from a class of 30 to prepare and serve food at the junior high school picnic. How many choices are possible?
9. All students at John Jay High School must take at least one of the school's three science courses: biology, chemistry, and physics. In order to project the future enrollment in these courses, the principal sent questionnaires to 297 junior high school students and got these results:

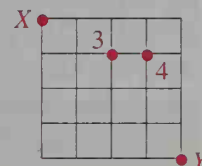
132 intend to take biology and chemistry.  
 107 intend to take chemistry and physics.  
 88 intend to take biology and physics.  
 43 intend to take only biology.  
 55 intend to take only chemistry.  
 38 intend to take only physics.

How many students intend to take all three science courses?

10. A radio show plays the top 10 musical hits of the previous week. During the first full week of last January, these 10 hits were chosen from 70 possibilities, and they were played in order of increasing popularity. How many possible orders were there? *Note:* Since calculators cannot evaluate large factorials, use the approximation given by the formula  $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ .
- B** 11. The locks on the gymnasium lockers have dials with numbers from 0 to 39. Each locker combination consists of 3 numbers.
  - a. Pythagoras remembers that his numbers are 8, 15, and 17, but he can't remember in which order they appear. If he can try one possibility every 10 s, what is the maximum amount of time that it would take him to find the right combination?
  - b. Hypatia cannot remember any of her combination numbers, but she does remember that exactly two of them are the same. If she can try one possibility every 10 s, what is the maximum amount of time that it would take her to find the right combination?
12. In how many ways can 8 jackets of different styles be hung:
  - a. on a straight bar?
  - b. on a circular rack?
13. A town council consists of 8 members including the mayor.
  - a. How many different committees of 4 can be chosen from this council?
  - b. How many of these committees include the mayor?
  - c. How many do not include the mayor?
  - d. Verify that the answer to part (a) is the sum of the answers to part (b) and part (c).
14. Repeat Exercise 13 if the council has 9 members including the mayor.
15. If you have a \$1 bill, a \$5 bill, a \$10 bill, and a \$20 bill, how many different sums of money can you make using one or more of these bills?

16. **Communications** In Morse code, letters, digits, and various punctuation marks are represented by a sequence of dots and dashes. Sequences can be from 1 unit to 6 units in length. How many such sequences are possible?
17. The Pizza Place offers pepperoni, mushrooms, sausages, onions, anchovies, and peppers as toppings for their regular plain pizza. How many different pizzas can be made?
18. a. How many 4-letter “words” can be formed by using the 8 letters of the word TRIANGLE?  
 b. How many of the “words” formed in part (a) have no vowels?  
 c. How many of the “words” formed in part (a) have at least one vowel?
19. Five boys and five girls stand in a line. How many arrangements are possible (a) if all of the boys stand in succession? (b) if the boys and girls stand alternately?
20. Answer Exercise 19 if the 5 boys and 5 girls stand in a circle instead of a line.
21. How many 5-digit numbers contain at least one 3?

22. In the diagram at the right, all paths go from  $X$  toward  $Y$  and consist only of south and east steps. For example, there are 3 paths that go from  $X$  to the point marked with a 3. Similarly, there are 4 paths that go from  $X$  to the point marked with a 4.



- a. Explain how the numbers 3 and 4 are obtained.
- b. Copy the grid and label each point with the number of paths that go from  $X$  to that point.

- C** 23. **Sports** In the World Series, two teams, A and B, play each other until one team has won 4 games. For example, the “word” ABBAAB represents a 6-game series in which team A wins games 1, 4, 5, and 6.

- a. Explain why the number of different 6-game series won by team A is  $\frac{5!}{3! 2!}$ .

- b. Without actually listing the various series between team A and team B, show that there are 70 different sequences of games possible.

24. **Sports** In the tennis championship between players A and B at Wimbledon, the first player to win 3 sets is champion. Find the number of different ways for player A to win the championship.



## 15-5 The Binomial Theorem; Pascal's Triangle

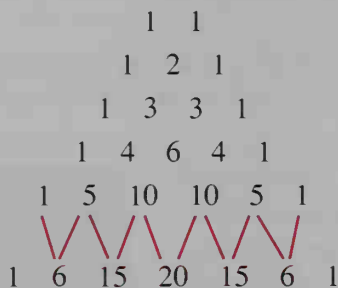
### Objective

To use the binomial theorem and Pascal's triangle.

Our goal in this section is to derive a formula for expanding  $(a + b)^n$  for positive integers  $n$ . For small values of  $n$  we get the following expansions:

$$\begin{aligned}(a + b)^1 &= 1a + 1b \\(a + b)^2 &= 1a^2 + 2ab + 1b^2 \\(a + b)^3 &= 1a^3 + 3a^2b + 3ab^2 + 1b^3 \\(a + b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \\(a + b)^5 &= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5\end{aligned}$$

From these examples we can see that the first term in the expansion of  $(a + b)^n$  is always  $1a^n b^0$ . In successive terms the exponents of  $a$  decrease by 1 and the exponents of  $b$  increase by 1, so that the sum of the two exponents in a term is always  $n$ . The coefficients of the terms also have a pattern, which can be seen by studying the array of numbers below. The first five rows of this array are like the array above except that the  $a$ 's,  $b$ 's, and plus signs have been omitted.



This array is called **Pascal's triangle**, named for the French mathematician Blaise Pascal (1623–1662). Because the numbers in the array are the coefficients of the terms in the expansion of  $(a + b)^n$ , they are called **binomial coefficients**. Notice that (except for the 1's) each number is the sum of the two numbers just above it. Hence, from the fifth row of the triangle, we can quickly form the sixth row, as shown above. You should now be able to write the expansion of  $(a + b)^6$ .

So far, it might seem that to get the numbers in the sixth row of Pascal's triangle you must first know the numbers in the fifth row, but this is not necessary. As the computations at the top of the next page show, each number in the sixth row can be calculated directly by formula.



Blaise Pascal

$$\text{6th row} \left\{ \begin{array}{ccccccc} {}_6C_0 & {}_6C_1 & {}_6C_2 & {}_6C_3 & {}_6C_4 & {}_6C_5 & {}_6C_6 \\ 1 & \frac{6}{1} & \frac{6 \cdot 5}{1 \cdot 2} & \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} & \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} & \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} & \frac{6!}{6!} \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array} \right.$$

You may be wondering why combinations have anything to do with the sixth row of Pascal's triangle and the expansion of  $(a + b)^6$ . To see why, consider  $(a + b)^6$  in factored form:

$$(a + b)^6 = (a + b)(a + b)(a + b)(a + b)(a + b)(a + b)$$

The term  $a^6$  in the expansion of  $(a + b)^6$  is obtained by multiplying the  $a$ 's in the 6 factors. The term containing  $a^5b$  is obtained by multiplying an  $a$  from 5 of the factors and a  $b$  from 1 of them. Since there are  ${}_6C_1 = 6$  ways to choose the factor that contributes the  $b$ , there are 6 of these  $a^5b$  terms. Likewise, the term containing  $a^4b^2$  is obtained by multiplying an  $a$  from 4 of the factors and a  $b$  from 2 of the factors. Since there are  ${}_6C_2 = 15$  ways to choose the 2 factors that contribute the  $b$ 's, there are 15 of these  $a^4b^2$  terms. A similar argument can be used to prove the following theorem.

### The Binomial Theorem

If  $n$  is a positive integer, then:

$$(a + b)^n = {}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + {}_nC_3 a^{n-3} b^3 + \cdots + {}_nC_n a^0 b^n$$

Equivalently:

$$(a + b)^n = 1a^n b^0 + \frac{n}{1} a^{n-1} b^1 + \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} b^3 + \cdots + 1a^0 b^n$$

#### Example

Give the first four terms in the expansion of  $(x - 2y)^{10}$  in simplified form.

#### Solution

First, we use the binomial theorem to find the first four terms in the expansion of  $(a + b)^{10}$ :

$$a^{10} + 10a^9b + \frac{10 \cdot 9}{1 \cdot 2} a^8b^2 + \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} a^7b^3, \text{ or} \\ a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3$$

Then, substituting  $x$  for  $a$  and  $-2y$  for  $b$  in the expression above, we obtain the first four terms in the expansion of  $(x - 2y)^{10}$ :

$$x^{10} + 10x^9(-2y) + 45x^8(-2y)^2 + 120x^7(-2y)^3, \text{ or} \\ x^{10} - 20x^9y + 180x^8y^2 - 960x^7y^3$$

## CLASS EXERCISES

1. **Visual Thinking** Study Pascal's triangle (page 590). Describe the symmetry that the numbers in the triangle have.
2. Give the seventh row of Pascal's triangle.
3. a. Find the first 4 numbers in the eighth row of Pascal's triangle.  
b. State the first 4 terms of the expansion of  $(x + y)^8$ .  
c. State the first 4 terms of the expansion of  $(x - y)^8$ .

In Exercises 4–6, use Pascal's triangle to give the expansion of each binomial.

4.  $(a - b)^3$
5.  $(a + b)^4$
6.  $(a - b)^4$
7. Find (a) the third term in the expansion of  $(x + y)^6$ , and (b) the fourth term in the expansion of  $(x + y)^9$ .

## WRITTEN EXERCISES

In Exercises 1–12, give the expansion of each binomial. Simplify your answers.

- |                                      |                                      |                                       |
|--------------------------------------|--------------------------------------|---------------------------------------|
| <b>A</b> 1. a. $(a + b)^3$           | b. $(20 + 1)^3$                      | c. $(20 - 1)^3$                       |
| 2. a. $(x + y)^4$                    | b. $(10 + 1)^4$                      | c. $(10 - 1)^4$                       |
| 3. a. $(a + b)^5$                    | b. $(a - b)^5$                       | c. $(2a + 1)^5$                       |
| 4. a. $(p + q)^6$                    | b. $(p - q)^6$                       | c. $(3p - 2)^6$                       |
| 5. a. $(x + y)^7$                    | b. $(x - y)^7$                       | c. $(x^2 - 2y)^7$                     |
| 6. a. $(a + b)^8$                    | b. $(a - b)^8$                       | c. $(2a - b^2)^8$                     |
| 7. $(x^2 - y^2)^3$                   | 8. $(2x^2 - 1)^4$                    | 9. $(x^2 + 1)^5$                      |
| 10. $\left(1 + \frac{x}{2}\right)^3$ | 11. $\left(x + \frac{1}{x}\right)^6$ | 12. $\left(2x - \frac{1}{x}\right)^8$ |

In Exercises 13–16, find the first four terms of the expansion of the given expression. Do not simplify your answers.

13.  $(a^2 - b)^{100}$
14.  $(3p + 2q)^{20}$
15.  $(\sin x + \sin y)^{10}$
16.  $(\sin x - \cos y)^{30}$
17. Find the value of  $(1.01)^5$  to the nearest hundredth by considering the expansion of  $(1 + 0.01)^5$ .
18. Find the value of  $(0.99)^5$  to the nearest hundredth by considering the expansion of  $(1 - 0.01)^5$ .
- B** 19. In the expansion of  $(a + b)^{12}$ , what is the coefficient (a) of  $a^8b^4$ ? (b) of  $a^4b^8$ ?
20. In the expansion of  $(a + b)^{20}$ , what is the coefficient (a) of  $a^{17}b^3$ ? (b) of  $a^3b^{17}$ ?



21. In the expansion of  $\left(x^2 + \frac{2}{x}\right)^{12}$ , find and simplify the constant term.
22. In the expansion of  $(a^3 - 2)^{10}$ , find the term containing  $a^{18}$ .
23. A municipal council consists of a mayor and  $n$  councilors. The council needs to choose a committee of  $k$  members.
- How many committees are possible if any council member can be chosen?
  - How many committees are possible if the mayor must be included?
  - How many committees are possible if the mayor must not be included?
  - Use parts (a), (b), and (c) to complete the following equation:

$${}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$$

- Use algebra to prove that the equation in part (d) is correct.
24. Rewrite the binomial theorem using sigma ( $\Sigma$ ) notation.

For Exercises 25 and 26, use the alternate form of Pascal's triangle shown at the right. Note the single 1 at the top of the triangle. This is sometimes called the "zeroth row" because it represents the expansion of  $(a + b)^0$ .

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

25. In what column of Pascal's triangle do the triangular numbers (see page 491) occur? Explain why they occur.
26. a. What special sequence of numbers results from summing the numbers in each row of Pascal's triangle?
- b. Consider the diagonals that originate at a 1 on the left side of Pascal's triangle and have a slope of 1. What special sequence of numbers results from summing the numbers along each diagonal?
27. **Investigation** Copy Pascal's triangle as given on page 590. Use a highlighting marker to color all the even numbers in the triangle. What patterns do you notice? (Note: You may need to add quite a few rows to the triangle before any patterns emerge.)
28. **Investigation** Repeat Exercise 27, but this time color all the numbers that are multiples of 3. What patterns do you notice?
29. a. Show that the expansion of  $(\cos \theta + i \sin \theta)^3$  simplifies to:
- $$(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \sin \theta \cos^2 \theta - \sin^3 \theta)$$
- b. Use De Moivre's theorem (see page 408) to find  $(\cos \theta + i \sin \theta)^3$ .
- c. Use parts (a) and (b) to show that:
- $$\begin{aligned}\cos 3\theta &= \cos^3 \theta - 3 \cos \theta \sin^2 \theta \\ \text{and } \sin 3\theta &= 3 \sin \theta \cos^2 \theta - \sin^3 \theta\end{aligned}$$
30. Use De Moivre's theorem and the expansion of  $(\cos \theta + i \sin \theta)^4$  to show that:

$$\begin{aligned}\cos 4\theta &= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta \\ \text{and } \sin 4\theta &= 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta\end{aligned}$$

In the binomial theorem, if we substitute 1 for  $a$  and  $x$  for  $b$ , we get:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

If  $n$  is not a positive integer, the right side is an infinite series with sum  $(1 + x)^n$  when  $|x| < 1$ . (A proof requires calculus.) Use this fact to do Exercises 31–34.

- C** 31. Show that  $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ . (As a check on your work, notice that the right side of the equation is an infinite geometric series with first term 1 and ratio  $-x$ . Thus, the series has sum  $\frac{1}{1 - (-x)} = (1 + x)^{-1}$  when  $|x| < 1$ .)
32. Show that  $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$ .
33. Show that  $(1 + x)^{1/2} \approx 1 + \frac{1}{2}x$  when  $|x|$  is small. Use the result to approximate  $\sqrt{1.04}$  and  $\sqrt{0.98}$ .
34. Show that  $(1 + x)^{1/3} \approx 1 + \frac{1}{3}x$  when  $|x|$  is small. Use the result to approximate  $\sqrt[3]{1.12}$  and  $\sqrt[3]{67}$ . (Hint:  $\sqrt[3]{67} = \sqrt[3]{64\left(1 + \frac{3}{64}\right)}$ )

## Chapter Summary

- If sets  $A$  and  $B$  are subsets of a universal set  $U$ , then:
  - the *intersection* of  $A$  and  $B$ , denoted  $A \cap B$ , consists of those elements in *both*  $A$  and  $B$ ,
  - the *union* of  $A$  and  $B$ , denoted  $A \cup B$ , consists of those elements in *either*  $A$  or  $B$ ,
  - the *complement* of  $A$ , denoted  $\bar{A}$ , consists of those elements of  $U$  that are *not* in  $A$ .

*Venn diagrams* are used to illustrate a universal set and its subsets, along with their intersections, unions, and complements.

- Four basic counting principles are presented.
  - The inclusion-exclusion principle:* For any sets  $A$  and  $B$ ,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

- The multiplication principle:* If an action can be performed in  $n_1$  ways, and for each of these ways another action can be performed in  $n_2$  ways, then the two actions can be performed together in  $n_1 n_2$  ways.
- The addition principle:* If two actions are mutually exclusive, and the first can be done in  $n_1$  ways and the second in  $n_2$  ways, then one action *or* the other can be done in  $n_1 + n_2$  ways.
- The complement principle:* If  $A$  is a subset of a universal set  $U$ , then

$$n(A) = n(U) - n(\bar{A}).$$

3. The choice of  $r$  objects from a set of  $n$  objects is called a *permutation* if the order of choosing is important and a *combination* if the order is unimportant. The number of possible choices in each case is given by:

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{and} \quad {}_nC_r = \frac{n!}{(n-r)!r!}$$

4. Let  $S$  be a set of  $n$  elements of  $k$  different types. Let  $n_1$  = the number of elements of type 1,  $n_2$  = the number of elements of type 2,  $\dots$ ,  $n_k$  = the number of elements of type  $k$ . Then the number of distinguishable permutations of the  $n$  elements is  $\frac{n!}{n_1! n_2! n_3! \cdots n_k!}$ .

5. For a positive integer  $n$ , the *binomial theorem* gives the expansion of  $(a + b)^n$ :

$$(a + b)^n = {}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + \cdots + {}_nC_n a^0 b^n$$

*Pascal's triangle* can be used to find the *binomial coefficients*,  ${}_nC_k$ .

### Key vocabulary and ideas

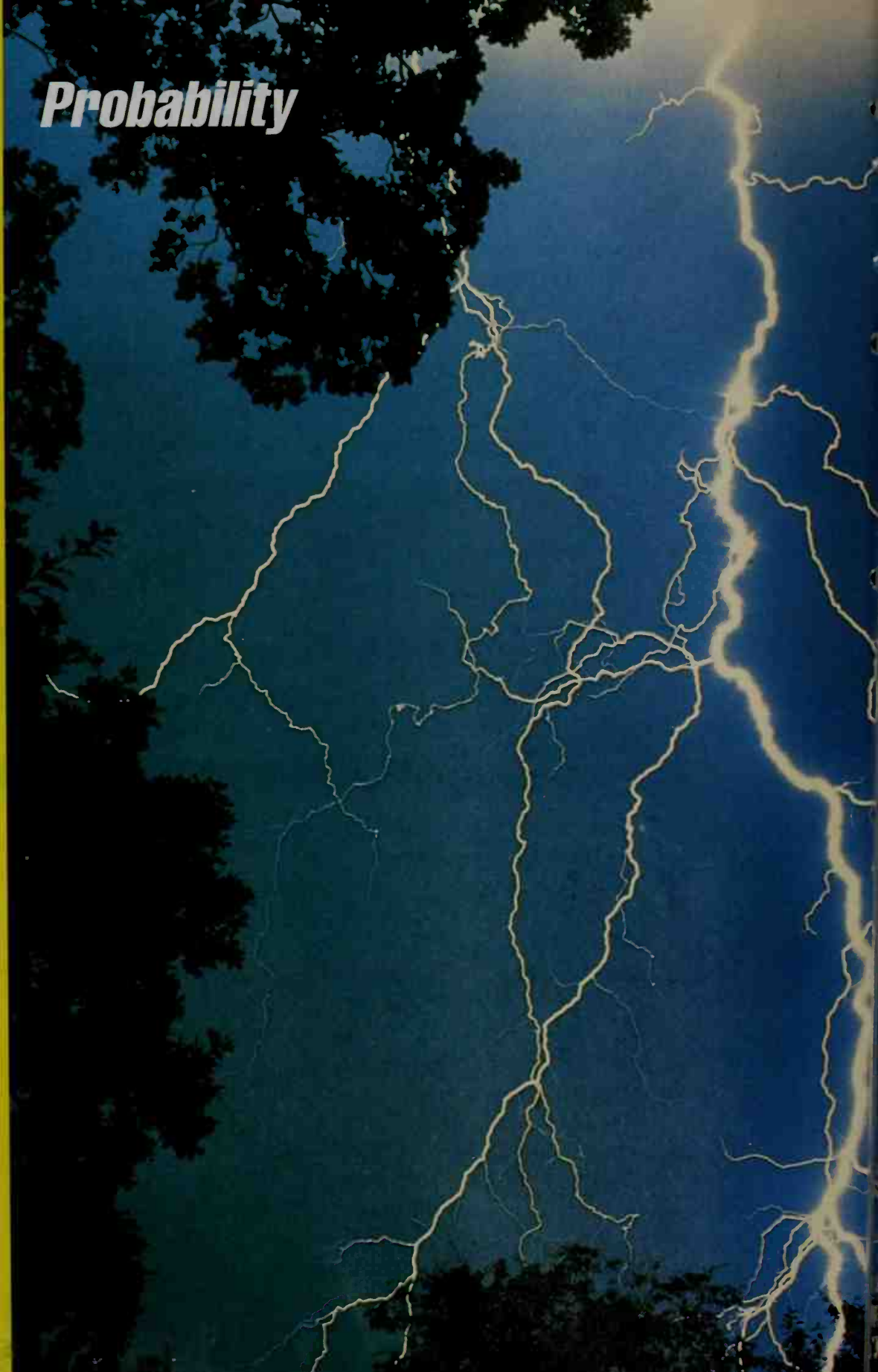
Venn diagram (p. 565)	complement principle (p. 574)
intersection (p. 565)	permutation, combination (p. 578)
union (p. 565)	permutations with repetition (p. 584)
complement (p. 566)	linear and circular permutations (p. 584)
inclusion-exclusion principle (p. 566)	Pascal's triangle (p. 590)
multiplication principle (p. 571)	binomial coefficient (p. 590)
addition principle (p. 573)	binomial theorem (p. 591)

## Chapter Test

- Of the 320 children at a movie, 190 buy popcorn and 245 buy something to drink. If 151 children buy both, how many buy neither? **15-1**
- A basketball team has 4 guards, 5 forwards, and 3 centers on its roster. In how many ways can a team consisting of 1 left guard, 1 right guard, 1 left forward, 1 right forward, and 1 center be formed? **15-2**
- A bank customer selects a password consisting of 4 different letters or 4 different digits. How many different passwords are possible?
- A bookshelf has space for 5 books. If 7 different books are available, how many different arrangements can be made on the shelf? **15-3**
- Six people meet at a party, and each pair of people shakes hands. How many handshakes are there?
- A boat has 3 red, 3 blue, and 2 yellow flags with which to signal other boats. All 8 flags are flown in various sequences to denote different messages. How many such sequences are possible? **15-4**
- Writing** Write a paragraph contrasting linear and circular permutations. Illustrate the difference between them with a specific example.
- In the expansion of  $(x - 2)^{15}$ , find the coefficients of  $x^{13}$  and  $x^{12}$ . **15-5**

16

# *Probability*



# Finding Probabilities

## 16-1 Introduction to Probability

### Objective

To find a sample space of an experiment and the probability of an event or either of two events.

Probability theory is the branch of mathematics that deals with uncertainty. Although the foundations of probability theory were laid in the 1500s and 1600s by mathematicians who were interested in questions about gambling, the subject has since become associated with many other fields, such as meteorology and genetics.

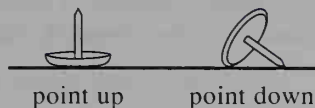
At its most basic level, probability theory assigns a number (between 0 and 1, inclusive) to an *event* as a means of indicating the *probability*, or likelihood, of the occurrence of the event. For example, if a weather report states that the probability of precipitation tomorrow is 60%, then in the past, whenever weather conditions similar to the current conditions existed, it rained 3 out of 5 times.

The probability of an event is determined either *empirically* or *theoretically*. The precipitation probability mentioned above is an example of an empirical probability because it is based on previous observations of the weather. Another example of empirical probability is found in the following activity.

### Activity

For this activity, you should work with a partner. You and your partner will need a cup and a thumbtack with a flat head. (All the thumbtacks used by the class should be alike.)

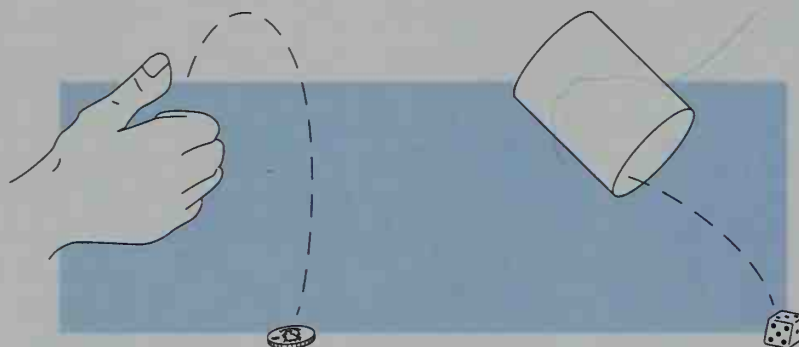
- Repeat the following at least 20 times: Place the thumbtack in the cup, shake it, and “pour” it onto your desk. Your partner should then record whether the tack lands “point up” or “point down,” as shown at the right.
- Based on your record of results from part (a), what would you say is the probability that the tack lands “point up”?
- Combine the results from part (a) for all the pairs of partners in the class. Based on the combined results, what would you say is the probability that the tack lands “point up”?
- Of the two probabilities from parts (b) and (c), which do you think is a better predictor of the tack’s “behavior”? Explain your reasoning.
- Describe how changing the shape of the tack would affect the probability that it lands “point up.”



◀ As Mark Twain reportedly said, “Everybody talks about the weather, but nobody ever does anything about it!” *Probability theory*, however, and records of similar conditions allow us to at least make reasonable weather predictions.



Although empirical probabilities are unavoidable in many applied areas of probability theory, in this chapter we will usually consider events whose probabilities can be determined simply by reasoning about the events. For example, by assuming that the mass of a coin is evenly distributed so that each of the two sides (which are called “heads” and “tails”) has an equal chance of turning up when the coin is flipped, we can say that the theoretical probability of getting “heads” is  $\frac{1}{2}$ . Likewise, the shape and composition of a die lead us to believe that each of the six faces has an equal chance of turning up, so the theoretical probability of rolling, say, a “3” is  $\frac{1}{6}$ .



At this point, let us generalize our discussion of theoretical probability. We refer to an action having various outcomes that occur unpredictably (such as tossing a coin or rolling a die) as an **experiment**. A **sample space** of an experiment is a set  $S$  such that each outcome of the experiment corresponds to exactly one element of  $S$ . An **event** is any subset of a sample space. If a sample space of an experiment contains  $n$  equally likely outcomes and if  $m$  of the  $n$  outcomes correspond to some event  $A$ , then the **probability** of event  $A$ , denoted  $P(A)$ , is:

$$P(A) = \frac{m}{n}$$

If all  $n$  outcomes correspond to event  $A$ , then the event is certain to occur and  $P(A) = \frac{n}{n} = 1$ . Similarly, if no outcomes correspond to event  $A$ , then the event is certain *not* to occur and  $P(A) = \frac{0}{n} = 0$ .

### Example 1

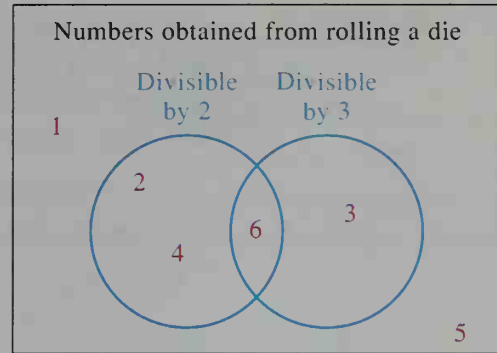
Suppose a die is rolled. Give a sample space for this experiment. Then find the probability of rolling a prime number.

### Solution

A sample space for the die-rolling experiment is  $\{1, 2, 3, 4, 5, 6\}$ . Of the six equally likely outcomes in the sample space, three outcomes (2, 3, and 5) correspond to the event “rolling a prime number.” Thus:

$$P(\text{prime number}) = \frac{3}{6} = \frac{1}{2}$$

To determine the probability of the occurrence of either one of two different events, we can use a Venn diagram (introduced in Section 15-1). For example, in the roll of a die, consider the events “rolling a number divisible by 2” and “rolling a number divisible by 3.” The Venn diagram at the right shows these two events. Notice that one number, 6, is common to both events, so that:



$$\begin{aligned}
 P\left(\begin{array}{c} \text{divisible} \\ \text{by 2 or 3} \end{array}\right) &= P\left(\begin{array}{c} \text{divisible} \\ \text{by 2} \end{array}\right) + P\left(\begin{array}{c} \text{divisible} \\ \text{by 3} \end{array}\right) - P\left(\begin{array}{c} \text{divisible} \\ \text{by 2 and 3} \end{array}\right) \\
 &= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{2}{3}
 \end{aligned}$$

Note that this result is based on the inclusion-exclusion principle (page 566).

### Probability of Either of Two Events

For any two events  $A$  and  $B$ ,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Two events  $A$  and  $B$  that cannot occur simultaneously are called **mutually exclusive** events. For example, in the roll of a die, the events “rolling a 3” and “rolling an even number” are mutually exclusive. Moreover:

$$\begin{aligned}
 P(3 \text{ or even number}) &= P(3) + P(\text{even number}) \\
 &= \frac{1}{6} + \frac{3}{6} = \frac{2}{3}
 \end{aligned}$$

This result, generalized below, is a special case of the result stated above.

### Probability of Either of Two Mutually Exclusive Events

If events  $A$  and  $B$  are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

The event “not  $A$ ” occurs when event  $A$  does not. Since events  $A$  and “not  $A$ ” are mutually exclusive and since one or the other is certain to occur, we have:

$$\begin{aligned}
 P(A \text{ or not } A) &= 1 \\
 P(A) + P(\text{not } A) &= 1 \\
 P(\text{not } A) &= 1 - P(A)
 \end{aligned}$$

**Example 2**

As shown below, a standard deck of playing cards consists of 52 cards, with 13 cards in each of four *suits* (clubs, spades, diamonds, and hearts). Clubs and spades are black cards, and diamonds and hearts are red cards. Also, jacks, queens, and kings are called *face cards*.

Clubs (♣): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

Spades (♠): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

Diamonds (♦): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

Hearts (♥): ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king

If the deck is well-shuffled, what is the probability that the top card is:

- a. a black ace?      b. not a black ace?      c. a diamond face card?

**Solution**

- a. Since the cards are well-shuffled, any one of them has the same chance of being the top card. With 2 black aces in the deck,

$$P(\text{black ace}) = \frac{2}{52} = \frac{1}{26}.$$

- b.  $P(\text{not a black ace}) = 1 - P(\text{black ace})$

$$= 1 - \frac{1}{26} = \frac{25}{26}$$

- c. With 3 diamond face cards in the deck,

$$P(\text{diamond face card}) = \frac{3}{52}.$$

In Example 2, notice that the card-choosing experiment has such sample spaces as:

$S_1$  = the set of 52 cards listed in Example 2

$S_2$  = {club, spade, diamond, heart}

$S_3$  = {black card, red card}

$S_4$  = {spade, non-spade}

We often refer to the elements of a sample space as *sample points*. For example, the outcome of choosing, say, the 7 of clubs corresponds to the sample point “club” in the sample space  $S_2$ , to the sample point “black card” in the sample space  $S_3$ , and to the sample point “non-spade” in the sample space  $S_4$ .

If you were asked for the probability of drawing a spade from a standard deck, sample space  $S_4$  would not be helpful, because its two sample points “spade” and “non-spade” are not equally likely. (Thus, it would be incorrect to say that the probability of a spade is  $\frac{1}{2}$ .) On the other hand, the four sample points

of sample space  $S_2$  are equally likely, so we can say  $P(\text{spade}) = \frac{1}{4}$ . We can also

use the 52 equally likely sample points of  $S_1$  and say  $P(\text{spade}) = \frac{13}{52} = \frac{1}{4}$ .

**Example 3**

Suppose a red die and a white die are rolled. What is the probability that the sum of the numbers showing on the dice is 9 or 10?

**Solution**

The table below gives a sample space consisting of 36 ordered pairs  $(r, w)$  where  $r$  and  $w$  are the numbers showing on the red die and the white die.

		White die					
		1	2	3	4	5	6
Red die	1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
	2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
	3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
	4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
	5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
	6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

$\text{Sum} = 9$        $\text{Sum} = 10$

Since four of the equally likely sample points give a sum of 9 and three give a sum of 10,  $P(\text{sum is 9 or 10}) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36}$ .

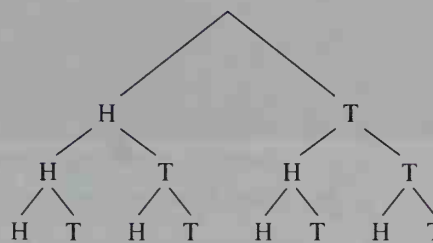
Sometimes it is convenient to use a tree diagram to obtain a sample space, as shown in the next example.

**Example 4**

Suppose you toss a coin three times. What is the probability that exactly two of the tosses result in “heads”?

**Solution**

The tree diagram at the right illustrates the experiment of tossing a coin three times and obtaining a “head” (H) or a “tail” (T) on each toss. By following each path of the tree, you get:



Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Thus,  $P(2 \text{ “heads”}) = P(\text{HHT or HTH or THH}) = \frac{3}{8}$ .

The chances of winning a sporting event are often given as *odds* instead of as a probability. If the odds in favor of a team winning are 6 to 5, this means that the team’s winning chances to its losing chances are in a 6 to 5 ratio. In other words, the probability of winning is  $\frac{6}{6+5} = \frac{6}{11}$ , and the probability of losing is  $\frac{5}{11}$ .

## CLASS EXERCISES

- As in Example 1, suppose a die is rolled. Find each probability.
  - $P(\text{perfect square})$
  - $P(\text{factor of } 60)$
  - $P(\text{negative number})$
- If a card is drawn at random from a standard deck of 52 cards, what is the probability of getting:
  - the queen of hearts?
  - a heart?
  - a queen?
  - a red card?
  - a face card?
  - a red face card?
- If two dice are rolled, what is the probability that both show the same number?
- If two dice are rolled, find the probability of getting:
  - a sum of 3
  - a sum of 4
  - a sum of 3 or 4
- Meteorology** If the probability of rain tomorrow is 40%, what is the probability of no rain tomorrow?
- Occupational Safety** If the probability of no accidents in a manufacturing plant during one month is 0.82, what is the probability of at least one accident?
- Discussion** In the solution of Example 3, another possible sample space is the set of the 11 possible sums for the two dice:

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Since two of these 11 possibilities correspond to a sum of 9 or 10, some people might be tempted to say that the probability of a sum of 9 or 10 is  $\frac{2}{11}$ . Discuss why this reasoning is incorrect.

- A penny, a nickel, and a dime are each tossed.
  - Does the set  $S = \{0, 1, 2, 3\}$ , which gives the four different numbers of “heads” that could come up, satisfy the definition of a sample space?
  - What is wrong with reasoning that since one of the four sample points in part (a) corresponds to 2 “heads,” then  $P(2 \text{ “heads”}) = \frac{1}{4}$ ?
- A card is picked at random from a standard deck. Explain why the set  $S = \{\text{club, spade, red card, face card}\}$  is *not* a sample space.
- Comment on the following reasoning:

There are 3 states whose names begin with the letter C (California, Colorado, and Connecticut); call them “C-states.” Likewise, there are 3 “O-states” (Ohio, Oklahoma, and Oregon). Thus, if a person is chosen at random from the U.S. population, that person has the same probability of being from a “C-state” as from an “O-state.”





## WRITTEN EXERCISES

For Exercises 1–4, suppose a card is drawn from a well-shuffled standard deck of 52 cards. Find the probability of drawing each of the following.

**A**

1. a. a black card                      b. a spade                      c. not a spade
2. a. a black face card              b. a black jack              c. not a black jack
3. a. a red diamond                  b. a black diamond              c. not a black diamond
4. a. a jack                              b. a jack or king              c. neither jack nor king
5. Mr. and Mrs. Smith each bought 10 raffle tickets. Each of their three children bought 4 tickets. If 4280 tickets were sold in all, what is the probability that the grand prize winner is:  
a. Mr. or Mrs. Smith?              b. one of the 5 Smiths?              c. none of the Smiths?
6. One of the integers between 11 and 20, inclusive, is picked at random. What is the probability that the integer is:  
a. even?                              b. divisible by 3?              c. a prime?
7. New York City is divided into five boroughs: Manhattan, Queens, the Bronx, Brooklyn, and Staten Island. Suppose that a New York City telephone number is randomly chosen.  
a. Explain why the probability that it is a Manhattan telephone number is *not*  $\frac{1}{5}$ .  
b. What do you need to know in order to find the correct probability in part (a)?
8. Suppose that a member of the U.S. Senate and a member of the U.S. House of Representatives are randomly chosen to be photographed with the President. Explain why  $\frac{1}{50}$  is the probability that the senator is from Iowa, and why  $\frac{1}{50}$  is *not* the probability that the representative is from Iowa.



In Exercises 9–12, use the table on page 601, which gives the 36 equally likely outcomes when two dice are rolled. Find the probability of each event.

9. a. Sum is 6.                      b. Sum is 7.                      c. Sum is 8.
10. a. Sum is even.                  b. Sum is 12.                  c. Sum is less than 12.
11. The two dice show different numbers.
12. The red die shows a greater number than the white die.

13. A die is rolled and a coin is tossed.
  - a. Make a tree diagram showing the 12 possible outcomes of this experiment.
  - b. Find the probability that the die's number is even and the coin is "heads."
14. The numbers 1, 2, 3, and 4 are written on separate slips of paper and placed in a hat. Two slips of paper are then randomly drawn, one after the other and without replacement.
  - a. Make a tree diagram showing the 12 possible outcomes of this experiment.
  - b. Find the probability that the sum of the numbers picked is 6 or more.
15. Suppose that the odds in favor of the National League's winning the All-Star Game are 4 to 3.
  - a. What is the probability that the National League wins?
  - b. What is the probability that the American League wins?
16. Suppose that the odds in favor of the incumbent's winning an election are 2 to 3.
  - a. What is the probability that the incumbent wins?
  - b. What is the probability that the incumbent's challenger wins?



- B** 17. Suppose you roll two dice, each of which is a regular octahedron with faces numbered 1 to 8.
- a. What is the probability that the sum of the numbers showing is 2?
  - b. What is the probability that the sum is 3?
  - c. What sum is most likely to appear?
18. Suppose you roll two dice, each of which is a regular dodecahedron with faces numbered 1 to 12.
- a. What is the probability that the sum of the numbers showing is 24?
  - b. What is the probability that the sum is 23?
  - c. What sum is most likely to appear?
19. From a group consisting of Alvin, Bob, Carol, and Donna, two people are to be randomly selected to serve on a committee.
- a. Give a sample space for this experiment.
  - b. Find the probability that Bob and Carol are selected.
  - c. Find the probability that Carol is not selected.
20. From a standard deck of cards, two cards are randomly drawn, one after the other and without replacement. The color of each card is noted.
- a. Give a sample space for this experiment.
  - b. Explain why two red cards are less likely to be drawn than a red card and then a black card.
21. **Reading** Explain how the inclusion-exclusion principle (see page 566) can be used to find the probability of either of two events (see page 599).

22. Of the 1260 households in a small town, 632 have dogs, 568 have cats, and 114 have both types of pet. If a household is chosen at random, what is the probability that the household has either type of pet?
23. The letters of the word TEXAS are arranged in a random order. What is the probability that the letters spell TAXES?
24. The letters of the word COSINE are arranged in a random order. What is the probability that the letters spell SONICE?
25. A number  $k$  is randomly chosen from  $\{-3, -2, -1, 0, 1, 2, 3, 4\}$ . What is the probability that the expression  $x^2 - 2x + k$  can be written as a product of two linear factors, each with integral coefficients?
26. A number  $c$  is randomly chosen from  $\{1, 2, 3, 4, 5, 6\}$ . What is the probability that the graph of  $y = x^2 - 4x + c$  intersects the  $x$ -axis?
27. If a 3-letter “word” is formed by randomly choosing 3 letters from the word OCEAN, what is the probability that it is composed only of vowels?
28. If a 3-letter “word” is formed by randomly choosing 3 letters from the word PAINTED, what is the probability that it is composed only of vowels?
29. A number between 100 and 999, inclusive, is chosen at random. What is the probability that it contains (a) no 0’s? (b) at least one 0?
30. A number between 1000 and 9999, inclusive, is chosen at random. What is the probability that it contains (a) no 9’s? (b) at least one 9?

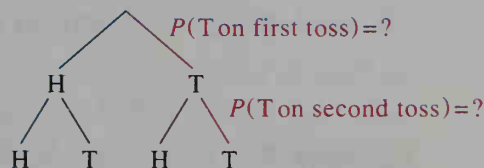
## 16-2 Probability of Events Occurring Together

**Objective** To find the probability of events occurring together and to determine whether two events are independent.

If an experiment involves two or more events occurring together (whether successively or simultaneously), a tree diagram is a useful way to calculate the probabilities of the possible outcomes.

### Activity

The tree diagram at the right illustrates an experiment in which a coin is tossed twice. (In the diagram, H and T represent “heads” and “tails,” respectively.)



- a. Use the tree diagram to give a sample space for the experiment.
- b. Use your answer to part (a) to find  $P(TT)$ .
- c. What are the probabilities indicated in red in the tree diagram?
- d. How could you obtain your answer to part (b) using your answers to part (c)?
- e. If the coin is tossed a third time, use the method from part (d) to find  $P(TTT)$ .

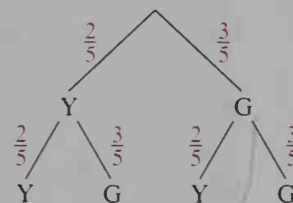
As the activity on the preceding page shows, if you know the probabilities that correspond to the branches of a tree diagram for some experiment, then you can find the probability of a particular outcome by multiplying the probabilities along the path leading to that outcome. We use this approach in the following examples.

### Example 1

Two yellow balls and three green balls are placed in a jar. One ball is randomly chosen, its color is noted, and the ball is put back in the jar. This procedure is repeated for a second ball. Find the probability that both balls are the same color.

### Solution

The probability of a yellow (Y) first ball is  $\frac{2}{5}$ , and the probability of a green (G) first ball is  $\frac{3}{5}$ . Because the first ball is put back in the jar before the second ball is chosen, the second ball has the same probabilities of being yellow and of being green as the first ball. These probabilities are shown next to the branches of the tree diagram at the right. Thus, the probability of getting two yellow balls is:



$$P(YY) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$$

Similarly:

$$P(YG) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}$$

$$P(GY) = \frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$$

$$P(GG) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$$

Note that the sum of these probabilities is 1.

From the above calculations, we find that

$$\begin{aligned} P(\text{both same color}) &= P(YY) + P(GG) \\ &= \frac{4}{25} + \frac{9}{25} = \frac{13}{25}. \end{aligned}$$

### Example 2

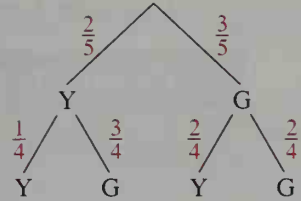
Rework Example 1 if the first ball is *not* put back in the jar before the second ball is chosen.

### Solution

In this situation, the probability of a yellow (Y) first ball is still  $\frac{2}{5}$ , and the probability of a green (G) first ball is still  $\frac{3}{5}$ . The probabilities for the second ball, however, are different this time.



If the first ball chosen is yellow, then there are 1 yellow and 3 green balls left, so that the probability of a yellow second ball is  $\frac{1}{4}$  and the probability of a green second ball is  $\frac{3}{4}$ . If the first ball is green,



however, then there are 2 yellow and 2 green balls left, so that the probability of a yellow second ball is  $\frac{2}{4}$  and the probability of a green second ball is also  $\frac{2}{4}$ . Thus:

$$\begin{aligned} P(\text{both same color}) &= P(YY) + P(GG) \\ &= \frac{2}{5} \cdot \frac{1}{4} + \frac{3}{5} \cdot \frac{2}{4} = \frac{8}{20} = \frac{2}{5} \end{aligned}$$

In Examples 1 and 2, we denote the probability that the second ball is yellow given that the first ball is yellow by:

$$P(\text{second is Y} \mid \text{first is Y})$$

This probability is called a **conditional probability**, because it gives the probability that the second ball is yellow on the condition that the first ball is yellow. Other conditional probabilities are considered below.

In Example 1:  $P(\text{second is G} \mid \text{first is G}) = \frac{3}{5}$

$P(\text{second is G} \mid \text{first is Y}) = \frac{3}{5}$

Note that these are equal.

In Example 2:  $P(\text{second is G} \mid \text{first is G}) = \frac{2}{4}$

$P(\text{second is G} \mid \text{first is Y}) = \frac{3}{4}$

Note that these are different.

In Example 1, as the calculations on the preceding page show, the probability that the second ball is green is  $\frac{3}{5}$  whether or not the first ball is green. Since the probability that the second ball is green does *not* depend on the color of the first ball, the two events “second ball is green” and “first ball is green” are called *independent events*. This is not the case, however, in Example 2 where the probability that the second ball is green *does* depend on whether or not the first ball is green.

In general, two events  $A$  and  $B$  are **independent** if and only if the occurrence of  $A$  does not affect the probability of the occurrence of  $B$ . In other words, events  $A$  and  $B$  are independent if and only if

$$P(B \mid A) = P(B).$$



Examples 1 and 2 suggest the following rules for events occurring together.

### Probability of Events Occurring Together

**Rule 1.** For any two events  $A$  and  $B$ ,

$$P(A \text{ and } B) = P(A) \cdot P(B | A).$$

**Rule 2.** If events  $A$  and  $B$  are independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

#### Example 3

A card is randomly drawn from a standard deck.

- Show that the events “jack” and “spade” are independent.
- Show how Rule 2 given above can be used to find the probability of drawing the jack of spades.

#### Solution

- a. Since 13 of the 52 cards are spades,  $P(\text{spade}) = \frac{13}{52}$ .

Since 1 of the 4 jacks is a spade,  $P(\text{spade} | \text{jack}) = \frac{1}{4}$ .

Thus, since  $P(\text{spade} | \text{jack}) = P(\text{spade})$ , the events “jack” and “spade” satisfy the condition for independence stated at the bottom of the preceding page.

- b. If we think of drawing the jack of spades as the simultaneous occurrence of the events “jack” and “spade,” Rule 2 gives:

$$P(\text{jack and spade}) = P(\text{jack}) \cdot P(\text{spade}) = \frac{1}{13} \cdot \frac{1}{4} = \frac{1}{52}$$

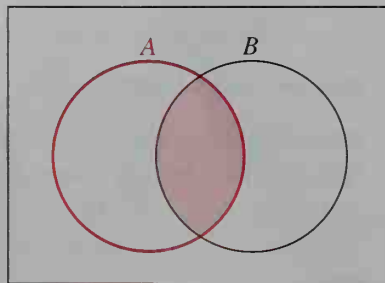
We can write Rule 1 given above as:

$$P(B | A) = \frac{P(A \text{ and } B)}{P(A)}$$

In this form, the rule gives us a means of calculating conditional probability. Another way to calculate  $P(B | A)$  can be seen by picturing events  $A$  and  $B$  as sets in a Venn diagram, as shown at the right. Since event  $A$  is known to have happened, we can ignore everything outside circle  $A$  in the diagram. The probability of  $B$  given  $A$  is just the fraction of  $A$ 's sample points that are in  $B$ . We therefore have the rule

$$P(B | A) = \frac{n(A \cap B)}{n(A)}$$

which is equivalent to the one above.



**Example 4**

Each student in a class of 30 studies one foreign language and one science. The students' choices are shown in the table below.

	Chemistry (C)	Physics (P)	Biology (B)	Totals
French (F)	7	4	3	14
Spanish (S)	1	6	9	16
Totals	8	10	12	30

- Find the probability that a randomly chosen student studies chemistry.
- Find the probability that a randomly chosen student studies chemistry given that the student studies French.
- Are the events “student studies chemistry” and “student studies French” independent?

**Solution**

- Since 8 of the 30 students study chemistry,  $P(C) = \frac{8}{30} = \frac{4}{15}$ .
- Of the 14 students who study French, 7 study chemistry. Thus:

$$P(C | F) = \frac{n(C \cap F)}{n(F)} = \frac{7}{14} = \frac{1}{2}$$

- Parts (a) and (b) show that  $P(C | F) \neq P(C)$ , so the events “student studies chemistry” and “student studies French” are *not* independent.

**CLASS EXERCISES**

- Reading** Find an instance where each of the two rules given at the top of the preceding page is used in Examples 1 or 2.
- In Example 3, are the events “jack” and “face card” independent?
- Consider the class of 30 students described in Example 4.
  - Find the probability that a randomly chosen student studies Spanish.
  - Find the probability that a randomly chosen student studies biology given that the student studies Spanish.
  - Are the events “student studies Spanish” and “student studies biology” independent?

**WRITTEN EXERCISES**

- A** 1. From a box containing 3 red balls and 5 green balls, 2 balls are randomly picked, one after the other and without replacement.
  - Draw a tree diagram showing the probabilities of each branch of the tree.
  - Are the events “second ball is red” and “first ball is red” independent?
  - Find the probability that both balls are the same color.
  - Find the probability that one ball is red and one is green.

2. Repeat Exercise 1 if the box contains 1 red ball and 3 green balls.
3. Repeat Exercise 1 if the first ball chosen is put back in the box and mixed with the other balls before the second ball is picked.
4. Repeat Exercise 2 if the box contains 1 red ball and 3 green balls and if the first ball chosen is put back in the box and mixed with the other balls before the second ball is picked.
5. Two cards are dealt from the top of a well-shuffled standard deck of cards.
  - a. Draw a tree diagram showing the probabilities of a heart (H) and non-heart (N) for each of the two cards.
  - b. Find  $P(HH)$ ,  $P(HN)$ ,  $P(NH)$ , and  $P(NN)$ .
6. A coin is slightly bent so that the probability of getting “heads” on a toss is 0.55 instead of 0.50.
  - a. Draw a tree diagram showing the probabilities for two tosses of the coin. Use H for “heads” and T for “tails.”
  - b. Find  $P(HH)$ ,  $P(HT)$ ,  $P(TH)$ , and  $P(TT)$ .
7. A card is randomly chosen from a standard deck.
  - a. Show that the events “face card” and “diamond” are independent.
  - b. Show that the events “red card” and “diamond” are *not* independent.
8. The following 6 cards are placed face down on a table: ace, 2, and 3 of hearts; ace and 2 of clubs; and ace of spades. One card is randomly chosen.
  - a. Are the events “ace” and “heart” independent?
  - b. Are the events “ace” and “club” independent?
9. A penny, a nickel, and a dime are tossed one after the other.
  - a. Draw a tree diagram showing the probabilities of a “head” (H) and “tail” (T) for each toss.
  - b. Find  $P(3 \text{ “heads”})$ ,  $P(\text{exactly } 2 \text{ “heads”})$ ,  $P(\text{exactly } 1 \text{ “head”})$ , and  $P(\text{no “heads”})$ .
10. A die is rolled 3 times.
  - a. Draw a tree diagram showing the probabilities of a six (S) and non-six (N) for each roll.
  - b. Find  $P(3 \text{ sixes})$ ,  $P(\text{exactly } 2 \text{ sixes})$ ,  $P(\text{exactly } 1 \text{ six})$ , and  $P(\text{no sixes})$ .
11. The letters of the word COMPUTER are written on separate slips of paper and placed in a hat. Three letters are then randomly drawn from the hat, one after the other and without replacement. To find the probability that all 3 letters are vowels, two methods can be used.

$$\text{Method 1: } P(3 \text{ vowels}) = \frac{\text{number of 3-letter words having 3 vowels}}{\text{number of 3-letter words}}$$

$$\begin{aligned} \text{Method 2: } P(3 \text{ vowels}) &= P(\text{first is vowel}) \times \\ &\quad P(\text{second is vowel} \mid \text{first is vowel}) \times \\ &\quad P(\text{third is vowel} \mid \text{first and second are vowels}) \end{aligned}$$

- a. Use both methods to find  $P(3 \text{ vowels})$ .
- b. Using whichever method you prefer, find the probability that all 3 letters are consonants.

12. In a math class with 13 girls and 11 boys, the teacher randomly selects four students to put homework problems on the chalkboard. What is the probability that all are girls? (*Hint*: Use one of the methods described in Exercise 11.)

In Exercises 13–18, use the table below, which gives the areas of concentration of the 400 students at a small college.

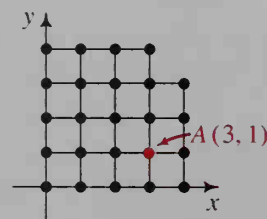
Area of concentration	Class				
	Freshman	Sophomore	Junior	Senior	Totals
Natural sciences	50	35	33	29	147
Social sciences	20	25	28	24	97
Humanities	40	40	39	37	156
Totals	110	100	100	90	400

13. If a student is selected at random, what is the probability that:
- the student's area of concentration is the natural sciences?
  - the student is a freshman in the social sciences?
14. If a sophomore is selected at random, what is the probability that his or her area of concentration is the humanities?
15. a. What is the probability that a senior's area of concentration is the natural sciences?  
b. What is the probability that a student in the natural sciences is a senior?
16. a. What is the probability that a junior's area of concentration is the humanities?  
b. What is the probability that a student in the humanities is a junior?
17. A student is selected at random. Are the events "student is a junior" and "student is in the humanities" independent?
18. A student is selected at random. Are the events "student is a junior" and "student is in the natural sciences" independent?
- B** 19. **Reading** Given a sample space  $S$  with subsets  $A$  and  $B$ , use  $n(S)$  to show that the rule  $P(B | A) = \frac{n(A \cap B)}{n(A)}$  is equivalent to the rule  $P(B | A) = \frac{P(A \text{ and } B)}{P(A)}$ .
20. **Reading** Read again the two rules given at the top of page 608. Explain why Rule 2 is a special case of Rule 1.



21. Three cards are dealt from a well-shuffled standard deck.
  - a. Complete the calculation:  $P(3 \text{ clubs}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{?}{?} = ?$
  - b. Find the probability that all 3 cards are red.
  - c. Find the probability that all 3 cards are aces.
  - d. Find the probability that none of the cards is an ace.
  - e. Find the probability that at least one of the cards is an ace.
  - f. Find the probability that either one or two of the cards is an ace.
22. Five cards are dealt from a well-shuffled standard deck.
  - a. Find the probability that all 5 cards are spades. (*Hint: See Exercise 21(a).*)
  - b. Find the probability that all 5 cards are of the same suit.
  - c. Find the probability that none of the cards is an ace.
  - d. Find the probability that at least one card is an ace.
23. Repeat Exercise 21 if each card is replaced before the next is drawn.
24. Repeat Exercise 22 if each card is replaced before the next is drawn.
25. Three people are randomly chosen. Find the probability of each event.
  - a. All were born on different days of the week.
  - b. At least two people were born on the same day of the week.
26. A die is rolled 3 times. Find the probability of each event.
  - a. All 3 numbers are different.
  - b. At least 2 of the numbers are the same.
27. A bag contains 3 red marbles and 2 black marbles. Two persons, A and B, take turns drawing a marble from the bag without replacing any of the marbles drawn. The first person to draw a black marble wins. What is the probability that person A, who draws first, wins?
28. Repeat Exercise 27 if the bag contains 4 red marbles and 1 black marble.

29. **Sports** The championship series of the National Basketball Association consists of a series of at most 7 games between two teams X and Y. The first team to win 4 games is the champion and the series is over. At any time before or after a game, the status of the series can be recorded as a point  $(x, y)$  on the grid shown at the right. The point  $A(3, 1)$ , for example, means that team X has won 3 games and team Y has won 1 game. From point A, the series can end in a championship for team X in 3 ways (X, YX, YYX). If you assume that team X has a probability of 0.6 of winning each and every remaining game, then the probability that team X becomes champion from point A is  $P(X) + P(YX) + P(YYX) = 0.6 + (0.4)(0.6) + (0.4)(0.4)(0.6) = 0.936$ .
  - a. Find the probability that team Y becomes champion from point A.
  - b. If team X has won 1 game and team Y has won 3 games, find the probability that team Y becomes champion.
  - c. If team X has won 2 games and team Y has won 1 game, find the probability that team X becomes champion.





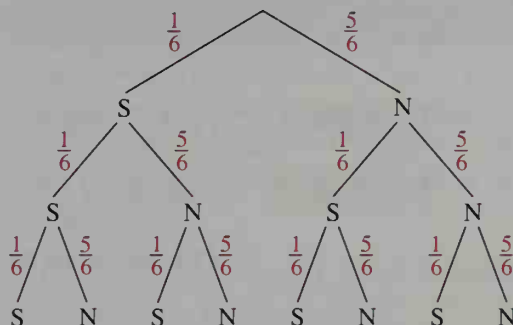
- C** 30. **Sports** Refer to Exercise 29 and suppose that the *home* team has a probability  $p$  of winning each game and a probability  $q = 1 - p$  of losing.
- If team X is the home team for games 1, 2, 6, and 7, what is the probability that team X wins the championship? Give the answer in terms of  $p$  only.
  - Use a computer or graphing calculator to find the approximate value of  $p$  that makes team X's probability of winning the championship 0.5.

## 16-3 The Binomial Probability Theorem

### Objective

To use the binomial probability theorem to find the probability of a given outcome on repeated independent trials of a binomial experiment and to approximate the probability when the trials are not independent.

Consider the experiment in which a die is rolled three times. On each roll, the probability of getting a six (S) is  $p = \frac{1}{6}$  and the probability of getting a non-six (N) is  $1 - p = \frac{5}{6}$ . These probabilities are shown in the tree diagram at the right. We can use the tree diagram to create the table below, which summarizes the results of the experiment.



	3 sixes	2 sixes			1 six			0 sixes
Outcome	SSS	SSN	SNS	NSS	SNN	NSN	NNS	NNN
Probability	$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$	$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}$	$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$	$\frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$	$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$	$\frac{5}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}$	$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$	$\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$
	$\left(\frac{1}{6}\right)^3$	$3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$			$3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$			$\left(\frac{5}{6}\right)^3$

Now compare:  $\downarrow$   $\downarrow$   $\downarrow$   $\downarrow$

$$[p + (1 - p)]^3 = p^3 + 3p^2(1 - p) + 3p(1 - p)^2 + (1 - p)^3$$

If the die were rolled 4 times instead of 3, we would have the following distribution of probabilities:

Outcome	4 sixes	3 sixes	2 sixes	1 six	0 sixes
Probability	$\left(\frac{1}{6}\right)^4$	$4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)$	$6\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$	$4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3$	$\left(\frac{5}{6}\right)^4$

$\downarrow$   $\downarrow$   $\downarrow$   $\downarrow$   $\downarrow$

$$[p + (1 - p)]^4 = p^4 + 4p^3(1 - p) + 6p^2(1 - p)^2 + 4p(1 - p)^3 + (1 - p)^4$$

The two illustrations on the preceding page can be generalized to give the following theorem, which is based on the binomial theorem (page 591).

### The Binomial Probability Theorem

Suppose an experiment consists of a sequence of  $n$  repeated independent trials, each trial having two possible outcomes,  $A$  or not  $A$ . If on each trial,  $P(A) = p$  and  $P(\text{not } A) = 1 - p$ , then the binomial expansion of  $[p + (1 - p)]^n$ ,

$${}_nC_n p^n + \cdots + {}_nC_k p^k (1 - p)^{n-k} + \cdots + {}_nC_0 (1 - p)^n,$$

gives the following probabilities for the occurrences of  $A$ :

Outcome	$n$ A's	$\cdots$	$k$ A's	$\cdots$	0 A's
Probability	${}_nC_n p^n$	$\cdots$	${}_nC_k p^k (1 - p)^{n-k}$	$\cdots$	${}_nC_0 (1 - p)^n$

**Example 1** A coin is tossed 10 times. What is the probability of exactly 4 “heads”?

**Solution** Substitute  $n = 10$ ,  $k = 4$ , and  $p = \frac{1}{2}$  in the theorem above.

$$\begin{aligned} P(4 \text{ out of } 10 \text{ tosses result in “heads”}) &= {}_{10}C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^6 \\ &= 210 \left(\frac{1}{2}\right)^{10} = \frac{210}{1024} \approx 0.205 \end{aligned}$$

**Example 2** In a survey of the 1000 students in a school, 950 indicated that they were right-handed. Find the probability that at least one of four randomly chosen students from the school is left-handed.

**Solution** Technically speaking, this is not a binomial experiment because the sampling of the four students is done without replacement. In this experiment the probability of the first student’s being right-handed is  $\frac{950}{1000} = 0.95$ .

The probability of the second student’s being right-handed is either  $\frac{949}{999}$  or  $\frac{950}{999}$  depending on whether the first student is right-handed. These two probabilities (as well as the probabilities of the third and fourth student’s being right-handed) are so close to 0.95 that we will use the binomial probability theorem even though the theorem assumes a constant 0.95 probability. Thus:

$$\begin{aligned} P(\text{at least one left-handed}) &= 1 - P(\text{all right-handed}) \\ &\approx 1 - {}_4C_4 (0.95)^4 \\ &\approx 1 - 0.815 = 0.185 \end{aligned}$$

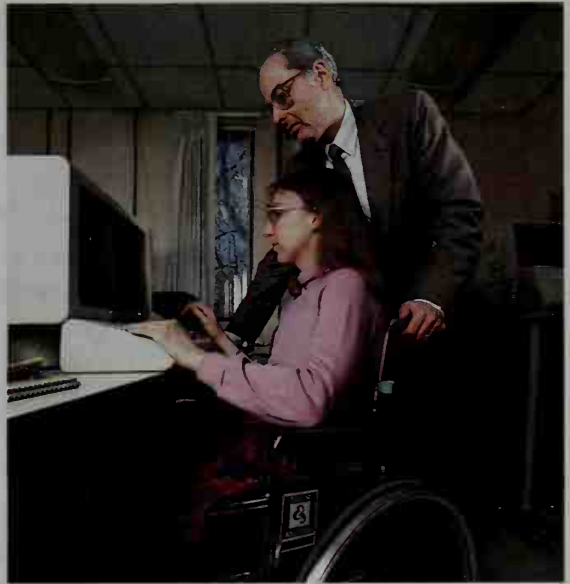
### Example 3

Of the computer operators who work for a large temporary employment agency, 60% have a car. If 5 of the computer operators are randomly chosen to work on a job that requires car transportation, what is the probability that at least 4 have cars?

### Solution

As was the case in Example 2, this is not a binomial experiment because the sampling of the 5 computer operators is done without replacement. Nevertheless, if there is a large number of computer operators, the probability of choosing each of the 5 is close enough to 0.6 that we can use the binomial probability theorem to obtain a good approximation of the probability that at least 4 have cars. Thus:

$$\begin{aligned} P(\text{at least 4 have cars}) &= P(\text{exactly 4 have cars}) + P(5 \text{ have cars}) \\ &\approx {}_5C_4(0.6)^4(0.4)^1 + {}_5C_5(0.6)^5 \\ &\approx 0.259 + 0.078 = 0.337 \end{aligned}$$



## CLASS EXERCISES

1. In the expansion of  $\left(\frac{1}{2} + \frac{1}{2}\right)^4$ , shown below, the first term represents the probability of getting 4 “heads” in 4 tosses of a coin. What probabilities do the other terms represent?

$$\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$$

2. **Discussion** In your own words, tell how to find the probability of getting exactly 3 fives when 4 dice are rolled.
3. A coin is tossed 9 times and comes up “heads” each time. What is the probability that it will come up “heads” on the next toss?
4.
  - a. A jar contains 50 red marbles and 50 white marbles. Two marbles are randomly drawn, one after the other and without replacement. What is the probability of drawing 2 red marbles?
  - b. Explain why the binomial probability theorem does not apply in part (a).
  - c. What does the binomial probability theorem give as an *approximation* of the probability in part (a)?

## WRITTEN EXERCISES

- A**
1. Make a table, like the ones on page 613, showing the 8 different ways in which “heads” (H) and “tails” (T) can occur if a coin is tossed 3 times. Then find the probability of getting:
 

a. 3 “heads”	b. 2 “heads”	c. 1 “head”	d. 0 “heads”
--------------	--------------	-------------	--------------
  2. Make a table, like the ones on page 613, showing the 16 different ways in which sixes (S) and non-sixes (N) can occur when a die is rolled 4 times. Then find the probability of getting:
 

a. 4 sixes	b. 3 sixes	c. 2 sixes	d. 1 six	e. 0 sixes
------------	------------	------------	----------	------------
  3. Consider the set of families with exactly 4 children. If  $P(\text{child is a boy}) = \frac{1}{2}$ , find the probability that one of these families, picked at random, has:
 

a. 4 boys	b. 3 boys	c. 2 boys	d. 1 boy	e. 0 boys
-----------	-----------	-----------	----------	-----------
  4. Suppose a coin is bent so that the probability of its coming up “heads” on any toss is  $\frac{2}{5}$ . If the coin is tossed 3 times, find the probability of getting:
 

a. 3 “heads”	b. 2 “heads”	c. 1 “head”	d. 0 “heads”
--------------	--------------	-------------	--------------
  5. What is the probability of getting exactly 2 fives in 4 rolls of a die?
  6. What is the probability of getting exactly 1 three in 7 rolls of a die?
  7.
    - a. If one card is drawn from a well-shuffled standard deck, what is the probability of drawing a spade?
    - b. If one card is drawn from each of two well-shuffled standard decks, what is the probability of drawing 0 spades? 1 spade? 2 spades?
  8. If one card is drawn from each of three well-shuffled standard decks, find the probability of drawing:
 

a. 3 spades	b. 2 spades	c. 1 spade	d. 0 spades
-------------	-------------	------------	-------------
  9. A quiz has 6 multiple-choice questions, each with 4 choices. If you guess at every question, what is the probability of getting:
 

a. all 6 questions right?	b. 5 out of 6 questions right?
---------------------------	--------------------------------
  10. A jar contains 4 red balls and 3 white balls, all the same size. Suppose you pull out a ball and note its color, put it back, and mix up the contents of the jar. If you do this twice more, find the probability of getting:
 

a. 0 red balls	b. 1 red ball	c. 2 red balls	d. 3 red balls
----------------	---------------	----------------	----------------
- B**
11. Eight out of every ten nutritionists recommend Brand X. If nutritionists A, B, and C are asked their opinions on Brand X, what is the probability that:
 

a. all three recommend Brand X?	b. none recommends Brand X?
c. at least one recommends Brand X?	
  12. In a certain high school, one third of the senior boys are at least 6 ft tall. In a randomly selected group of 7 senior boys, what is the probability that:
 

a. all are less than 6 ft tall?	b. none are less than 6 ft tall?
c. all but one are less than 6 ft tall?	

13. In the *random-number table* shown below, the probability that each digit, 0–9, occurs in any given position is 0.1. We can use the table to perform *Monte Carlo simulations* in order to find certain probabilities empirically.

49487	52802	28667	62058	87822	14704	18519	17889	45869	14454
29480	91539	46317	84803	86056	62812	33584	70391	77749	64906
25252	97738	23901	11106	86864	55808	22557	23214	15021	54268
02431	42193	96960	19620	29188	05863	92900	06836	13433	21709
69414	89353	70724	67893	23218	72452	03095	68333	13751	37260

For example, to use Monte Carlo simulations to find the probability of getting 0 “heads” in 3 tosses of a coin, first associate the digits 0–4 with the outcome “heads” and the digits 5–9 with the outcome “tails.” Then the first row of the random-number table represents 16 simulations of 3 tosses of a coin.

494	87	5	280	2	28	667	620	58	8	782	2	14	704	185	19	1	788	9	45	869	144	54
HTH	TT		THTH	H	HTTTT		THH	TT		TTHH	H	HH	THH	HTT	HT		HTTTT		HT	TTT		HHH

- Of the 16 simulations, 5 produce 0 “heads.” Thus, based on 16 simulations,  $P(0 \text{ “heads”}) = 0.3125$ . Compare this probability to the theoretical probability of getting 0 “heads” in 3 tosses of a coin (see Exercise 1(d)).
  - Begin with the last 2 digits of the first row and perform 14 more simulations. Using all 30 simulations, find the probability of getting 0 “heads.”
  - How many simulations can be done using the whole table? Based on these simulations, what is the probability of getting 0 “heads”?
  - Discussion** When you use Monte Carlo simulations to approximate the theoretical probability of an event, will increasing the number of simulations always improve the accuracy of the approximation? Explain.
14. a. **Discussion** In order to use Monte Carlo simulations (see Exercise 13) to find the probabilities asked for in Exercises 2–12, how should the digits 0–9 be assigned to the possible outcomes for each exercise? (*Hint:* It may be helpful to eliminate some digits. Cross off all occurrences of those digits in the table before actually doing a simulation.)
- Perform Monte Carlo simulations for Exercise 2 and Exercise 7. Compare the probabilities you find to the theoretical probabilities found using the binomial probability theorem.
15. a. **Sports** A basketball player’s free-throw percent is .750. What is the probability that she scores on exactly 4 of her next 5 free throws?
- Discussion** Using the binomial probability theorem to complete part (a) assumes that the probability of a successful free throw is always .750. Is this assumption valid? Explain.





16. **Investigation** Let  $n = 6$  and  $P(A) = 0.5$  in the binomial probability theorem. Plot the points  $(k, P(k \text{ occurrences of } A))$  for  $k = 0, 1, \dots, 6$ , and describe the shape of a smooth curve that can be drawn through the points. What happens to the shape of this curve for values of  $P(A)$  other than 0.5?
17. In a package of tomato seeds, 9 seeds out of 10 sprout on the average. What is the probability that of the first 10 seeds planted, 1 does not sprout?
18. One out of every 5 boxes of Rice Toasties has a secret message decoder ring. You buy 5 boxes hoping to get at least 2 rings. What are your chances?



**Parts of Exercises 19 and 20 require the use of a calculator or computer. If you use a computer, you may need to write your own short programs.**

19. a. If  $n$  people are randomly chosen, for what value of  $n$  is the probability of at least two having the same birthday equal to 1?
- b. Suppose you guess that the probability in part (a) is 0.5 if  $n = 10$ . To test the guess, use the probability that no two of 10 people share a birthday:

$$\begin{aligned}
 P(\text{no match}) &= P\left(\begin{array}{c} \text{2nd person} \\ \text{does not} \\ \text{match first} \end{array}\right) \times P\left(\begin{array}{c} \text{3rd person does} \\ \text{not match either} \\ \text{of first two} \end{array}\right) \times \cdots \times P\left(\begin{array}{c} \text{10th person does} \\ \text{not match any} \\ \text{of first nine} \end{array}\right) \\
 &= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{356}{365} = \frac{364P_9}{365^9}
 \end{aligned}$$

Thus,  $P(\text{at least one match}) = 1 - P(\text{no match}) = 1 - \frac{364P_9}{365^9}$ .

Use a calculator to evaluate the above expression for  $P(\text{at least one match})$ .

- c. The probability that of 10 randomly chosen people at least two have the same birthday is less than 0.5. Use the method of part (b) to determine how many people are needed for the probability to be 0.5.

- C** 20. a. What is the probability that in a group of  $n$  randomly chosen people, at least one has the same birthday as you?
- b. Find the value of  $n$  that makes the probability in part (a) approximately 0.5.



## //// COMPUTER EXERCISE

Write a program that does Monte Carlo simulations (see Exercise 13) of the binomial probability theorem. Use the computer's random-number generator to simulate  $N$  independent trials with probability  $P$  that a given outcome occurs on each trial. Count the number of times  $T$  that the outcome occurs. Repeat the simulation of  $N$  trials many times and count how often  $T = 0, T = 1, \dots, T = N$ . Convert these tallies to approximate probabilities by dividing by the number of simulations.

## 16-4 Probability Problems Solved with Combinations

### Objective

To use combinations to solve probability problems.

In Example 1 we show two methods of finding the probability that five cards randomly drawn from a standard deck are all hearts. Method 1 uses conditional probability. Method 2, which uses combinations, is presented because it can be used to solve many problems that are not readily solved using conditional probability.

### Example 1

Five cards are drawn at random from a standard deck. Find the probability that all 5 cards are hearts.

### Solution

**Method 1** Since there are 13 hearts in the 52-card deck, the probability that the first card drawn is a heart is  $\frac{13}{52}$ . With 12 hearts among the 51 cards remaining, the probability that the second card drawn is a heart is  $\frac{12}{51}$ . Continuing in this way, we have:

$$\begin{aligned}P(\text{all hearts}) &= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \\&= \frac{33}{66,640} \approx 0.000495\end{aligned}$$

**Method 2** The experiment of choosing 5 cards from 52 has  ${}_{52}C_5$  equally likely outcomes. Of these, there are  ${}_{13}C_5$  combinations that contain 5 (of the 13) hearts. Thus:

$$P(\text{all hearts}) = \frac{{}_{13}C_5}{{}_{52}C_5} = \frac{1287}{2,598,960} \approx 0.000495$$

### Example 2

Five cards are drawn at random from a standard deck. Find the probability that exactly 2 are hearts.

### Solution

In this example it is impractical to use a card-by-card approach based on conditional probability, as in Method 1 of Example 1. The difficulty is that we do not know whether the hearts will appear as the first and second cards, as the first and third cards, or as some other two cards. Instead, we determine the number of 5-card combinations that contain exactly 2 hearts. There are  ${}_{13}C_2$  choices for the 2 hearts, and there are  ${}_{39}C_3$  choices for the other 3 cards. By the multiplication principle (page 571), the number of 5-card combinations containing exactly 2 hearts is  ${}_{13}C_2 \cdot {}_{39}C_3$ . Since there are  ${}_{52}C_5$  possible 5-card combinations, we have:

$$P\left(\begin{array}{l} 2 \text{ hearts and} \\ 3 \text{ non-hearts} \end{array}\right) = \frac{{}_{13}C_2 \cdot {}_{39}C_3}{{}_{52}C_5} = \frac{78 \cdot 9139}{2,598,960} \approx 0.274$$

**Example 3**

Thirteen cards are dealt from a well-shuffled standard deck. What is the probability that the 13 cards contain exactly 4 aces and exactly 3 kings?

**Solution**

1. There are  ${}_4C_4$  choices for the aces.
2. There are  ${}_4C_3$  choices for the kings.
3. Since the remaining 6 cards must be chosen from the 44 cards that are neither aces nor kings, there are  ${}_{44}C_6$  choices for the remaining cards.
4. There are  ${}_{52}C_{13}$  choices for the 13 cards.
5. Using these results and the multiplication principle, we have:

$$P\left(\begin{array}{l} 13 \text{ cards with exactly} \\ 4 \text{ aces and exactly 3 kings} \end{array}\right) = \frac{{}_4C_4 \cdot {}_4C_3 \cdot {}_{44}C_6}{{}_{52}C_{13}} = \frac{1 \cdot 4 \cdot 7,059,052}{635,013,559,600} \approx 0.0000445$$

**CLASS EXERCISES**

1. Three marbles are picked at random from a bag containing 4 red marbles and 5 white marbles. Match each event with its probability.

*Events:*

- a. All 3 marbles are red.

- b. Exactly 2 marbles are red.

- c. Exactly 1 marble is red.

- d. No marble is red.

*Probabilities:*

$$\frac{{}_4C_1 \cdot {}_5C_2}{{}_9C_3}$$

$$\frac{{}_4C_2 \cdot {}_5C_1}{{}_9C_3}$$

$$\frac{{}_4C_3 \cdot {}_5C_0}{{}_9C_3}$$

$$\frac{{}_4C_0 \cdot {}_5C_3}{{}_9C_3}$$

2. Five cards are drawn at random from a standard deck. Match each event with its probability.

*Events:*

- a. All 4 aces are chosen.

- b. No aces are chosen.

- c. Exactly 4 diamonds are chosen.

- d. Four aces and one jack are chosen.

*Probabilities:*

$$\frac{{}_4C_0 \cdot {}_{48}C_5}{{}_{52}C_5}$$

$$\frac{{}_4C_4 \cdot {}_{48}C_1}{{}_{52}C_5}$$

$$\frac{{}_4C_4 \cdot {}_4C_1}{{}_{52}C_5}$$

$$\frac{{}_{13}C_4 \cdot {}_{39}C_1}{{}_{52}C_5}$$

3. Five cards are drawn at random from a standard deck.
- a. You can find the probability of getting at least 1 ace by calculating the sum:

$$P\left(\begin{smallmatrix} \text{exactly} \\ 1 \text{ ace} \end{smallmatrix}\right) + P\left(\begin{smallmatrix} \text{exactly} \\ 2 \text{ aces} \end{smallmatrix}\right) + P\left(\begin{smallmatrix} \text{exactly} \\ 3 \text{ aces} \end{smallmatrix}\right) + P(4 \text{ aces})$$

It is far easier, however, to find  $1 - P(\underline{\quad ? \quad})$ .

- b. Find the probability of getting at least 1 ace using the method suggested in part (a).

## WRITTEN EXERCISES

In Exercises 1 and 2, leave your answers in terms of factorials unless directed otherwise by your teacher.

- A** 1. Five cards are dealt from a well-shuffled standard deck. What is the probability of getting:
- a. all hearts?                      b. no hearts?                      c. at least one heart?
2. Thirteen cards are dealt from a well-shuffled standard deck. What is the probability of getting:
- a. all clubs?                      b. no clubs?                      c. at least one club?
3. A bag contains 5 red marbles and 3 white marbles. If 2 marbles are randomly drawn, one after the other and without replacement, what is the probability that the number of red marbles is 0? 1? 2? (Check to see that the sum of the probabilities is 1.)
4. Repeat Exercise 3 if the bag contains 6 red marbles and 2 white marbles; that is, find the probabilities of drawing 0, 1, and 2 red marbles.
5. Free concert tickets are distributed to 4 students chosen at random from 8 juniors and 12 seniors in the school orchestra. What is the probability that free tickets are received by:
- a. 4 seniors?  
b. exactly 3 seniors?  
c. exactly 2 seniors?  
d. exactly 1 senior?  
e. no seniors?
6. A town council consists of 8 Democrats, 7 Republicans, and 5 Independents. A committee of 3 is chosen by randomly pulling names from a hat. What is the probability that the committee has:
- a. 2 Democrats and 1 Republican?  
b. 3 Independents?  
c. no Independents?  
d. 1 Democrat, 1 Republican, and 1 Independent?



A committee of 3 people is to be randomly selected from the 6 people Archibald (A), Beatrix (B), Charlene (C), Denise (D), Eloise (E), and Fernando (F). The sample space below lists the 20 possible committees.

{A, B, C}	{A, C, E}	{B, C, D}	{B, E, F}
{A, B, D}	{A, C, F}	{B, C, E}	{C, D, E}
{A, B, E}	{A, D, E}	{B, C, F}	{C, D, F}
{A, B, F}	{A, D, F}	{B, D, E}	{C, E, F}
{A, C, D}	{A, E, F}	{B, D, F}	{D, E, F}

In Exercises 7–14, find the probability in each of two ways:

Method 1: Use the sample space given above.

Method 2: Use combinations.

### Sample

Find the probability that Archibald and Beatrix are on the committee.

### Solution

**Method 1** Since 4 of the 20 sample points include A and B,

$$P(A \text{ and } B) = \frac{4}{20} = \frac{1}{5}.$$

$$\text{Method 2 } P(A \text{ and } B) = \frac{{}^2C_2 \cdot {}^4C_1}{{}^6C_3} = \frac{1 \cdot 4}{20} = \frac{1}{5}$$

7. Find the probability that Eloise is on the committee.
  8. Find the probability that Eloise and Fernando are on the committee.
  9. Find the probability that either Eloise or Fernando is on the committee.
  10. Find the probability that neither Eloise nor Fernando is on the committee.
  11. Find the probability that Archibald is on the committee and Beatrix is not.
  12. Find the probability that Archibald and Beatrix are on the committee but Charlene is not.
  13. Find the probability that Denise is on the committee given that Archibald is.
  14. Find the probability that Denise is on the committee given that neither Archibald nor Beatrix is.
- B** 15. Thirteen cards are dealt from a well-shuffled standard deck. What is the probability of getting:
- a. all red cards?
  - b. 7 diamonds and 6 hearts?
  - c. at least 1 face card?
  - d. all face cards?
16. Thirteen cards are dealt from a well-shuffled standard deck. What is the probability of getting:
- a. all cards from the same suit?
  - b. 7 spades, 3 hearts, and 3 clubs?
  - c. all of the 12 face cards?
  - d. at least one diamond?
17. A committee of 4 is chosen at random from a group of 5 married couples. What is the probability that the committee includes no two people who are married to each other?



18. The letters of the word ABRACADABRA are written on separate slips of paper and placed in a hat. Five slips of paper are then randomly drawn, one after the other and without replacement. What is the probability of getting:

- a. all A's?
- b. both R's?
- c. at least one B?
- d. one of each letter?

19. **Manufacturing** A quality control inspector randomly inspects 4 microchips in every lot of 100. If one or more microchips are defective, the entire lot is rejected for shipment. Suppose a lot contains 10 defective microchips and 90 acceptable ones. What is the probability that the lot is rejected?

20. **Manufacturing** A lot of 20 television sets consists of 6 defective sets and 14 good ones. If a sample of 3 sets is chosen, what is the probability that the sample contains:

- a. all defective sets?
- b. at least one defective set?

- C** 21. A standard deck of 52 cards is shuffled and the cards are dealt face up one at a time until an ace appears.
- a. What is the probability that the first ace appears on the third card?
  - b. Show that the probability of getting the first ace on or before the ninth card is greater than 50%.



## COMMUNICATION: Reading

### Vocabulary in Mathematics

Like most disciplines, mathematics has a special vocabulary. Many of the terms found in a book like this are used only in mathematics. A few examples are *polynomial*, *logarithm*, *asymptote*, and *cosine*. Other terms, however, are used in other disciplines or in everyday discourse with meanings different from their mathematical meanings.

1. Consider the word *experiment*. Explain the difference between its meaning in probability and its meaning in science.
2. Select one of the words *degree*, *function*, *radical*, *polar*, *series*, and look it up in a dictionary. Write down two or three meanings for the word, including its mathematical meaning. For each meaning, write a sentence that uses the word with that meaning.

# Applications of Probability

## 16-5 Working with Conditional Probability

**Objective** To solve problems involving conditional probability.

In this section we will consider the probability of a certain cause when a certain effect is observed. For example, a flu can cause symptoms such as high fever and sore throat, but these can be symptoms of other disorders besides flu. Example 1 examines the probability that a person who has a fever (an effect) also has the flu (a possible cause of the fever).

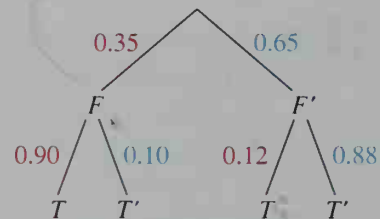
### Example

During a flu epidemic, 35% of a school's students have the flu. Of those with the flu, 90% have high temperatures. However, a high temperature is also possible for people without the flu; in fact, the school nurse estimates that 12% of those without the flu have high temperatures.

- Incorporate the facts given above into a tree diagram.
- About what percent of the student body have a high temperature?
- If a student has a high temperature, what is the probability that the student has the flu?

### Solution

- a. In the tree diagram at the right,  $F$  and  $F'$  represent the events "has the flu" and "does not have the flu," and  $T$  and  $T'$  represent "has a high temperature" and "does not have a high temperature." The probabilities in red come directly from the description in the example. The probabilities in blue are deduced from the fact that all the branches from any given point of a tree must have probabilities that total 1.



- b. To find  $P(T)$ , the probability that a student has a high temperature, we add the probabilities of the two paths leading to a  $T$ :

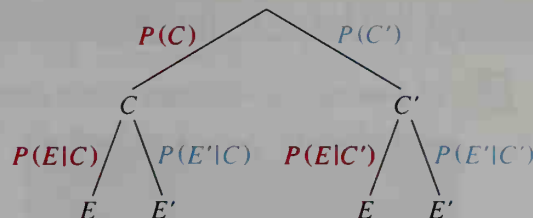
$$\begin{aligned} P(T) &= P(F \text{ and } T) + P(F' \text{ and } T) \\ &= 0.35 \times 0.90 + 0.65 \times 0.12 \\ &= 0.315 + 0.078 = 0.393 \end{aligned}$$

Thus, 39.3% of the student body have a high temperature.

- c. Since a high temperature already exists, we consider only the *portion* of the students who have high temperatures. Part (b) has shown that this portion, 0.393, is the sum of 0.315 (those with high temperatures and the flu) and 0.078 (those with high temperatures and no flu). Thus:

$$P(F | T) = \frac{P(F \text{ and } T)}{P(T)} = \frac{0.315}{0.393} \approx 0.802$$

The technique used in the preceding example can be generalized. Suppose we have a situation in which there is a cause  $C$  and an effect  $E$ . Also suppose we know the probability of the occurrence of  $C$  and the probabilities of the occurrence of  $E$  given the occurrence or nonoccurrence of  $C$ . (These three probabilities are shown in red in the tree diagram above.) The problem is to find the probability that  $C$  has occurred given that  $E$  has.



Let us first symbolically list what we know and can immediately deduce:

1.  $P(C)$  and therefore  $P(C') = 1 - P(C)$
2.  $P(E|C)$  and therefore  $P(E' | C) = 1 - P(E | C)$
3.  $P(E | C')$  and therefore  $P(E' | C') = 1 - P(E | C')$

Then, to find  $P(C | E)$ , we take the following two steps:

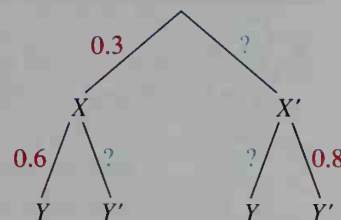
Step 1. Find  $P(E) = P(C \text{ and } E) + P(C' \text{ and } E)$

$$= P(C) \cdot P(E | C) + P(C') \cdot P(E | C')$$

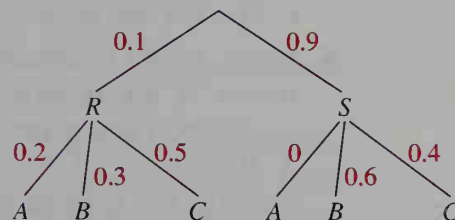
Step 2. Find  $P(C | E) = \frac{P(C \text{ and } E)}{P(E)}$ . Each of these is calculated in Step 1.

## CLASS EXERCISES

1. a. Copy the tree diagram at the right and find the missing probabilities.  
 b. The question mark at the lower left refers to  $P(Y' | X)$ . To what probabilities do the other two question marks refer?  
 c. Find  $P(X \text{ and } Y)$ .  
 d. Find  $P(X' \text{ and } Y)$ .  
 e. Find  $P(Y)$ .  
 f. Find  $P(X|Y)$ .



2. Use the tree diagram at the right to find each probability.  
 a.  $P(A|R)$   
 b.  $P(A|S)$   
 c.  $P(R \text{ and } C)$   
 d.  $P(S \text{ and } C)$   
 e.  $P(C)$   
 f.  $P(R|C)$

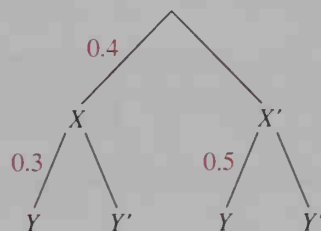


3. **Reading** Study the example in this section. Given the results of parts (b) and (c) of the example, show two different ways to find the probability that a student who has a high temperature does *not* have the flu.

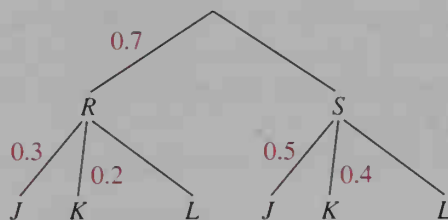
## WRITTEN EXERCISES

- A** 1. Use the tree diagram at the left below to find each probability.

- a.  $P(X')$                       b.  $P(Y' | X)$                       c.  $P(Y' | X')$                       d.  $P(X \text{ and } Y)$   
 e.  $P(X' \text{ and } Y)$                       f.  $P(Y)$                       g.  $P(X | Y)$                       h.  $P(X' | Y)$



Ex. 1



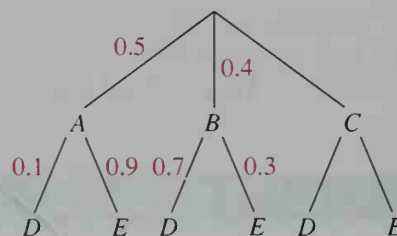
Ex. 2

2. Use the tree diagram at the right above to find each probability.

- a.  $P(S)$                       b.  $P(L | R)$                       c.  $P(L | S)$                       d.  $P(J \text{ and } R)$   
 e.  $P(J \text{ and } S)$                       f.  $P(J)$                       g.  $P(R | J)$                       h.  $P(S | J)$

3. Use the tree diagram at the right to find each sum of probabilities.

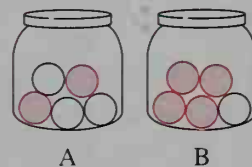
- a.  $P(A \text{ and } D) + P(A \text{ and } E)$   
 b.  $P(B \text{ and } D) + P(B \text{ and } E)$   
 c.  $P(C \text{ and } D) + P(C \text{ and } E)$



Exs. 3, 4

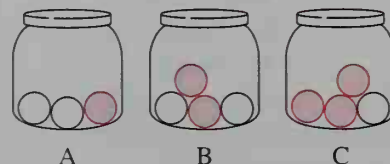
4. Refer to the tree in Exercise 3 and suppose that event  $D$  cannot possibly happen if event  $C$  happens. Find  $P(A | D)$ ,  $P(B | D)$ , and  $P(C | D)$ .

5. Jar A contains 2 red balls and 3 white balls. Jar B contains 4 red balls and 1 white ball. A coin is tossed. If it shows "heads," a ball is randomly picked from Jar A; if it shows "tails," a ball is randomly picked from Jar B.



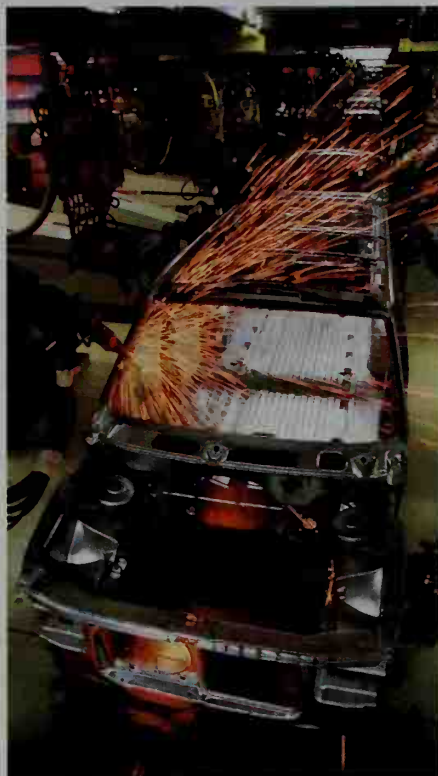
- a. Draw a tree diagram showing the probabilities of each jar and then the probabilities of picking a red ball or a white ball.  
 b. Find the probability of picking a red ball.  
 c. If a red ball is picked, find the probability that it came from Jar A.

6. Jars A, B, and C contain red and white balls as shown. A die is rolled. If an even number comes up, a ball is randomly picked from Jar A. If a "1" or a "3" comes up, a ball is randomly picked from Jar B. If a "5" comes up, a ball is randomly picked from Jar C.



- a. Incorporate the facts given above into a tree diagram.  
 b. Find the probability of picking a red ball.  
 c. If a red ball is picked, what is the probability that it came from Jar A? from Jar B? from Jar C?

7. **Manufacturing** Machine A produces 60% of the ball bearings manufactured by a factory and Machine B produces the rest. Five percent of Machine A's bearings fail to have the required precision, and two percent of Machine B's bearings fail.
- Incorporate the facts given above into a tree diagram.
  - What percent of the bearings fail to have the required precision?
  - If a bearing is inspected and fails to have the required precision, what is the probability that it was produced by Machine A?
8. **Manufacturing** Five percent of the welds on an automobile assembly line are defective. The defective welds are found using an X-ray machine. The machine correctly rejects 92% of the defective welds and correctly accepts all of the good welds.
- Incorporate the facts given above into a tree diagram.
  - What percent of the welds are accepted by the machine?
  - Find the probability that an accepted weld is defective.
9. **Insurance** An auto insurance company charges younger drivers a higher premium than it does older drivers because younger drivers as a group tend to have more accidents. The company has 3 age groups: Group A includes those under 25 years old, 22% of all its policyholders. Group B includes those 25–39 years old, 43% of all of its policyholders. Group C includes those 40 years old or older. Company records show that in any given one-year period, 11% of its Group A policyholders have an accident. The percentages for groups B and C are 3% and 2%, respectively.
- What percent of the company's policyholders are expected to have an accident during the next 12 months?
  - Suppose Mr. X has just had a car accident. If he is one of the company's policyholders, what is the probability that he is under 25?
10. **Insurance** Suppose the insurance company of Exercise 9 not only classifies drivers by age, but in the case of drivers under 25 years old, it also notes whether they have had a driver's education course. One quarter of its policyholders under 25 have had driver's education and 5% of these have an accident in a one-year period. Of those under 25 who have not had driver's education, 13% have an accident within a one year period. A 20-year-old woman takes out a policy with this company and within one year she has an accident. What is the probability that she did *not* have a driver's education course?



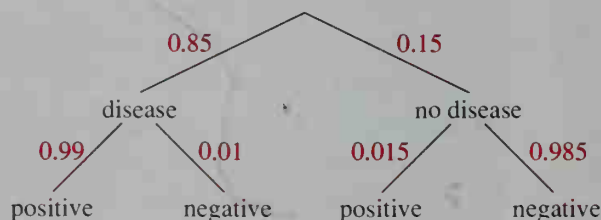


- B** 11. **Medicine** A medical research lab proposes a screening test for a disease. In order to try out this test, it is given to 100 people, 60 of whom are known to have the disease and 40 of whom are known not to have the disease. A positive test indicates the disease and a negative test indicates no disease. Unfortunately, such medical tests can produce two kinds of errors:

- (1) A *false negative* test: For the 60 people who do have the disease, this screening test indicates that 2 do *not* have it.
- (2) A *false positive* test: For the 40 people who do not have the disease, this screening test indicates that 10 *do* have it.

- a. Which of the false tests do you think is more serious? Why?
- b. Incorporate the facts given above into a tree diagram. (Be sure to convert the given integers into probabilities.)
- c. Suppose the test is given to a person not in the original group of 100 people. It is not known whether this person has the disease, but the test result is positive. What is the probability that the person really does have the disease?
- d. Suppose the test is given to a person whose disease status is unknown. If the test result is negative, what is the probability that the person does *not* have the disease?

12. **Medicine** Part (c) of Exercise 11 indicates that about 85% of those who test positive really do have the disease, so that 15% of those who test positive do not have it. This 15% error may seem high, but people with a positive screening test are usually given a more thorough diagnostic test. Even the diagnostic test can yield errors but they are much less likely than the screening test, as the diagram shows.



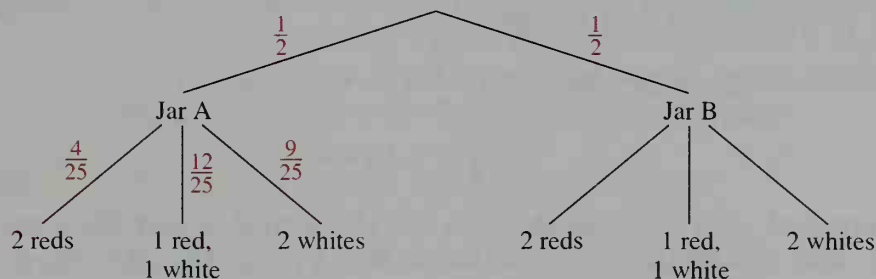
- a. What is the probability that the diagnostic test gives:
    - (1) a false negative result?
    - (2) a false positive result?
  - b. What is the probability that:
    - (1) the diagnostic test gives the correct result?
    - (2) a person with a positive diagnostic test has the disease?
    - (3) a person with a negative diagnostic test does not have the disease?
13. The children of a math professor play two games that use dice. In one game, two dice are rolled and the sum of the numbers on the dice is called out. In the other game, a single die is rolled and its number is called out. The professor hears the children in another room call out the number 2, and knowing that they play the two games about equally often, the professor is able to calculate the probability they are playing the two-dice game. What is this probability?
14. Solve Exercise 13 if the children call out:
- a. the number 4
  - b. the number 7
  - c. the number 1

15. Suppose you have two pairs of dice. One pair is fair, but each die of the other pair is weighted so that a six comes up with probability  $\frac{1}{4}$  instead of the usual  $\frac{1}{6}$ . If you randomly choose one pair of dice, roll them, and obtain two sixes, what is the probability that you rolled the weighted dice?
16. Suppose you have 2 coins. One of them is fair, but the other has two “heads.” You choose one coin at random, flip it  $n$  times, and get “heads” each time.
- Find the probability that the coin is two-headed for  $n = 1$ ,  $n = 2$ , and  $n = 10$ .
  - In terms of  $n$ , find the probability that the coin is two-headed.
- C** 17. Refer to the two jars pictured in Exercise 5. Suppose that someone randomly picks one of the jars, but you don’t know which one. Before guessing which jar the “mystery jar” is, you have your choice of doing either of the following experiments.

Experiment 1: Pick 2 balls from the “mystery jar” and note their colors.  
Replace the first ball before choosing the second.

Experiment 2: Pick 2 balls from the “mystery jar” and note their colors.  
Do *not* replace the first ball before choosing the second.

The tree diagram for Experiment 1 looks like this:



- Copy and complete the diagram.
- Suppose you performed Experiment 1 and got 2 red balls. Then you would no doubt guess that they came from Jar B. Which jar would you guess if you got 2 white balls? 1 red and 1 white ball? What is the probability that you would guess the correct jar if you performed the first experiment?
- Make a tree diagram for Experiment 2.
- If you performed Experiment 2, which jar would you guess if you got 2 red balls? 2 white balls? 1 red and 1 white ball? What is the probability that you would guess correctly if you performed the second experiment?
- Which experiment gives you the better chance of guessing correctly?
- Suppose a third experiment allows you to pick just one ball and then guess the jar. How likely are you to guess correctly in this experiment? Compare your answer with the probability of being correct in the first or second experiment. Are you surprised?

## 16-6 Expected Value

### Objective

To find expected value in situations involving gains and losses and to determine whether a game is fair.

In this section we will consider probabilistic situations involving gains or losses. For example:

1. What can you expect to win or lose in various games of chance?
2. What is the value of a \$1 ticket in a \$1 million lottery?
3. Should someone who is 18 years old pay \$160 for collision damage insurance on his or her \$1500 car?

Let us begin with the first question. Consider a simple game in which a die is rolled and you win points from, or lose points to, another player as follows.

Event	Die shows 1, 2, or 3	Die shows 4 or 5	Die shows 6
Gain or loss	+10 points	-13 points	-1 point
Probability	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

To calculate the *expected value* of this game, you must multiply each gain or loss by its probability and then add the products.

$$\begin{aligned}\text{Expected value} &= 10 \cdot \frac{3}{6} + (-13) \cdot \frac{2}{6} + (-1) \cdot \frac{1}{6} \\ &= +\frac{3}{6} = 0.5 \text{ point}\end{aligned}$$

The expected value of 0.5 point means that if you were to play this game many times, your *average gain per game* would be 0.5 point. The other player would expect to lose 0.5 point per game.

In general, if a given situation involves various payoffs (gains or losses of points, money, time, and so on), then its **expected value** is calculated as follows.

Payoff	$x_1$	$x_2$	$x_3$	$\cdots$	$x_n$
Probability	$P(x_1)$	$P(x_2)$	$P(x_3)$	$\cdots$	$P(x_n)$

$$\text{Expected value} = x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + x_3 \cdot P(x_3) + \cdots + x_n \cdot P(x_n)$$

If the expected value of a game is 0, then the game is called a **fair game**. The dice game previously discussed is not a fair game because the expected value is 0.5 point and not 0. An example of a fair game is a coin toss in which you win a point if the coin comes up “heads” and you lose a point if the coin comes up “tails.” The expected value is:

$$(1 \text{ point}) \times \frac{1}{2} + (-1 \text{ point}) \times \frac{1}{2} = 0$$

**Example 1**

If the sum of two rolled dice is 8 or more, you win \$2; if not, you lose \$1.

- Show that this is not a fair game.
- To have a fair game, the \$2 winnings should instead be what amount?

**Solution**

Make a table of payoffs. For the probabilities, use the table on page 601.

Event	Sum $\geq 8$	Sum $< 8$
Payoff	\$2	−\$1
Probability	$\frac{15}{36}$	$\frac{21}{36}$

- Expected value of game =  $2\left(\frac{15}{36}\right) + (-1)\left(\frac{21}{36}\right) = +\frac{1}{4} = \$0.25$
- To have a fair game, let  $x$  be the winnings for a sum greater than or equal to 8 and find the value of  $x$  that produces an expected value of 0.

$$\text{Expected value} = x\left(\frac{15}{36}\right) + (-1)\left(\frac{21}{36}\right) = 0$$

$$15x - 21 = 0$$

$$x = \$1.40$$

**Example 2**

In a certain state's lottery, six numbers are randomly chosen without repetition from the numbers 1 to 40. If you correctly pick all 6 numbers, only 5 of the 6, or only 4 of the 6, then you win \$1 million, \$1000, or \$100, respectively. What is the value of a \$1 lottery ticket?

**Solution**

First we must find the probabilities of winning:

$$P(\text{all 6 correct}) = \frac{{}^6C_6}{{}^{40}C_6} \approx 0.00000026$$

$$P(5 \text{ of 6 correct}) = \frac{{}^6C_5 \cdot {}^{34}C_1}{{}^{40}C_6} \approx 0.000053$$

$$P(4 \text{ of 6 correct}) = \frac{{}^6C_4 \cdot {}^{34}C_2}{{}^{40}C_6} \approx 0.0022$$

Then we multiply these probabilities by their associated gains, temporarily ignoring the cost of the \$1 ticket:

Expected gain

$$\begin{aligned} &\approx 1,000,000(0.00000026) + 1000(0.000053) + 100(0.0022) \\ &= 0.26 + 0.053 + 0.22 \approx \$0.53 \end{aligned}$$

When we consider the \$1 cost of the ticket, the expected value of the ticket is about  $\$0.53 - \$1 = -\$0.47$ .

**Example 3**

An 18-year-old student must decide whether to spend \$160 for one year's car collision damage insurance. The insurance carries a \$100 deductible, which means that when the student files a damage claim, the student must pay \$100 of the damage amount, with the insurance company paying the rest (up to the value of the car). Because the car is only worth \$1500, the student consults with an insurance agent who draws up a table of possible damage amounts and their probabilities based on the driving records for 18-year-olds in the region.

Event	Accident costing \$1500	Accident costing \$1000	Accident costing \$500	No accident
Payoff	\$1400	\$900	\$400	\$0
Probability	0.05	0.02	0.03	0.90

What is the expected value of this insurance?

**Solution**

We temporarily ignore the \$160 cost and calculate the expected payoff from the table above.

$$\begin{aligned}
 \text{Expected payoff} &= 1400(0.05) + 900(0.02) + 400(0.03) + 0(0.90) \\
 &= 70 + 18 + 12 + 0 \\
 &= \$100
 \end{aligned}$$

Now we consider the \$160 cost and find that:

$$\text{Expected value} = 100 - 160 = -\$60$$

While an insurance policy with an expected value of  $-\$60$  may not seem like a "fair game" between the student and the insurance company, the company uses the \$60 to pay for operating expenses, salaries, and profit.

**CLASS EXERCISES**

Find the expected payoff.

1.

Payoff	5	10
Probability	0.6	0.4

2.

Payoff	3	1	-1
Probability	0.5	0.1	0.4

3. Players A and B play a game in which a die is rolled and A wins 2 points from B if a 5 or 6 appears. Otherwise, B wins 1 point from A. Decide if this is a fair game.
4. **Reading** The game in Example 1 can be made fair by changing the winnings from \$2 to \$1.40 (see part (b)). Find a different way to make the game fair.



5. Suppose you toss 3 coins and win payoffs as shown in the table below. Complete the table and find your expected payoff.

Number of "heads"	3	2	1	0
Payoff	\$5	\$3	\$1	-\$9
Probability	?	?	?	?

6. **Discussion** Discuss why the student in Example 3 may want the car collision damage insurance even though its expected value to the student is negative.

## WRITTEN EXERCISES

Find the expected payoff.

**A**

1. 

Payoff	9	7	-5
Probability	0.1	0.3	0.6

2. 

Payoff	6	3	-5
Probability	0.2	0.1	0.7

3. 

Payoff	60	52	50
Probability	0.4	0.5	0.1

4. 

Payoff	13	-7	-12
Probability	0.4	0.2	0.4

For Exercises 5–8, decide if each game is a fair game. If not, state which player has the advantage.

- A die is rolled. If the number that shows is odd, player A wins \$1 from player B. If it is a 6, A wins \$2 from B. Otherwise B wins \$3 from A.
- A box contains 2 red balls and 1 white ball. Two balls are randomly chosen without replacement. If both are red, player A wins \$5 from player B. Otherwise B wins \$2 from A.
- Two dice are rolled. If the sum is 6, 7, or 8, player A wins \$5 from player B. Otherwise B wins \$4 from A.
- Two dice are rolled. If the sum of the numbers showing on the dice is odd, player A wins \$1 from player B. If both dice show the same number, A wins \$3 from B. Otherwise B wins \$3 from A.
- Suppose you play a game in which you make a bet and then draw a card from a well-shuffled deck that includes the standard 52 cards as well as 2 jokers. If you draw a joker, you keep your bet and win \$5; if you draw a face card, you keep your bet and win \$2; and if you draw any other card, you lose your bet. What is your expected gain or loss on this game if your bet is \$1?
- Suppose you have \$10 to bet on the game described in Exercise 9. Is your expected gain or loss any different if you bet the whole \$10 on one game rather than betting \$1 at a time on 10 successive games?

- B** 11. **Test Taking** On a multiple-choice test, a student is given five possible answers for each question. The student receives 1 point for a correct answer and loses  $\frac{1}{4}$  point for an incorrect answer. If the student has no idea of the correct answer for a particular question and merely guesses, what is the student's expected gain or loss on the question?
12. **Test Taking** Suppose you are taking the multiple-choice test described in Exercise 11. Suppose also that on one of the questions you can eliminate two of the five answers as being wrong. If you guess at one of the remaining three answers, what is your expected gain or loss on the question?
13. A box contains 3 red balls and 2 green balls. Two balls are randomly chosen without replacement. If both are green, you win \$2. If just one is green, you win \$1. Otherwise you lose \$1. What is your expected gain or loss?
14. In the carnival game "chuck-a-luck," you pick a number from 1 to 6 and roll 3 dice in succession. If your number comes up all 3 times, you win \$3; if your number comes up twice, you win \$2; if it comes up once, you win \$1; otherwise you lose \$1. What is your expected gain or loss? (*Hint*: Make a table like the one in Class Exercise 5.)
15. Rewrite the definition of expected value (page 630) using sigma notation.
16. Rewrite the definition of expected value (page 630) as a dot product of two vectors.
17. **Farming** A dairy farmer estimates that next year the farm's cows will produce about 25,000 gallons of milk. Because of variation in the market price of milk and the cost of feeding the cows, the profit per gallon may vary with the probabilities given in the table below. Estimate the profit on the 25,000 gallons.



Gain per gallon	\$1.10	\$.90	\$.70	\$.40	\$0.00	−\$.10
Probability	0.30	0.38	0.20	0.06	0.04	0.02

18. **Insurance** At many airports, a person can pay only \$1 for a \$100,000 life insurance policy covering the duration of the flight. In other words, the insurance company pays \$100,000 if the insured person dies from a possible flight crash; otherwise the company gains \$1 (before expenses). Suppose that past records indicate 0.45 deaths per million passengers. How much can the company expect to gain on one policy? on 100,000 policies?

19. **Business** A construction company wants to submit a bid for remodeling a school. The research and planning needed to make the bid cost \$4000. If the bid is accepted, the company would make \$26,000. Would you advise the company to spend the \$4000 if the bid has only a 20% probability of being accepted? Explain your reasoning.
20. **Consumer Economics** Suppose the warranty period for your family's new television is about to expire and you are debating about whether to buy a one-year maintenance contract for \$35. If you buy the contract, all repairs for one year are free. Consumer information shows that 12% of the televisions like yours require an annual repair that costs \$140 on the average. Would you advise buying the maintenance contract? Explain your reasoning.
21. A lottery has one \$1000 prize, five \$100 prizes, and twenty \$10 prizes. What is the expected gain from buying one of the 2000 tickets sold for \$1 each?
22. In a state lottery, five numbers are randomly chosen from the numbers 1 to 30. If you pick all 5 numbers, you win \$100,000; and if you pick 4 of the 5 numbers, you win \$100. What is the value of a \$1 lottery ticket?
23. Players A and B are playing a game in which A wins a point every time a coin lands "heads" and B wins a point every time the coin lands "tails." (No points are lost by either player.) The first person to reach 3 points wins \$100. If A currently has 2 points and B has 1 point, what is A's expected gain? (*Hint:* Make a tree diagram showing the ways in which the game can be finished. From the diagram, determine the probability that A wins.)
24. Suppose you play a game in which you make a bet, toss a coin, and either win an amount equal to your bet or lose your bet depending on whether you correctly call "heads" or "tails." Also suppose you begin with a \$1 bet and double your bet on each toss until you win once and leave the game or until you have lost \$15. What can you expect to win with this betting strategy?

- C** 25. A die is rolled repeatedly until a "1" appears.
- Complete the table.

Number of rolls until "1" appears	1	2	3	4	5	...	$n$	...
Probability	$\frac{1}{6}$	$\frac{5}{6} \cdot \frac{1}{6}$	$\left(\frac{5}{6}\right)^2 \cdot \frac{1}{6}$	?	?	...	?	...

- Express the expected number of rolls as the sum of an infinite series. Then factor  $\frac{1}{6}$  from the sum.
- It can be proved (most easily with calculus) that

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Use this to show that the expected number of rolls until a "1" appears is 6.

## Chapter Summary

1. The probability of an event is a number (between 0 and 1, inclusive) that indicates the likelihood of the event's occurrence. Probabilities can be determined either empirically or theoretically.
2. An *experiment* is any action having various outcomes that occur unpredictably. A set  $S$  is a *sample space* of an experiment if each outcome of the experiment corresponds to exactly one element of  $S$ . Any subset of  $S$  is an *event*.
3. If the sample space of an experiment consists of  $n$  equally likely outcomes,  $m$  of which correspond to event  $A$ , then the probability of  $A$  is  $P(A) = \frac{m}{n}$ . The event "not  $A$ " occurs whenever event  $A$  does not, and  $P(\text{not } A) = 1 - P(A)$ .
4. Events  $A$  and  $B$  are *mutually exclusive* if they cannot occur simultaneously.
5. For any two events  $A$  and  $B$ ,  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ , or, if  $A$  and  $B$  are mutually exclusive,  $P(A \text{ or } B) = P(A) + P(B)$ .
6. For two events  $A$  and  $B$ , the probability that  $B$  occurs on the condition that  $A$  occurs is denoted  $P(B|A)$  and is called a *conditional probability*.
7. Events  $A$  and  $B$  are *independent* if the occurrence of one does not affect the probability of the other's occurrence, that is,  $P(B|A) = P(B)$ .
8. For any two events  $A$  and  $B$ ,  $P(A \text{ and } B) = P(A) \cdot P(B|A)$ , or, if  $A$  and  $B$  are independent,  $P(A \text{ and } B) = P(A) \cdot P(B)$ .
9. The binomial probability theorem gives the probabilities of 0, 1, 2, . . . ,  $n$  occurrences of the event  $A$  in  $n$  repeated independent trials.
10. If a given situation involves payoffs  $x_1, x_2, \dots, x_n$  with corresponding probabilities  $P(x_1), P(x_2), \dots, P(x_n)$ , then the situation's *expected value* is:

$$x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + x_3 \cdot P(x_3) + \cdots + x_n \cdot P(x_n)$$

### Key vocabulary and ideas

probability (p. 597)	odds (p. 601)
empirical probability (p. 597)	conditional probability (p. 607)
theoretical probability (p. 597, p. 598)	independent events (p. 607)
experiment, sample space, event (p. 598)	$P(A \text{ and } B)$ (p. 608)
probability of event $A$ , or $P(A)$ (p. 598)	binomial probability theorem (p. 614)
$P(A \text{ or } B)$ (p. 599)	Monte Carlo simulation (p. 617)
mutually exclusive events (p. 599)	expected value (p. 630)
$P(\text{not } A)$ (p. 599)	fair game (p. 630)

## Chapter Test

1. **Writing** Write a paragraph in which you discuss the difference between, and give examples of, empirical and theoretical probability. 16-1

2. An experiment consists of randomly drawing a card from a standard deck and rolling a die. Find the probability that:
  - a. the card is a spade and the die shows a 5
  - b. the card is *not* a spade and the die shows a 5
  - c. the card is *not* a spade and the die does *not* show a 5
3. A fish bowl contains slips of paper numbered 1 through 9. Two slips are drawn, one after the other and without replacement. Find the probability of the each of the following events. 16-2
  - a. Both numbers are odd.
  - b. The first number drawn is 6.
  - c. The first number drawn is greater than 6 and the second number is less than 4.
4. For a certain brand of marigold seeds, the seeds sprout on average 9 out of 10 times. If 5 seeds are selected at random and planted, find the probability that: 16-3
  - a. all 5 seeds sprout
  - b. at least 3 of the 5 seeds sprout
5. A Central School PTA committee is to consist of 3 teachers and 3 parents, who are to be chosen at random from the 24 teachers and 145 parents involved in the PTA. If half of the teachers and 99 parents are women, find the probability that the committee has: 16-4
  - a. only female members
  - b. only 1 parent and only 1 teacher who are women
6. Ninety-five percent of the sneakers manufactured by a shoe company have no defects. In order to find the 5% that do have defects, inspectors carefully look over every pair of sneakers. Still, the inspectors sometimes make mistakes because 8% of the defective pairs pass inspection and 1% of the good pairs fail the inspection test. 16-5
  - a. Incorporate the facts given above into a tree diagram.
  - b. What percent of the pairs of sneakers pass inspection?
  - c. If a pair of sneakers passes inspection, what is the probability that it has a defect?
7. Suppose you play a game in which 5 coins are tossed simultaneously. If 1, 2, 3, or 4 “heads” occur, you win \$1 for each “head.” If all “heads” or all “tails” occur, however, you lose \$20. 16-6
  - a. Copy and complete the following table.

Number of “heads”	0	1	2	3	4	5
Payoff	−\$20	?	?	?	?	?
Probability	$\frac{1}{32}$	?	?	?	?	?

- b. What is the game’s expected payoff?





# ■ Descriptive Statistics

## 17-1 Tables, Graphs, and Averages

### Objective

To display a set of data using a stem-and-leaf plot, a histogram, a frequency polygon, or a cumulative frequency polygon and to find the mean, median, and mode.

The subject of statistics can be divided into two parts, *descriptive statistics* and *inferential statistics*. **Descriptive statistics** involves collecting, organizing, and summarizing numerical facts, called *data*, about some group. **Inferential statistics** involves making inferences or decisions based on the data. The first and second parts of this chapter deal with descriptive and inferential statistics, respectively.

### Tables and Graphs

Shown below are the mathematics scores for the 50 seniors in a Midwestern high school who took a standardized achievement test.

480	520	670	580	500	570	700	540	690	500
520	450	570	540	620	450	460	790	580	610
550	630	460	470	510	630	510	720	490	540
690	510	500	710	520	480	750	450	670	610
320	680	530	510	350	400	400	640	430	480

To make some sense out of these scores, we can organize them into a **stem-and-leaf plot**. The first digit of each score is called the *stem* and the second digit is called the *leaf*. The result of recording the first column of scores (480, 520, 550, 690, and 320) is shown at the right. (Notice that all the leaves for a given stem are simply written one after the other.) Two complete stem-and-leaf plots are shown below. The ordered plot (on the right) is useful when calculating certain statistics, as you will see.

stem	leaf
3	2
4	8
5	2 5
6	9
7	

3|2 represents  
a score of 320.

3	2 5
4	8 5 6 7 5 8 0 6 0 5 9 3 8
5	2 5 2 1 7 0 3 8 4 1 0 1 2 7 1 4 8 0 4
6	9 3 8 7 2 3 4 9 7 1 1
7	1 0 5 9 2

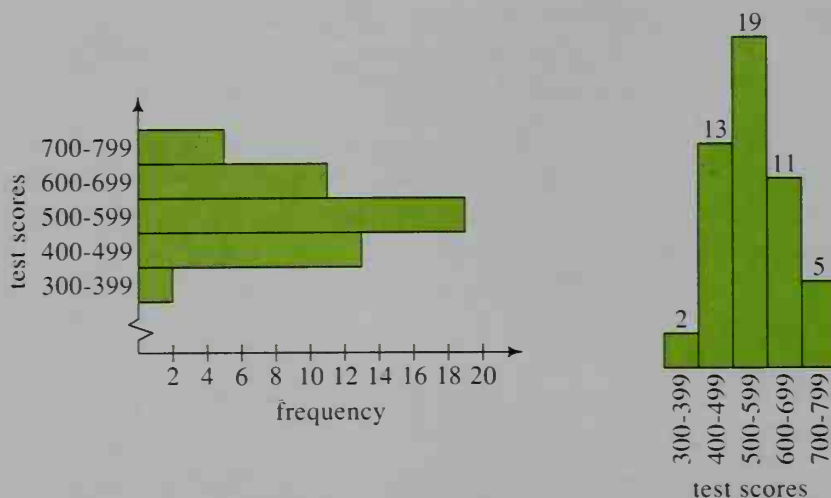
Scores recorded by column

3	2 5
4	0 0 3 5 5 5 6 6 7 8 8 8 9
5	0 0 0 1 1 1 1 2 2 2 3 4 4 4 5 7 7 8 8
6	1 1 2 3 3 4 7 7 8 9 9
7	0 1 2 5 9

Scores rearranged in increasing order

◀ One can record the weights and heights of zebras in a small random sample, then use statistics to make a confident prediction about the average weights and heights in the population as a whole.

The test score data can also be organized and displayed in a **histogram** with either horizontal or vertical bars, as shown below. Notice that the histogram with horizontal bars looks very much like the stem-and-leaf plots at the bottom of the preceding page. Also notice that a frequency axis is not necessary when the frequency is noted on or near each bar.



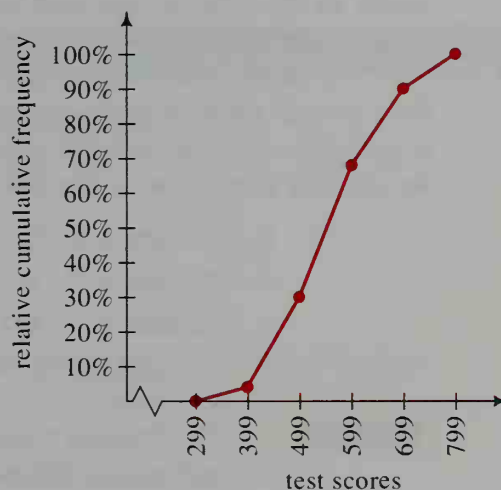
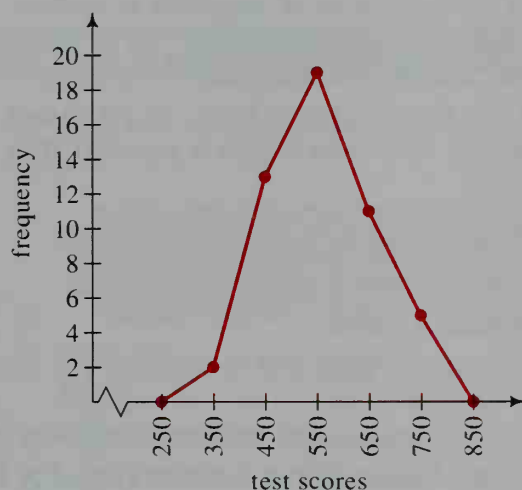
The test score data can also be given in a **frequency table**, as shown at the left below. Notice that the column labeled “frequency” gives the numbers of scores in each *class*, or group, of scores. The column labeled “cumulative frequency,” on the other hand, gives the number of scores (from the entire set of scores) less than or equal to the largest score in each class. For example, the frequency for the class 400–499 is 13 (because there are 13 scores in the 400’s), but the cumulative frequency for the same class is  $2 + 13 = 15$  (because there are a total of 15 scores less than or equal to 499).

Frequency Table		
Class	Frequency	Cumulative frequency
300–399	2	2
400–499	13	15
500–599	19	34
600–699	11	45
700–799	5	50
Total = 50		

Relative Frequency Table		
Class	Relative frequency	Relative cumulative frequency
300–399	4%	4%
400–499	26%	30%
500–599	38%	68%
600–699	22%	90%
700–799	10%	100%
Total = 100%		

The table on the right at the bottom of the preceding page is like the one on the left except that it gives *relative frequencies* (expressed as percents). For example, while the left table shows that 13 out of 50 scores are in the class 400–499, the right table records this as  $\frac{13}{50} = 26\%$ . Likewise, while the left table shows that the cumulative frequency for the class 400–499 is 15, the right table records this as a relative cumulative frequency of  $\frac{15}{50} = 30\%$ .

Using the frequency tables, we can construct the **frequency polygon** shown at the left below. Its vertical axis can be labeled either as integer frequencies, as shown, or as relative frequencies in percents. Either way, the horizontal axis of the graph shows the middle score for each class. Note that a frequency polygon is usually drawn so that it begins and ends on the horizontal axis.



Using the frequency tables, we can also construct the **cumulative frequency polygon** shown at the right above. Although its vertical axis can be labeled either in integers or in percents, the use of percents is most common. The horizontal axis shows the largest score for each class.

## Averages

Sometimes we use a single number, called an *average*, to represent an entire set of data. In statistics the three most common averages are the *mean*, the *median*, and the *mode*. Because each average points to some “center” of the data, each is called a *measure of central tendency*.

The **mean** of a set of data is the sum of the data divided by the number of items of data. The mean is the familiar average that you may use when talking about your “test average.” If the data are given in a list  $x_1, x_2, x_3, \dots, x_n$ , then the mean, which is denoted  $\bar{x}$  (read “ $x$  bar”), is given by:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{\sum_{i=1}^n x_i}{n}$$



The formula for the mean given at the bottom of the preceding page can be used to find the mean of the 50 test scores given on page 639. Since the sum of the scores is 27,480, the mean is  $\frac{27,480}{50} = 549.6$ .

The **median** of a set of data is found by arranging the data in increasing or decreasing order. If the number of items of data is odd, then the median is the middle number in the ordered set. For example, the median of the data 1, 3, 3, 5, 6, 8, 9 is 5. If the number of items of data is even, then the median is the mean of the two middle numbers. For example, the median of the data 2, 3, 3, 5, 8, 9 is 4, which is the mean of 3 and 5.

If a set of data is given in a stem-and-leaf plot with ordered leaves, then the median of the data can be found easily. For example, the median of the 50 test scores in the ordered stem-and-leaf plot at the bottom of page 639 is halfway between the 25th and 26th scores; it is 525.

The **mode** of a set of data is the one item of data that occurs most often. If there are two (or three or more) items of data that occur most often, the set of data is called *bimodal* (or *trimodal* or *multimodal*). If no item of data is repeated, then we say that there is no mode. For example:

1. The data 1, 3, 3, 5, 8, 9 have mode 3.
2. The data 1, 3, 3, 5, 8, 8 have modes 3 and 8.
3. The data 1, 2, 3, 5, 8, 9 have no mode.
4. The mode of the 50 test scores on page 639 is 510.

### Example

The frequency table below summarizes the numbers of siblings (brothers and sisters) for each of the 25 students in a statistics class. For the data, find (a) the mean, (b) the median, and (c) the mode.

Number of siblings	0	1	2	3	4	5
Number of students	2	10	8	4	0	1

### Solution

$$\begin{aligned}
 \text{a. Mean} &= \frac{\text{total number of siblings}}{25} \\
 &= \frac{0 \cdot 2 + 1 \cdot 10 + 2 \cdot 8 + 3 \cdot 4 + 4 \cdot 0 + 5 \cdot 1}{25} \\
 &= \frac{43}{25} = 1.72
 \end{aligned}$$

- b. Since the data consist of 25 numbers, the middle one is the thirteenth. Reading from the bottom row of the table, we see that twelve ( $2 + 10$ ) students have either 0 or 1 siblings. Thus, the thirteenth student has 2 siblings, which is the median number.
- c. The mode of the given data is 1 because this number occurs most frequently (10 times).



Part (a) of the preceding example suggests an alternate way of calculating the mean of a set of data. If the data are given in a frequency table like this:

Item of data	$x_1$	$x_2$	$\cdots$	$x_n$
Frequency	$f_1$	$f_2$	$\cdots$	$f_n$

then the mean is given by:

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^n x_i f_i}{\sum_{i=1}^n f_i}$$

## CLASS EXERCISES

1. Give the mean, median, and mode of each set of data.

a. 1, 2, 3, 5, 5

b. 3, 4, 4, 7, 8, 8, 10, 11

c. 1, 2, 3

d.

Observation	0	1	2	3	4
Frequency	3	2	5	8	2

2. **Investigation** Record the number of siblings that each member of your class has. Tabulate the data and draw a histogram. Then calculate the mean, median, and mode of the data.
3. a. Is it possible for the mean of a set of integers not to be an integer?  
 b. Is it possible for the median of a set of integers not to be an integer?  
 c. Is it possible for the mode of a set of integers not to be an integer?
4. In a set of data, suppose the smallest number (not a mode) is replaced with an even smaller number. How are the mean, median, and mode affected?

## WRITTEN EXERCISES

In Exercises 1–4, find the mean, median, and mode for each set of data.

**A**

1. The numbers of mice born in 9 different litters were:

5, 7, 6, 3, 8, 6, 4, 6, 4

2. The test scores for a statistics student were:

85, 74, 92, 87, 84, 78, 90

3. The numbers of items produced in 1 h by 12 different machines were:

0, 1, 3, 5, 5, 5, 7, 9, 9, 11, 15, 99

4. The heights in centimeters of the members of a college basketball team are:

185, 189, 191, 193, 193, 195, 196, 198, 198, 200

5. a. The number of trials required by 20 different puppies to learn a trick is recorded in the stem-and-leaf plot at the left below. Rearrange the data in increasing order, and then find the median and the mode of the data.
- b. The stem-and-leaf plot at the left below has just three stems (0, 1, and 2). In order to spread out the data more, you can draw a plot with 6 stems, as shown at the right below. The stem 2 is used for data from 20 to 24, while the dot below the 2 is the stem for data from 25 to 29. The other dots have similar meanings. Complete this plot.

0	9 7 9 4 8
1	8 2 3 0 9 1 6 4 5 6 3
2	1 2 8 5

Stem-and-leaf plot  
with 3 stems

0
•
1
•
2
•

Stem-and-leaf plot  
with 6 stems

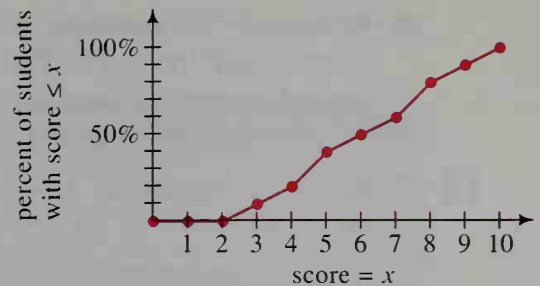
6. **Meteorology** The maximum Fahrenheit temperatures at the municipal airport for each of the 30 days of September were:
- 82°, 75°, 78°, 83°, 81°, 74°, 80°, 74°, 73°, 62°, 57°, 64°, 68°, 70°, 76°, 67°, 63°, 60°, 55°, 58°, 62°, 59°, 55°, 53°, 53°, 56°, 61°, 55°, 52°, 60°
- a. Summarize the data in a stem-and-leaf plot having stems 5, 6, 7, and 8.
- b. Summarize the data in a stem-and-leaf plot with 8 stems. (See part (b) of Exercise 5.)
- c. Find the median and mode of these 30 temperatures.

7. **Sports** The 14 American League baseball teams had team batting averages for one season as shown in the table at the right.

American League team	Batting average	Home runs
Baltimore	.252	129
Boston	.277	108
California	.256	145
Chicago	.271	94
Cleveland	.245	127
Detroit	.242	116
Kansas City	.261	101
Milwaukee	.259	126
Minnesota	.276	117
New York	.269	130
Oakland	.261	127
Seattle	.257	134
Texas	.263	122
Toronto	.260	142

- a. To summarize the batting-average data in a stem-and-leaf plot, what stems would you use?
- b. Using your answer to part (a), draw a stem-and-leaf plot.
- c. What is the median of the data?
8. **Sports** The 14 American League baseball teams had total home runs for one season as shown in the table at the right.
- a. To summarize the home-run data in a stem-and-leaf plot, what stems would you use?
- b. Using your answer to part (a), draw a stem-and-leaf plot.
- c. What is the median of the data?

9. A 10-question true-false quiz was given to 50 students and the number of questions correct ranged from 3 to 10, as shown in the cumulative frequency polygon at the right.



- a. What percent of the students had scores less than or equal to 7? greater than 8?
- b. What is the median score?
10. Draw a frequency polygon and a cumulative frequency polygon for the following data: 20, 21, 22, 24, 24, 25, 25, 25, 27, 28.
11. **Physiology** The table below gives the results of a driver-education experiment. The experiment measures the time between the appearance of a stimulus on a screen and a student's reaction of depressing a brake pedal.

Time (to the nearest 0.1 s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
Frequency	1	4	5	4	3	1	2

- a. Draw a histogram for the data.
- b. Find the mean, median, and mode.
- c. Draw a cumulative frequency polygon for the data.
12. The table below gives the number of questions answered correctly by 30 students on the 20-question written portion of a driver education test.

Number correct	12	13	14	15	16	17	18	19	20
Frequency	1	0	3	3	3	5	6	5	4

- a. Draw a histogram for the data.
- b. Find the mean, median, and mode.
- c. Draw a cumulative frequency polygon for the data.
13. Five workers on an assembly line have hourly wages of \$8.00, \$8.00, \$8.50, \$10.50, and \$12.00.
- a. Find the mean, median, and mode.
- b. If the hourly wage of the highest paid worker is raised to \$20.00 per hour, how are the mean, median, and mode affected?
14. A teacher told her principal that her class of 10 astronomy students had done well on the last test by getting a modal score of 88. The students, however, thought that the test was difficult because the mean score was 68. Make up 10 scores that fit this description and find their median.
15. **Writing** For some set of data, suppose that you are given only a frequency polygon (with integer labels on the frequency axis). Write a paragraph in which you explain how to estimate the mean of the data from the graph. Then apply your method to the frequency polygon on page 641.

16. **Research** A cumulative frequency polygon is sometimes called an *ogive* (pronounced “oh-jive”). This word is also used in architecture. Find out what the architectural meaning of ogive is and explain how this meaning relates to the mathematical graph of the same name.

- B** 17. **Consumer Economics** A consumer organization collected last week’s prices for a chef’s salad at 100 restaurants. The prices are given below.

Price in cents	Number of restaurants
325–374	8
375–424	10
425–474	30
475–524	20
525–574	12

Price in cents	Number of restaurants
575–624	10
625–674	6
675–724	2
725–774	1
775–824	1

- Draw a frequency polygon for the data.
  - Calculate the mean, median, and mode using the middle price in each class as the representative price for the class. (That is, use 350 (cents), 400, 450, 500, and so on.)
18. Each of the 50 students in two mathematics classes was asked to keep track of the number of hours spent doing mathematics homework during one week. The results are given below.

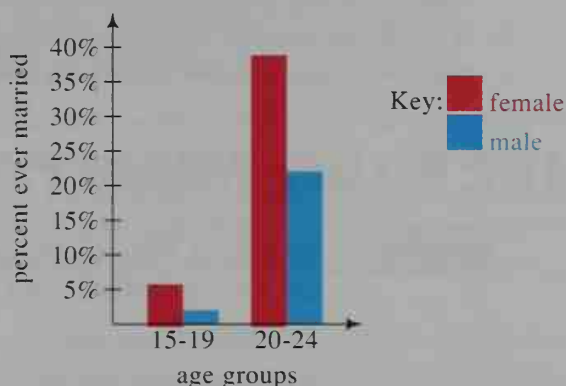
Hours spent on mathematics homework	0–2	3–5	6–8	9–11	12–14
Number of students	3	11	17	15	4

- Draw a frequency polygon for the data.
  - Calculate the mean, median, and mode using the middle number of hours for each homework category (1, 4, 7, 10, and 13).
19. **Research** Find a frequency table in a newspaper or magazine. Calculate the mean, median, and any modes.
20. Suppose you told a friend your “average” weekly earnings for the 12 weeks of summer. Could your friend calculate your total summer earnings if your “average” was (a) a mean? (b) a median? (c) a mode?
21. A pollster reports that for 100 families interviewed, the mean number of children per family was 2.038, the median was 1.9, and the mode was 1.82. Explain why each of these figures must be wrong.

22. **Geography** The table below gives data on the percent of the U.S. population ever married.

Age group	15–19	20–24	25–29	30–34	35–44	45–54
Percent of females ever married	6	39	71	84	92	95
Percent of males ever married	2	22	57	75	89	94

- a. At the right is part of a comparative histogram showing two side-by-side bars for each of the age groups. Complete the histogram.



- b. **Writing** Write a paragraph explaining what the data show.

23. **Nutrition** The table below gives data on the recommended energy intake, in kilocalories (Cal), for average females and males in various age groups.

Age group	11–14	15–18	19–22	23–50	51–75	≥76
Energy needs (in Cal) for females	2200	2100	2100	2000	1800	1600
Energy needs (in Cal) for males	2700	2800	2900	2700	2400	2050

- a. Draw a comparative histogram showing two side-by-side bars for each of the age groups. (See Exercise 22 for an example of such a histogram.)
- b. **Writing** Write a paragraph explaining what the data show.

24. For a group of 10 teenagers, the mean age is 17.1, the median is 16.5, and the mode is 16. If a 21-year-old joins the group, give the mean, median, and mode for the ages of the 11 people.





25. **Economics** The distribution of salaries in a company is shown below. The median salary is \$14,500 and the mean salary is \$17,800.

Job	5 executives	15 supervisors	80 production workers
Salary range	\$40,200–\$87,500	\$15,800–\$25,000	\$9200–\$18,700

- If the 15 supervisors are each given a \$1000 raise and no one else gets a raise, what are the new mean and median?
- If the 80 production workers each get a \$1000 raise and no one else gets a raise, what are the new mean and median?

## 17-2 Box-and-Whisker Plots

### Objective

To draw a box-and-whisker plot for a set of data and to use box-and-whisker and stem-and-leaf plots to compare sets of data.

### Drawing Box-and-Whisker Plots

The stem-and-leaf plot below shows the ages of winners of the Academy Award for Best Actress from 1928 to 1989. Marlee Matlin was 21 at the time that she received an Oscar for her 1986 role in *Children of a Lesser God*. Since 21 is the youngest age of an Oscar-winning actress, it is called the *lower extreme* of the data. The *upper extreme* of the data is 80, Jessica Tandy's age when she won an Oscar for her 1989 role in *Driving Miss Daisy*. You might want to update the data with the winners of the Academy Award for Best Actress since 1989. You can find the information in an almanac.

Ages of Academy Award Winners, 1928–1989

Best Actresses

2	1 2 4 4 4 4 4
•	5 6 6 6 6 6 6 6 7 7 7 8 8 9 9 9
3	0 0 0 0 1 1 2 3 3 3 4 4 4 4 4 4
•	5 5 6 7 7 8 8 8
4	0 1 1 1 1 1 2
•	5 8 9
5	
•	
6	0 1 1 2
•	
7	4
•	
8	0

2|1 represents  
an age of 21.

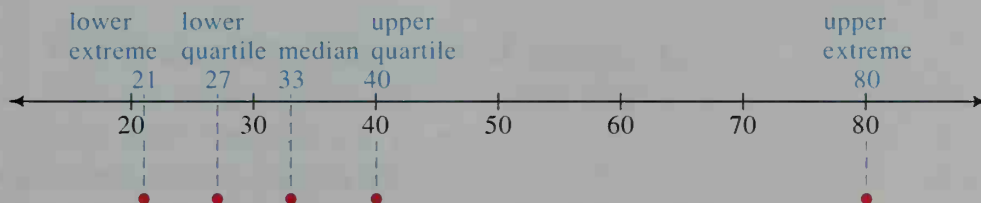


The stem-and-leaf plot shows 63 ages (one more than expected because there were two winners in 1968). If these 63 ages are arranged from left to right in increasing order, the median is the 32nd age in the list. The median of the ages to the left of the median position is called the *lower quartile*, and the median of the ages to the right of the median position is called the *upper quartile*.

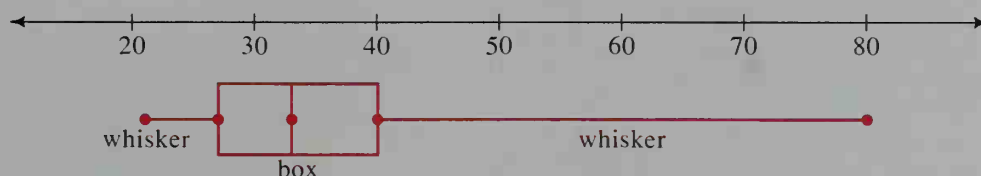
	Lower extreme	Lower quartile	Median	Upper quartile	Upper extreme
Position in list	1	16	32	48	63
Age	21	27	33	40	80

Note that the positions of the median, quartiles, and upper extreme shift as new data are added to the list. For example, if the age of the 1990 winner is included, the median would be halfway between the 32nd and 33rd ages, the lower quartile would be halfway between the 16th and 17th ages, the upper quartile would be halfway between the 48th and 49th ages, and the upper extreme would be at the 64th age.

We can display the data above in a **box-and-whisker plot**. To do so, we first plot the median, quartiles, and extremes not on, but just below, a number line.



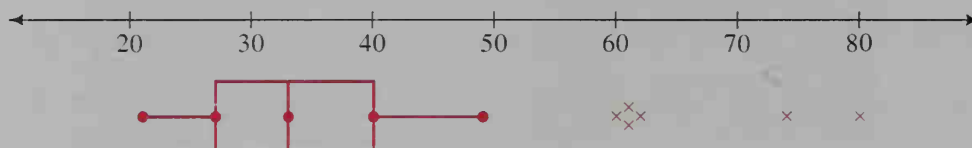
We then draw a *box* extending from the lower quartile to the upper quartile and *whiskers* extending from the quartiles to the extremes. We also draw a line through the box at the median as shown.



The difference between the two extremes is called the **range**; in this case, the range is  $80 - 21 = 59$ . About 50% of the data lie in the interval of the box, from the lower quartile 27 to the upper quartile 40. The length of this interval, which is  $40 - 27 = 13$ , is called the *interquartile range*.

If you refer to the stem-and-leaf plot on the preceding page, you will see that the 6 oldest ages are separated from the cluster of the remaining data. These 6 ages (60, 61, 61, 62, 74, and 80) are called *outliers*.

Although mathematicians do not always agree on how far out an item of data must be in order to be considered an outlier, we will define an outlier to be any item of data whose distance to the nearer quartile exceeds 1.5 times the interquartile range. For the age data, then,  $1.5 \times \text{interquartile range} = 1.5 \times 13 = 19.5$ , so that any age more than 19.5 from the box in the box-and-whisker plot on the preceding page is an outlier. We therefore modify the plot by letting the whiskers include only the data that are not outliers and by using small x's to show the outliers.



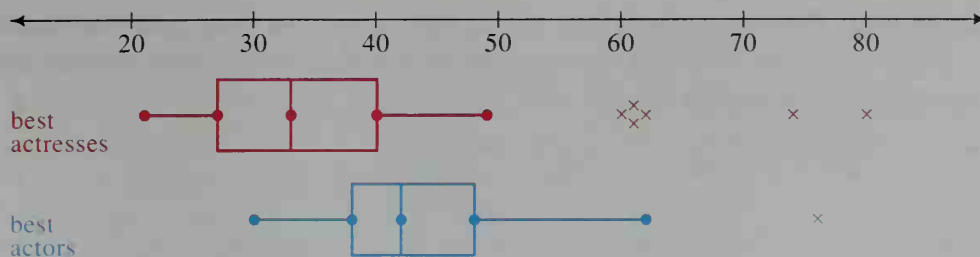
## Comparing Sets of Data

How do the ages of winners of the Academy Award for Best Actor compare with the ages of Oscar-winning actresses? One way to make this comparison is with the back-to-back stem-and-leaf plot shown below. The youngest age for a best actor is 30 (Marlon Brando, *On the Waterfront*, 1954) and the oldest age is 76 (Henry Fonda, *On Golden Pond*, 1981).

Ages of Academy Award Winners, 1928–1989

Best Actors		Best Actresses	
	2	1	2 4 4 4 4 4
	•	5	6 6 6 6 6 6 6 7 7 7 8 8 9 9 9
4 4 3 3 2 2 2 1 0	3	0	0 0 0 0 1 1 2 3 3 3 4 4 4 4 4 4
9 9 8 8 8 8 8 7 7 5 5 5 5	•	5	5 5 6 7 7 8 8 8
4 4 3 3 3 3 3 2 2 1 1 1 1 0 0 0 0	4	0	1 1 1 1 1 2
9 9 9 8 8 8 7 7 6 6 5	•	5	8 9
3 2 1 1	5		
6 6 6 5	•		
2 1 0	6	0	1 1 2
	•		
	7	4	
6	•		
	8	0	
Lower quartile:	38		27
Median:	42		33
Upper quartile:	48		40

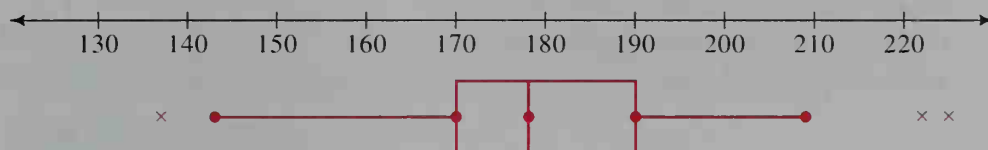
The stem-and-leaf plot above makes it appear that best actors tend to be older than best actresses. Another way to make this comparison is with side-by-side box-and-whisker plots, as shown at the top of the next page.



The apparent difference in the ages of best actors and best actresses can be shown by statistical methods to be *statistically significant*. This means that it is highly unlikely for the difference in ages to be due merely to chance.

## CLASS EXERCISES

For Exercises 1–4, use the box-and-whisker plot shown below. It gives the weights in pounds of the 42 players on a high school football team.

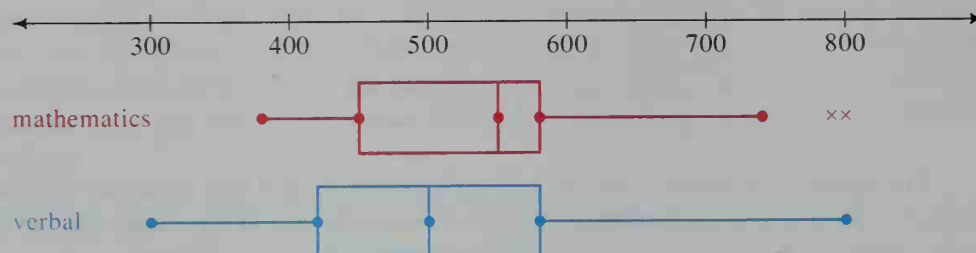


1. What are the extremes? What is the range?
2. What are the lower and upper quartiles? What is the interquartile range?
3. What is the median?
4. Explain why the weight of 225 lb is an outlier. What are the other outliers?

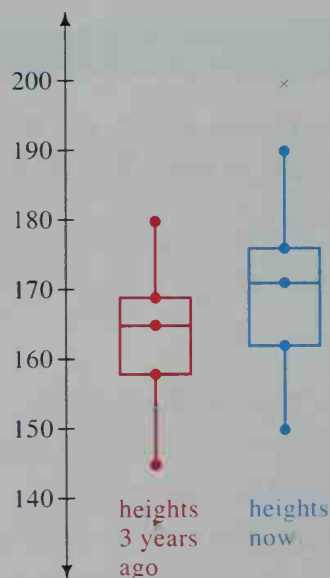
## WRITTEN EXERCISES

- A** 1. a. Recently, several states with small populations, such as Vermont, had only one representative in the U.S. House of Representatives. On the other hand, California had 52 representatives, more than any other state. The median number of representatives was 6, the lower quartile was 2, and the upper quartile was 10. With these facts, draw a box-and-whisker plot. Show whiskers extending to the extremes of the data.
- b. Would Vermont's number of representatives be an outlier? What about California?
2. a. Several U.S. presidents, including George Washington, had no children. John Tyler, the tenth U.S. president, had 15 children (from two marriages), more than any other president. The median number of children of a president is 3, and the lower and upper quartiles are 2 and 5, respectively. With these facts, draw a box-and-whisker plot. Show whiskers extending to the extremes of the data.
- b. Would the number of Tyler's children be an outlier? Explain.

3. **Writing** The mathematics and verbal SAT scores for seniors at Van Buren High School are shown in the box-and-whisker plots below. Write a paragraph explaining what the plots tell you.



4. **Writing** Box-and-whisker plots can be displayed on a vertical scale as well as on a horizontal one. The two plots shown at the right compare the heights in centimeters of this year's senior class with their heights three years ago when they were freshmen. Write a paragraph explaining what the plots tell you.

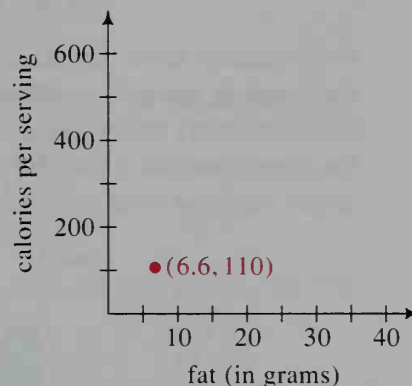


**Nutrition** For Exercises 5–8 at the top of the next page, use the table shown below, which gives nutritional information about some of the menu items at a restaurant.

Menu item	Calories per serving	Protein (in grams)	Carbohydrates (in grams)	Fat (in grams)
Regular salad	110	7.1	6.2	6.6
Chef salad	230	20.5	7.5	13.3
Fish sandwich	440	13.8	37.9	26.1
Chicken sandwich	490	19.2	39.8	28.6
Regular hamburger	260	12.3	30.6	9.5
Regular cheeseburger	310	15.0	31.2	13.8
Quarter-pound hamburger	410	23.1	34.0	20.7
Quarter-pound cheeseburger	520	28.5	35.1	29.2
Deluxe hamburger	560	25.2	42.5	32.4



5. Draw a stem-and-leaf plot and a box-and-whisker plot for the number of calories in one serving of each menu item.
6. Draw a stem-and-leaf plot and a box-and-whisker plot for the grams of fat (rounded to the nearest gram) in one serving of each menu item.
7. Draw a back-to-back stem-and-leaf plot and a side-by-side box-and-whisker plot comparing grams of protein and grams of carbohydrates (each rounded to the nearest gram).
8. The point (6.6, 110), plotted in the graph at the right, indicates that a regular salad has 6.6 grams of fat and 110 calories. Copy the graph and plot a point for each of the other menu items. Does the graph suggest any trends to you? (*Note:* The graph is an example of a *scatter plot*, which will be discussed in Section 18-1.)



**Research** In Exercises 9–14, use an almanac to find data for each of the 50 states. Then draw stem-and-leaf and box-and-whisker plots.

9. Per pupil expenditure for education
10. Percent of eligible voters who actually voted in the last presidential election
11. Mean family income
12. Salary of governor
13. Comparison of land areas of states east and west of the Mississippi River (Consider Louisiana and Minnesota to be west of the Mississippi, since most of each state's area is west of the river.)
14. Comparison of populations of states east and west of the Mississippi River
15. **Research** From an almanac, select data that you find interesting. Then draw stem-and-leaf and box-and-whisker plots of the data.

## 17-3 Variability

### **Objective**

To find the variance and standard deviation of a set of data and to convert data to standard values.

A **statistic** is a number that describes some characteristic of a set of data. For example, the mean, median, and mode (see Section 17-1) are statistics used to describe the center of a set of data, and the range and interquartile range (see Section 17-2) are statistics used to describe the spread of data about the center.

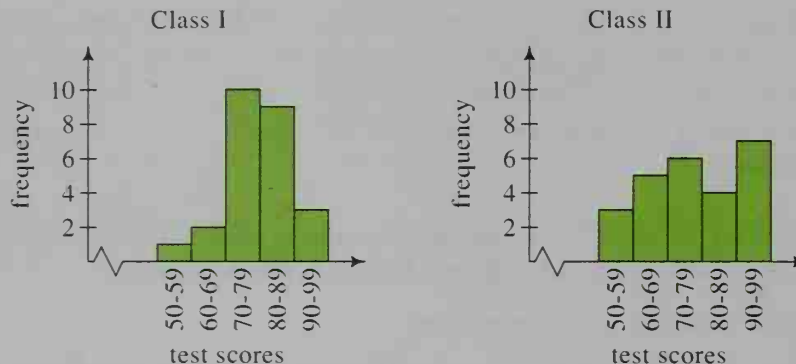
In this section we will discuss other statistics that are used to describe the spread of data about the center. These statistics are called *measures of dispersion*, because they indicate the amount of variability in the data.

Consider the algebra test scores, given below, for two classes of 25 students.

Class I: 50, 64, 68, 70, 70, 72, 73, 75, 75, 76, 78, 78, 79,  
80, 81, 82, 82, 82, 85, 88, 88, 89, 90, 92, 93

Class II: 56, 57, 58, 60, 61, 65, 67, 68, 70, 71, 74, 75, 77,  
79, 82, 87, 88, 89, 92, 95, 96, 97, 98, 99, 99

Both classes have the same mean score, 78.4. Moreover, both classes have the same range, since  $93 - 50 = 99 - 56 = 43$ . Even though the ranges are the same, the histograms below show that the scores for Class I are packed more closely to the mean than the scores for Class II. It seems that the students in Class I are more alike—at least with respect to algebra—than the students in Class II.



Two statistics used to describe the spread of data about the mean are called the *variance* and the *standard deviation*. The **variance**, denoted  $s^2$  or  $\sigma^2$  (sigma squared), is the mean of the squares of the deviations from  $\bar{x}$ .

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \quad (1)$$

The **standard deviation**, denoted  $s$  or  $\sigma$ , is the positive square root of the variance.

$$s = \sqrt{\text{variance}} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}} \quad (2)$$

The formulas for variance and standard deviation are sometimes written with a denominator of  $n - 1$  instead of  $n$ . The reasons for this are best left to a full course in statistics. If you have a calculator that gives you these statistics, check to see whether it uses the formulas given above by reading its instruction booklet or by doing a calculation (as in Example 1 on the next page).

Formulas (1) and (2) can be expressed in equivalent forms, given at the top of the next page, that are often easier to use if you need to do the calculations for variance and standard deviation by hand.

$$\text{variance: } s^2 = \frac{\sum_{i=1}^n x_i^2}{n} - \bar{x}^2 = \text{mean of squares} - \text{square of mean} \quad (1a)$$

$$\text{standard deviation: } s = \sqrt{\frac{\sum_{i=1}^n x_i^2}{n} - \bar{x}^2} \quad (2a)$$

You can check for yourself that the standard deviations of the algebra test scores given on the preceding page are:

Class I: standard deviation  $\approx 9.6$

Class II: standard deviation  $\approx 14.5$

The smaller standard deviation for Class I indicates that the test scores for this class are packed more closely to the mean.

**Example 1** Compute the variance and standard deviation of the data 3, 5, 6, 7, 9.

**Solution** We first find the mean:

$$\bar{x} = \frac{3 + 5 + 6 + 7 + 9}{5} = 6$$

Since  $\bar{x}$  is an integer, formulas (1) and (1a) are equally easy to use. We will illustrate the use of both in finding the variance and standard deviation of the data.

Using formula (1):

$x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
3	-3	9
5	-1	1
6	0	0
7	1	1
9	3	9

20 = sum

$$\begin{aligned} \text{variance} = s^2 &= \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n} \\ &= \frac{20}{5} = 4 \end{aligned}$$

$$\text{standard deviation} = s = \sqrt{4} = 2$$

Using formula (1a):

$x_i$	$x_i^2$
3	9
5	25
6	36
7	49
9	81

200 = sum

$$\begin{aligned} \text{variance} = s^2 &= \frac{\sum_{i=1}^n x_i^2}{n} - \bar{x}^2 \\ &= \frac{200}{5} - 6^2 = 4 \end{aligned}$$

$$\text{standard deviation} = s = \sqrt{4} = 2$$

If data appear in a frequency table instead of a list, then we need to modify formula (1a). Suppose  $n$  items of data are given as  $r$  distinct numbers  $x_1, \dots, x_r$ , and each  $x_i$  occurs with frequency  $f_i$ . Formula (1a) then becomes:

$$\text{variance: } s^2 = \frac{\sum_{i=1}^r x_i^2 \cdot f_i}{n} - \bar{x}^2 \quad (1b)$$

**Example 2** Find the mean and standard deviation for the data in the table below.

$x_i = \text{item of data}$	3	6	7
$f_i = \text{frequency}$	2	5	3

**Solution**

The number of distinct data items is  $r = 3$ .

The total number of data items is  $n = 2 + 5 + 3 = 10$ .

Therefore the mean is:

$$\bar{x} = \frac{\sum_{i=1}^r x_i f_i}{n} = \frac{3 \cdot 2 + 6 \cdot 5 + 7 \cdot 3}{10} = \frac{57}{10} = 5.7$$

Using formula (1b) to find the variance, we have:

$$\begin{aligned} s^2 &= \frac{\sum_{i=1}^r x_i^2 \cdot f_i}{n} - \bar{x}^2 = \frac{3^2 \cdot 2 + 6^2 \cdot 5 + 7^2 \cdot 3}{10} - 5.7^2 \\ &= \frac{18 + 180 + 147}{10} - 5.7^2 \\ &= 34.5 - 32.49 \\ &= 2.01 \end{aligned}$$

Therefore the standard deviation is  $s = \sqrt{2.01} \approx 1.42$ .

## Standard Value

As you will see in the next section, it is useful to convert a set of data into a related set having a mean of 0 and a standard deviation of 1. Each item of data  $x$  is converted to a **standard value**  $z$  by the formula:

$$z = \frac{x - \bar{x}}{s}$$

Written in words, the conversion formula is

$$\text{standard value} = \frac{\text{item of data} - \text{mean}}{\text{standard deviation}},$$

from which you can see that the standard value gives the number of standard deviations between an item of data and the mean.

**Example 3**

The mean and standard deviations of the algebra test scores on page 654 are as follows:

Class I: mean = 78.4, standard deviation  $\approx 9.6$

Class II: mean = 78.4, standard deviation  $\approx 14.5$

For each class, give the standard value of the scores 92 and 68.

**Solution**

For a score of 92:

$$\text{Class I: } z \approx \frac{92 - 78.4}{9.6} \approx 1.4$$

$$\text{Class II: } z \approx \frac{92 - 78.4}{14.5} \approx 0.9$$

A score of 92 is 1.4 and 0.9 standard deviations above the mean for Class I and Class II, respectively.

For a score of 68:

$$\text{Class I: } z \approx \frac{68 - 78.4}{9.6} \approx -1.1$$

$$\text{Class II: } z \approx \frac{68 - 78.4}{14.5} \approx -0.7$$

A score of 68 is 1.1 and 0.7 standard deviations below the mean for Class I and Class II, respectively.

**CLASS EXERCISES**

1. a. What does the following information tell you about two different basketball teams?

Team A: mean height = 192 cm, standard deviation = 4 cm

Team B: mean height = 192 cm, standard deviation = 8 cm

- b. Which team do you think has a better chance of getting rebounds?

2. Complete the table at the left below and then the calculations at the right below in order to find the standard deviation of the data 1, 7, 9, 15.

$x_i$	$x_i^2$
1	?
7	?
9	?
15	?
sum = ?	? = sum

$$\text{mean} = \bar{x} = ?$$

$$\text{variance} = s^2 = \frac{\sum_{i=1}^n x_i^2}{n} - \bar{x}^2 = ?$$

$$\text{standard deviation} = s = \sqrt{?} \approx ?$$

3. If each of the  $x_i$  in Exercise 2 were decreased by 1, tell how the mean and standard deviation would change.
4. The mean and standard deviation of a set of data are  $\bar{x} = 10$  and  $s = 4$ . Give the standard values of the following numbers from the set.

a. 14

b. 18

c. 20

d. 8

e. 10



5. **Reading** In Example 1, suppose the mean had not been an integer. To compute the variance, which formula, (1) or (1a), would have been easier to use?
6. **Discussion** Is it possible to have a variance of zero? Explain.

## WRITTEN EXERCISES

In Exercises 1 and 2, compute the mean and standard deviation for each set of data.

- A** 1. The number of grammatical errors on five daily French quizzes:  
10, 8, 7, 5, 5
2. The number of fish caught on each day of vacation: 3, 8, 7, 0, 4, 7, 13
3. a. Each number in row B is 10 more than the corresponding number in row A. Compare the mean and standard deviation of each row of numbers.

Row A	1	2	3	4	5
Row B	11	12	13	14	15

- b. Estimate the mean and standard deviation of the numbers 21, 22, 23, 24, 25.
4. a. Each number in row B is four times the corresponding number in row A. Compare the mean and standard deviation of each row of numbers.

Row A	5	5	7	7	8	10
Row B	20	20	28	28	32	40

- b. Estimate the mean and standard deviation of the numbers 50, 50, 70, 70, 80, 100.
5. A teacher gives a test in which the average of the test scores is 68 and the standard deviation is 14. Realizing that the test was probably too difficult, the teacher decides to scale the tests by adding 10 points to all scores. What is the mean and standard deviation of the scaled test scores?
6. A group of students has a mean height of 66 in. with a standard deviation of 4 in. If the heights had been measured in centimeters instead of inches, what would be the mean and standard deviation? Use the fact that 1 in.  $\approx$  2.5 cm.
7. A set of test scores has a mean of 78 and a standard deviation of 6. Give the standard value for each of the following test scores:  
a. 90                      b. 75                      c. 78                      d. 70
8. For all of the apartments in a large building, the number of kilowatt hours of electricity used last month had a mean of 760 and a standard deviation of 80. Give the standard value for each of the following usages:  
a. 720                      b. 860                      c. 600                      d. 776

9. Suppose the means for the mathematics and verbal scores on a standardized achievement test are 500 and 504, respectively, and the standard deviations are 100 and 98, respectively. If a student has a mathematics score of 610 and a verbal score of 580, give the standard values of the student's scores.
10. Sue's scores on her first and second physics tests were 82 and 88, respectively. On which test did she do better relative to the rest of the class? Use the following information:

Test I: mean = 72, standard deviation = 5

Test II: mean = 80, standard deviation = 6

11. The mathematics achievement scores of 45 physics students are compared with the mathematics achievement scores of 45 students selected at random. Which group of students do you think has the greater mean score? the greater standard deviation in its scores? Give your reasons.
12. The typing rates (in words per minute) of 25 professional typists are compared with the typing rates of 25 randomly selected people who own typewriters. Which group do you think has the greater standard deviation of typing rates? Give your reasons.
13. Four people each think of an integer from 1 to 9. What are the smallest and largest possible standard deviations of the four numbers?
- B** 14. **Consumer Economics** Thirty samples of milk purchased at various stores were tested for bacteria. The bacteria count per milliliter in these 30 samples is given in the table below. Find the mean and standard deviation of the data.

$x_i$ = bacteria count	3	4	5	6	7	8
$f_i$ = frequency	1	3	6	9	7	4

15. Twenty puppies were taught to sit and stay on command. Find the mean and standard deviation of the number of trials required before they learned to do this.

Number of trials	Number of puppies
7	1
8	2
9	5
10	4
11	4
12	4



- C** 16. Prove that adding a constant  $c$  to each item in a set of data does not change the standard deviation.
17. Prove that multiplying each item in a set of data by a positive constant  $c$  makes the standard deviation  $c$  times as large.
18. Derive formula (1a) on page 655 from formula (1) on page 654.



**Gertrude Mary Cox (1900–1978)**

Born and educated in Iowa, Gertrude Mary Cox studied psychological statistics and later coordinated the teaching of statistical theory and methodology at the University of North Carolina and North Carolina State College.

Her 1950 book *Experimental Designs* is a classic textbook on the design and analysis of replicated experiments.

Cox organized conferences in the South on plant and animal science, agricultural economics, quality control, and taste testing. She also helped develop statistical programs for Egypt and Thailand.

## 17-4 The Normal Distribution

### Objective

*To recognize uniform, skewed, and normal distributions, to determine for a normal distribution the percent of data within a given interval, and to find percentiles.*

As we have seen in previous sections, histograms have various shapes according to the distribution of data. The distribution of the last digits of 500 different telephone numbers shown in figure (a) below is an example of a *uniform distribution*. Each digit occurs about 10% of the time.

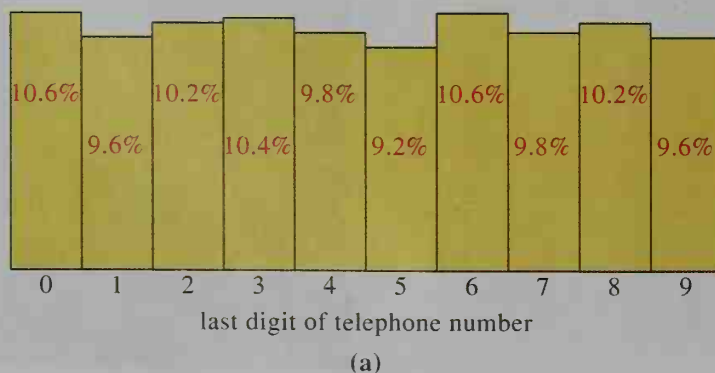
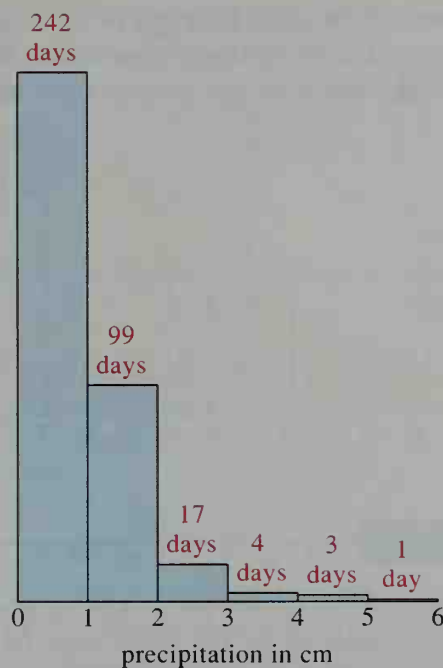


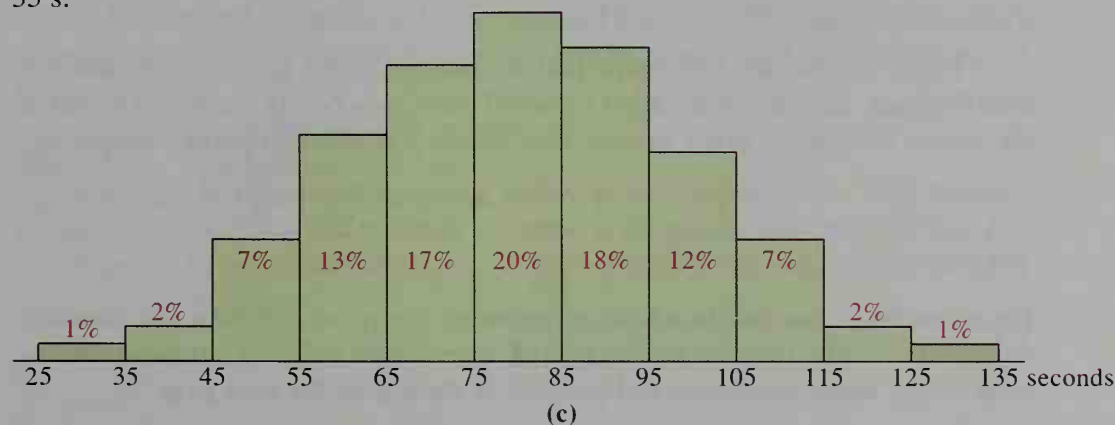
Figure (b) at the right shows the distribution of precipitation for one year in a Midwestern city. On 242 days of the year the precipitation was less than 1 cm, on 99 days it was greater than or equal to 1 cm but less than 2 cm, and so on. This distribution is an example of a *skewed distribution* with a hump at one end and a long tail at the other.

In this section, we will consider a third type of distribution, called the *normal distribution*. The normal distribution is extremely important in that it occurs in a wide variety of data, including heights and weights of people, dimensions of manufactured goods, test scores, blood cholesterol levels, and times for marathon races.

Consider the following experiment: A psychologist asks 100 children to solve a certain puzzle and records their solution times. The results are shown in figure (c) below. This histogram gives the percent of children who completed the puzzle in the indicated time interval. Each interval includes its left endpoint but not its right endpoint. Thus, 1% of the children solved the puzzle in a time greater than or equal to 25 s, but less than 35 s.

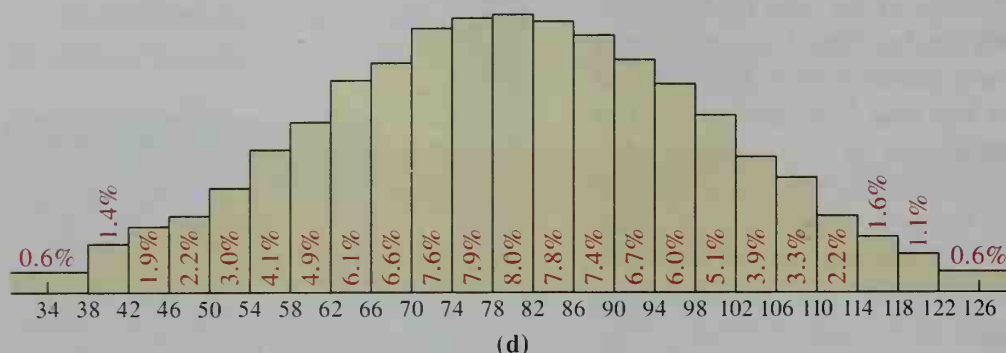


(b)



(c)

Each of the class intervals in figure (c) is 10 units wide. If the psychologist had observed a much larger population, the data would be more specific, and it would make sense to use smaller class intervals, as shown in figure (d) below.



### Activity

Use figure (d) to answer the following questions.

- What appears to be the mean time for the puzzle-solving experiment?
- If the standard deviation of the experiment is  $s = 20$  s, approximately what percent of the data is within 1 standard deviation of the mean?

If a smooth curve is drawn through the tops of the rectangles in figure (d), we get figure (e) below.

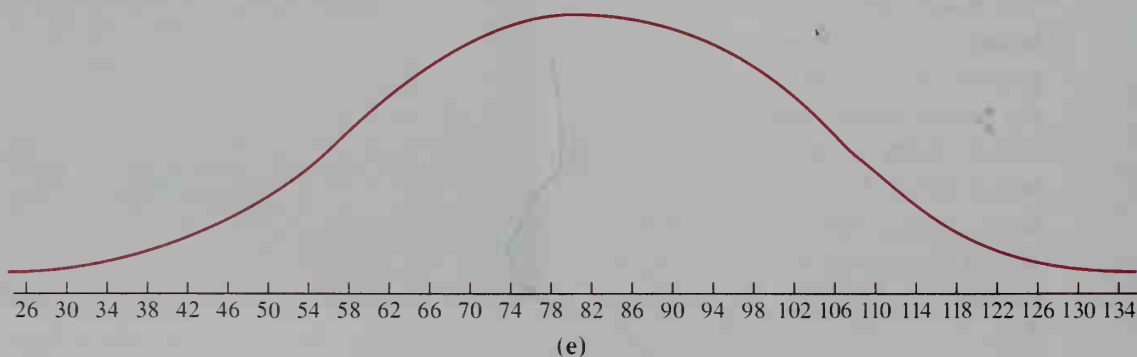
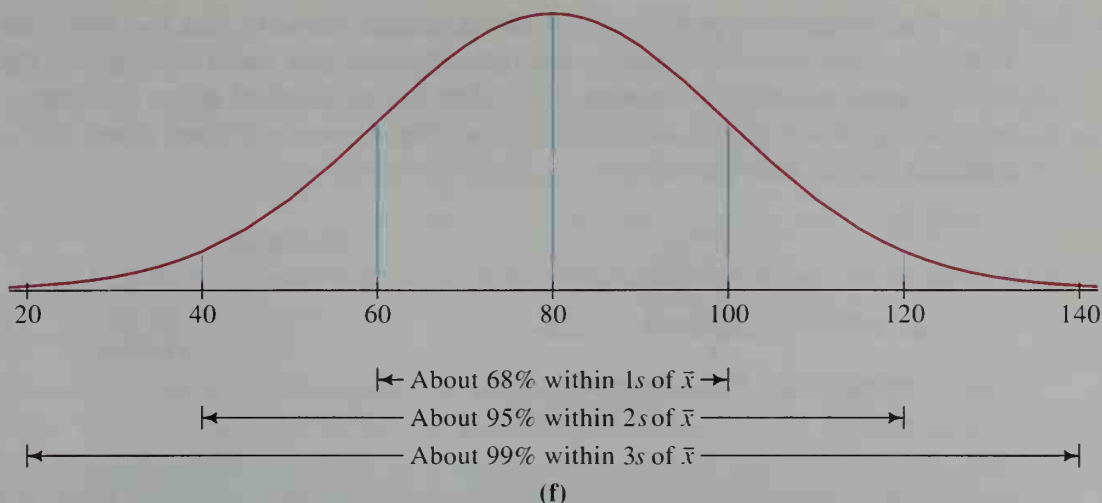


Figure (e) has the bell shape that is common to the graphs of all **normal distributions**. Such a graph, called a *normal curve*, reaches its maximum height at the mean. Moreover, every normal distribution has these important properties:

- About 68% of the distribution is within 1 standard deviation of the mean.
- About 95% of the distribution is within 2 standard deviations of the mean.
- About 99% of the distribution is within 3 standard deviations of the mean.

For example, if the puzzle-solving experiment has mean  $\bar{x} = 80$  s and standard deviation  $s = 20$  s, then the percentages of times within 1, 2, and 3 standard deviations of the mean are shown in figure (f) at the top of the next page.

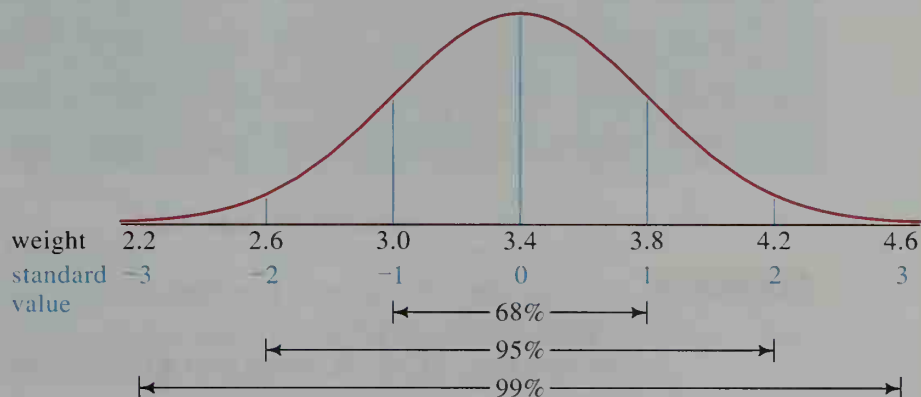




### Example 1

Over the last 10 years, the mean weight  $\bar{x}$  of newborn babies in a large metropolitan hospital has been 3.4 kg and the standard deviation  $s$  has been 0.4 kg. Use this information to sketch a normal curve showing the weights at 1, 2, and 3 standard deviations from the mean. Also show the percents of the weights within 1, 2, and 3 standard deviations of the mean.

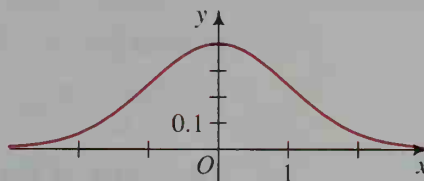
### Solution



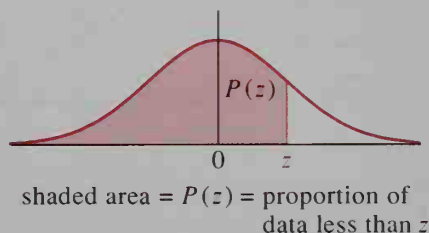
The **standard normal distribution** is the normal distribution having a mean of 0 and a standard deviation of 1. The standard normal distribution is particularly important because any normal distribution can be related to it through the use of standard values.

The graph of the standard normal distribution, called the *standard normal curve*, is shown at the right. Its equation is:

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



The standard normal curve has the important property that the total area under the curve (and above the  $x$ -axis) is 1. Also, the area under the curve to the left of a number  $z$  is the proportion of the data having standard values less than  $z$ , as indicated in the figure at the left below. These proportions are given in the standard normal table below.



Examples:

$$P(-2.0) = 0.0228$$

$$P(-0.3) = 0.3821$$

$$P(0.5) = 0.6915$$

$$P(1.7) = 0.9554$$

$z$	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
-3	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000+
-2	.0228	.0179	.0139	.0107	.0082	.0062	.0047	.0035	.0026	.0019
-1	.1587	.1357	.1151	.0968	.0808	.0668	.0548	.0446	.0359	.0287
-0	.5000	.4602	.4207	.3821	.3446	.3085	.2743	.2420	.2119	.1841
0	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159
1	.8413	.8643	.8849	.9032	.9192	.9332	.9452	.9554	.9641	.9713
2	.9772	.9821	.9861	.9893	.9918	.9938	.9953	.9965	.9974	.9981
3	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000-

### Standard Normal Table

#### Example 2

Use the data of Example 1 to find the approximate percent of newborn babies having weights:

a. less than 4.0 kg   b. more than 4.0 kg   c. between 3.0 kg and 4.0 kg

#### Solution

We must first convert each weight to its standard value so that the standard normal distribution can be used.

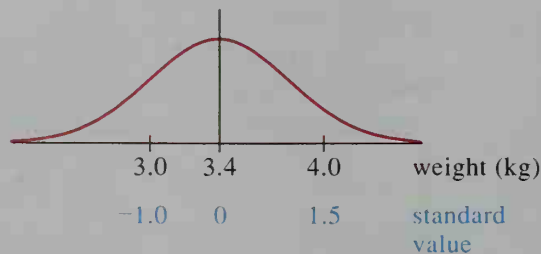
For 4.0 kg:

$$z = \frac{4.0 - 3.4}{0.4} = 1.5$$

For 3.0 kg:

$$z = \frac{3.0 - 3.4}{0.4} = -1.0$$

The graph at the right shows these weights and their standard values.



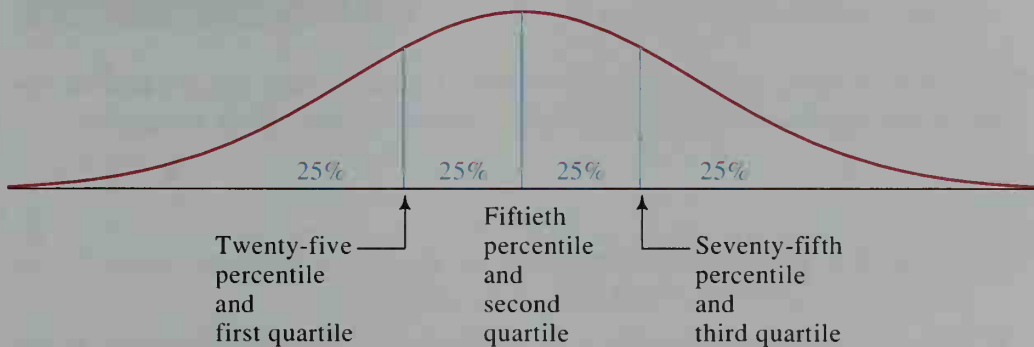
- From the standard normal table,  $P(1.5) = 0.9332$ . Thus, about 93% of the babies weigh less than 4.0 kg.
- About  $100\% - 93\% = 7\%$  of the babies weigh more than 4.0 kg.
- The proportion of weights between 3.0 kg and 4.0 kg is the same as the proportion of the standard normal values between  $-1.0$  and  $1.5$ :

$$P(1.5) - P(-1.0) = 0.9332 - 0.1587 = 0.7745$$

Thus, about 77% of the babies weigh between 3.0 kg and 4.0 kg.

## Percentiles

Sometimes a set of data is arranged in ascending order and divided into 100 equal parts. The 99 points that divide the data are called **percentiles**. Percentiles are often used in reporting scores on a standardized test. If you score at the 70th percentile, for instance, you know that 70% of all people taking the test had a score less than or equal to yours. Similarly, the 25th percentile separates the bottom 25% of the test scores. The 25th percentile, 50th percentile (the median), and 75th percentile are also called the first quartile, the second quartile, and the third quartile. As the diagram below shows, the quartiles divide the population into quarters.



The next example shows that we can use the standard normal table to find various percentiles for any set of data having a normal distribution.

**Example 3** The scores on a standardized test are normally distributed with mean  $\bar{x} = 50$  and standard deviation  $s = 10$ . Find the 90th percentile.

**Solution** When we look at the standard normal table to find a number close to  $90\% = 0.90$ , we find 0.9032, which corresponds to the standard value  $z = 1.3$ . Using the formula for standard value, we have:

$$z = \frac{x - \bar{x}}{s}$$

$$1.3 = \frac{x - 50}{10}$$

$$x = 63$$

Thus, a score of 63 is approximately the 90th percentile.

## CLASS EXERCISES

1. **Discussion** For each of the following sets of data, discuss whether you expect the distribution to be uniform, skewed, or normal. Then collect the data to check your expectation. (*Note:* If your class size is small, you may not get the type of distribution that you expect.)
  - a. The number of times in a day that students in this class use their lockers
  - b. The heights of females in this class
  - c. The last digit of the home street addresses of students in this class
  - d. The ages, in months, of students in this class
  - e. The shoe sizes of males in this class
  - f. The number of books that students have brought with them to this class.



For Exercises 2–7, use the standard normal table to find the percent of the data in the standard normal distribution satisfying the given inequality.

2.  $z < 1.2$
3.  $z < -1.8$
4.  $z > 0.6$
5.  $z > -1.3$
6.  $0 < z < 1$
7.  $-1 < z < 1$
8. Explain how the standard normal table can be used to show that about 95% of the data is within 2 standard deviations of the mean.
9. In a certain large school district, the set of all standardized mathematics scores is normally distributed with mean  $\bar{x} = 500$  and standard deviation  $s = 100$ .
  - a. Make a sketch of the distribution showing the scores at 1, 2, and 3 standard deviations from the mean. Also show the percents of the scores within 1, 2, and 3 standard deviations of the mean.
  - b. Find the percent of scores that are:  
(1) less than 300      (2) greater than 700      (3) between 400 and 600
  - c. Find the score at the 90th percentile.
  - d. Find the score at the first quartile.
10. The reaction times of all people in a psychology experiment were normally distributed with a mean  $\bar{x} = 2$  s and a standard deviation  $s = 0.5$  s.
  - a. Make a sketch of the distribution showing the reaction times at 1, 2, and 3 standard deviations from the mean. Also show the percents of the times within 1, 2, and 3 standard deviations of the mean.
  - b. Find the percent of reaction times that are:  
(1) less than 0.5 s      (2) greater than 3 s      (3) between 1.5 s and 2.5 s
  - c. Find the reaction time at the 20th percentile.
  - d. Find the reaction time at the third quartile.

## WRITTEN EXERCISES

When necessary to use the standard normal table, round standard values to the nearest tenth.

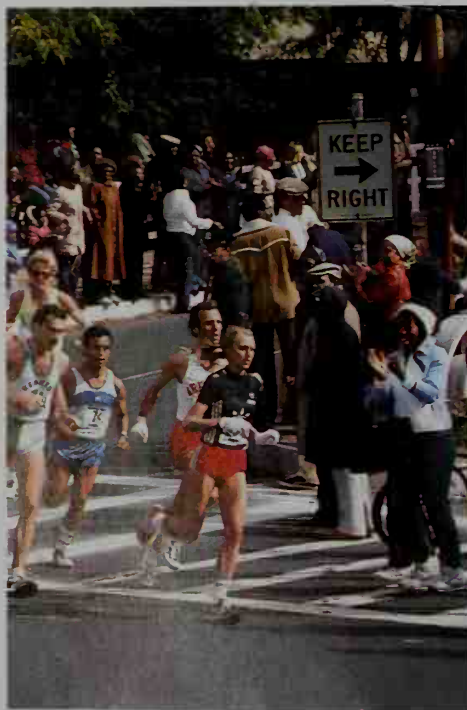
**A**

1. On a standardized aptitude test, scores are normally distributed with mean  $\bar{x} = 100$  and standard deviation  $s = 10$ . Find the percent of scores that are:
  - a. less than 95
  - b. greater than 115
  - c. between 90 and 110
  - d. between 105 and 125
2. **Horticulture** At a tree nursery, the heights of the trees are normally distributed with a mean of 200 cm and a standard deviation of 20 cm. Find the percent of the trees with heights that are:
  - a. less than 150 cm
  - b. greater than 220 cm
  - c. between 170 cm and 230 cm
  - d. between 160 cm and 190 cm
3. The standardized verbal scores of students entering a large university are normally distributed with mean  $\bar{x} = 600$  and standard deviation  $s = 80$ . Find the percent of scores that are:
  - a. less than 500
  - b. greater than 640
  - c. between 600 and 720
  - d. between 760 and 800
4. **Operations Research** The number of letters handled daily by a post office is normally distributed with a mean of 20,000 letters and a standard deviation of 100 letters. Find the percent of the days on which the post office handles:
  - a. more than 20,200 letters
  - b. fewer than 19,900 letters
  - c. between 19,950 and 20,050 letters
  - d. between 20,100 and 20,150 letters
5. **Operations Research** At a very large factory, employees must report to work at 8 A.M. The arrival times of the employees are normally distributed with mean 7:52 A.M. and standard deviation 4 min.
  - a. On a typical day, what percent of the employees are late?
  - b. What percent of the employees arrive between 7:59 and 8:00 A.M.?
  - c. What percent of the employees arrive before 7:45 A.M.?
6. **Manufacturing** A light bulb manufacturing company claims that a 100 watt bulb has a mean life of 750 h with a standard deviation of 70 h. Find the percent of light bulbs that have a life:
  - a. less than 610 h
  - b. more than 940 h
  - c. between 700 h and 800 h
  - d. between 730 h and 770 h
7. **Manufacturing** A manufacturer makes ball bearings that must have diameters between 17.0 mm and 18.0 mm. Quality control procedures indicate that the bearings produced have a mean diameter of 17.6 mm and a standard deviation of 0.3 mm. What percent of these ball bearings fail to meet the specifications?
8. **Transportation** An airline has room for 262 passengers on its New York to Los Angeles flight. Because some people who book the flight do not show up, the airline books 290 people on the flight. If the mean number of “no-shows” is 45 people and the standard deviation is 10 people, what percent of these flights will be unable to seat all the passengers who do show up?



9. Suppose that in a large school district, the standardized mathematics test scores have a mean of 500 and a standard deviation of 100. Find the score at:
  - a. the 80th percentile
  - b. the first quartile
10. At a certain university, the combined mathematics plus verbal standardized test scores of incoming freshmen had a mean of 1250 and a standard deviation of 70. Find the combined score at:
  - a. the 90th percentile
  - b. the third quartile
- B** 11. The professor teaching a large freshman course decides to give a grade of A to the top 10% of the final exam scores and a grade of B to the next 30%. If the mean score was 72 and the standard deviation was 9, what were the minimum scores needed for an A and a B?
12. Refer to Exercise 11 and suppose the professor decided that 5% will fail and 10% will get a grade of D. Find (a) the minimum passing grade and (b) the minimum grade needed to get a C.
13. **Manufacturing** A 2 L beverage bottle is filled from a machine that fills the bottle with a mean of 2.01 L and a standard deviation of 0.002 L.
  - a. What percent of the bottles actually contain more than 2 L of beverage?
  - b. Ten percent of the bottles contain less than     ? L.
  - c. Eighty percent of the bottles contain more than     ? L.
14. **Physiology** The weights of all students in a certain grade in a school district are normally distributed with  $\bar{x} = 65$  kg and  $s = 5$  kg.
  - a. The “middle half” have weights between     ? kg and     ? kg.
  - b. The lightest 10% have weights less than     ? kg.
  - c. The heaviest 5% have weights greater than     ? kg.
15. **Physiology** The reaction times for a particular task are normally distributed with a mean of 0.92 s and a standard deviation of 0.24 s. A reaction time in excess of 1.2 s is considered too slow. What percent of the people tested had reaction times that were too slow?

- C** 16. **Sports** Those wanting to participate in a state marathon race must first qualify by running in a regional marathon. The times of the 3750 regional runners are normally distributed with a mean of 198 min 36 s and a standard deviation of 23 min 14 s. If there are to be only 600 runners in the state marathon, what is the slowest time that will qualify a regional runner for the state race?



17. Let  $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ . The

graph of  $f$  is the standard normal curve, shown at the right.

a. **Visual Thinking** Explain how the shape of the curve implies that the region in red has an area of 0.5.

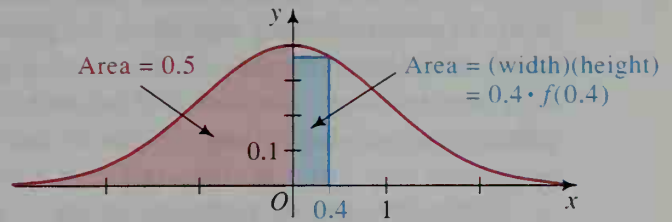
b. Use a calculator to find the area of the region in blue.

c. Use parts (a) and (b) to approximate  $P(0.4)$ .

d. **Visual Thinking** Examine the graph to determine whether the approximation found in part (c) is too large or too small. Then compare the approximation with the value of  $P(0.4)$  given in the standard normal table.

e. Suppose the region in blue has a height of  $f(0)$  instead of  $f(0.4)$ . Draw a graph to illustrate this. Then repeat parts (b), (c), and (d).

f. **Investigation** Find and use another method of approximating  $P(0.4)$  that improves on the approximations found so far.



## ■ Inferential Statistics

### 17-5 Sampling

#### Objective

*To recognize different types of sampling procedures, to identify their limitations, and to estimate population characteristics based on samples.*

In statistics, the word *population* does not necessarily refer to people. It may, for example, refer to a set of stereo receivers sold or to a set of airline luggage claim tickets issued. A **population** is the entire set of individuals or objects in which we are interested, whether they be stereo receivers whose prices are recorded or claim tickets whose numbers are crosschecked. A subset of the population is called a **sample**, and the number of objects in a sample is called the **sample size**.



The process of selecting a sample that is representative of the total population is called **sampling**. **Sampling theory** is the branch of statistics that deals with the questions that arise when a sample is taken. How should the sample be selected? How large should the sample size be? How reliable are the conclusions drawn from the sample? The answers to such questions often depend upon the circumstances under which the sampling is done, the resources available to do the sampling, and the results desired from the sampling.

We can distinguish between *probability sampling* and *nonprobability sampling*. In **nonprobability sampling**, the procedures for selecting a sample are not based on random processes. Examples of nonprobability sampling are *convenience sampling*, *judgment sampling*, and *sampling by questionnaire*. Such sampling procedures may produce a sample that is biased; that is, the sample and the total population may differ in important characteristics.

Consider, for example, a quality control inspector who inspects the top few apples in several crates to determine the percent of the fruit that is bruised. This is called **convenience sampling** because it is more convenient to sample the top apples than the bottom ones. The resulting sample is not representative of all the apples because apples near the bottom are more likely to be bruised.

In **judgment sampling**, one or more experts select a representative sample using their subjective judgment. It is quite likely that no two experts will agree on what is truly “representative.”



One problem that occurs with **sampling by questionnaire** is that conclusions are based on voluntary responses. This is also true of other voluntary responses, such as phoned responses to questions that are broadcast on live television or radio. Usually only those who feel strongly about an issue will respond. Another problem with questionnaires is that the questions may be biased. For example, consider whether you agree with statements (a) and (b) below.

- (a) It is best to buy the least expensive brand.
- (b) It is best to buy the cheapest brand.

Some people might agree with statement (a) but disagree with (b) because “cheapest” also may mean “poorest quality.”

A famous example of a biased poll occurred in 1936. A popular magazine mailed straw-poll ballots to telephone and automobile owners asking whether they favored Roosevelt or Landon for president. The poll showed that Landon would be victorious by a landslide, yet Roosevelt won the actual election. The problem, of course, was that the magazine’s telephone and automobile owners were not representative of the population in 1936. The sample was biased toward voters who preferred Landon. Can you see why this is an example of convenience sampling?

In **probability sampling**, the procedures for selecting a sample are based on random processes. The most basic type of probability sampling, called **simple random sampling**, is a procedure in which every element of the population has an equal and independent chance of being chosen. This can be done by any chance method, such as flipping a coin or drawing lots, but random numbers generated by a computer or taken from a table of random digits are most frequently used.

For example, suppose a school has 960 students and the school newspaper wants to poll 60 randomly selected students. Although the 960 student names could be written on slips of paper and put in a box so that 60 names could be chosen at random, that would be tedious. It would be easier to have a computer generate 60 three-digit numbers from 001 to 960 and then take the 60 corresponding students from an alphabetized list.

Another type of probability sampling is **stratified random sampling**, in which the entire population is broken into groups, called *strata*, chosen because the individuals in each *stratum* have some common property. For example, you might have two strata (male and female) or four strata (freshmen, sophomores, juniors, and seniors). The only requirement for choosing strata is that you know what percent of the entire population each stratum represents. You then take a simple random sample of each stratum and combine the results.

### Example

Southbrook High School has 960 students and 40 teachers. The school newspaper interviews 60 students and 12 teachers to see whether they favor changing a school policy. The results are given in the table below.

Stratum	Population size	Sample size	Number favoring change
Students	960	60	32
Teachers	40	12	4
Total	1000	72	36

Estimate the percent of the school population favoring change.

### Solution

First note that 50% of the sample (36 out of 72) are in favor of change. Nevertheless, it would be a mistake to estimate that 50% of the school population favors change because the teachers are disproportionately represented in the sample. Instead, we take the following steps.

**Step 1** For each stratum in the sample, we find the proportion of people favoring change:

$$\bar{p}_1 = \text{sample proportion of students in favor} = \frac{32}{60} = \frac{8}{15}$$

$$\bar{p}_2 = \text{sample proportion of teachers in favor} = \frac{4}{12} = \frac{1}{3}$$

**Step 2** To estimate  $p$ , the proportion of the whole population favoring change, we take a weighted average of  $\bar{p}_1$  and  $\bar{p}_2$  by assigning weights according to the number in each stratum:

$$\begin{aligned}\bar{p} = \text{estimate of } p &= \frac{960}{1000}\bar{p}_1 + \frac{40}{1000}\bar{p}_2 \\ &= \frac{96}{100} \cdot \frac{8}{15} + \frac{4}{100} \cdot \frac{1}{3} = \frac{788}{1500} \approx 52.5\%\end{aligned}$$

Thus, about 52.5% of the students and teachers favor change.



## CLASS EXERCISES

In Exercises 1–6, state whether the sampling procedure described is an example of (a) convenience sampling, (b) judgment sampling, (c) sampling by questionnaire, (d) simple random sampling, or (e) stratified random sampling. Mention any limitations that the sampling procedure might have.

1. A television rating organization mails questionnaires to subscribers of a television magazine. On the basis of responses to this questionnaire, it names the most popular shows.
2. A popular financial magazine would like to survey the techniques of successful managers. A managerial consultant is asked to select a group of 50 successful managers to participate in the survey.
3. A manufacturing company would like to determine the approximate market share of a certain product. A representative of the company is asked to stand in front of a certain grocery store and ask the first 100 people who go into the store whether they use the product.
4. A television cable company with 20,000 subscribers would like to determine its most popular services. To do so, the company's computer is programmed to select 1000 subscribers at random. What these subscribers watch is then electronically monitored.
5. In order to determine the public response to a proposed change in regulations concerning small businesses, a questionnaire is being developed. Two alternative questions differing by just one word are suggested for the survey.
  - (a) Do you regret the disappearance of the friendly corner grocery store?
  - (b) Do you regret the disappearance of the old corner grocery store?
6. An automobile manufacturer sends out questionnaires to all people who purchased one of its cars 10 years ago. Of those who responded, 90% reported that they still use the cars. The manufacturer cites this result as evidence of vehicle durability.
7. Senior citizens are 20% of Littleton's voting population. In a poll of 100 citizens, half of whom were senior citizens, 30 senior citizens voted yes and 20 nonsenior citizens voted yes. Estimate the percent of the whole voting population who would vote yes.





## WRITTEN EXERCISES

In Exercises 1–4, state any errors that you think might occur in the sampling situation discussed.

**A**

1. A senator explains a vote in favor of a new bill by stating that 60% of the mail received favored the bill.
2. A city newspaper asks readers to mail in a form on which they can choose one of three ways to finance improvements to the zoo. Based on the responses, the newspaper reports that financing the improvements by raising the admission charge is the first choice of the majority of the citizens.
3. A radio talk show host invites listeners to telephone the station and talk about their feelings on a proposed highway to be built in their county.
4. A newspaper reporter randomly stops people going in a grocery store and asks, “Does your family use newspaper coupons?”
5. In Farmington High School, there are 360 ninth- or tenth-grade students and 320 eleventh- or twelfth-grade students. A poll shows that 12 out of 40 ninth- or tenth-grade students intend to vote for Lahey as student council president. It also shows that 24 out of 40 eleventh- or twelfth-grade students intend to vote for Lahey. Estimate the percent of the student body favoring Lahey.
6. Twenty-five percent of a city’s employees live in the city and 75% live in the surrounding suburbs. The mayor conducts a small survey asking 10 employees who live in the city and 10 who live in the suburbs if they would prefer increases in pay or increases in benefits. Eight of the 10 city dwellers and four of the 10 suburbanites preferred an increase in pay. Estimate the percent of city employees preferring an increase in pay to an increase in benefits.

**B**

7. Estimate the total number of years of service of 200 factory workers based on a sample of size  $n = 5$  where the years of service reported are 15, 8, 20, 5, 12.
8. **Manufacturing** In one hour, a factory packaged 150 boxes of light bulbs. Estimate the total number of defective light bulbs packaged that hour based on a sample of 8 boxes containing the following numbers of defective bulbs: 5, 3, 5, 1, 2, 9, 4, 3.
9. **Ecology** The State Conservation Department wants to estimate the number of deer in a mountainous area. It captures and tags 80 deer and then releases them. Later it captures 156 deer and finds that 12 are tagged. Estimate the number of deer in the area.



10. **Ecology** The number of fish in a pond is unknown; call it  $x$ . Initially,  $c$  fish are caught, marked, and released. Later on,  $d$  fish are caught, and  $m$  of them are found to be marked. Estimate the value of  $x$  in terms of  $c$ ,  $d$ , and  $m$ .

For Exercises 11 and 12, use the table below, which gives both the distribution (by class) of students at a small college and the responses of 25 randomly selected members of each class to two survey questions, one about meals and one about the required English course.

Stratum	Stratum size	Sample size	Pleased with meals	Pleased with English course
Freshmen	300	25	15	20
Sophomores	250	25	12	14
Juniors	250	25	10	18
Seniors	200	25	9	16

11. Estimate the percent of the student body pleased with the meals at the college.  
12. Estimate the percent of the student body pleased with the English course.  
13. **Research** Read about the 1948 presidential election, for which most poll-based predictions were incorrect. Report on what happened and why.

### //// COMPUTER EXERCISE

Write a program that randomly selects a specified number of samples of a specified size from the set of 50 test scores on page 639. For each sample selected, have the program calculate the mean and display it in a histogram of sample means. Then, holding the number of samples constant, run the program for samples of size 5, 10, 15, and 20. (If possible, avoid having to enter the test score data on each run. In BASIC, for example, you can use READ and DATA statements.) Knowing that the mean of all 50 test scores is 549.6, write a paragraph in which you state any conclusions that you draw from comparing the histograms from the four runs.

## 17-6 Confidence Intervals for Surveys and Polls

### **Objective**

*To use a sample proportion to find a confidence interval for the corresponding population proportion.*

When you know all the relevant information about a population, you can use probability theory to make deductions about a sample. For example, knowing the contents of a standard 52-card deck, you can make probability predictions about a 5-card sample that you are about to deal. On the other hand, there are many situations in which you know all about a sample but not about the population from which it comes. In this case, you can use statistics to make inferences about the population based on the sample.

In everyday situations, samples often take the form of a survey or poll. For example, suppose that in a recent state poll of 100 randomly selected voters, 56 indicated that they were going to vote in favor of Proposition 1. Now, 100 voters are not very many compared with an entire state's voting population. If 560 out of 1000 had favored Proposition 1 instead of just 56 out of 100, we could be more confident of the poll's result. Suppose also that a different poll showed that as many as 60 out of 100 voters favor Proposition 1, and another showed that only 49 out of 100 favor it. The question is, how close is the sample proportion favoring Proposition 1 to the population proportion favoring it? At this point, we need to introduce some notation:



Let  $p$  = population proportion having a given characteristic.

Let  $\bar{p}$  = sample proportion having a given characteristic.

For example, if a poll of 15 doctors shows that 12 recommend daily exercise, then  $\bar{p} = \frac{12}{15} = 0.8$ . Likewise, if 76 out of 200 voters surveyed say that they favor candidate Smith, then  $\bar{p} = \frac{76}{200} = 0.38$ .

You can see that  $\bar{p}$  will vary from sample to sample and will depend partly on the sample size  $n$ . Moreover, an advanced theorem in statistics states three things about the values of  $\bar{p}$ :

1. The values of  $\bar{p}$  are normally distributed.
2. The mean of this distribution is  $p$ .
3. The standard deviation of this distribution is approximately  $\sqrt{\frac{\bar{p}(1 - \bar{p})}{n}}$ .

Since  $\bar{p}$  is normally distributed, we can apply the 68%-95%-99% normal rule given at the bottom of page 662. For instance, 95% of the time,  $\bar{p}$  is within 2 standard deviations of its mean,  $p$ . Since we use  $\bar{p}$  to estimate  $p$ , the following equivalent statement is more useful: 95% of the time,  $p$  is within 2 standard deviations of  $\bar{p}$ . This interval in which  $p$  lies 95% of the time is called a **95% confidence interval** and is given by:

$$\bar{p} - 2 \times (\text{standard deviation}) < p < \bar{p} + 2 \times (\text{standard deviation})$$

$$\bar{p} - 2\sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} < p < \bar{p} + 2\sqrt{\frac{\bar{p}(1 - \bar{p})}{n}}$$

**Example 1**

Suppose that 56 out of 100 voters polled favor Proposition 1. What is the 95% confidence interval for  $p$ , the unknown proportion of the population favoring Proposition 1?

**Solution**

**Step 1**  $\bar{p} = \frac{56}{100} = 0.56$

**Step 2** The standard deviation of  $\bar{p}$  is:

$$\sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} = \sqrt{\frac{(0.56)(0.44)}{100}} \approx 0.05$$

**Step 3** The 95% confidence interval for  $p$  is:

$$0.56 - 2(0.05) < p < 0.56 + 2(0.05) \\ 0.46 < p < 0.66$$

In Example 1, other samples of 100 voters will yield other 95% confidence intervals. Most of these confidence intervals (about 95% of them) will contain  $p$ , but a few of them (about 5%) will not. For this reason, we can be 95% confident that the unknown value of  $p$  is between 0.46 and 0.66. Even so, the 95% confidence interval from 0.46 to 0.66 is rather wide and does not pinpoint  $p$  to any great extent. (In fact, we cannot tell whether a majority favor Proposition 1.) Example 2 shows that we can obtain a narrower confidence interval by taking a larger sample.

**Example 2**

Suppose that 560 out of 1000 voters polled favor Proposition 1. Find a 95% confidence interval for  $p$ .

**Solution**

**Step 1**  $\bar{p} = \frac{560}{1000} = 0.56$

**Step 2** The standard deviation of  $\bar{p}$  is:

$$\sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} = \sqrt{\frac{(0.56)(0.44)}{1000}} \approx 0.016$$

**Step 3** The 95% confidence interval for  $p$  is:

$$0.56 - 2(0.016) < p < 0.56 + 2(0.016) \\ 0.528 < p < 0.592$$

The results of Examples 1 and 2 are summarized in the table at the right. Notice that the sample of size 1000 gives a much narrower confidence interval than the sample of size 100. In fact, with the larger sample, we can be quite confident that a majority of the voters favors Proposition 1, since the smaller endpoint of the sample's 95% confidence interval, 0.528, is greater than 0.5. Bear in mind, however, that the larger sample may be more costly and time consuming to obtain than the smaller one.

Sample size	95% confidence interval
100	$0.46 < p < 0.66$
1000	$0.528 < p < 0.592$



Sometimes a 95% confidence interval is not enough. In testing new medical drugs or procedures, a *99% confidence interval* may be required before the new drug or procedure is approved for general use. A **99% confidence interval** is based on  $p$  being within 3 standard deviations of  $\bar{p}$ :

$$\bar{p} - 3 \times (\text{standard deviation}) < p < \bar{p} + 3 \times (\text{standard deviation})$$

$$\bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} < p < \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

### Example 3

A new, medically approved wheelchair was judged to be an improvement over a standard wheelchair by 320 out of 400 patients in a survey.

- Find a 99% confidence interval for  $p$ , the proportion of the wheelchair-using population for whom the new wheelchair would be an improvement.
- If 72% of the wheelchair-using population has found the standard wheelchair to be acceptable, does it appear that manufacturing the new wheelchair is a good idea?



### Solution

a. **Step 1**  $\bar{p} = \frac{320}{400} = 0.80$

**Step 2** The standard deviation of  $\bar{p}$  is:

$$\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = \sqrt{\frac{(0.80)(0.20)}{400}} = 0.02$$

**Step 3** The 99% confidence interval for  $p$  is:

$$0.80 - 3(0.02) < p < 0.80 + 3(0.02) \\ 0.74 < p < 0.86$$

- Since 0.72 lies outside the 99% confidence interval, the new wheelchair is probably a little better than the standard wheelchair. However, what the new one will cost, how widely available it will be, and other related questions must be considered in deciding whether to proceed with its manufacture.

## CLASS EXERCISES

- If  $\bar{p} = 0.4$  and  $n = 24$ , find the standard deviation of  $\bar{p}$ .
  - Find both a 95% and a 99% confidence interval for  $p$ .
- If  $\bar{p} = 0.5$  and  $n = 25$ , find the standard deviation of  $\bar{p}$ .
  - Find both a 95% and a 99% confidence interval for  $p$ .



3. a. If 50 out of 100 people surveyed favor brand X, find  $\bar{p}$  and the standard deviation of  $\bar{p}$ .  
b. Find a 95% confidence interval for  $p$ , the proportion of the population favoring brand X.
4. **Reading** Does the formula for the standard deviation of  $\bar{p}$  depend on the sample size or the population size?
5. Consider the two polls described below. Is there any difference in the 95% confidence intervals of these two polls?  
  
Poll A: 23 out of 100 randomly chosen students in Central High School favor having an additional gym class in place of a study period.  
Poll B: 23 out of 100 randomly chosen high school students from around the state favor having an additional gym class in place of a study period.

## WRITTEN EXERCISES

In Exercises 1–6, find  $\bar{p}$  and the standard deviation of  $\bar{p}$ . Also, give a 95% confidence interval for  $p$ , the proportion of the entire population having the indicated characteristic.

- A**
1. Seven out of 10 dentists surveyed recommend Smile toothpaste.
  2. Six out of 10 doctors polled recommend eating apples.
  3. Thirty-six out of 100 voters surveyed favor Proposition 4.
  4. Sixteen out of 25 students questioned prefer summer to winter.
  5. Twenty-five out of 400 flashbulbs tested were defective.
  6. Eighty out of 100 students have after-school jobs.

In Exercises 7–12, find a 99% confidence interval for  $p$ .

7. Of 1000 adults randomly selected for reading tests, 30 were illiterate.
8. Of 1600 eligible voters who were polled, 1280 favored the proposal.
9. Of 100 workers surveyed, 9 have been promoted.
10. In a television poll, 10 out of 100 television sets were tuned to the concert.
11. Of 250 students interviewed, 200 prefer tennis to racquetball.
12. In a survey, 150 out of 200 people preferred brand A to other brands.
13. If you wish to halve the standard deviation of  $\bar{p}$ , you must multiply the sample size by   ?  .
14. If you wish to divide the standard deviation of  $\bar{p}$  by 10, you must multiply the sample size by   ?  .

- B**
15. **Polling** After conducting a poll, a national magazine reported in an article that 64% of the 1000 voters surveyed approved a proposed law. The article also reported a “margin of error of plus or minus 3 percentage points.” Is this 3% margin of error associated with a 95% or a 99% confidence interval?

16. **Polling** A city newspaper reported in an article that 55% of the city's registered voters intend to vote for Connelly for mayor. The article noted that this figure was based on a telephone survey of 380 voters and that there was a 5% margin of error in this estimate. Does this margin of error correspond to a 95% or a 99% confidence interval?
17. In a poll of 240 college freshmen, 144 said that they exercise regularly. Find both a 95% and a 99% confidence interval for  $p$ , the proportion of the population of college freshmen who exercise regularly.
18. An audit of 840 randomly selected tax returns shows that 30% require payment of additional taxes. Find both a 95% and a 99% confidence interval for  $p$ , the proportion of tax returns requiring additional payments.
19. a. Graph  $f(\bar{p}) = \bar{p}(1 - \bar{p})$ .  
b. Find the maximum value of  $f$ .  
c. The standard deviation of  $\bar{p}$  is always less than or equal to  $\sqrt{\frac{1}{4n}}$ . Explain.
20. Use Exercise 19 to prove that the widest possible 95% confidence interval is:



$$\bar{p} - \sqrt{\frac{1}{n}} < p < \bar{p} + \sqrt{\frac{1}{n}}$$

For Exercises 21–23, use the following sample.

**Sample**

A researcher wants to ensure that 95% of the time  $\bar{p}$  is within 0.1 of  $p$ . How large a sample is needed?

**Solution**

The 95% confidence interval is to be  $\bar{p} - 0.1 < p < \bar{p} + 0.1$ .

$$\sqrt{\frac{1}{n}} < 0.1 \quad \leftarrow \text{See Exercise 20.}$$

$$\frac{1}{n} < 0.01$$

$$n > 100$$

Thus, a sample size greater than 100 is needed.

21. **Polling** A pollster wants a 95% confidence interval in which  $p$  is within 0.01 of  $\bar{p}$ . How large a sample is needed?
22. **Polling** Seeking to find the proportion of the population of voters who favor an increased budget, a pollster wants a 95% confidence interval of the form  $\bar{p} - 0.03 < p < \bar{p} + 0.03$ . How large a sample is needed?

23. **Manufacturing** A quality control inspector wishes to estimate the percent  $p$  of computer chips that are defective. Suppose that  $|p - \bar{p}| < 0.05$  is a requirement. How large a sample is needed for a 95% confidence interval?
- C** 24. **Polling** If only 18 out of 50 voters surveyed favor Article VII, find an 80% confidence interval for the proportion of all voters favoring it.

## Chapter Summary

- We can display a set of data in a *histogram*, *frequency polygon*, *cumulative frequency polygon*, *stem-and-leaf plot*, or *box-and-whisker plot*.
- The *mean*, *median*, and *mode* are three measures of central tendency. For the data  $x_1, x_2, \dots, x_n$ :
  - The *mean*, denoted  $\bar{x}$ , is the sum of the data divided by the number of items of data.
  - If the data are arranged in increasing or decreasing order, the *median* is the middle item of data if  $n$  is odd and is the mean of the two middle items of data if  $n$  is even.
  - The *mode* is the one item of data that occurs most often. A set of data can have more than one mode or no mode at all.
- The *range*, *interquartile range*, *variance*, and *standard deviation* are all measures of dispersion. Definitions are given on pages 649 and 654.
- A set of data  $x_1, x_2, \dots, x_n$  having mean  $\bar{x}$  and standard deviation  $s$  can be converted to *standard values*  $z_1, z_2, \dots, z_n$  by the formula  $z_i = \frac{x_i - \bar{x}}{s}$ .
- A *normal distribution* has a bell-shaped graph called a *normal curve*. In a normal distribution, about 68%, 95%, and 99% of the data are within 1, 2, and 3 standard deviations of the mean, respectively. The *standard normal distribution* has mean 0 and standard deviation 1. Any normal distribution can be related to the standard normal distribution through standard values.
- A set of data arranged in ascending order can be divided into 100 equal parts by *percentiles*. The median is the 50th percentile, the *first quartile* is the 25th percentile, and the *third quartile* is the 75th percentile.
- A sample should be chosen so that it is representative of the whole population. Methods of sampling include *nonprobability sampling*, such as *convenience sampling*, *judgment sampling*, and *sampling by questionnaire*, and *probability sampling*, such as *simple random sampling* and *stratified random sampling*.
- A *confidence interval* indicates how well a sample proportion  $\bar{p}$  estimates the true population proportion  $p$ . Confidence intervals have the form

$$\bar{p} - k \cdot \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}} < p < \bar{p} + k \cdot \sqrt{\frac{\bar{p}(1 - \bar{p})}{n}}$$

where the value of  $k$  is the number of standard deviations related to the confidence level selected. See Examples 1–3 on pages 676 and 677.

### Key vocabulary and ideas

stem-and-leaf plot (p. 639)	normal distribution (p. 662)
histogram (p. 640)	normal curve (p. 662)
frequency polygon (p. 641)	standard normal distribution (p. 663)
cumulative frequency polygon (p. 641)	percentiles (p. 665)
mean, median, mode (pp. 641–642)	methods of sampling (pp. 669–671)
lower quartile, upper quartile (p. 649)	95% confidence interval (p. 675)
box-and-whisker plot (p. 649)	99% confidence interval (p. 677)
standard value (p. 656)	

## Chapter Test

1. **Writing** In a paragraph, discuss *descriptive* and *inferential* statistics. **17-1**
2. **a.** Draw a histogram for the real-estate sales data shown in the table.

Price ( $\times \$1000$ )	70–89.9	90–109.9	110–129.9	130–149.9
Number of sales	4	9	11	6

- b.** Using the middle price (80, 100, and so on) as the representative price for each class, find the mean, median, and mode.
3. The stem-and-leaf plot shown at the right gives the page lengths of a book's 24 chapters. **17-2**

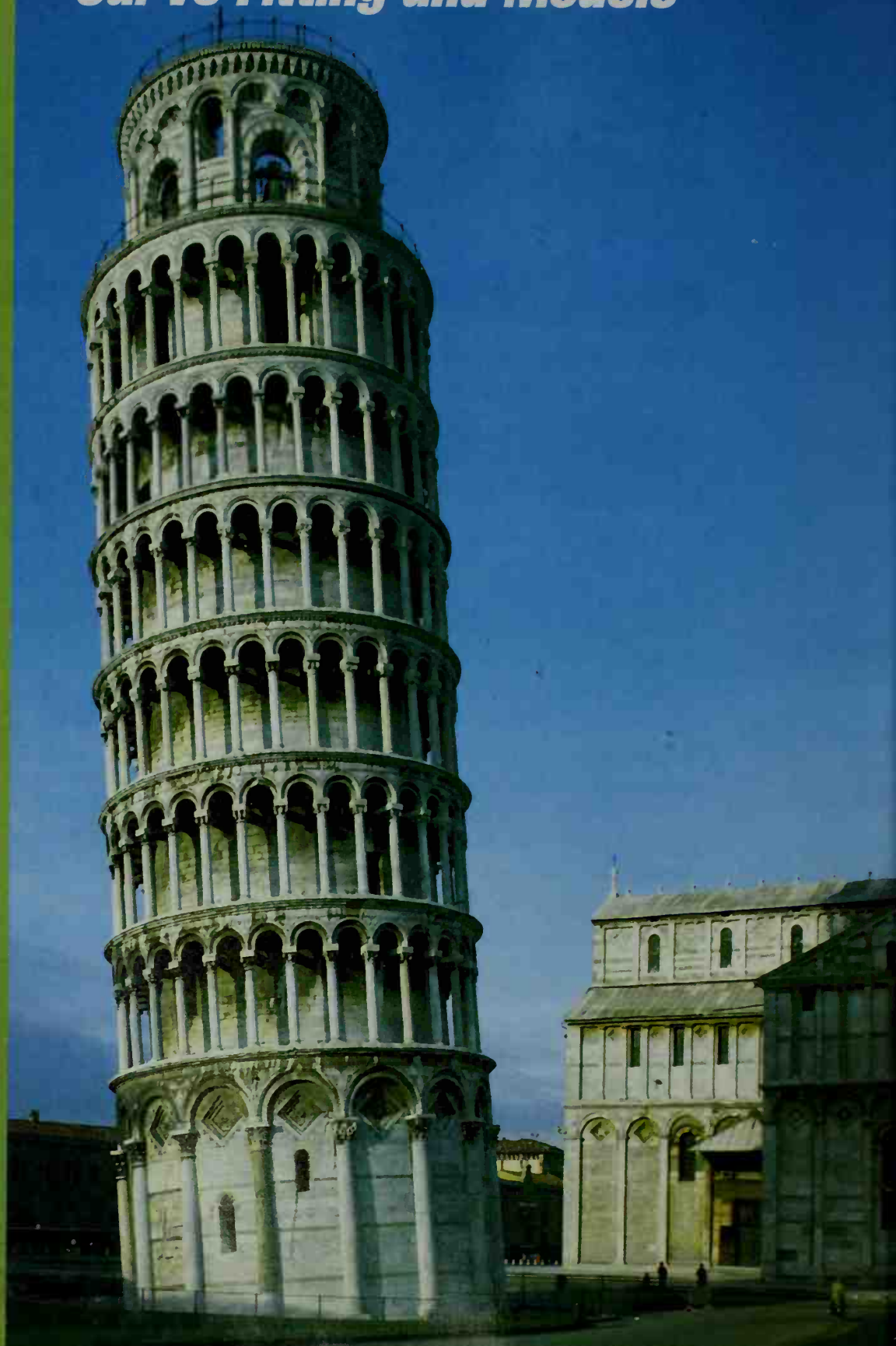
1	8 9 9
2	0 1 2 2 3 3 5 5 5 7 8 8 8 9
3	1 1 3 4 4 6

Find (a) the median, lower quartile, and upper quartile, and (b) the range and interquartile range. (c) Draw a box-and-whisker plot.
4. Zina scored an 80 on both an English test and a math test. The mean score in the English class was 70, with a standard deviation of 5. In the math class, the mean score was 65, with a standard deviation of 10. For which test did Zina rank higher with respect to her class? **17-3**
5. Calculate the standard deviation of: 4, 7, 10, 11, 12, 14, 15, 21, 32.
6. At a large university, the mathematics-plus-verbal standardized test scores of incoming students are normally distributed with a mean score of 1200 and a standard deviation of 100. Find the percent of incoming students with scores (a) greater than 1300 and (b) less than 1000. Find (c) the 90th percentile and (d) the third quartile of the scores. **17-4**
7. At a certain college, 62% of the students live on campus. Of 50 randomly selected on-campus students, 27 favor a change in housing rules. Of 50 randomly selected off-campus students, 14 favor the change. Estimate the percent of all students favoring the change. **17-5**
8. In a survey of 600 school children, 486 said that their favorite sandwich filling is peanut butter. Find 95% and 99% confidence intervals for  $p$ , the true proportion of school children who prefer peanut butter. **17-6**



# 18

## *Curve Fitting and Models*





# 18-1 Introduction to Curve Fitting; the Least-Squares Line

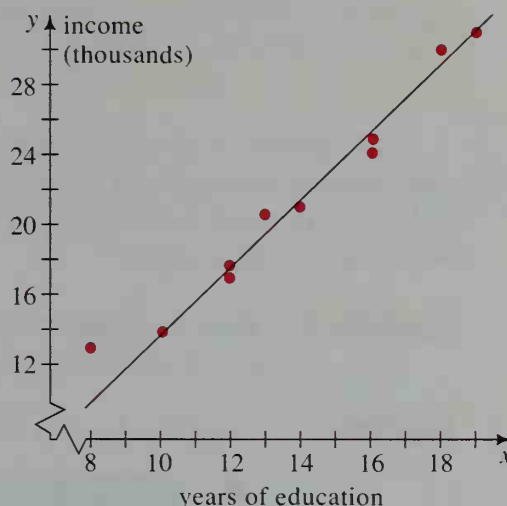
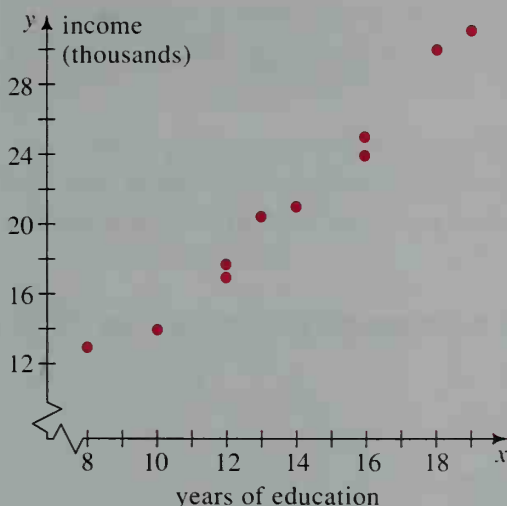
## Objective

To find the line of best fit for a set of data and to find the correlation coefficient for a set of data.

Fitting curves to a set of data points is a major goal of this chapter. The process of finding an equation of a curve that fits observed data is called *curve fitting*, and the equation is said to *model* the data. We will begin our study by learning to fit a line to a set of data points.

For example, ten 25-year-olds are surveyed to find their years of formal education ( $x$ ) and their annual incomes ( $y$ ). The data are given in the table below and the ten points  $(x, y)$  are graphed in a *scatter plot*, as shown at the left below.

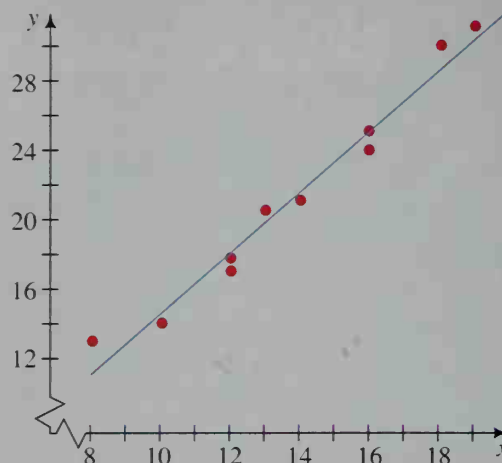
Years of education	8	10	12	12	13	14	16	16	18	19
Income (in thousands)	13	14	17	17.5	20.6	21	24	25	30	31



It is possible to fit a line to the scatter plot at the left above by using just your eye and a piece of black thread. Holding the thread taut, move the thread until most of the points lie on or close to the thread. One such line is shown in black at the right above. Some of the points lie on this line, while others are above it and still others are below it. Associated with every point is its vertical distance to the line. The sum of all the distances is a measure of how well the line fits the points. The smaller the sum of the distances, the better the fit.

◀ Will the Leaning Tower of Pisa ever fall down? A structural engineer might wish to collect and analyze data on the past shifting of the tower. If the data conformed to some mathematical model, the engineer might even predict, with some precision, future shifts.

To find the line that best fits the data, statisticians prefer to work with the sum of the *squares* of these distances, rather than the vertical distances themselves. The line that minimizes the sum of these squares is called the *least-squares line*. It is shown in blue in the figure at the right.



### Properties of the Least-Squares Line

1. The line contains the point  $(\bar{x}, \bar{y})$ .

2. The line has slope  $= \frac{\overline{xy} - \bar{x} \cdot \bar{y}}{s_x^2}$

$$m = \frac{(\text{mean of the products}) - (\text{product of the means})}{(\text{standard deviation of } x\text{'s})^2}$$

An equation of the least-squares line is computed routinely by many computer software packages and scientific calculators. Example 1 illustrates how this can be done by hand. In this example, just four points are used in order to make the calculations easier to follow.

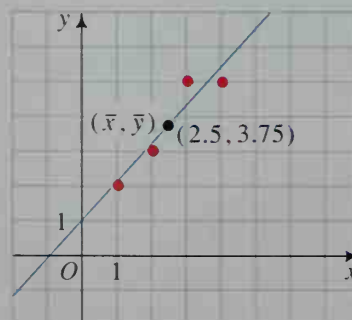
#### Example 1

Find an equation of the least-squares line for the data  $(x, y) = (1, 2)$ ,  $(2, 3)$ ,  $(3, 5)$ , and  $(4, 5)$ .

#### Solution

Organize the data in a table.

$x$	$y$	$xy$	$x^2$
1	2	2	1
2	3	6	4
3	5	15	9
4	5	20	16
Sums	10	15	43
			30



To find an equation of the least-squares line, we must compute the means of the  $x$ - and  $y$ -coordinates and the mean of the product of the  $x$ - and  $y$ -coordinates. Then we compute the standard deviation of the  $x$ -coordinates.

$$\text{Since } n = 4: \bar{x} = \frac{10}{4} = 2.5 \quad \bar{y} = \frac{15}{4} = 3.75 \quad \overline{xy} = \frac{43}{4} = 10.75$$

$$s_x = \sqrt{\frac{\sum_{i=1}^n x_i^2 - n(\bar{x})^2}{n}} = \sqrt{\frac{30 - 4(2.5)^2}{4}} = \sqrt{1.25} \approx 1.118$$

The slope of the least-squares line is:

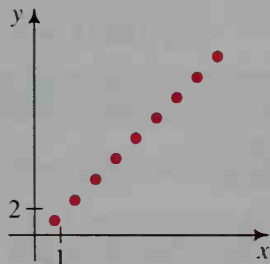
$$\frac{\overline{xy} - \bar{x} \cdot \bar{y}}{s_x^2} = \frac{10.75 - (2.5)(3.75)}{1.25} = 1.1$$

Since the line contains  $(\bar{x}, \bar{y}) = (2.5, 3.75)$ , an equation of the least-squares line is:

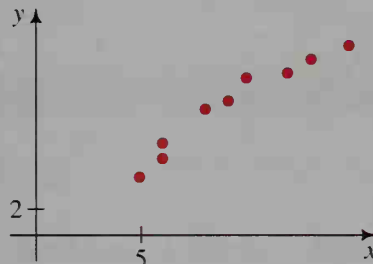
$$y - 3.75 = 1.1(x - 2.5), \text{ or } y = 1.1x + 1$$

## The Correlation Coefficient

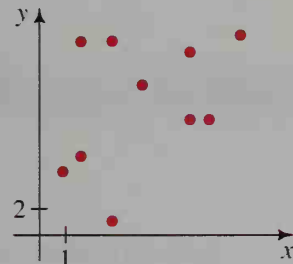
In order to measure how closely points tend to cluster about the least-squares line, statisticians use a *correlation coefficient*, denoted  $r$ . If the data fit perfectly on a line with positive slope, the correlation coefficient is said to be  $+1$ . If the data fit perfectly on a line with negative slope, the correlation coefficient is said to be  $-1$ . If the points tend not to lie on any line, then the correlation coefficient is close to 0. Some calculators will compute the correlation coefficient for you.



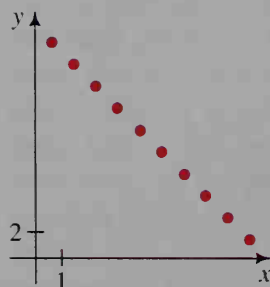
Perfect positive  
correlation  
 $r = +1$



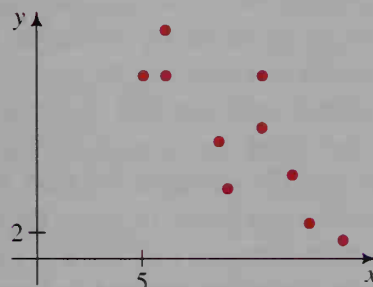
Strong positive  
correlation  
 $r \approx 0.91$



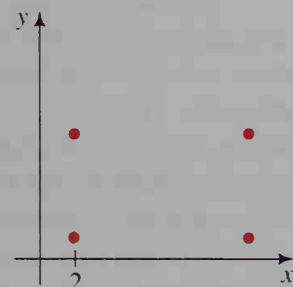
Weak positive  
correlation  
 $r \approx 0.38$



Perfect negative  
correlation  
 $r = -1$



Strong negative  
correlation  
 $r \approx -0.85$



No correlation  
 $r = 0$

## The Correlation Coefficient

The correlation coefficient for a set of data  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  is given by:

$$\begin{aligned} r &= \frac{\overline{xy} - \bar{x} \cdot \bar{y}}{s_x s_y} \\ &= \frac{(\text{mean of the products}) - (\text{product of the means})}{(\text{product of the standard deviations})} \end{aligned}$$

**Example 2** Find the correlation coefficient for the data in Example 1.

**Solution** Most of the needed calculations can be found in the solution to Example 1. We need only calculate  $\sum_{i=1}^4 y_i^2$  and  $s_y$  in order to find the correlation coefficient.

$$\sum_{i=1}^4 y_i^2 = 2^2 + 3^2 + 5^2 + 5^2 = 63$$

$$s_y = \sqrt{\frac{\sum_{i=1}^n y_i^2 - n(\bar{y})^2}{n}} = \sqrt{\frac{63 - 4(3.75)^2}{4}} = \sqrt{1.6875} \approx 1.299$$

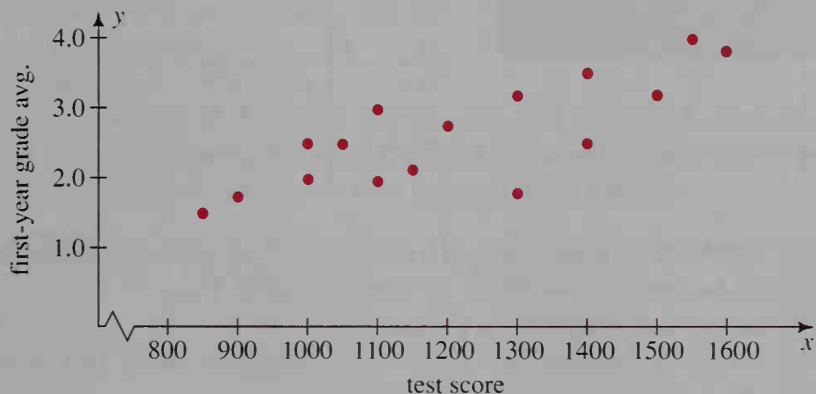
Therefore:

$$r = \frac{\overline{xy} - \bar{x} \cdot \bar{y}}{s_x s_y} \approx \frac{10.75 - (2.5)(3.75)}{(1.118)(1.299)} \approx 0.95$$

If two variables are highly correlated, you should not infer that there is a cause-and-effect relationship between them. For example, the reading level of children aged four to ten years is highly correlated to shoe size. The reason, of course, is not that bigger feet help children to read better but that bigger feet and increased reading level are both related to age.

As another example, consider the following situation. A certain state university administers a test to each of its incoming freshmen. The university finds that there is a 0.80 correlation coefficient between the test scores and first-year grade averages. The positive correlation indicates that higher test scores generally yield higher first-year averages, but good test scores do not themselves cause good averages. Rather, the scores and the averages are both influenced by other variables such as knowledge, ability, and drive. In short, the 0.80 correlation between test scores and first-year averages does not indicate a cause-and-effect relationship. Nevertheless, it does indicate a relationship and is therefore useful, along with other data, in predicting a student's academic success at the university.

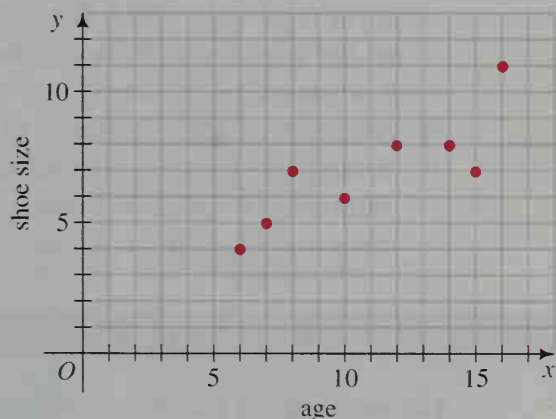
To show how such a prediction might be done, consider the scatter plot below. A relatively strong correlation, such as 0.80, indicates that the points tend to cluster around the least-squares line. With a computer or calculator, we find that an equation of the least-squares line is  $y = 0.00263x - 0.541$ . Then we can use it to predict  $y$  (a student's first-year grade average) from  $x$  (the student's test score). For example, if a freshman at the university has a total test score of 1200, then the predicted first-year average is about 2.6. This predicted value is approximate and should be regarded as the center of a range of values. Advanced statistics books present formulas for finding the range of values. For our purposes it is sufficient to be aware that this prediction and those predictions called for in Exercises 13–16 are approximate.



## CLASS EXERCISES

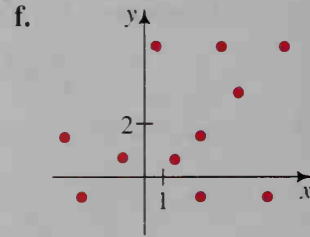
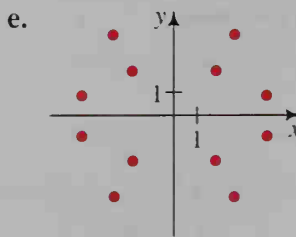
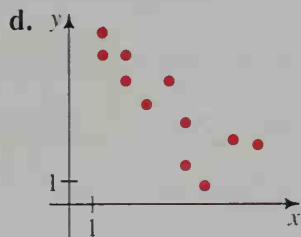
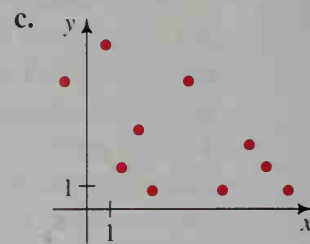
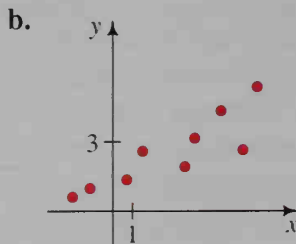
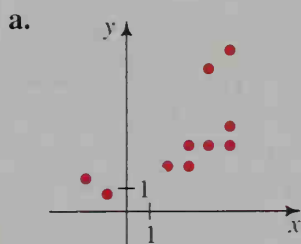
The shoe sizes of eight boys are plotted versus their ages.

1. Use your eye to fit a line to the data.
  - a. What is the slope of this line?
  - b. What is the  $y$ -intercept of the line?
  - c. What is an equation of the line?
2. a. Use the coordinates of the eight points  $(x, y)$  shown at the right to find  $\bar{x}$ ,  $\bar{y}$ ,  $\overline{xy}$ ,  $s_x$ , and  $s_y$ .
  - b. Name one point that the least-squares line contains.
  - c. Find the slope of the least-squares line.
  - d. Find an equation of the least-squares line.
  - e. Find the value of  $r$ , the correlation coefficient.
3. **Discussion** How do you think the *least-squares line* got its name?





4. **Visual Thinking** Six scatter plots are shown below. Match each scatter plot to one of the following estimated values of  $r$ : 0.8, 0.6, 0.2, 0, -0.2, -0.6, -0.8.



5. **Visual Thinking** For each pair of variables, tell whether you think the correlation will be positive, negative, or approximately zero.
- The height and weight of a person
  - The height of a person and the average height of his or her parents
  - The value of an automobile and its age
  - The incidence of flu and the outside temperature
  - The height of a person and the number of years of formal education that person has completed
6. **Reading** Refer to the least-squares line for predicting annual income at age 25 from the years of formal education (page 684).
- Estimate the slope of this line. (Note that the axes have different scales.)
  - In what units is the slope measured?
  - For every year of formal education, about how much does annual income increase?
7. **Discussion** The correlation between the number of students in the Chesterfield High School and the number of firefighters in the Chesterfield Fire Department is 0.9. Can you conclude that a change in the student population will cause a change in the number of firefighters? Give your reasons.

## WRITTEN EXERCISES

For each pair of variables, tell whether you think the correlation will be positive, negative, or approximately zero. Briefly give your reasons.

**A**

- The number of hours spent studying statistics and the grade on a statistics test
- The total number of absences in a chemistry course and the average of all the grades in that course

3. The weight of a car and the car's fuel economy
4. The blood pressure of an adult and his or her weight
5. The blood pressure of an adult and his or her age
6. Standardized test mathematics scores and standardized test verbal scores
7. The supply of an item and the demand for that item
8. The number of hours of sleep and the ability to memorize



Display each set of data in a scatter plot and find the correlation coefficient. Also, find an equation of the least squares line and draw the line on your scatter plot. You may find a calculator helpful.

9.

$x$	$y$
1	1
1	2
3	3
3	4

10.

$x$	$y$
1	4
2	3
2	1
3	0

11.

$x$	$y$
1	6
2	6
3	5
4	3
5	0

12.

$x$	$y$
-2	-1
-1	-1
0	0
2	3
6	4

13. In a ten-week course, the final grades A, B, C, D, and F were given the scores  $y = 4, 3, 2, 1$ , and  $0$ , respectively. The least-squares line for predicting scores from  $x$ , the number of absences, was  $y = 3.5 - 0.5x$ . What letter grade can you predict for a student with (a) 1 absence? (b) 3 absences? (c) 8 absences?
14. When the sales volume in hundreds of units is plotted against  $x$ , the money spent on advertising in thousands of dollars, researchers obtained a least-squares line with the equation  $y = 14x + 0.7$ . What average sales volumes (in hundreds of units) can you predict for the following amounts (in thousands of dollars) spent on advertising? a. 10      b. 5      c. 7

Find an equation of the least-squares line, then use it to complete the table.

**B** 15.

$x$ = amount of nitrogen (pounds per acre)	111	169	230	287	335	?
$y$ = amount of alfalfa (tons per acre)	2	3	4	5	6	7

16.

$x$ = standardized test math score	400	500	500	600	700	800	550
$y$ = college first-year math avg.	50	65	70	80	85	95	?

17. **Writing** In each of Exercises 15 and 16 you made a prediction. Which of these predictions do you think has a better chance of being correct? Write a few sentences explaining your reasoning.

The table shows statistics for the ten players of a college basketball team in its opening game of the season. Use the data to complete Exercises 18–20.

minutes played	32	30	25	21	21	18	16	15	12	10
points scored	22	15	11	8	6	4	8	10	4	6
fouls committed	4	3	5	2	3	2	2	1	0	1



Use computer software to make a scatter plot, draw the least-squares line, find its equation, and find the correlation coefficient for each set of data.

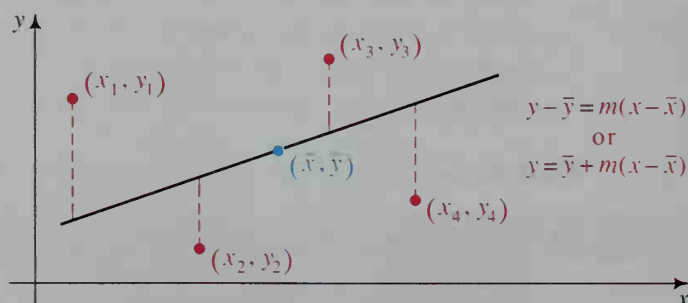
	$x$	$y$
18.	minutes	points
19.	minutes	fouls
20.	fouls	points



21. **Writing** Refer to the results of Exercise 20. Does the positive correlation between fouls committed and points scored suggest a cause-and-effect relationship between fouls and points? As a coach, would you advise playing more defensively and thus committing more fouls as a way of increasing your players' points scored? Write a short paragraph explaining your reasoning.



22. The diagram at the right shows a scatter plot of a set of data and a line with slope  $m$  that contains the point  $(\bar{x}, \bar{y})$ .



- a. For each point  $(x_i, y_i)$ , let  $d_i$  represent the vertical distance between the point and the line, that is,  $d_i = |y_i - \bar{y}|$ . Show that:

$$d_i = |(y_i - \bar{y}) - m(x_i - \bar{x})|$$

b. Let  $D = \sum_{i=1}^n d_i^2$ . Show that:

$$D = m^2 \sum_{i=1}^n (x_i - \bar{x})^2 - 2m \sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x}) + \sum_{i=1}^n (y_i - \bar{y})^2$$

c. Note that  $\bar{x}$ ,  $\bar{y}$ , and each  $x_i$ ,  $y_i$  are known, so that  $D$  is a quadratic function of  $m$ , the slope of the line. Show that  $D$  is minimized when

$$m = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}.$$

d. In part (c), divide the numerator and denominator by  $n$ , then expand the product  $(y_i - \bar{y})(x_i - \bar{x})$ , and show that the value of  $m$  that minimizes  $D$  is:

$$\frac{\overline{xy} - \bar{x} \cdot \bar{y}}{s_x^2}$$

*Hint:* Remember that  $\bar{x}$  and  $\bar{y}$  are constants. Thus,

$$\frac{\sum_{i=1}^n \bar{x}y_i}{n} = \frac{\bar{x} \sum_{i=1}^n y_i}{n} = \bar{x} \cdot \bar{y}.$$

## 18-2 Fitting Exponential Curves

### **Objective**

To find an exponential function that models certain sets of data.

In Section 18-1, we fitted *lines* to data. In this section and the next, we will fit *curves* to data. Knowing what equations and curves are good candidates for fitting a set of data takes experience. However, in this section, we will consider only the exponential curve,  $y = ab^x$ , which you studied in Chapter 5. This curve and a translation of it can be used to model exponential growth ( $b > 1$ ) and exponential decay ( $0 < b < 1$ ), including the following phenomena.

**Exponential Models,  $y = ab^x$**

1. Increasing costs due to inflation (See Example 1.)
2. Decreasing values due to depreciation (See Exercise 9.)
3. Population growth (See Exercises 10 and 11.)
4. Learning experiments (See Exercise 16.)
5. Performance in contests (See Exercise 17.)

One way to see if an exponential equation fits a set of data  $(x, y)$  is to plot the points  $(x, \log y)$ . If the points  $(x, \log y)$  lie on a line, then the original data points  $(x, y)$  lie on an exponential curve. The reasoning behind this method follows.

If  $y = ab^x$ , then

$$\log y = \log a + x \log b$$

$$\log y = (\log b)x + \log a$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \log y = & mx & + k \end{array}$$

Thus,  $\log y$  is a *linear* function of  $x$ .

Since the steps above are reversible, we have proved the following:

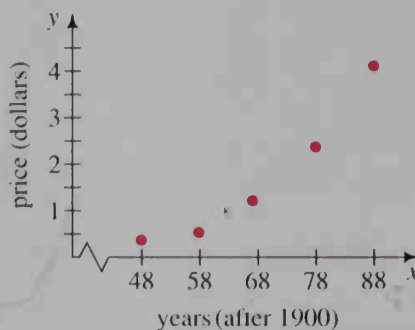
### Test for an Exponential Model

The points  $(x, y)$  lie on an exponential curve,  $y = ab^x$ , if and only if the points  $(x, \log y)$  lie on a line.

#### Example 1

The average price of an adult ticket to the movies in the United States has increased, as the entries in the table below show.

$x = \text{year}$	$y = \text{price}$
1948	\$0.36
1958	\$0.68
1967	\$1.22
1978	\$2.34
1988	\$4.11



- Use the movie price data above to plot the log of the price  $y$  versus the year  $x$ . Represent the years 1948, 1958, . . . as  $x = 0, 10, \dots$
- Fit a line to your plot in part (a) and find an equation of the line.
- Find an exponential model for the price data.
- Use the equation found in part (c) to predict the price of a movie ticket in the year 2000.

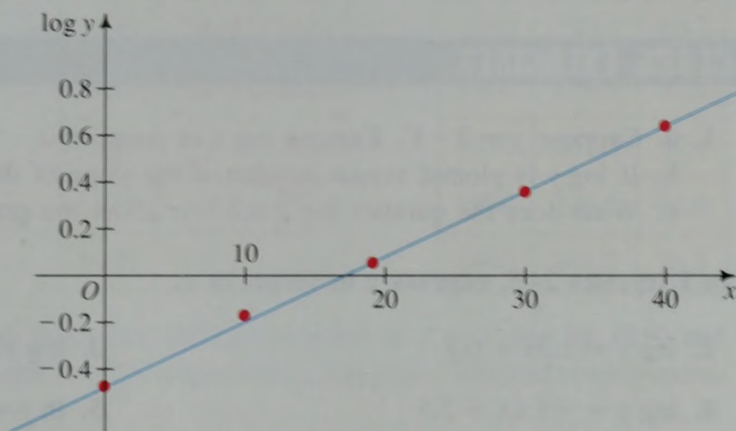




**Solution**

- a. Find the logarithm of each price.

$x$	0	10	19	30	40
$y$	0.36	0.68	1.22	2.34	4.11
$\log y$	-0.4437	-0.1675	0.0864	0.3692	0.6138



- b. Applying the least-squares method from Section 18-1, we obtain the following values for the slope and the vertical intercept of the least-squares line.

$$\text{slope} \approx 0.0265$$

$$\text{vertical intercept} \approx -0.4331$$

$$\text{Therefore, } \log y \approx 0.0265x - 0.4331.$$

- c. If
- $\log y \approx 0.0265x - 0.4331$
- , then:

$$y \approx 10^{0.0265x - 0.4331} = 10^{-0.4331} \cdot (10^{0.0265})^x$$

$$y \approx (0.369)1.063^x$$

- d. For the year 2000,
- $x = 52$
- and
- $y \approx (0.369)1.063^{52} \approx \$8.85$
- .

In the year 2000, a movie ticket will cost about \$9.00.

### Fitting an Exponential Curve $y = ab^x$ to a Set of Data

1. Compute  $(x, \log y)$  for each data point.
2. Fit a least-squares line to the set of data points in Step 1.
3. Convert the equation in Step 2 to give  $y$  as a function of  $x$ .

There are calculators and computer software that can give you the least-squares line needed in Step 2. Some will also carry out Step 3. If you do not have such a calculator or software available, proceed as shown in the next example.

**Example 2** Find  $y$  in terms of  $x$  if  $\log y = 2x + 3$ .

**Solution** If  $\log y = 2x + 3$ , then  $y = 10^{2x+3}$ . Therefore:

$$\begin{aligned}y &= 10^{2x} \cdot 10^3 \\y &= (10^2)^x \cdot 10^3 \\y &= (1000) \cdot 100^x\end{aligned}$$

## CLASS EXERCISES

1. a. Suppose  $y = 2 \cdot 3^x$ . Express  $\log y$  in terms of  $x$ .  
b. If  $\log y$  is plotted versus  $x$ , what is the slope of the resulting line?  
c. What does the number  $\log 2$  tell you about the graph?

In Exercises 2–5, express  $y$  in terms of  $x$ .

2.  $\log y = 1.3x + 0.8$

3.  $\log y = \frac{1}{2}x + 2$

4.  $\log y = -1.4x + 2.5$

5.  $\ln y = 1.5x + 4$

## WRITTEN EXERCISES

In Exercises 1–6, express  $y$  in terms of  $x$ .

**A**

1.  $\log y = 1.6x + 1.8$

2.  $\log y = 2.41x + 0.75$

3.  $\log y = -0.4x + 2.3$

4.  $\log y = -1.05x - 1.92$

5.  $\ln y = 0.5x + 3.2$

6.  $\ln y = 4.5 - 2x$

In Exercises 7 and 8, fit an exponential curve to the data.

7.

$x$	1	3	4	5
$y$	3.00	6.75	10.13	15.19

8.

$x$	2	2.5	4	5.5
$y$	3.63	3.81	4.39	5.07

9. **Consumer Economics** Each year after he bought his new car, Mr. Brown kept track of the market value of the car.

$x$ = year after purchase	1	2	3	4	5	6
$y$ = market value (in dollars)	12,000	9600	7700	6200	4900	3900

- a. Fit an exponential curve to the data.
- b. Predict the value of the car when it is 10 years old.
- c. Estimate the amount that Mr. Brown paid for the car.

The tables give the population  $P$  in millions and the population density  $D$  in people per square mile for the United States.

year	$P$	$D$
1790	3.9	4.5
1800	5.3	6.1
1810	7.2	4.3
1820	9.6	5.5
1830	12.9	7.4
1840	17.1	9.8
1850	23.2	7.9

year	$P$	$D$
1860	31.4	10.6
1870	39.8	13.4
1880	50.2	16.9
1890	62.9	21.2
1900	76.0	25.6
1910	92.0	31.0
1920	105.7	35.6

year	$P$	$D$
1930	122.8	41.2
1940	131.7	44.2
1950	150.7	50.7
1960	179.3	50.6
1970	203.3	57.4
1980	226.5	64.0
1990	248.7	70.3

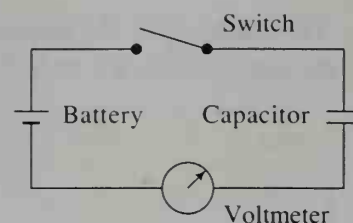
10. a. Plot  $\log P$  versus time. Let 1800 correspond to  $x = 0$ , and let 1810 and 1820 be  $x = 10$  and  $x = 20$ , respectively. The year 1790 will correspond to  $x = -10$ .  
 b. Fit a line to your plot in part (a) and find its equation.  
 c. **Geography** Use your equation in part (b) to find an equation giving population as an exponential function of  $x$ , the number of years since 1800. Use your equation to predict the United States population in the year 2000.
11. a. **Research** Even though the population of the United States increased between 1800 and 1810, the population density decreased because the United States acquired a vast land area when it made the Louisiana Purchase in 1803. How do you account for the other two decreases in population density shown in the table?  
 b. **Discussion** Assuming that the future population grows exponentially, can you assume that population density will also grow exponentially? Explain your answer.
12. **Astronomy** The observed brightness of stars is classified by magnitude. The table shows how the difference  $d$  in the magnitudes of two stars is related to the ratio of their brightness  $r$ . Comparing a first magnitude star with a sixth magnitude star, we have the magnitude difference  $d = 6 - 1 = 5$ , and we can see from the table that the first magnitude star appears 100 times as bright.

$d$	0	0.5	0.75	1.0	1.5	2	3	4	5	6	10	15
$r$	1	1.6	2	2.5	4	6.3	16	40	100	251	$10^4$	$10^6$

- a. By plotting  $\log r$  versus  $d$ , show that  $d \approx 2.5 \log r$ .
- b. Express  $r$  in terms of  $d$ .
- c. The difference in magnitudes of the brightness of the sun and the full moon is 14. What is their approximate brightness ratio?

13. **Electronics** An electrical circuit containing a capacitor has voltage 1.52 V. When the electrical circuit is broken, the capacitor gradually discharges. At various times  $t$  (in minutes) after the circuit is broken, the voltage  $V$  (in volts) is read on a voltmeter. The table below shows the data obtained.

$t$	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
$V$	1.52	1.25	1.01	0.82	0.67	0.54	0.45	0.37



- Plot  $(t, \log V)$ . Find an equation of the resulting straight line in the form  $\log V = mt + k$ .
- Solve the equation from part (a) for  $V$  in terms of  $t$ .

Exercises 14–17 concern exponential decay for which the graph has a horizontal asymptote  $y = c$ . An equation of such a curve has the form

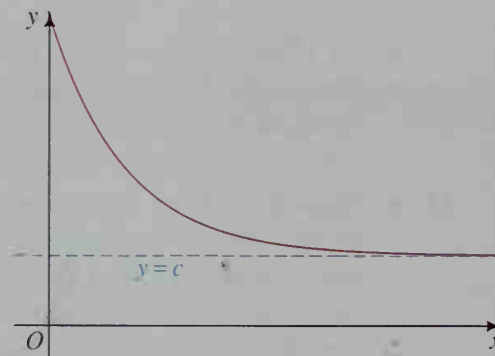
$$y - c = ab^x, \text{ where } 0 < b < 1.$$

To find values for  $a$ ,  $b$ , and  $c$ , proceed as follows:

Step 1 Plot the points  $(x, y)$  and visually find the value of  $c$ .

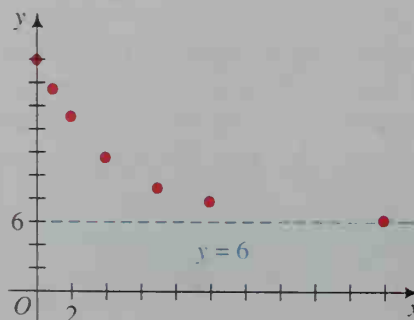
Step 2 Plot  $(x, \log(y - c))$ . Express  $\log(y - c)$  as a linear function of  $x$ .

Step 3 Solve the equation in Step 2 for  $y$ .



- B** 14. The scatter plot below suggests exponential decay with a horizontal asymptote  $y = 6$ .

$x$	$y$
0	20.0
1	17.2
2	15.0
4	11.7
7	9.0
10	7.5
20	6.2



- Plot  $\log(y - 6)$  versus  $x$  and find an equation of the resulting line.
- Find  $y$  as a function of  $x$ .



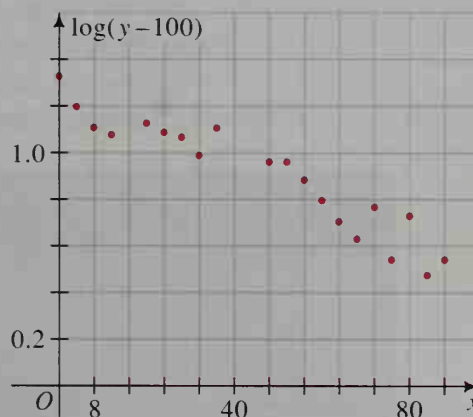
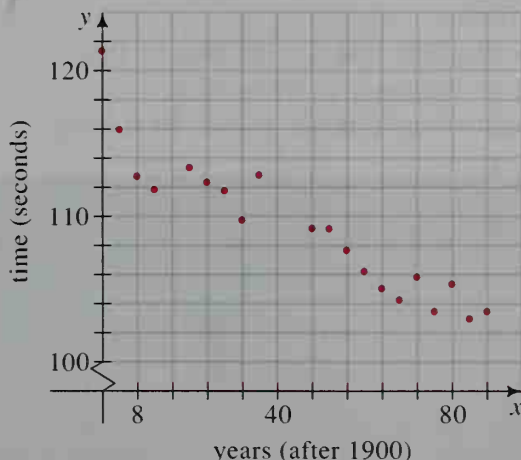
15. **Thermodynamics** A cup of hot tea just poured at  $158^{\circ}\text{F}$  slowly cools over time  $t$  (in minutes) and its temperature  $T$  is recorded, as shown in the table.

$t$	0	10	30	50	70	90	110	120	125	130
$T$	158	132.8	105.8	92.3	84.2	79.2	76.1	75	74.7	74.5

- Estimate the temperature of the room.
  - If the points  $(t, T)$  were plotted, what would be the equation of the horizontal asymptote?
  - Find an equation that models the data.
16. Each student in a typing class is tested at various times in the course and the average number of errors for the class is recorded.

$t$ = time of testing (days)	2	10	14	21	30	45	63	70	91
$y$ = avg. no. of errors	45.2	36.1	30.2	23.1	18.7	11.0	5.6	4.3	2.4

- Plot these points and connect them to make a “learning curve.” What is the equation of the horizontal asymptote?
  - Find an equation that models the data.
17. **Sports** The scatter plot at the left below shows the winning times  $y$  (in seconds) for the Olympic men’s 800-meter run. (Due to World War I and World War II, there were no Olympics in 1916, 1940, and 1944.) It suggests exponential decay with a horizontal asymptote at  $y = 100$ . The scatter plot at the right below shows the ordered pairs  $(x, \log(y - 100))$ .
- Imagine a line fitted to the data in the scatter plot on the left. What impossible event does the  $x$ -intercept suggest?
  - Fit a line by eye to the plot at the right below. Write an equation of this line. Then write an exponential equation that expresses  $y$  in terms of  $x$ .



18. **Visual Thinking** Suppose that  $y = ab^x$  and the points  $(x, \log y)$  are plotted. If a line through these points has negative slope, what can you say about  $b$ ?



## 18-3 Fitting Power Curves

**Objective** To fit a power curve to a set of data.

Sometimes data points  $(x, y)$  collected experimentally do not appear to lie on a line or an exponential curve. In such a case, the points may lie along a power curve,  $y = ax^m$ . Power curves occur often in geometric and scientific applications. Here are three illustrations.

1. The quadratic function  $y = ax^2$  is often used when  $y$  is the area of a figure, or a price based on the area, and  $x$  is a linear dimension of the figure. For example, the cost  $y$  of paving a square parking area  $x$  meters on a side is modeled by  $y = ax^2$  and we say that  $y$  varies as the square of  $x$ .
2. The cubic function  $y = ax^3$  is often used when  $y$  is the volume or weight of some object and  $x$  is a linear dimension of the object. In this case, we say that  $y$  varies as the cube of  $x$ .
3. The power function  $y = ax^{1.5}$  is used to model the periods of orbiting satellites and planets. (See Exercises 16 and 17.)

To see if a set of points  $(x, y)$  lies on a power curve, you can plot the points  $(\log x, \log y)$  and see whether they lie on a line. The following reasoning shows why the technique works.

If  $y = ax^m$ , then  $\log y = \log a + m \log x$ .

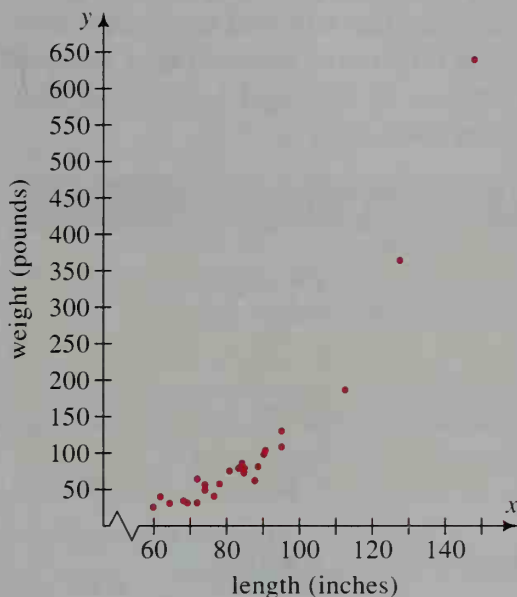
Since  $(\log y) = k + m(\log x)$ , it follows that  $(\log y)$  is a linear function of  $(\log x)$ .

### Example 1

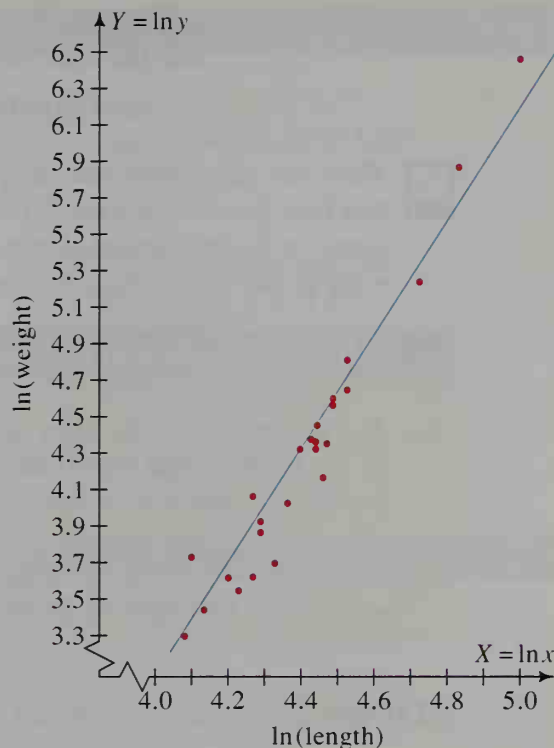
Two biologists hope to find an equation that models how an alligator's weight depends on its length. After measuring and weighing 25 alligators, they plot the points  $(x, y) = (\text{length}, \text{weight})$ . (See figure (a) at the top of the next page.) When they take natural logarithms of length and weight, the points  $(\ln x, \ln y)$  seem to have a linear relationship. (See figure (b) at the top of the next page.)

- a. Find an equation of a line that fits the data in figure (b).
- b. Use the equation to express  $y$  in terms of  $x$ .
- c. Estimate the weight of an alligator that is 10 ft long.





(a) weight versus length



(b)  $\ln(\text{weight})$  versus  $\ln(\text{length})$

### Solution

- a. First notice that the axes of the scatter plot in figure (b) do *not* intersect at the origin  $(0, 0)$ . The  $X = (\ln x)$ -axis begins at 4.0 and the  $Y = (\ln y)$ -axis begins at 3.3.

Fit a line by sight to the data. One such line is shown in blue. To find the slope of the line, use two points that are far apart. For example, the line appears to go through the points  $(4.1, 3.4)$  and  $(5.0, 6.2)$ , so we can approximate the slope as follows:

$$m \approx \frac{6.2 - 3.4}{5.0 - 4.1} = \frac{2.8}{0.9} \approx 3.1$$

Using the slope 3.1 and the point  $(5.0, 6.2)$ , we can write:

$$\frac{Y - 6.2}{X - 5.0} = 3.1$$

$$\text{or } Y = 3.1X - 9.3$$

$$\text{or } \ln y = 3.1 \ln x - 9.3$$

- b. If  $\ln y = 3.1 \ln x - 9.3$ , then  $y = e^{3.1 \ln x - 9.3}$ .

$$y = e^{3.1 \ln x} \cdot e^{-9.3}$$

$$= e^{\ln x^{3.1}} \cdot e^{-9.3} \quad \leftarrow 3.1 \ln x = \ln x^{3.1}$$

$$\approx x^{3.1} \cdot (0.000091) \quad \leftarrow e^{\ln N} = N$$

(Solution continues on the next page.)

- c. A 10 ft alligator is 120 in. long. Therefore, substitute  $x = 120$  into the equation of part (b).

$$y = 0.000091x^{3.1} = 0.000091(120)^{3.1} \approx 254 \text{ lb}$$



There are calculators and computer software that will find the power curve that best fits a set of data. To understand fully the process of fitting a power curve, it is recommended that you proceed as described below. You may wish to use a calculator or computer to check your answers.

### Fitting a Power Curve $y = ax^m$ to a Set of Data

1. Plot the points  $(X, Y) = (\log x, \log y)$  for each data point.
2. Fit a line to the set of data points obtained in Step 1.
3. Find an equation of this line:

$$Y = mX + k$$

$$\log y = m \log x + k$$

4. Convert the equation in Step 3 to give  $y$  as a function of  $x$ .

**Example 2** Find  $y$  in terms of  $x$  if  $\log y = 2.1 \log x + 1.5$ .

#### Solution

$$\log y = 2.1 \log x + 1.5$$

$$y = 10^{2.1(\log x) + 1.5}$$

$$= 10^{2.1(\log x)} \cdot 10^{1.5}$$

$$= 10^{\log x^{2.1}} \cdot 10^{1.5}$$

$$\approx (x^{2.1})(31.6)$$

$$y = 31.6x^{2.1}$$

#### Activity

The time  $t$  required for a pendulum to swing back and forth once is called the *period* of the pendulum.

1. Make two pendulums by hanging different weights from two strings of the same length. With a stopwatch, time ten back-and-forth swings of the pendulum with the lighter weight, and then divide the sum of the times by ten to get the period of the pendulum. Do the same for the other pendulum. Does it seem that a pendulum's period depends on the weight of the pendulum?
2. Vary the length  $l$  of a pendulum's string, measuring the length from the pivot point to the center of the pendulum weight. Find the period  $t$  of the pendulum for various values of  $l$  and make a table of ordered pairs  $(l, t)$ .
3. Plot  $\log t$  versus  $\log l$ , and use the resulting line to find an equation giving  $t$  in terms of  $l$ .



## CLASS EXERCISES

1. a. Suppose  $y = 4x^3$ . Express  $\log y$  in terms of  $\log x$ .  
b. If  $\log y$  is plotted versus  $\log x$ , what is the slope of the resulting line?

Express  $y$  in terms of  $x$ .

2.  $\log y = 5 \log x + 2$
3.  $\log y = 2.5 \log x + 3.5$
4.  $\ln y = -3 \ln x + 1$
5.  $\ln y = 2 \ln x + 0.5$
6. **Discussion** The cost  $y$  of a cheese pizza depends on the diameter  $x$  of the pizza. If  $y = ax^m$ , what do you think the value of  $m$  is? Explain.
7. **Discussion** The weight  $y$  of a great blue whale depends on its length  $x$ . If  $y = ax^m$ , what do you think the value of  $m$  is? Explain.

## WRITTEN EXERCISES

Express  $y$  in terms of  $x$ .

- A**
1.  $\log y = 4 \log x + 1$
  2.  $\log y = 2.5 \log x + 1.3$
  3.  $\log y = -\log x + 0.7$
  4.  $\log y = -2.6 \log x - 0.7$
  5.  $\ln y = 1.6 \ln x + 3.2$
  6.  $\ln y = -0.4 \ln x + 4.2$

Fit a power curve to the data in Exercises 7 and 8.

7.

$x$	2.0	3.0	5.0	6.0
$y$	4.0	13.5	62.5	108.0

8.

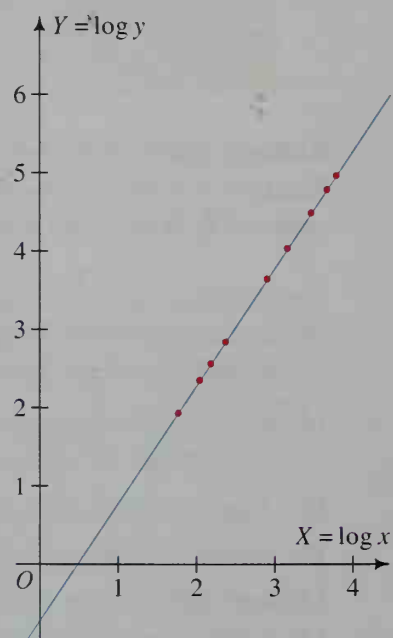
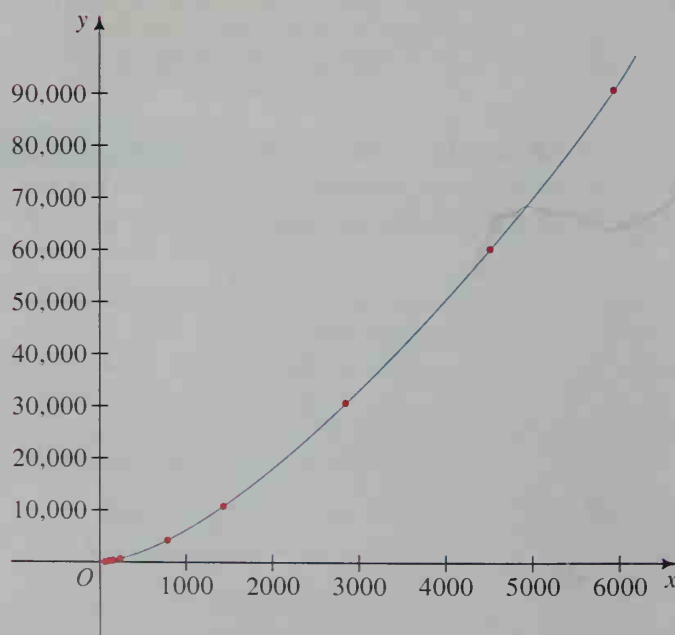
$x$	0.9	1.5	2.4	3.0
$y$	4.94	1.78	0.70	0.44

**Writing** Each of the following sets of data  $(x, y)$  can be modeled by a power function  $y = ax^m$ . Using what you know about geometry, conjecture what the value of  $m$  is in each case. Then write a few sentences justifying your answer.

$x$	$y$
9. a. length of bear footprint b. length of bear footprint	area of the footprint weight of the bear
10. a. diameter of a spherical tank b. diameter of a spherical tank	capacity of the tank amount of paint needed to paint it
11. diagonal of a television screen	area of the television screen
12. weight of a wooden cube	length of the cube's diagonal
13. cost of a circular rug	diameter of the rug
14. surface area of a sphere	volume of the sphere
15. weight of an elephant	surface area of the elephant's ear

- B 16. Astronomy** Johannes Kepler (1571–1630) spent years trying to discover a relationship between the time for a planet to revolve around the sun and the distance of the planet from the sun. A table and graph of the data for the nine planets are shown below. (Uranus, Neptune, and Pluto were not known in Kepler's time.) Also included is a graph of the logarithms of the data, which appears to be a straight line.

Planet	$x$ = avg. dist. from sun (millions of kilometers)	$y$ = time of orbit (days)
Mercury	57.9	88
Venus	108.2	225
Earth	149.6	365
Mars	227.9	687
Jupiter	778.3	4333
Saturn	1427	10,759
Uranus	2871	30,685
Neptune	4497	60,189
Pluto	5913	90,800





- Use the graph at the right on the preceding page to find a linear equation for  $Y$  in terms of  $X$ .
- Use your answer to part (a) to find an equation giving  $y$  in terms of  $x$ .
- Suppose that some time in the future a new planet is discovered and its average distance from the sun is estimated at 7000 million kilometers. Use your equation in part (b) to estimate its time of orbit around the sun.

17. **Astronomy** In 1610, Galileo discovered how the time,  $T$ , required for each of Jupiter's satellites to revolve about Jupiter is related to the average distance,  $a$ , of the satellite from Jupiter. By plotting  $\log T$  versus  $\log a$ , see if you can discover the relationship.

Satellite	$a$ (kilometers)	$T$ (hours)
Io	422,000	42.5
Europa	671,000	85.2
Ganymede	1,070,000	171.7
Callisto	1,883,000	400.5



18. **Sports** The winning times (in seconds) for various men's races (in meters) in the 1988 Olympics are given below.

$x$ = distance (m)	100	200	400	800	1500	5000	10,000
$y$ = time (s)	9.92	19.75	43.87	103.45	215.96	791.70	1641.46

- Plot  $\log y$  versus  $\log x$ . Find an equation of the resulting line.
  - Find an equation that gives  $y$  in terms of  $x$ .
  - If there had been a 1000-meter race in these Olympics, what would you predict the winning time would have been?
19. **Forestry** Forest managers keep records giving the number of board feet  $B$  (in hundreds of board feet) of lumber produced by harvested white pine trees with diameter  $d$  (in inches). Consider the following data for nine white pine.

$d$	15	16	17	18	19	20	21	22	23
$B$	18.68	22.85	27.61	33.00	39.06	45.84	53.38	61.72	70.90

- Plot  $\log B$  versus  $\log d$  and show that you get a line with slope about 3.
- Writing** Explain why part (a) shows that volume varies approximately with the cube of the diameter.
- Writing** Explain why the result of part (b) is not surprising if you think of a white pine as a cylinder-like object whose height  $h$  is proportional to the radius  $r$ .

20. **Sports** A statistician for a college basketball team keeps track of every shot the team's players attempt during games, and then computes for each player statistics that show how well the player shoots at various distances from the basket. The figures below are for the team's leading scorer, a 6'3" guard.

$D$ = dist. from basket (ft)	1–5	6–10	11–15	16–20	21–25	26–30	over 30
$P$ = % of shots made	70	43	30	23	19	17	15

- Make two plots: (1)  $\log P$  versus  $D$  and (2)  $\log P$  versus  $\log D$ . Use the middle value of each distance interval. Thus when  $D$  is in the interval from 1 to 5 ft, use  $D = 3$  and  $P = 70$ . For  $D$  over 30 ft, use  $D = 33$ .
  - Which of your plots seems to fit the data better? Use the better model to find an equation that gives  $P$  in terms of  $D$ .
  - Suppose the player whose statistics are given above attempts a shot from half-court (45 ft). How likely is it that the player will make the shot?
  - Discussion** Is it appropriate to use your model for half-court and full-court distances? Why or why not?
21. The table gives data on some famous roller coasters. Find a power function  $y = ax^m$  that gives the roller coaster's maximum speed ( $y$ ) as a function of its greatest single vertical drop ( $x$ ).

Roller coaster		$x$ = vertical drop (ft)	$y$ = max. speed (mi/h)
Texas Giant	Arlington, TX	137	62
The Beast	Cincinnati, OH	141	65
American Eagle	Gurnee, IL	147	66
Hercules	Allentown, PA	157	68

22. **Consumer Economics** The cost of making a plain pizza of diameter  $x$  can be modeled by an equation of the form  $y = ax^2 + k$ , where  $k$  is a constant that accounts for fixed costs, such as electricity for the ovens, rent, and so forth.
- Explain why the term  $ax^2$  makes sense.
  - Use the data below to write two equations in the unknowns  $a$  and  $k$ . Then solve for  $a$  and  $k$ . Give an equation for  $y$  in terms of  $x$ .

$x$ = diameter (in.)	10	15
$y$ = cost (dollars)	4	7

- Use the equation from part (b) to predict the cost of a 12 in. pizza.
23. **Research** Go to a pizza shop and find the costs of various sizes of plain pizza. Decide whether your data can be modeled by an equation like the one in Exercise 22(b).

## 18-4 Choosing the Best Model

### Objective

To choose the model that best describes a set of data.

The purpose of this section is to give you the opportunity to choose from among several models the one that best describes a set of data. Some of the most common models are the following:

1. Linear Function

Least-squares line  $y - \bar{y} = m(x - \bar{x})$

This line is discussed on page 684.

2. Exponential Function

$$y = ab^x$$

A list of exponential growth and decay phenomena is given on page 691.

3. Power Function

$$y = ax^m$$

Situations involving  $m = 2, 3$ , and  $1.5$  are discussed on page 698.

4. Functions with Asymptotic Behavior

Exponential:  $y - c = ab^x$

If, after having analyzed the problem, you're still not sure what model best fits a set of data, try one of the following strategies:

1. Graph the data and see what type of curve the graph suggests.
2. Graph  $(x, y)$ ,  $(x, \log y)$ , and  $(\log x, \log y)$  with a graphing calculator or computer and determine which looks most linear.

The equations above are not the only types of models used to describe real world situations. Difference equations and equations using matrices and calculus are also important. In many situations, a system of equations instead of a single equation must be applied. You should also know that there are many phenomena that at present seem to have no mathematical model at all. Where such phenomena seem to be chaotic in their behavior, some modeling progress has recently been made by using iteration of functions, a topic to be discussed in Chapter 19.

### MIXED EXERCISES

Model the data in each exercise. To do this, you must choose an appropriate model and then write an equation.

In Exercises 1–4, write an equation that models the data in part (a), then use your equation to answer part (b).

- A** 1. a. **Biology** A scientist caught, measured, weighed, and then released eight Maine landlocked salmon. The data were then recorded in a table.

$x = \text{length (in.)}$	5.5	10.6	15	17	19.6	22	25	28
$y = \text{weight (lb)}$	0.1	0.4	1.0	1.6	2.5	3.5	5.4	7.4

- b. Predict the weight of a salmon that is 12 in. long.

2. a. **Economics** The table below shows the average salaries (in thousands of dollars) of major league baseball players for selected years from 1967 to 1989.

year	1967	1976	1979	1982	1984	1986	1989
salary ( $\times \$1000$ )	19.0	51.5	114	241	329	413	497

- b. Estimate the average salary in the year 1970.
3. a. **Physics** The boiling point of water increases with atmospheric pressure. At sea level, where the atmospheric pressure is about 760 mm of Hg (mercury), water boils at  $100^\circ\text{C}$ . The table below shows the boiling point of water for different values of atmospheric pressure.

$P$ = pressure (mm of Hg)	149	234	355	526	760	1075
$T$ = boiling point ( $^\circ\text{C}$ )	60	70	80	90	100	110

- b. When the atmospheric pressure is 620 mm of Hg, at what temperature does water boil?
4. a. **Oceanography** Various depths (in meters) below the water surface and corresponding water pressures (in pounds per square inch) at those depths are given in the table below.

$d$ = depth (m)	10	20	30	40	50	60
$P$ = pressure (lb/in. <sup>2</sup> )	14.7	29.4	44.1	58.8	73.5	88.2

- b. Predict the water pressure at a depth of 13 m.
5. *The Guinness Book of World Records* (1990 edition) says that the world's largest pumpkin has a circumference of 11 ft  $11\frac{1}{4}$  in. and weighs 671 lb.

- a. **Reading** Which function would you choose to model a pumpkin's weight as a function of its circumference? Explain your answer.
- b. Use the given data to find an equation that gives pumpkin weight as a function of circumference.



- c. Predict the weight of a pumpkin whose *diameter* is 18 in.
- d. **Writing** In one or two sentences, explain why your prediction in part (c) might be a poor estimate of the actual weight of a pumpkin with diameter 18 in.

- B** 6. **Economics** The Consumer Price Index (CPI), published by the United States Bureau of Labor Statistics, compares the present cost of goods and services with their cost at another time. The period 1982–1984 is considered the base period and is assigned an index of 100. All other costs are expressed as a percent of the costs in the 1982–1984 period.

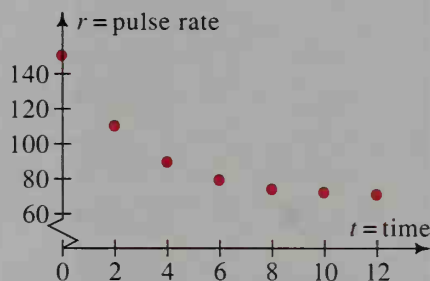
year	1945	1955	1965	1975	1985
CPI	18.0	26.8	31.5	53.8	107.6

- What model do you think will best fit the data? Explain.
  - Find an equation that models the data above.
  - Discussion** Use your equation to predict the CPI in the year 2000. What factors might affect the accuracy of your prediction?
7. **Economics** The gross national product (GNP) is the total sum spent on goods and services by a country in any year. The table below gives the GNP in billions of dollars for the United States for various years.

year	1900	1920	1940	1960	1980	1988
GNP	19.0	92.0	100	507	2732	4881

- Research** The GNP declined sharply in the early 1930s and rose dramatically in the early 1940s. To what historical events would you attribute these changes?
- Find an equation that models the data above.
- Use your equation to predict the GNP in the year 2000.

8. **Medicine** A doctor measures the pulse rate of a patient who has just completed vigorous exercise. The results shown in the graph at the right and the table below suggest that the graph of pulse rate versus time has a horizontal asymptote at  $r = 70$ .



$t$ = time after stopping (min)	0	2	4	6	8	10	12
$r$ = pulse rate (beats/min)	150	110	90	78	73	72	71

Find an equation that gives pulse rate in terms of the time elapsed since the exercise ended.



9. a. **Thermodynamics** The table shows the temperatures of a cold glass of water left to warm in a room. Study the data and give the approximate temperature of the room.

$t$ = time (min)	0	20	40	60	80	100	110	120
$T$ = temperature ( $^{\circ}\text{C}$ )	3.5	8.1	12.2	15.4	17.0	18.2	18.6	18.9

- b. Plot  $T$  versus  $t$  and find an equation of the horizontal asymptote.  
 c. Plot  $\log(19 - T)$  versus  $t$  and find an equation of the resulting line.  
 d. Use your answer to part (c) to find  $T$  in terms of  $t$ .
10. **Manufacturing** Last year, a chair manufacturer marketed a new style of chair. As the manufacturer produced more and more chairs, the production cost per chair decreased as shown below.

$x$ = no. of chairs	50	300	800	2000	3000
$y$ = cost per chair	\$517	\$387	\$250	\$162	\$152

- a. Study the data. What seems to be the limit of the manufacturing cost per chair as the number of chairs becomes very large?  
 b. Find an equation that models the data.  
 c. Predict the cost per chair when 500 chairs have been produced.
11. **Sports** The table gives the average speed of a runner for various distances.

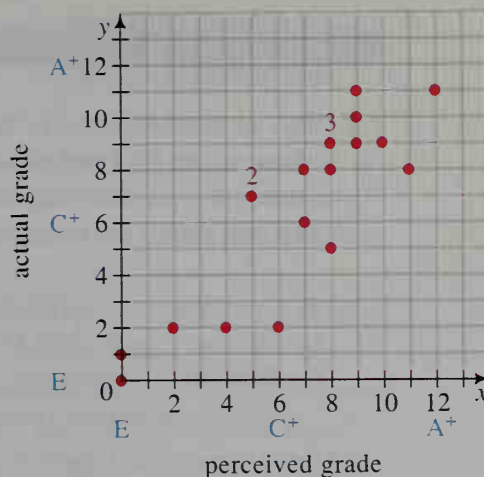
$x$ = distance (m)	$y$ = speed (m/s)
100	8.7
200	8.6
400	7.7
800	6.6
1500	6.1
3000	5.7

If the data were plotted, the line  $y = 5$  would be an approximate asymptote. Find an equation that models the data.



12. **Research** Use an almanac or other reference book to research two data sets that you think might be linearly related. Make a scatter plot and fit a line to the data. Finally, use the line to explore a question that the data might suggest.

13. An English teacher returns students' papers with many comments, but never a letter grade. The data shown in the scatter plot correlate the grade each student *thought* he or she had received with the actual grade that the teacher put in the grade book. The numbers from 0 to 12 represent letter grades from E to A<sup>+</sup>. A (2) or (3) above a point indicates that 2 or 3 students, respectively, have the same data point.



- According to the scatter plot, how many students *thought* they got a B on their papers? Find the average (mean) grade that these students *actually* received.
  - Find an equation of the least-squares line that fits the data. Use this equation to predict an actual grade for a student who thinks she got a B. Compare this with your answer to the second question of part (a).
14. **Research** From reference books, magazines, or newspapers, collect data that can be represented as ordered pairs. Plot the data, fit a curve to the data, and write a report that explains what you have done and summarizes your findings.

## Chapter Summary

- The process of finding an equation of a curve that fits observed data is called *curve fitting*. The equation is said to *model* the data.
- A *scatter plot* provides a visual picture of any relationship between two variables. The line that best fits a set of data points is called the *least-squares line*. The *correlation coefficient* is a number between  $-1$  and  $1$  that measures how closely the observed points tend to cluster about the least-squares line.
- The points  $(x, y)$  lie on an *exponential curve*,  $y = ab^x$ , if and only if the points  $(x, \log y)$  lie on a line. Exponential growth and decay phenomena are modeled by exponential functions. An exponential model with a horizontal asymptote has the form  $y - c = ab^x$ .
- The points  $(x, y)$  lie on a *power curve*,  $y = ax^m$ , if and only if the points  $(\log x, \log y)$  lie on a line. Power curves occur often in geometric and scientific applications.

### Key vocabulary and ideas

curve fitting (p. 683)

model (p. 683)

scatter plot (p. 683)

least-squares line (p. 684)

correlation coefficient (pp. 685–686)

exponential curve  $y = ab^x$  (p. 691)

fitting an exponential curve (p. 693)

exponential model  $y - c = ab^x$  (p. 696)

power curve  $y = ax^m$  (p. 698)

fitting a power curve (p. 700)

## Chapter Test

1. The weather bureau for a certain city correlated the average annual temperature ( $^{\circ}\text{F}$ ) and annual precipitation (in.) for the last 40 years. Letting  $x$  represent temperature and  $y$  represent precipitation, the statisticians found the following:  $\bar{x} = 55$ ,  $\bar{y} = 40$ ,  $\bar{xy} = 2240$ ,  $s_x = 7$ , and  $s_y = 8$ . 18-1
  - a. Calculate the correlation coefficient.
  - b. Find an equation of the least-squares line.
  - c. **Writing** Evaluate the usefulness of the least-squares line in predicting the annual precipitation of the city for a given temperature.
2. Let  $x$  represent a student's score on the SAT portion of the College Entrance Exams and let  $y$  represent that person's score on the Achievement portion of the Exams. If  $x$  and  $y$  have a high correlation, can you infer that a high score on the SAT's causes a high score on the Achievements? Explain.
3. Express  $y$  in terms of  $x$  if  $\log y = 3x + 2$ . 18-2
4. The table gives the per capita personal income  $y$ , in dollars, for various years,  $x$ .

$x$	1935	1945	1955	1965	1975	1985
$y$	474	1223	1881	2773	5851	13,896

- a. Fit an exponential curve to the data.
  - b. Predict the per capita personal income in 1995.
5. Express  $y$  in terms of  $x$  if  $\log y = 1.7 \log x - 2$ . 18-3
  6. **Writing** Let  $x$  represent the length of a diagonal of a square rug and let  $y$  represent the cost of the rug. The data  $(x, y)$  can be modeled by a power function  $y = ax^m$ . Using what you know about geometry, conjecture what the value of  $m$  is. Then write a few sentences justifying your answer.
  7. The following table gives the length of the year,  $y$ , in Earth-days for three planets, and the mean distance,  $d$ , in miles, of these planets from the sun. 18-4

	$d$ (in miles)	$y$ (in Earth-days)
Mercury	36,000,000	88
Venus	67,250,000	225
Mars	141,700,000	687

- a. Find an equation that gives  $y$  in terms of  $d$ .
- b. Use your equation to predict the distance of Earth from the sun.

1. An analysis of two test items revealed that 62 of 100 students responded correctly to the first item, while 48 responded correctly to the second. If 37 students got both items correct, how many students missed both questions?
2. How many different sequences can be formed using one consonant followed by one vowel (a, e, i, o, u) followed by a different consonant?
3. To get an A on her report, Kyoko knows her teacher expects her to use at least 4 sources. In how many ways can she choose 4 books from a list of 9 books?
4. In how many ways can a judge choose a first-place winner, a second-place winner, and a third-place winner from 9 entries submitted in an art show?
5. How many permutations are there of the letters of the word BANANA?
6. Give the binomial expansion of  $\left(2x - \frac{1}{2}\right)^4$ .
7. Suppose a card is drawn from a well-shuffled standard deck of 52 cards. Find the probability of drawing each of the following:
  - a. a diamond                      b. not a diamond                      c. a red club
  - d. not a red club                      e. a 10                      f. an ace or a king
8. Three cards are randomly chosen from a well-shuffled standard deck.
  - a. What is the probability that all 3 cards are spades?
  - b. Consider the events “at least one ace is drawn,” “all cards are black,” and “none of the cards is the ace of spades.” Which events are independent?
9. What is the probability of getting exactly 2 threes in 5 rolls of a die?
10. Given that the first card randomly drawn from a standard deck is a diamond, what is the probability that the second card randomly drawn is also a diamond? a spade?
11. In a manufacturing plant, Assembly Line A produces 60% of the plant’s output and Assembly Line B produces the rest. Four percent of Line A’s output is defective, and seven percent of Line B’s output is defective.
  - a. What percent of the plant’s output is defective?
  - b. What is the probability that a randomly selected item found to be defective is from Line A?
12. On a game show, a contestant spins a giant wheel that is divided into 10 equal segments, 4 marked \$50, 3 marked \$100, 2 marked \$500, and 1 marked \$1000. What is the expected payoff of a spin?
13. The table below shows the number of customers of Sheffield Village (the frequency) making telephone payments on different days of the month (the items of data). Draw a histogram for this data. Then find the mean, median, and mode.

Day of month	1	2	3	4	5	6	7	8	9	10
No. of customers	9	3	0	2	5	9	5	9	8	15

14. The age at which each of the United States presidents from George Washington to George Bush began his term is listed below. Draw (a) a stem-and-leaf plot and (b) a box-and-whisker plot of the data.
- 57, 61, 57, 57, 58, 57, 61, 54, 68, 51, 49, 64, 50, 48, 65, 52, 56, 46, 54, 49, 51, 47, 55, 55, 54, 42, 51, 56, 55, 51, 54, 51, 60, 62, 43, 55, 56, 61, 52, 69, 64
15. Compute the mean and standard deviation for the following set of data: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100
16. The number of chocolate chips in each cookie sampled from a bag of cookies is approximately normally distributed with a mean of 25 and a standard deviation of 4. Find the approximate percent of cookies with the number of chocolate chips being:
- a. fewer than 18                      b. at least 20                      c. between 22 and 27
17. A polling organization is trying to determine the level of support for a tax measure. Describe any errors that might occur when the organization randomly calls 100 private homes during business hours.
18. Of 1000 university students surveyed, 430 said they would use the library for studying during final exam week. Find a 99% confidence interval for  $p$ , the true proportion of all university students who would use the library.
19. Consider this set of data:  $(x, y) = (0, 12), (1, 10), (2, 6), (3, 6),$  and  $(4, 2)$
- a. Draw a scatter plot of the data, and find the correlation coefficient.
- b. Find an equation of the least squares line, and draw the line on your scatter plot. You may find a calculator helpful.
20. Tell whether the relationships between the two quantities would most likely be modeled by a linear function, an exponential function, a power function, or a function with asymptotic behavior. Give reasons for your answers.
- a. The diameter of a pipe and the volume of the pipe.
- b. The cost of a calculator and the sales tax on the calculator.
- c. The amount of time a tea kettle has been heated and the temperature of the water in the tea kettle.
21. Fit a power curve to the data in the following table.

$x$	1.5	2.0	2.5	3.0
$y$	6.7	15.9	31.2	54.2

22. The following table gives the population size,  $P$ , of bacteria at various regular time intervals,  $t$ .

$t$	0	1	2	3	4	5	6	7
$P$	1	2	4	8	12	21	37	61

- a. What model do you think will best fit the data? Explain.
- b. Find an equation that models the data above.
- c. Predict the population size when  $t = 10$ .



# Careers in Genetics and Statistics

The twentieth century has seen unprecedented progress in our ability to understand, control, and cure disease. Many fields of study contribute to medical research, and the methods of data analysis play an important role.

## GENETICS

Genetics, the study of heredity, encompasses a number of areas of research. Examples include population genetics, which uses statistical analysis to find patterns in gene distribution and the occurrence of genetic diseases; microbial genetics, which explores the process of mutation by studying rapidly multiplying microorganisms; and cytogenetics, which looks at hereditary activity at the cellular level.

**Agnes Stroud-Lee**, a University of Chicago Ph.D., has done research in cytogenetics at the Los Alamos National Laboratory. By observing the effects of radiation and toxic chemicals on chromosomes, she sought clues to abnormalities in genetic material. Her results have improved our understanding of birth defects and cancer. "I have made inroads," she says, "... other scientists can use my work to further their pursuit of science." Ms. Stroud-Lee retired in 1979.



*Agnes Stroud-Lee*

## STATISTICS

Statisticians are experts in the design of experiments, surveys, and questionnaires used to generate accurate and useful data, and in the meaningful analysis of data. Their skills are indispensable to medical and health research.

The National Center for Health Statistics is an important source of information for researchers and policymakers. NCHS analysis of data reveals connections between such varied health factors as socioeconomic status, type of health care used, nutrition, exposure to risks, and age.

**Joe Fred Gonzalez, Jr.**, a mathematical statistician for NCHS whose work includes developing estimation procedures and national sample survey designs, always found math "challenging and intellectually stimulating." His desire to understand applications of mathematics led him to an M.S. degree in statistics from the George Washington University. Believing in the value of education, Mr. Gonzalez is also a part-time university and college professor.



*Joe Fred Gonzalez, Jr.*

## Preference Testing

French fries or baked? Milk or cream? Country-western or rock-and-roll? Vanilla or chocolate? Most everyone at some time has been asked to state a preference. In fact, the study of people's preferences plays an important part in decision-making. Marketing strategies, for instance, often rely heavily on determining consumer likes and dislikes. To get an idea of how this might be done, let's explore one of the simplest techniques for evaluating preferences.

### Materials:

a random number generator

a graphing calculator or computer software

### The Sign Test

How can you quantify an opinion? If you have ever participated in a survey, you probably have seen a scale like the one shown below.

strongly dislike = 1    dislike = 2    neutral = 3    like = 4    strongly like = 5

To determine whether most people prefer one item over another, you need a way of comparing people's responses. Here is one method. Suppose that *everyone* in a given population rated two items of interest using the scale above. A small portion of the results might look like this:

Person	Item #1 rating	Item #2 rating	Item #1 – Item #2	Sign of the difference
1	4	3	1	+
2	4	5	-1	-
3	3	3	0	omit data
4	5	2	3	+

Notice that when there is a tie, we omit the piece of data. Now, let  $p$  denote the population proportion of + signs out of the total number of signs. If  $p > 0.5$ , then the population must have rated Item #1 higher than Item #2. Why? However, as you learned in Chapter 17, using a random sample to represent a population is often far more reasonable than using the entire population. For this reason, statisticians have developed a rule known as the *sign test*. We can use a simplified version of the sign test to determine whether people significantly prefer Item #1. First, find the *critical value*  $z$  using the following formula.

$$z = \frac{2x - n}{\sqrt{n}},$$

where  $n$  is the number of pairs left in the sample *after discarding ties*, and  $x$  is the number of + signs in the sample. For  $n \geq 10$ , we can say that the following is true.

If  $z > -1.645$ , then  $p > 0.5$ . Thus, the population probably prefers Item #1. If  $z \leq -1.645$ , then  $p \leq 0.5$ . Thus, the population probably does not prefer Item #1.

If  $n < 10$ , then the sample size is too small for the test to be valid. Are you ready to give the sign test a try?

### Conduct Your Survey.

- An essential first step is to prepare a well-designed study. Think of two ratable things that you'd like to compare, and label them Item #1 and Item #2. The items needn't necessarily be commodities. They could be, for example, two viewpoints, such as attitudes towards an issue before and after a debate. Order your rating scale carefully according to your choice of topics.
- Next, describe the population that you wish to survey. If at all possible, use one of the methods suggested in Section 17-5 to select a simple random sample. Otherwise, choose a sample that you believe to be random. Instruct the people in your sample to rate each of the two items with the scale that you've developed. Be sure that each person responds independently. After you've collected all of your data, calculate the value of  $n$ . You may need to add a few more people to your sample so that  $n \geq 10$ .
- Illustrate your data by graphing on the same set of axes the ordered pairs

$$(x, y) = (\text{Item \#1 rating}, \text{Item \#2 rating})$$

and the line  $y = x$ . Notice that data points below the line  $y = x$  indicate a preference for Item #1.

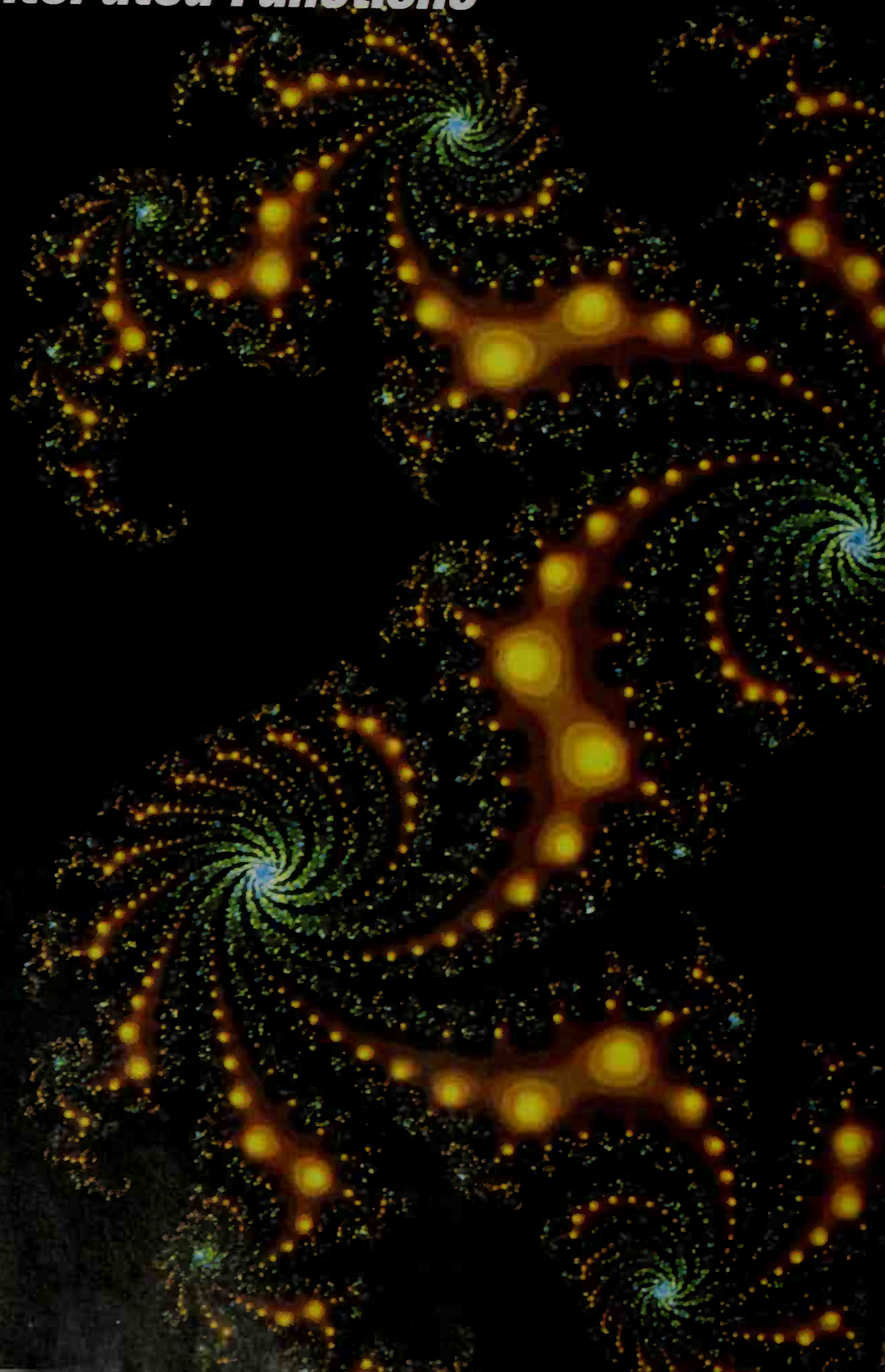
- Use the sign test to determine whether your population significantly prefers Item #1. Based on your choice of a population, why do you think your results turned out as they did? Compare the outcome with the actual sample proportion  $\frac{x}{n}$ . Does there seem to be any conflicting information? Remember that even though a preference may exist in a sample, it may not be statistically significant enough to represent the population.

### Report Your Results.

Any well-written statistical report should contain a section of summary statistics. For example, supply as much demographic information (mean age, number of males and females, years of education completed, and so forth) as possible about your sample. You should also provide a tally of ratings along with a table listing each item's rating and the corresponding sign of the difference between the ratings. Don't forget to include your graph as part of your data description. Explain why your choice of a scale fits the items you choose to rate. Justify your method of sample selection. State your values of  $x$ ,  $n$ , and  $z$ , and cite the sign test rule you followed to obtain your conclusion. Finally, use your demographic data to help interpret your results.

# 19

## *Limits, Series, and Iterated Functions*





# Limits

## 19-1 Limits of Functions

### Objective

To find the limit of a function or the quotient of two functions and to determine whether a function is continuous.

### The Limit as $x$ Approaches $\infty$ or $-\infty$

In Chapter 13 we discussed limits of infinite sequences and saw, for example, that the sequence

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}, \dots$$

has limit 1. We expressed this fact by writing:

$$(1) \quad \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$$

Likewise, for the function  $f(x) = \frac{x+1}{x}$ , we write

$$(2) \quad \lim_{x \rightarrow \infty} \frac{x+1}{x} = 1$$

and say “the limit of  $\frac{x+1}{x}$  as  $x$  approaches infinity is 1.” The only difference between statements (1) and (2) is that the sequence  $t_n = \frac{n+1}{n}$  is defined only for positive integers  $n$ , whereas the function  $f(x) = \frac{x+1}{x}$  is defined for all real numbers  $x \neq 0$ .

Because  $f(x)$  is defined for negative as well as positive numbers, we can also talk about “the limit of  $f(x)$  as  $x$  approaches negative infinity.” For the case in which  $f(x) = \frac{x+1}{x}$ , this limit is again 1, and we write:

$$\lim_{x \rightarrow -\infty} \frac{x+1}{x} = 1$$

These symbols mean that  $\frac{x+1}{x}$  can be made as close to 1 as we like just by considering negative values of  $x$  with large enough absolute value. For example, when  $x = -1,000,000$ ,

$$\frac{x+1}{x} = \frac{-999,999}{-1,000,000} \approx 1.$$

◀ The *Julia set* shown at the left is produced by iterating the complex polynomial function  $f(z) = z^2 + 0.32 + 0.043i$ . In a Julia set, as in all *fractals*, the larger patterns repeat at smaller levels of scale.



A function  $f$  for which  $|f(x)|$  becomes arbitrarily large as  $x$  approaches  $\infty$  or  $-\infty$  has no finite limit. It instead has an infinite limit, either  $\infty$  or  $-\infty$ . Although we say that an infinite limit exists, we recognize that  $\infty$  and  $-\infty$  are not numbers, and so we must define finite and infinite limits differently. Formal definitions of these limits are given on pages 722 and 723. The following examples, however, should help to clarify the concept of an infinite limit.

### Example 1

Evaluate: a.  $\lim_{x \rightarrow \infty} x^{1/3}$

b.  $\lim_{x \rightarrow -\infty} x^{1/3}$

### Solution

a. Since the value of  $x^{1/3}$  becomes arbitrarily large as  $x$  becomes arbitrarily large,  $\lim_{x \rightarrow \infty} x^{1/3} = \infty$ .

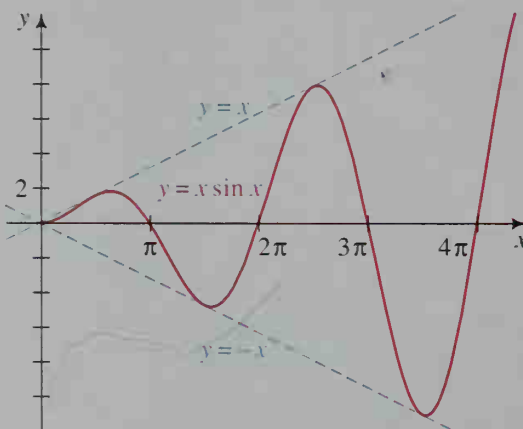
b. When  $x$  is negative, the value of  $x^{1/3}$  is negative. As  $x$  approaches negative infinity,  $|x^{1/3}|$  becomes arbitrarily large. Thus,  $\lim_{x \rightarrow -\infty} x^{1/3} = -\infty$ .

### Example 2

Explain why  $\lim_{x \rightarrow \infty} x \sin x \neq \infty$ .

### Solution

Since the value of the sine function oscillates between 1 and  $-1$ , the graph of  $y = x \sin x$  oscillates between the lines  $y = x$  and  $y = -x$  (see below). In particular, as  $x$  approaches infinity, the graph of  $y = x \sin x$  crosses the  $x$ -axis infinitely often, so there is no value of  $x$  beyond which the value of  $x \sin x$  becomes—and stays—arbitrarily large. Thus,  $\lim_{x \rightarrow \infty} x \sin x \neq \infty$ .



## The Limit as $x$ Approaches a Real Number $c$

To determine the behavior of a function  $f(x)$  as  $x$  approaches a real number  $c$ , we consider the following two limits:

- (1)  $\lim_{x \rightarrow c^+} f(x)$ , read “the limit of  $f(x)$  as  $x$  approaches  $c$  from the right”
- (2)  $\lim_{x \rightarrow c^-} f(x)$ , read “the limit of  $f(x)$  as  $x$  approaches  $c$  from the left”

The meaning of limits of this type is discussed in the next two examples.

**Example 3**

Using the graph of  $f(x)$  shown below, find  $\lim_{x \rightarrow 2^+} f(x)$  and  $\lim_{x \rightarrow 2^-} f(x)$ .

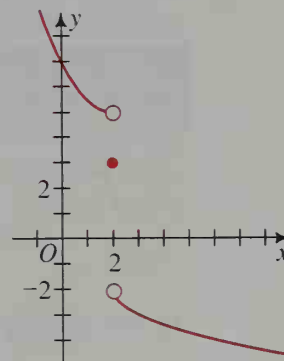
**Solution**

The fact that  $f(2) = 3$  has nothing to do with the solution. To evaluate  $\lim_{x \rightarrow 2^+} f(x)$ , we consider the value of  $f(x)$  for  $x$  near, but greater than, 2. Since the value of  $f(x)$  gets closer and closer to  $-2$  as  $x$  approaches 2 from the right,

$$\lim_{x \rightarrow 2^+} f(x) = -2.$$

Also, since the value of  $f(x)$  gets closer and closer to 5 as  $x$  approaches 2 from the left,

$$\lim_{x \rightarrow 2^-} f(x) = 5.$$

**Example 4**

If  $f(x) = \frac{x^2 - 4}{x - 2}$ , describe the behavior of  $f(x)$  near  $x = 2$ .

**Solution**

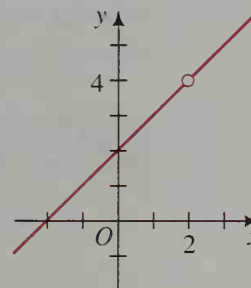
The fact that  $f(2)$  is undefined has nothing to do with the solution. We wish to determine what happens to  $f(x)$  as  $x$  gets closer to 2. Substituting values of  $x$  near 2 gives the values of  $f(x)$  shown below. These values suggest that  $f(x)$  approaches 4 as  $x$  approaches 2 from the right or the left.

$f(2.1) = 4.1$	$f(1.9) = 3.9$
$f(2.01) = 4.01$	$f(1.99) = 3.99$
$f(2.001) = 4.001$	$f(1.999) = 3.999$
$\downarrow$	$\downarrow$
$\lim_{x \rightarrow 2^+} f(x) = 4$	$\lim_{x \rightarrow 2^-} f(x) = 4$

We can confirm these limits by examining the graph of  $f(x)$  near the point where  $x = 2$ . The graph is easier to sketch if we first reduce  $f(x)$ :

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$$

The graph of  $f(x)$  is a line; however, since  $x = 2$  is not in the domain of  $f$ , the graph has a “hole” at the point  $(2, 4)$ . Be aware that a graphing calculator display might not show the hole.



Notice in Example 4 that  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = 4$ . In this case we can speak of “the limit of  $f(x)$  as  $x$  approaches 2” and write  $\lim_{x \rightarrow 2} f(x) = 4$ . In Example 3, however,  $\lim_{x \rightarrow 2^+} f(x) = -2$  and  $\lim_{x \rightarrow 2^-} f(x) = 5$ . Since the right-hand and left-hand limits are different,  $\lim_{x \rightarrow 2} f(x)$  does not exist. In general:

$\lim_{x \rightarrow c} f(x)$  exists if and only if  $\lim_{x \rightarrow c^+} f(x)$  and  $\lim_{x \rightarrow c^-} f(x)$  exist and agree.

## Continuous Functions

The easiest limits to evaluate often involve a *continuous function*. Roughly speaking, a function is continuous if you can draw its graph without lifting your pencil from the paper. For example, any polynomial function, such as the cubic function shown at the right, is continuous.

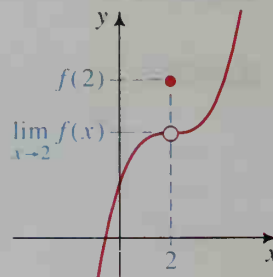
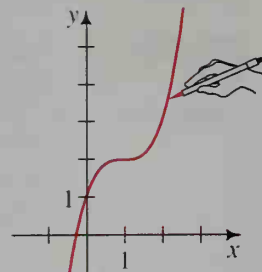
Formally speaking, a function  $f(x)$  is **continuous** at a real number  $c$  if:

$$\lim_{x \rightarrow c} f(x) = f(c)$$

In this definition, note that there are *three* conditions for continuity at  $x = c$ :

1.  $\lim_{x \rightarrow c} f(x)$  must exist.
2.  $f(c)$  must exist.
3. (1) and (2) must be equal.

A function can fail to be continuous at  $x = c$  in different ways. The function of Example 3 on page 719 is discontinuous at  $x = 2$  because  $\lim_{x \rightarrow 2} f(x)$  does not exist. The function of Example 4 on page 719 is discontinuous at  $x = 2$  because  $f(2)$  does not exist. The function graphed at the right is discontinuous at  $x = 2$  because, although both  $\lim_{x \rightarrow 2} f(x)$  and  $f(2)$  exist,  $\lim_{x \rightarrow 2} f(x) \neq f(2)$ .



## The Limit of a Quotient of Two Functions

The following theorem is useful for limit problems that involve a quotient of two functions. (A proof is difficult and will not be given.) Since the theorem applies whether we consider the limit as  $x$  approaches  $\infty$ ,  $-\infty$ , or a real number  $c$ , the theorem uses the general notation “lim” instead of  $\lim_{x \rightarrow \infty}$ ,  $\lim_{x \rightarrow -\infty}$ , or  $\lim_{x \rightarrow c}$ .

### Quotient Theorem for Limits

If  $\lim f(x)$  and  $\lim g(x)$  both exist, and  $\lim g(x) \neq 0$ , then:

$$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$$

#### Example 5

Evaluate  $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}}{x^2-1}$ .

#### Solution

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1}}{x^2-1} = \frac{\lim_{x \rightarrow 3} \sqrt{x+1}}{\lim_{x \rightarrow 3} (x^2-1)} = \frac{\sqrt{4}}{8} = \frac{1}{4}$$

Evaluating  $\lim \frac{f(x)}{g(x)}$  is not always as simple as Example 5 suggests. Sometimes we need to use special techniques, described below.

### Techniques for Evaluating $\lim \frac{f(x)}{g(x)}$

1. If possible, use the quotient theorem for limits.
2. If  $\lim f(x) = 0$  and  $\lim g(x) = 0$ , try the following techniques.
  - a. Factor  $g(x)$  and  $f(x)$  and reduce  $\frac{f(x)}{g(x)}$  to lowest terms.
  - b. If  $f(x)$  or  $g(x)$  involves a square root, try multiplying both  $f(x)$  and  $g(x)$  by the conjugate of the square root expression.
3. If  $\lim f(x) \neq 0$  and  $\lim g(x) = 0$ , then either statement (a) or (b) below is true.
  - a.  $\lim \frac{f(x)}{g(x)}$  does not exist.
  - b.  $\lim \frac{f(x)}{g(x)} = \infty$  or  $\lim \frac{f(x)}{g(x)} = -\infty$ .
4. If  $x$  is approaching infinity or negative infinity, divide the numerator and the denominator by the highest power of  $x$  in the denominator.
5. If all else fails, you can guess  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  by evaluating  $\frac{f(x)}{g(x)}$  for very large values of  $x$ , and you can guess  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  by evaluating  $\frac{f(x)}{g(x)}$  for  $x$ -values very near  $x = c$ . These limits can also be guessed by using a graphing calculator to examine the graph of  $y = \frac{f(x)}{g(x)}$  for very large values of  $x$ , or for  $x$ -values very near  $x = c$ . (Remember: A graphing calculator might not show points of discontinuity.)

**Example 6** Evaluate: **a.**  $\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - 1}$       **b.**  $\lim_{x \rightarrow 0} \frac{1 - \sqrt{1+x}}{x}$

#### Solution

**a.** Using technique (2a), we have:

$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{x+3}{x+1} = \frac{4}{2} = 2$$

**b.** Using technique (2b), we have:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \sqrt{1+x}}{x} &= \lim_{x \rightarrow 0} \frac{1 - \sqrt{1+x}}{x} \cdot \frac{1 + \sqrt{1+x}}{1 + \sqrt{1+x}} \\ &= \lim_{x \rightarrow 0} \frac{-x}{x(1 + \sqrt{1+x})} = \lim_{x \rightarrow 0} \frac{-1}{1 + \sqrt{1+x}} = -\frac{1}{2} \end{aligned}$$



The techniques for evaluating limits described in part (5) of the chart on page 721 are also useful for *confirming* limits found using the algebraic methods described in parts (1)–(4). In the examples that follow, a graphing calculator is used both to help evaluate a limit (Example 7(b)), and to confirm a limit found algebraically (Example 8).

### Example 7

Evaluate: a.  $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$       b.  $\lim_{x \rightarrow 1} \frac{1}{x-1}$

### Solution

In each case we recognize that technique (3) applies. In addition, we can use the methods suggested in technique (5) to help evaluate these limits.

a. As  $x$  approaches 1, the values of  $(x-1)^2$  are positive and approach 0 so that the reciprocals of these values approach infinity. For example:

$$\text{When } x = 0.999, \text{ we have } \frac{1}{(x-1)^2} = \frac{1}{(-0.001)^2} = 1,000,000.$$

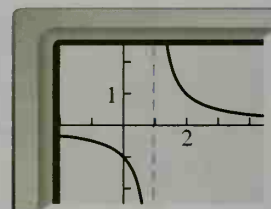
$$\text{When } x = 1.001, \text{ we have } \frac{1}{(x-1)^2} = \frac{1}{(0.001)^2} = 1,000,000.$$

$$\text{Thus, } \lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty.$$

b. A graph of the equation  $y = \frac{1}{x-1}$  shows

that  $\lim_{x \rightarrow 1} \frac{1}{x-1}$  does not exist because:

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$



### Example 8

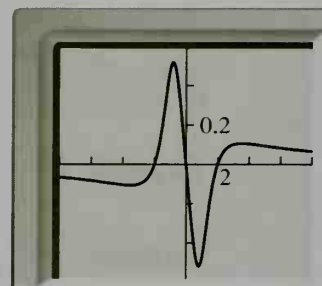
Evaluate  $\lim_{x \rightarrow \infty} \frac{x^3 - 4x}{2x^4 + 5}$ .

### Solution

Using technique (4), we divide numerator and denominator by  $x^4$  and get:

$$\lim_{x \rightarrow \infty} \frac{x^3 - 4x}{2x^4 + 5} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{4}{x^3}}{2 + \frac{5}{x^4}} = \frac{0 - 0}{2 + 0} = 0$$

A graph of  $y = \frac{x^3 - 4x}{2x^4 + 5}$  confirms this limit.



## Formal Definitions of Limit

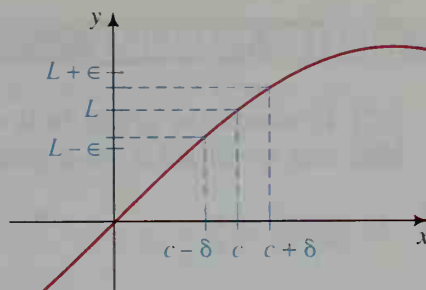
Although the idea of a limit is fairly easy to understand on an intuitive level, it is quite difficult to express formally. Therefore, read carefully the three formal definitions on page 723. The first two, in which the limit  $L$  is a real number, define finite limits. The last one defines an infinite limit.



1.  $\lim_{x \rightarrow c} f(x) = L$  means that for any small positive number  $\epsilon$  (Greek epsilon), there is a positive number  $\delta$  (Greek delta) such that

$$|f(x) - L| < \epsilon$$

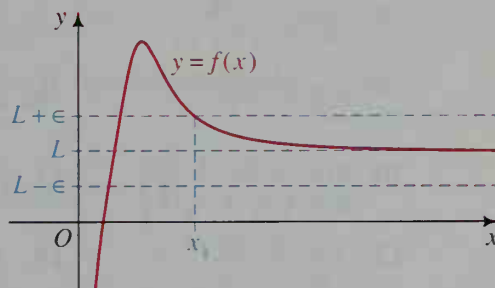
whenever  $x$  is in the domain of  $f$  and  $0 < |x - c| < \delta$ .



2.  $\lim_{x \rightarrow \infty} f(x) = L$  means that for any small positive number  $\epsilon$ , there is a value of  $x$ , call it  $x_1$ , such that

$$|f(x) - L| < \epsilon$$

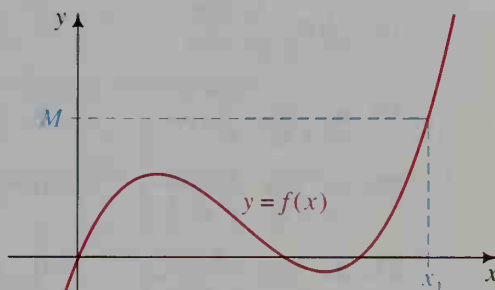
whenever  $x$  is in the domain of  $f$  and  $x > x_1$ . ( $\lim_{x \rightarrow -\infty} f(x) = L$  is similarly defined.)



3.  $\lim_{x \rightarrow \infty} f(x) = \infty$  means that for any large positive number  $M$ , there is a value of  $x$ , call it  $x_1$ , such that

$$f(x) > M$$

whenever  $x$  is in the domain of  $f$  and  $x > x_1$ . (Other statements, such as  $\lim_{x \rightarrow -\infty} f(x) = \infty$ , are similarly defined.)



## CLASS EXERCISES

1. Read each equation aloud and explain what it means.

a.  $\lim_{x \rightarrow \infty} \frac{2x^2}{x^2 + 1} = 2$    b.  $\lim_{x \rightarrow -\infty} \frac{3x + 1}{2x - 5} = \frac{3}{2}$    c.  $\lim_{x \rightarrow 2^+} f(x) = 3$    d.  $\lim_{x \rightarrow 2^-} g(x) = \infty$

In Exercises 2–4, refer to the graph of  $y = f(x)$  shown at the right below.

2. Evaluate  $\lim_{x \rightarrow \infty} f(x)$ ,  $\lim_{x \rightarrow -\infty} f(x)$ ,  $\lim_{x \rightarrow 2} f(x)$ , and  $f(2)$ .

3. Explain why  $f(x)$  is discontinuous at  $x = 2$ .

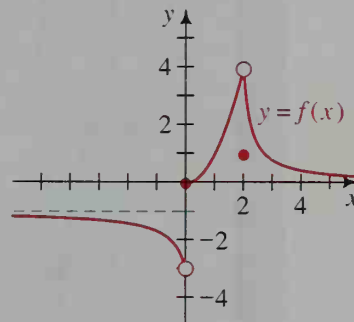
4. a. Evaluate  $\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow 0^-} f(x)$ .

- b. Explain why  $\lim_{x \rightarrow 0} f(x)$  does not exist.

5. **Reading** Express the quotient theorem for limits in words.

6. Evaluate each limit.

a.  $\lim_{x \rightarrow \infty} \frac{5x^2 - 3x + 1}{7x^2 + 9}$    b.  $\lim_{x \rightarrow 1^+} \frac{1}{x - 1}$    c.  $\lim_{x \rightarrow 1^-} \frac{1}{x - 1}$    d.  $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$



# WRITTEN EXERCISES



Evaluate the given limit or state that it does not exist. You may wish to use a computer or a graphing calculator to confirm your answers.

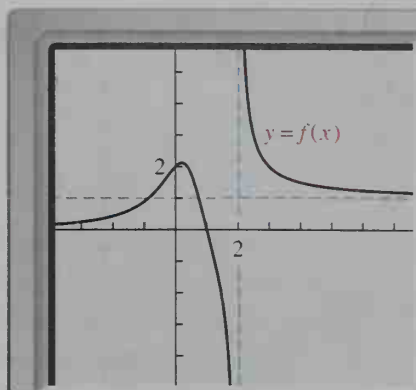
- A**
1.  $\lim_{x \rightarrow \infty} \frac{3x - 5}{4x + 9}$
  2.  $\lim_{x \rightarrow \infty} \frac{2x^2 - 7x}{3x^2 + 5}$
  3.  $\lim_{x \rightarrow -\infty} \frac{8x^2 - 7x + 5}{4x^2 + 9}$
  4.  $\lim_{x \rightarrow -\infty} \frac{5x^3}{7x^3 + 8x^2}$
  5.  $\lim_{x \rightarrow \infty} \frac{(x^2 + 1)(x^2 - 1)}{2x^4}$
  6.  $\lim_{x \rightarrow \infty} \frac{x^2 \cos \frac{1}{x}}{2x^2 - 1}$
  7.  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$
  8.  $\lim_{x \rightarrow 3} \frac{2x^2 - 6x}{x - 3}$
  9.  $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{2x - 4}$
  10.  $\lim_{x \rightarrow 4} \frac{x - 4}{x^2 - x - 12}$
  11.  $\lim_{x \rightarrow 0^+} \frac{x + 1}{x}$
  12.  $\lim_{x \rightarrow -1^+} \frac{x}{x + 1}$
  13.  $\lim_{x \rightarrow 3^+} \frac{x - 4}{x - 3}$
  14.  $\lim_{x \rightarrow 3^-} \frac{2x - 6}{x^2 - 3x}$
  15.  $\lim_{x \rightarrow -2^-} \frac{x^2 + 4x + 4}{x^2 + 3x + 1}$
  16.  $\lim_{x \rightarrow -2^+} \frac{3x + 6}{2x + 4}$
  17.  $\lim_{x \rightarrow 0} \frac{x}{1 - \sqrt{1 - x}}$
  18.  $\lim_{x \rightarrow 0} \frac{2 - \sqrt{4 - x}}{x}$
  19. a.  $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$
  - b.  $\lim_{x \rightarrow 0^-} \frac{|x|}{x}$
  - c.  $\lim_{x \rightarrow 0} \frac{|x|}{x}$
  20. a.  $\lim_{x \rightarrow 1^+} \frac{x - 3}{x^2 - 1}$
  - b.  $\lim_{x \rightarrow 1^-} \frac{x - 3}{x^2 - 1}$
  - c.  $\lim_{x \rightarrow 1} \frac{x - 3}{x^2 - 1}$
  21. a.  $\lim_{x \rightarrow 2^+} \frac{x - 2}{\sqrt{x^2 - 4}}$
  - b.  $\lim_{x \rightarrow 2^-} \frac{x - 2}{\sqrt{x^2 - 4}}$
  - c.  $\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x^2 - 4}}$
  22. a.  $\lim_{x \rightarrow 3^+} \lfloor x \rfloor$
  - b.  $\lim_{x \rightarrow 3^-} \lfloor x \rfloor$
  - c.  $\lim_{x \rightarrow 3} \lfloor x \rfloor$

(Note:  $\lfloor x \rfloor$  is the greatest integer function. See Exercise 21 on page 124.)

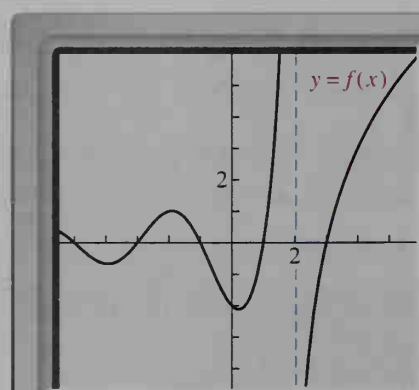
**Visual Thinking** Use the given function's graph to evaluate each limit.

- a.  $\lim_{x \rightarrow \infty} f(x)$       b.  $\lim_{x \rightarrow -\infty} f(x)$       c.  $\lim_{x \rightarrow 2^+} f(x)$       d.  $\lim_{x \rightarrow 2^-} f(x)$

**B** 23.



24.



In Exercises 25–30, evaluate the given limit.

$$25. \lim_{x \rightarrow 0} \frac{1 - \sqrt{x^2 + 1}}{x^2}$$

$$26. \lim_{x \rightarrow -\infty} (\sqrt{4x^2 - 4x} - 2x)$$

$$27. \lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$$

$$28. \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - x)$$

$$29. \text{ a. } \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h}$$

$$\text{ b. } \lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h}$$

$$\text{ c. } \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$30. \text{ a. } \lim_{h \rightarrow 0} \frac{(1+h)^3 - 1}{h}$$

$$\text{ b. } \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h}$$

$$\text{ c. } \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

Determine whether each function is continuous. If it is discontinuous, state where any discontinuities occur.

$$31. f(x) = \begin{cases} 2 - x^2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

$$32. f(x) = \begin{cases} x^2 + 1 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

$$33. f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

$$34. f(x) = \frac{x-1}{1-x}$$

Determine values for  $a$  and  $b$  so that each function is continuous.

$$35. f(x) = \begin{cases} x^2 & \text{if } x \leq -1 \\ ax + b & \text{if } -1 < x < 1 \\ -x^2 & \text{if } x \geq 1 \end{cases}$$

$$36. f(x) = \begin{cases} 1 - 2x & \text{if } x \leq -2 \\ ax + b & \text{if } -2 < x < 1 \\ 3x + 2 & \text{if } x \geq 1 \end{cases}$$

Evaluate each limit or state that it does not exist.

$$37. \lim_{x \rightarrow \infty} (43,987)^{1/x}$$

$$38. \lim_{x \rightarrow 0} 2^{1/x}$$

$$39. \lim_{x \rightarrow 0} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$$

$$40. \lim_{x \rightarrow -\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}}$$

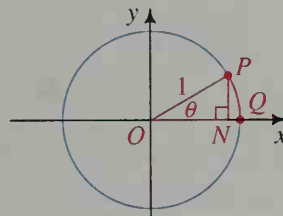
$$41. \lim_{x \rightarrow 0^+} \sin \frac{1}{x}$$

$$42. \lim_{x \rightarrow \infty} \sin \frac{1}{x}$$

43. a. In the diagram at the right, explain why arc  $PQ$  has length  $\theta$ , if  $\theta$  is in radians.

b. **Visual Thinking** Use the diagram to explain why

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$



Evaluate each limit. The result of part (b) of Exercise 43 may be helpful.

$$44. \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta}$$

$$45. \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\theta}$$

$$46. \lim_{t \rightarrow 0} \frac{1 - \cos t}{t^2}$$

$$47. \lim_{x \rightarrow \infty} x \sin \frac{1}{x}$$

(Hint: Multiply by  $\frac{1 + \cos t}{1 + \cos t}$ .)

(Hint: Let  $t = \frac{1}{x}$ . As  $x \rightarrow \infty$ ,  $t \rightarrow 0$ .)

## 19-2 Graphs of Rational Functions

### Objective

To sketch the graph of a rational function.

A **rational function** has the form  $y = \frac{f(x)}{g(x)}$  where  $f(x)$  and  $g(x)$  are polynomials and  $g(x) \neq 0$ . In the following Activity, you will use a computer or a graphing calculator to learn about several features of the graph of a rational function.

### Activity



- Use a computer or a graphing calculator to graph  $y = \frac{x^2 - 9}{x^2 - 4}$ .
- Use the TRACE feature to find the  $x$ - and  $y$ -intercepts of the graph.
- How can you find the graph's  $x$ - and  $y$ -intercepts algebraically?
- Give equations of all lines that appear to be asymptotes of the graph.
- How can you find the equations of the asymptotes algebraically?

### Procedure for Graphing the Rational Function $y = \frac{f(x)}{g(x)}$

- Locate the zeros of  $f(x)$ . A zero of  $f(x)$  will be an  $x$ -intercept of the graph of  $y = \frac{f(x)}{g(x)}$  unless it is also a zero of  $g(x)$ .
- Locate the zeros of  $g(x)$ . These values, of course, are not in the domain of  $y = \frac{f(x)}{g(x)}$ . Usually, vertical lines through these values are asymptotes. If  $f(x)$  and  $g(x)$  have a zero in common, however, reduce  $\frac{f(x)}{g(x)}$  to lowest terms before determining vertical asymptotes.
- Using the techniques of Section 2-3, perform a sign analysis of  $\frac{f(x)}{g(x)}$ , which should be in lowest terms. (Essentially, the sign changes at each zero of  $f(x)$  or  $g(x)$  unless the zero is a double zero, in which case the sign remains the same.)
- If  $x = c$  is a vertical asymptote, then the graph will approach infinity as it nears the asymptote in a region where the values of the function are positive. The graph will approach negative infinity as it nears the asymptote in a region where the values of the function are negative.
- You can find nonvertical asymptotes by considering  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  and  $\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)}$ . If  $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = c$ , then  $y = c$  is a horizontal asymptote.

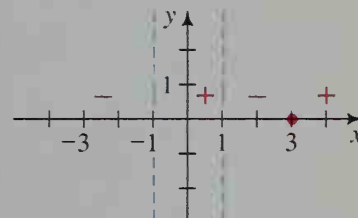
**Example 1** Sketch the graph of  $y = \frac{x-3}{x^2-1}$ .

**Solution** We use the procedure described on the preceding page, as follows.

**Step 1** The numerator  $x-3$  has a zero at  $x=3$ , which is not a zero of the denominator. Hence 3 is an  $x$ -intercept of the graph.

**Step 2** Since  $x^2-1 = (x+1)(x-1)$ , the denominator has zeros at  $x=-1$  and  $x=1$ . Hence the graph has vertical asymptotes  $x=-1$  and  $x=1$ .

**Step 3** A sign analysis of  $\frac{x-3}{(x+1)(x-1)}$  shows that the values of the function are negative for  $x < -1$  and  $1 < x < 3$ , and that they are positive for  $-1 < x < 1$  and  $x > 3$ .

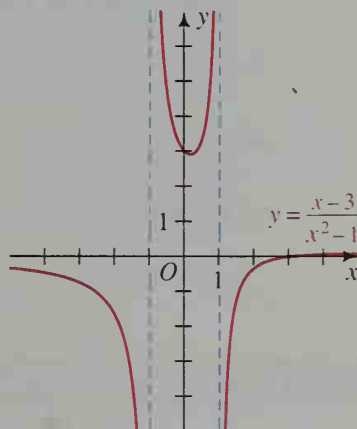


**Step 4** Since the values of the function are negative in the region immediately to the right of the asymptote  $x=1$ , the graph approaches negative infinity as it nears  $x=1$  from the right. Since the values of the function are positive in the region to the left of the asymptote, the graph approaches infinity as it nears  $x=1$  from the left. Similar reasoning shows that the graph approaches infinity as it nears  $x=-1$  from the right, and that the graph approaches negative infinity as it nears  $x=-1$  from the left.

**Step 5** Since  $\lim_{x \rightarrow \infty} \frac{x-3}{x^2-1} = 0$

and  $\lim_{x \rightarrow -\infty} \frac{x-3}{x^2-1} = 0$ ,

the graph has the  $x$ -axis as a horizontal asymptote at the left (where the graph will approach from below) and at the right (where the graph will approach from above). The graph is at the right.



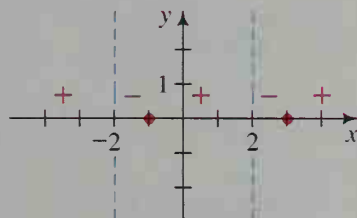
**Example 2** Sketch the graph of  $y = \frac{x^2-2x-3}{x^2-4}$ .

**Solution** This time we will shorten the steps taken in Example 1.

**Steps 1–3**  $\frac{x^2-2x-3}{x^2-4} = \frac{(x-3)(x+1)}{(x-2)(x+2)}$

- (1)  $x$ -intercepts:  $-1, 3$
- (2) vertical asymptotes:  $x = \pm 2$
- (3) sign analysis at the right

(Solution continues on the next page.)





### Steps 4–5

$$(4) \lim_{x \rightarrow -2^-} \frac{x^2 - 2x - 3}{x^2 - 4} = \infty$$

$$\lim_{x \rightarrow -2^+} \frac{x^2 - 2x - 3}{x^2 - 4} = -\infty$$

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 2x - 3}{x^2 - 4} = \infty$$

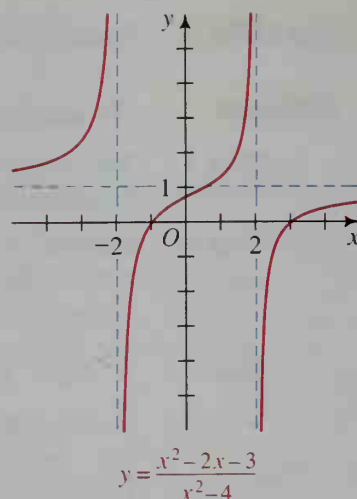
$$\lim_{x \rightarrow 2^+} \frac{x^2 - 2x - 3}{x^2 - 4} = -\infty$$

$$(5) \lim_{x \rightarrow \infty} \frac{x^2 - 2x - 3}{x^2 - 4} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 2x - 3}{x^2 - 4} = 1$$

horizontal asymptote:  $y = 1$

The graph is shown at the right.



## CLASS EXERCISES



For each function, give the  $x$ -intercepts and the equations of any vertical or horizontal asymptotes. You may wish to use a computer or a graphing calculator to confirm your answers.

$$1. y = \frac{x - 2}{(x - 1)(x - 3)}$$

$$2. y = \frac{x}{x^2 - 4}$$

$$3. y = \frac{x^2 - 4}{x}$$

$$4. y = \frac{x^2 - 9}{x^2 + 9}$$

$$5. y = \frac{2x^2}{x^2 - 6}$$

$$6. y = \frac{x^3}{x^2 - 1}$$

7. **Discussion** Let  $f(x)$  and  $g(x)$  be polynomial functions. Suppose that for  $x = c$ ,  $f(c) = g(c) = 0$ . At  $x = c$ , what happens to the graph of  $y = \frac{f(x)}{g(x)}$ ?

## WRITTEN EXERCISES



In Exercises 1–15, sketch the graph of each function. Show vertical and horizontal asymptotes and  $x$ -intercepts. You may wish to use a computer or a graphing calculator to confirm your answers.

**A**

$$1. y = \frac{x - 1}{x + 1}$$

$$2. y = \frac{x}{x - 2}$$

$$3. y = \frac{x}{x^2 - 1}$$

$$4. y = \frac{2x^2}{x^2 - 9}$$

$$5. y = \frac{x^2 + 4}{x^2 - 4}$$

$$6. y = \frac{x^2 - 4}{x^2 + 4}$$

$$7. y = \frac{x - 4}{(x - 1)(x + 2)}$$

$$8. y = \frac{x + 3}{(x + 1)(x - 3)}$$

$$9. y = \frac{12}{x^2 + 2}$$

$$10. y = \frac{12x}{x^2 + 2}$$

$$11. y = \frac{12x^2}{x^2 + 2}$$

$$12. y = \frac{3x^2}{x^2 - 3x}$$

**B** 13.  $y = \frac{x^2 - 2x}{x^2 - 4}$

14.  $y = \frac{x^2 - 16}{x^2 - 6x + 8}$

15.  $y = \frac{x^2 - x - 2}{x^2 - 2x + 1}$

16. **Writing** Write a paragraph in which you explain why the line  $y = x$  is an asymptote for the graph of  $y = \frac{x^3}{x^2 + 1}$ . Then sketch the graph.

17. **Writing** Write a paragraph in which you explain why the line  $y = -x$  is an asymptote for the graph of  $y = \frac{1 - x^2}{x}$ . Then sketch the graph.

**C** 18. Consider the function  $f(x) = \frac{\sin x}{x}$ , where  $x$  is in radians.

a. What is the domain of  $f$ ?

b. What are the zeros of  $f$ ?

c. Evaluate  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow 0} f(x)$ .

d. Sketch the graph of  $y = f(x)$ .

## Series

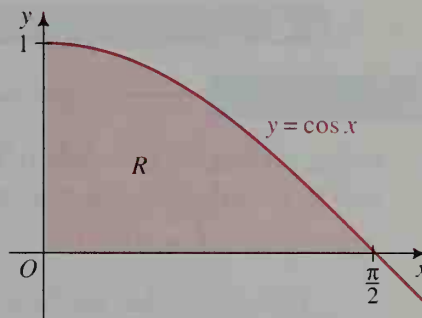
### 19-3 Using Technology to Approximate the Area Under a Curve



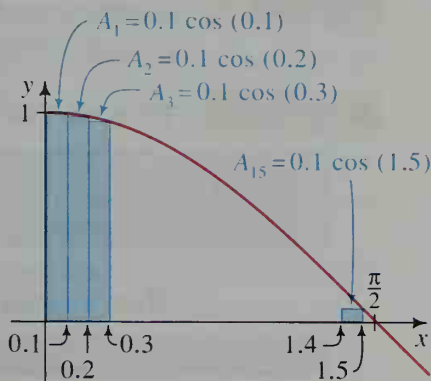
#### Objective

To use technology to approximate the area under a curve.

In this section we will use technology to approximate the areas of regions in a coordinate plane. If the region is bounded by the graph of a function  $y = f(x)$ , the  $x$ -axis, and a pair of vertical lines  $x = a$  and  $x = b$ , then we will speak of finding the *area under the curve*  $y = f(x)$  from  $x = a$  to  $x = b$ . For instance, consider the region  $R$  shown at the right. In this case we want to approximate the area under the cosine curve from  $x = 0$  to  $x = \frac{\pi}{2}$ .



The area of region  $R$  can be approximated by finding the areas of rectangles having base vertices at  $0, 0.1, 0.2, \dots, 1.5$ , as shown at the right. Since the height of the leftmost rectangle is  $\cos(0.1)$ , its area is  $0.1 \cos(0.1)$ . Likewise, the next rectangle has a height equal to  $\cos(0.2)$  and an area equal to  $0.1 \cos(0.2)$ , the one after has a height equal to  $\cos(0.3)$  and an area equal to  $0.1 \cos(0.3)$ , and so on. Note that the sum of the areas of all the rectangles is less than the area of region  $R$  because each rectangle is completely contained in  $R$ .



The calculations involved in approximating the area of region  $R$  are ideally suited for a computer or programmable calculator. The program below, written in BASIC, approximates the area of  $R$  by keeping a running sum of the areas of the rectangles we've discussed.

```
10 LET A = 0
20 FOR X = .1 TO 1.5 STEP .1
30 LET Y = COS (X)
40 LET A = A + .1 * Y
50 NEXT X
60 PRINT "THE AREA IS APPROXIMATELY "; A
70 END
```

When the program is run, we get the following result.

```
THE AREA IS APPROXIMATELY .950200462
```

A better approximation can be obtained if we use rectangles having base vertices 0, 0.01, 0.02, . . . , 1.57, instead of 0, 0.1, 0.2, . . . , 1.5. All that we need to do is to change lines 20 and 40 as shown below.

```
20 FOR X = .01 TO 1.57 STEP .01
40 LET A = A + .01 * Y
```

These changes give the following result.

```
THE AREA IS APPROXIMATELY .994995334
```

### Activity

Using rectangles of width 0.001, modify and run the program given above to obtain a third approximation of the area of region  $R$ .

The results that we have obtained suggest that the area approximations are approaching 1. In calculus, it is possible to show that the area of region  $R$  is *exactly* 1 without using approximations.

### Example

Using rectangles of width 0.01, modify and run the program given above to approximate the area under the curve  $y = 2^x$  from  $x = 0$  to  $x = 1$ .

### Solution

First, we change lines 20–40 as shown below.

```
20 FOR X = .01 TO 1 STEP .01
30 LET Y = 2 ^ X
40 LET A = A + .01 * Y
```

Then, running the modified program, we get the following result.

```
THE AREA IS APPROXIMATELY 1.44770082
```

## CLASS EXERCISES

For Exercises 1 and 2, tell how you would modify the computer program on the preceding page to approximate each area.

1. The area under the curve  $y = \sin x$  from  $x = 0$  to  $x = \pi$
2. The area between the curves  $y = \sin x$  and  $y = -\sin x$  from  $x = 0$  to  $x = \pi$
3. In the program on the preceding page, suppose we change line 20 as follows.

```
20 FOR X = 0 TO 1.5 STEP .1
```

- a. Draw a graph, like the one at the bottom of page 729, to illustrate how this program change affects the rectangles used to approximate the area of  $R$ .
- b. **Visual Thinking** Will the approximation from part (a) be less than or greater than the actual area? Explain.

## WRITTEN EXERCISES

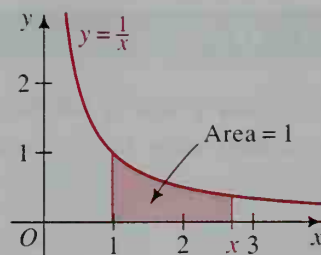


Many of these Written Exercises require the use of a computer or programmable calculator.

For Exercises 1–4, modify and run the program on the preceding page to approximate each area. Use rectangles of width 0.01.

- A** 1. The area under the curve  $y = \tan x$  from  $x = 0$  to  $x = \frac{\pi}{4}$
2. The area under the curve  $y = \sin x$  from  $x = 0$  to  $x = \pi$
3. The area under the curve  $y = x^2$  from  $x = 0$  to  $x = 1$
4. The area under the curve  $y = x^3$  from  $x = 0$  to  $x = 1$
5. Consider the area under the curve  $y = x^n$  from  $x = 0$  to  $x = 1$ .
  - a. When  $n = 1$ , you know from geometry that the area is  $\frac{1}{2}$ .
  - b. When  $n = 2$ , Exercise 3 suggests that the area is  $\frac{1}{3}$ . When  $n = 3$ , Exercise 4 suggests that the area is  $\frac{1}{4}$ . Make a conjecture for  $n = 4$ . Then use a computer or programmable calculator to check your guess.
6. Consider the regions  $A$ ,  $B$ , and  $C$ , each of which is bounded by the graph of  $y = \frac{1}{x}$ , the  $x$ -axis, and a pair of vertical lines. These pairs of vertical lines are  $x = 1$  and  $x = 2$  for  $A$ ,  $x = 1$  and  $x = 3$  for  $B$ , and  $x = 1$  and  $x = 6$  for  $C$ .
  - a. Use a computer or programmable calculator to approximate the areas of regions  $A$ ,  $B$ , and  $C$ . Compare your answers with  $\ln 2$ ,  $\ln 3$ , and  $\ln 6$ .
  - b. What relationship exists among the areas of regions  $A$ ,  $B$ , and  $C$ ? (*Hint:*  $\ln 2 + \ln 3 = \ln \frac{?}{?}$ .) What conclusion can you draw about areas under the curve  $y = \frac{1}{x}$ ?

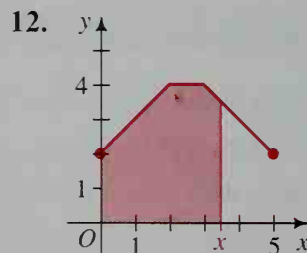
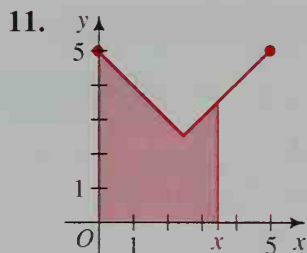
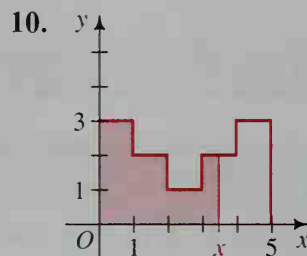
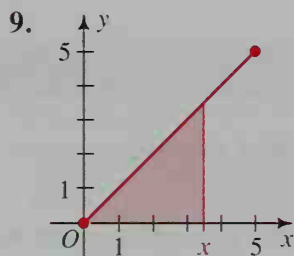
- B** 7. In Exercise 6, you found that the area under the curve  $y = \frac{1}{x}$  from  $x = 1$  to  $x = 2$  was less than 1, and that the area from  $x = 1$  to  $x = 3$  was greater than 1. Thus, there must be a number  $x$  between 2 and 3 so that the area is exactly 1. Use a computer or programmable calculator to approximate  $x$ .



8. The area under the curve  $y = \frac{4}{1+x^2}$  from  $x = 0$  to  $x = 1$  is a special number.

Use a computer or programmable calculator to find this number.

The area  $A(x)$  of each shaded region depends only on the right-hand boundary  $x$ , which can vary from 0 to 5. Sketch the graph of  $A(x)$  for  $0 \leq x \leq 5$ .



**Visual Thinking** Find each area using the fact that the area under the curve  $y = \cos x$  from  $x = 0$  to  $x = \frac{\pi}{2}$  is 1. Also give a geometric argument to defend your answer.

13. The area under the curve  $y = \cos x$  from  $x = -\frac{\pi}{2}$  to  $x = \frac{\pi}{2}$
14. The area between the curves  $y = \cos x$  and  $y = -\cos x$  from  $x = -\frac{\pi}{2}$  to  $x = \frac{\pi}{2}$
15. The area under the curve  $y = 2 \cos x$  from  $x = 0$  to  $x = \frac{\pi}{2}$
16. The area under the curve  $y = \sin x$  from  $x = 0$  to  $x = \pi$
17. The area under the curve  $y = |\cos x|$  from  $x = 0$  to  $x = \pi$



18. a. What do you think is the exact area between the  $x$ -axis and the curve  $y = \cos x$  from  $x = 0$  to  $x = \pi$ ?
- b. Modify and run the program on page 730 to approximate the area described in part (a).
- c. Does the result of part (b) agree with your answer to part (a)? If not, use a geometric argument to explain why.

## 19-4 Power Series

### Objective

To use the power series of a given function to find an infinite series for a functional value or for a related function.

In Section 13-5, we discussed the infinite geometric series

$$a + ax + ax^2 + ax^3 + \cdots + ax^n + \cdots.$$

It is a special case of a **power series**

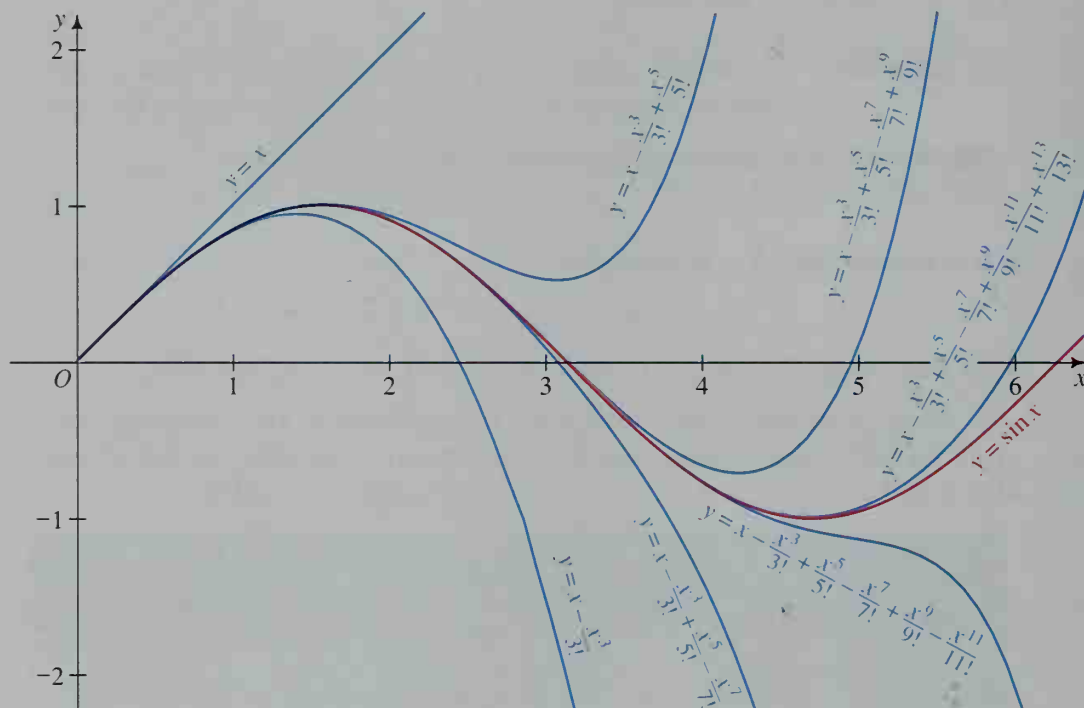
$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots.$$

Note that the coefficients of a power series need not all be the same.

Some extremely useful power series are given below. The *interval of convergence* for each series is also given. Note that several of the series involve  $n!$ , where  $n! = 1 \times 2 \times 3 \times \cdots \times n$  for positive integral values of  $n$  and  $0! = 1$ .

Power series	Interval of convergence
(1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	all real $x$
(2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	all real $x$
(3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all real $x$
(4) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$-1 \leq x \leq 1$
(5) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leq 1$

Power series are used by some calculators and computers to give approximations. For example, a computer may have a subroutine that evaluates several terms of the power series for  $\sin x$ . If the computer encounters the number  $\text{SIN}(2)$  in a program, it will substitute  $x = 2$  into the series. The computer is programmed to evaluate enough terms so that the result will be accurate to a predetermined number of decimal places. The diagram below shows that  $\sin 2$  is quite closely approximated by just three terms of the sine series. To evaluate  $\sin 4$  or  $\sin 5$ , however, more terms are needed.



The following example shows how series (1) through (5) on the preceding page can be used to find other series.

### Example

Find an infinite series for: a.  $\sin 1$

b.  $\sin(-x)$

### Solution

a. In series (3), we substitute 1 for  $x$  and obtain:

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

b. In series (3), we substitute  $-x$  for  $x$  and obtain:

$$\begin{aligned} \sin(-x) &= -x - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \cdots \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \end{aligned}$$

Notice that the series for  $\sin(-x)$  is the opposite of the series for  $\sin x$ .

In addition to power series, the infinite  $p$ -series (so called because the  $n$ th term,  $\frac{1}{n^p}$ , involves the power  $p$ ) is important in calculus. The following theorem, which Exercises 19–21 ask you to prove, gives the values of  $p$  for which the  $p$ -series converges and diverges.

### Convergence and Divergence of the $p$ -series

The  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

The  $p$ -series in which  $p = 1$ ,  $\sum_{n=1}^{\infty} \frac{1}{n}$ , is called the **harmonic series**.

### CLASS EXERCISES

- Visual Thinking** Suppose you want to use the power series for  $\sin x$  to obtain a reasonable approximation of  $\sin 4$ . What minimum number of terms in the power series would you use? Explain your answer by referring to the graph on page 734.
- Reading** The result of part (b) of the example on the preceding page agrees with the fact that  $\sin(-x) = \underline{\quad ? \quad}$  for all  $x$ .

### WRITTEN EXERCISES

For Exercises 1–18, refer to series (1)–(5) on page 733.

- A**
- Write infinite series in expanded form for  $\cos \frac{\pi}{4}$  and  $\sin \left(-\frac{\pi}{4}\right)$ .
  - Write infinite series in expanded form for  $e$  and  $e^{-1}$ .
  - Substitute  $x = 1$  in the series for  $\ln(1 + x)$  to get an infinite series for  $\ln 2$ .
    - Why can't you substitute  $x = 2$  to get an infinite series for  $\ln 3$ ?
  - Substitute  $x = 1$  in the series for  $\tan^{-1} x$  to get an infinite series for a number involving  $\pi$ .

In Exercises 5–10, find a power series for each function. State the interval of convergence in each case.

5.  $e^{-x}$

6.  $e^{-x^2}$

7.  $\tan^{-1} 2x$

8.  $\ln(1 - x)$

9.  $\sin x^2$

10.  $\cos 2x$

- B** 11. Use the sine series to show that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .
12. Use the cosine series to find  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ .
13. Use series (5) to find  $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$ .
14. Use series (1), (2), and (3) to prove that  $e^{i\theta} = \cos \theta + i \sin \theta$ .
15. Use Exercise 14 to show that  $e^{i\pi} = -1$  and  $e^{2i\pi} = 1$ . (Note: These identities were discovered by the Swiss mathematician Leonhard Euler. Notice the combination of arithmetic (the number  $-1$ ), algebra (the number  $i$ ), geometry (the number  $\pi$ ), and analysis (the number  $e$ ).)
16. Use Exercise 14 to show that  $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ .
17. Use Exercise 14 to show that  $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ .
18. Show that the expressions for  $\sin \theta$  and  $\cos \theta$  given in Exercises 16 and 17 satisfy the equation  $(\sin \theta)^2 + (\cos \theta)^2 = 1$ .

The following exercises constitute a proof of the theorem on page 735.

- C** 19. Show that the harmonic series diverges by rewriting the series as
- $$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$
- and showing that the terms within each pair of parentheses total more than  $\frac{1}{2}$ .
20. Using a technique like the one in Exercise 19, show that the  $p$ -series converges if  $p > 1$ .
21. Show that the  $p$ -series diverges if  $p < 1$ .



**Benoit Mandelbrot (1924– )**

Born in Poland and educated in France, Benoit Mandelbrot is best known for developing the geometry of *fractals*, objects having similar features at many levels of scale.

Mandelbrot's work with fractals has application to many fields, helping to describe erratic stock market fluctuations, weather patterns, turbulence in fluid flow, disturbances in heart rhythms, and the irregular structures of coastlines, clouds, and galaxies.

Fractals have even been applied to filmmaking, where computer-generated fractals are used to create alien landscapes that look real.

# Iterated Functions

## 19-5 Analyzing Orbits

### Objective

To analyze orbits for iterations of a given function.

In this section and the next, we will discuss a branch of mathematics called *dynamical systems*. Generally speaking, a dynamical system is some situation that is undergoing change, whether the change is as orderly as the motion of a planet or as erratic as the fluctuations of a stock market. The goal is to predict the behavior of a dynamical system.

We can easily create a dynamical system by starting with an initial value, called a *seed*, and repeatedly applying a function to it.

That is, given a seed  $x_0$  and a function  $f$ , we will examine the sequence

$$\begin{aligned}x_0 \\x_1 &= f(x_0) \\x_2 &= f(x_1) = f(f(x_0)) \\x_3 &= f(x_2) = f(f(f(x_0))) \\&\vdots\end{aligned}$$

This sequence is called the *orbit* of  $x_0$ . Notice that the terms of the sequence involve successive *iterations* of  $f$ . To identify iterations more easily, we will use superscripts:  $f^2(x_0)$  will mean  $f(f(x_0))$ ,  $f^3(x_0)$  will mean  $f(f(f(x_0)))$ , and so on. Thus, the  $(n + 1)$ st term in the orbit of  $x_0$  is:

$$x_n = f^n(x_0)$$

Note that  $n$  indicates the number of times that  $f$  has been applied to  $x_0$ ; it does *not* indicate a power of  $f$ .

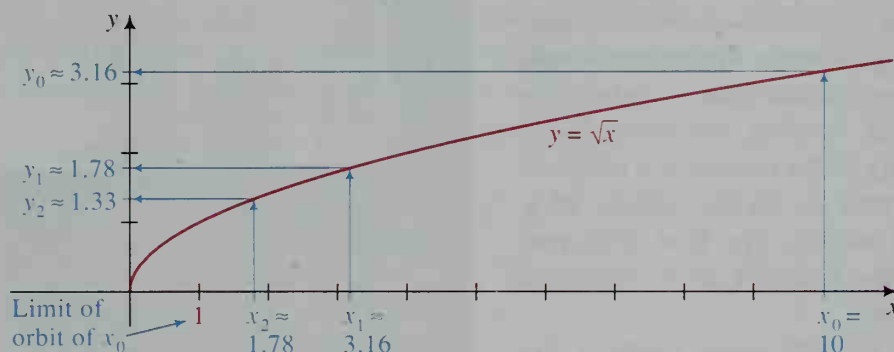


### Activity 1

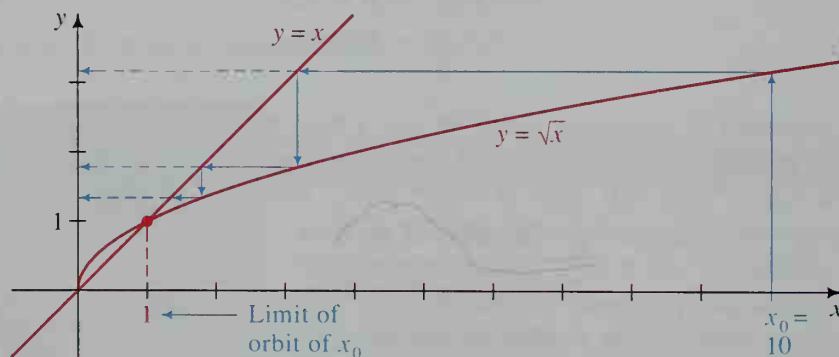
- Using a calculator, obtain decimal approximations for the numbers in the sequence  $10, \sqrt{10}, \sqrt{\sqrt{10}}, \sqrt{\sqrt{\sqrt{10}}}, \dots$ , which is the orbit of  $x_0 = 10$  for iterations of  $f(x) = \sqrt{x}$ . What seems to be the limit of the orbit?
- Repeat part (a), this time using  $x_0 = 0.1$ .



Although Activity 1 shows that we can examine an orbit by looking at *numbers*, we can also examine an orbit by looking at a *graph*. For example, we can use the graph of  $y = \sqrt{x}$ , shown below, to examine the orbit of  $x_0 = 10$  for iterations of the square root function. Substituting  $x_0$  into  $y = \sqrt{x}$ , we obtain the corresponding  $y$ -value,  $y_0 \approx 3.16$ . Then, letting  $x_1 = y_0$  and substituting  $x_1$  into  $y = \sqrt{x}$ , we obtain  $y_1 \approx 1.78$ . Letting  $x_2 = y_1$  and substituting  $x_2$  into  $y = \sqrt{x}$ , we obtain  $y_2 \approx 1.33$ . Continuing in this way, we see that the orbit of  $x_0 = 10$  tends toward 1. We can write this observation as  $\lim_{n \rightarrow \infty} f^n(10) = 1$  where  $f(x) = \sqrt{x}$ .



Since each  $y$ -value that is obtained becomes the next  $x$ -value in the iteration of a function, we can use the line  $y = x$  to simplify the graphical analysis of a given seed's orbit. Consider the diagram below, where we start with  $x_0 = 10$  on the  $x$ -axis and move vertically to the curve, horizontally to the line, vertically to the curve, horizontally to the line, and so on. A diagram like this, with the dotted lines omitted, is called a *web diagram* (because it looks like a cobweb).



### Example 1

Use (a) a calculator and (b) a web diagram to analyze the orbit of  $x_0 = 1$  for iterations of the function  $f(x) = 2 \sin x$ .

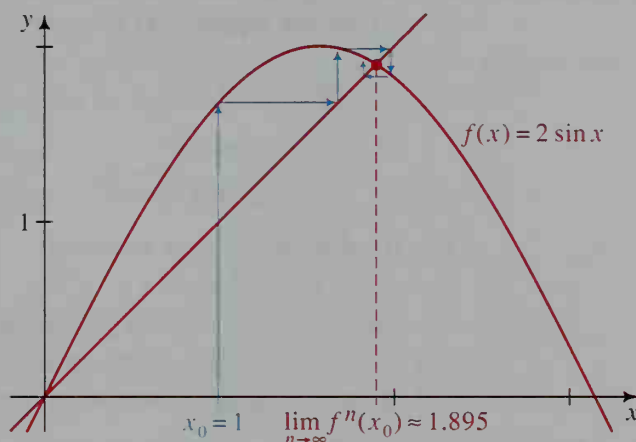
### Solution

- With your calculator in radian mode, enter the seed and iterate the function to produce the sequence at the top of the next page.

$$\begin{aligned}
 x_0 &= 1 \\
 x_1 &= 2 \sin x_0 = 2 \sin 1 \approx 1.683 \\
 x_2 &= 2 \sin x_1 \approx 2 \sin 1.683 \approx 1.987 \\
 x_3 &= 2 \sin x_2 \approx 2 \sin 1.987 \approx 1.829 \\
 &\vdots
 \end{aligned}$$

You should find a limiting value of approximately 1.895.

- b. The web diagram below shows that  $\lim_{n \rightarrow \infty} f^n(1) \approx 1.895$ . Notice that this limit is the  $x$ -coordinate of the point of intersection of the graphs of  $y = 2 \sin x$  and  $y = x$ .

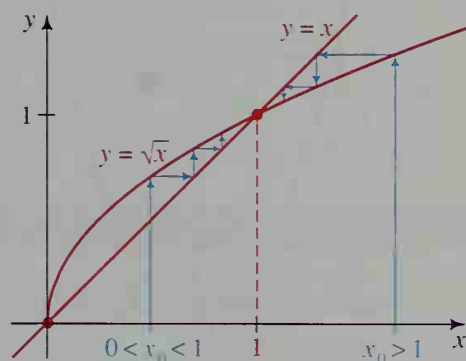


## Activity 2

Use a web diagram to analyze the orbit in part (b) of Activity 1 on page 737. Does using some other seed between 0 and 1 affect the analysis?

Activity 2 indicates that for any seed  $x_0$  such that  $0 < x_0 < 1$ , the orbit of  $x_0$  for iterations of  $f(x) = \sqrt{x}$  tends toward 1. The same is true for any seed  $x_0 > 1$ , as the web diagram at the right indicates. We have thus accounted for all seeds except  $x_0 = 0$  and  $x_0 = 1$ . Since  $\sqrt{0} = 0$  and  $\sqrt{1} = 1$ , the orbit of each seed is constant (that is, the orbit of 0 is 0, 0, 0, . . . , and the orbit of 1 is 1, 1, 1, . . . ). The numbers 0 and 1 are therefore called *fixed points* of  $f$ . In general, a fixed point of a function  $f$  is any value of  $x$  satisfying the equation

$$f(x) = x.$$



Although 0 and 1 are both fixed points, they differ in one important aspect. The orbit of any seed near 1 moves toward 1, while the orbit of any seed near 0 moves away from 0. Consequently, 1 is called an *attracting* fixed point, and 0 is called a *repelling* fixed point.

**Example 2** Analyze all orbits for iterations of the function  $f(x) = \frac{1}{2}x^2 - 1$ .

**Solution**

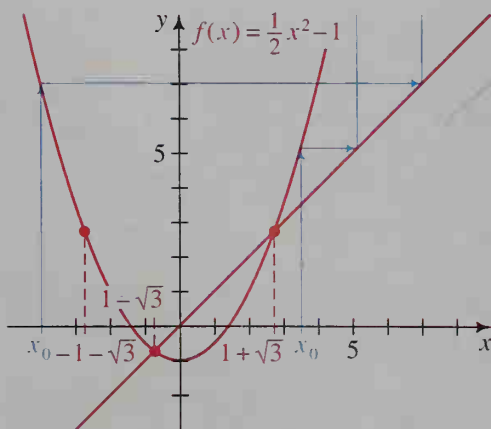
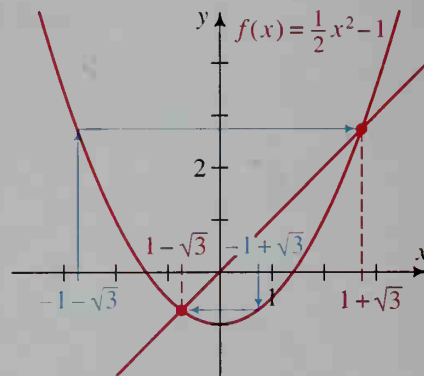
**Step 1** We first find the fixed points of  $f$  by solving the equation  $f(x) = x$ :

$$\begin{aligned}\frac{1}{2}x^2 - 1 &= x \\ x^2 - 2x - 2 &= 0 \\ x &= 1 \pm \sqrt{3}\end{aligned}$$

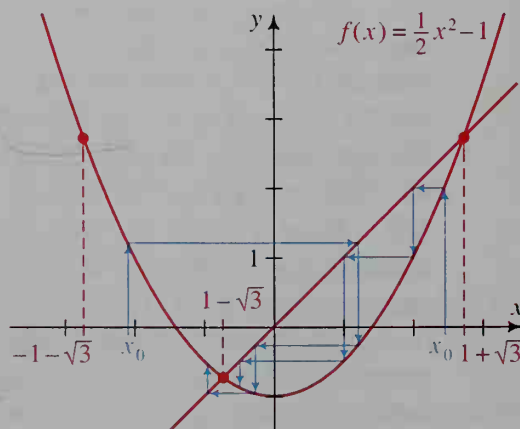
The fixed points of  $f$  are therefore  $x_0 = 1 + \sqrt{3}$  and  $x_0 = 1 - \sqrt{3}$ .

Moreover, since  $f(-x) = f(x)$  for all  $x$ , we can see that  $x_0 = -1 - \sqrt{3}$  and  $x_0 = -1 + \sqrt{3}$  each become fixed after one iteration of  $f$ . (See the web diagram above.) They are therefore called *eventually fixed* points.

**Step 2** We now draw web diagrams, like the ones below, to examine the orbits of seeds other than the fixed and eventually fixed points. (Note that different scales are used for the two web diagrams.) The web diagram at the left below indicates that if  $|x_0| > 1 + \sqrt{3}$ , the orbit of  $x_0$  tends toward infinity. On the other hand, the web diagram at the right below indicates that if  $|x_0| < 1 + \sqrt{3}$ , the orbit of  $x_0$  tends to  $1 - \sqrt{3}$ .



If  $|x_0| > 1 + \sqrt{3}$ , then  $\lim_{n \rightarrow \infty} f^n(x_0) = \infty$ .

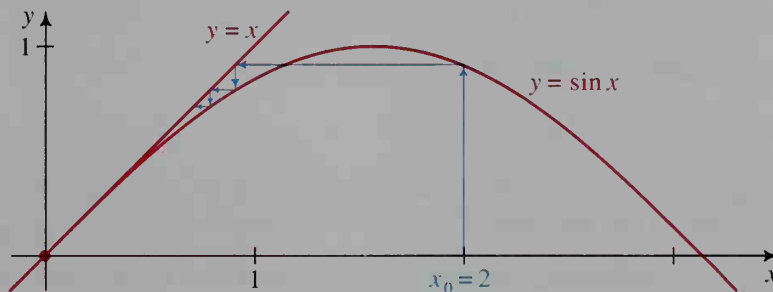


If  $|x_0| < 1 + \sqrt{3}$ , then  $\lim_{n \rightarrow \infty} f^n(x_0) = 1 - \sqrt{3}$ .

## CLASS EXERCISES

In Exercises 1–4, find the first four terms in the orbit of  $x_0$  for iterations of the given function  $f$ .

- $f(x) = x + 2$ ,  $x_0 = 5$
- $f(x) = 1 - x$ ,  $x_0 = 0$
- $f(x) = 3x + 2$ ,  $x_0 = -1$
- $f(x) = \frac{1}{10}x + 1$ ,  $x_0 = 1$
- Give an example showing the difference between  $f^2(x_0)$  and  $[f(x_0)]^2$ .
- The web diagram below shows the orbit of  $x_0 = 2$  for iterations of the sine function. What is the limit of this orbit?
  - If you set your calculator in radian mode and compute the orbit in part (a), does it seem that the limit is approached rapidly or slowly?



- Visual Thinking** Describe where the fixed points of a function  $f$  occur on the graph of  $f$ .
- Reading** Tell whether each of the two fixed points in Example 2 is attracting or repelling.
- Consider the function  $f(x) = x^2$ .
  - What are the fixed points of  $f$ ?
  - Does  $f$  have any eventually fixed points? If so, what are they?
  - Using web diagrams, find  $\lim_{n \rightarrow \infty} f^n(x_0)$  if  $|x_0| > 1$  and if  $|x_0| < 1$ .
  - Based on your answers to parts (a) and (c), tell whether each fixed point of  $f$  is attracting or repelling.
- Repeat Exercise 9 for the function  $f(x) = x^3$ .

## WRITTEN EXERCISES

In Exercises 1–4, find the first four terms in the orbit of  $x_0$  for iterations of the given function  $f$ .

- A**  $f(x) = 2x - 1$ ,  $x_0 = 5$
- $f(x) = \sqrt{x + 1}$ ,  $x_0 = 0$
- $f(x) = \frac{1}{x}$ ,  $x_0 = 2.5$
- $f(x) = \frac{x}{2} - 2$ ,  $x_0 = 20$

In Exercises 5–10, determine the fixed points of the given function  $f$ .

5.  $f(x) = 5x - 16$

6.  $f(x) = 3 - \frac{1}{2}x$

7.  $f(x) = x^2 + \frac{1}{4}$

8.  $f(x) = x^2 - 1$

9.  $f(x) = 2 + \sqrt{x}$

10.  $f(x) = \frac{1}{2}\left(x + \frac{4}{x}\right)$

11. Use a calculator to examine the orbit of  $x_0 = 235$  for iterations of the function  $f(x) = \log_{10} x$ . Describe what happens and explain why.

12. Use a calculator to examine the orbit of  $x_0 = 101$  for iterations of the function  $f(x) = \sqrt{x - 1}$ . Describe what happens and explain why.

13. a. Draw a web diagram to analyze the orbit of  $x_0 = 6$  for iterations of the function  $f(x) = \frac{1}{2}x + 1$ . Then find  $\lim_{n \rightarrow \infty} f^n(x_0)$ .

b. If you change  $x_0$ , how does the limit change?

14. Repeat Exercise 13 if  $f(x) = 1 - \frac{2}{3}x$ .

15. a. Using a calculator set in radian mode, examine the orbit of  $x_0 = 1$  for iterations of the function  $f(x) = \cos x$ . Approximate  $\lim_{n \rightarrow \infty} f^n(x_0)$ .

b. Using web diagrams, convince yourself that the limit found in part (a) applies to *any* seed  $x_0$ .

16. Repeat Exercise 15 if  $f(x) = \frac{1}{2} \cos x$ .

**B** 17. Consider the function  $f(x) = \ln(x + 1)$ .

a. What is  $\lim_{n \rightarrow \infty} f^n(x_0)$  for any positive seed  $x_0$ ?

b. What is  $\lim_{n \rightarrow \infty} f^n(x_0)$  for any seed  $x_0$  such that  $-1 < x_0 < 0$ ?

18. Consider the function  $f(x) = x - x^2$ .

a. What is  $\lim_{n \rightarrow \infty} f^n(x_0)$  for any seed  $x_0$  such that  $0 < x_0 < 1$ ?

b. What is  $\lim_{n \rightarrow \infty} f^n(x_0)$  for any seed  $x_0$  such that  $x_0 < 0$  or  $x_0 > 1$ ?

19. Analyze all orbits for iterations of the function  $f(x) = 2x - x^2$ .

20. Analyze all orbits for iterations of the function  $f(x) = 2 \sin x$ . (*Hint:* The orbit given in Example 1 on page 738 tends toward one of the fixed points of  $f$ . There are two other fixed points.)

21. Consider the set of linear functions  $f(x) = mx + k$ . For what values of  $m$  do iterations of  $f$  produce orbits that have a finite limit? (*Note:* Exercises 13 and 14 are two cases.)

22. Consider the set of quadratic functions  $f(x) = x^2 + c$ .

a. For what values of  $c$  does  $f$  have one or more fixed points?

b. Given any value of  $c$  for which  $f$  has *no* fixed points, find  $\lim_{n \rightarrow \infty} f^n(x_0)$  for any seed  $x_0$ .



23. The expression below is an example of a *continued fraction*.

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

- a. By iterating the function  $f(x) = 1 + \frac{1}{x}$  for the seed  $x_0 = 1$ , you can find the value of the continued fraction. Approximate this value to the nearest hundredth.
- b. Try seeds other than  $x_0 = 1$ . Does changing the seed affect  $\lim_{n \rightarrow \infty} f^n(x_0)$ ? Explain.
24. To the nearest hundredth, find the value of the continued fraction

$$2 - \frac{1}{2 - \frac{1}{2 - \dots}}$$

by iterating the function  $f(x) = 2 - \frac{1}{x}$  for the seed  $x_0 = 2$ .

25. The “divide and average” algorithm for finding the square root of a positive number  $N$  is as follows:
1. Let the seed  $x_0$  be any approximation value of  $\sqrt{N}$ .
  2. Compute the orbit of  $x_0$  by iterating the function  $f(x) = \frac{x + \frac{N}{x}}{2}$ .
- a. Use the algorithm to find the first four approximations for  $\sqrt{10}$ . Begin with  $x_0 = 3$ .
- b. Show that a fixed point of  $f$  is  $\sqrt{N}$ . Using web diagrams, also show that this is an attracting fixed point for any positive seed  $x_0$ .
26. **Investigation** Consider the function  $f$  defined for positive integers as shown below:

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

The orbit of  $x_0 = 4$  for iterations of  $f$  is:

$$4, 2, 1, 4, 2, 1, 4, 2, 1, \dots$$

Since the orbit repeats the same three numbers over and over, the orbit is called *periodic* with *fundamental period* 3. Similarly, the orbit of  $x_0 = 5$  is:

$$5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, \dots$$

This orbit is called *eventually periodic*. Examine and describe the orbits of other seeds. (You may wish to write a computer program that generates a given seed's orbit. If you do, be sure to try  $x_0 = 27$ .)

## 19-6 Applications of Iterated Functions

**Objective** To use iterated functions to model money and population growth.

The iteration of a function is closely related to recursion formulas, which we discussed in Section 13-2. Example 1 compares these ideas.

**Example 1** An investor initially deposits \$2000 in an account that pays 6% annual interest, and each year thereafter the investor deposits \$1000 in the account. Define the sequence of yearly account balances (a) by using a recursion formula and (b) by using an iterated function.

**Solution** It is helpful to see the first few dollar amounts in the sequence.

$$x_0 = 2000$$

$$x_1 = 1.06(2000) + 1000 = 3120$$

$$x_2 = 1.06(3120) + 1000 = 4307.20$$

$$\vdots$$

a. The sequence can be defined recursively as follows:

$$x_0 = 2000$$

← initial amount

$$x_n = 1.06(x_{n-1}) + 1000$$

← recursion formula

b. The sequence can also be defined as the orbit of  $x_0 = 2000$  for iterations of the function  $f(x) = 1.06x + 1000$ .

An investment program like that in Example 1 is called an *annuity*. To see how changes in the interest rate or in the deposited amounts affect an annuity, a computer program like the one on page 481 is very helpful.

Example 1 dealt with the dynamics of money growth. Let us now look at the dynamics of population growth. Consider a population in some environment, such as bass in a lake, lions on a savanna, or owls in a forest. From one generation to the next, the population typically changes. Moreover, the environment in which the population exists can usually sustain the population up to some maximum number, or *capacity*. If we define the capacity to be 1, then we can always express the population as some fraction of the capacity. For example, to say that a population is 0.62 means that the population is at 62% of capacity.



When a population is low (that is, near 0% of capacity), the next generation tends to be larger because ample food is usually available, disease is not easily spread, and so on. However, when a population is high (that is, near 100% of capacity), the next generation tends to be smaller because food becomes scarce, disease is more likely, and so on. Therefore, a reasonable mathematical model for predicting the size of the next generation of a population would involve the following quantities:

- (1) the current population,  $x$  (where  $0 < x < 1$ )
- (2) the room available for growth,  $1 - x$

One of the simplest such growth models assumes that the next generation is jointly proportional with  $x$  and  $1 - x$ . For example, if the current population is 0.62 and the room for growth is  $1 - 0.62 = 0.38$ , then the model predicts that the next generation is some positive proportionality constant times the product of 0.38 and 0.62. Thus, the model is based on the function

$f(x) = cx(1 - x)$ , where  $c$  is a positive proportionality constant.

The function  $f(x) = cx(1 - x)$  is called a *logistic* function. It has become very important not only to biologists studying population growth but also to mathematicians studying dynamical systems.



Using a computer or programmable calculator and the program given below, we can iterate  $f(x) = cx(1 - x)$  for various proportionality constants  $c$  and various seeds  $x_0$ , as Example 2 shows.

```

10 PRINT 'WHAT IS THE PROPORTIONALITY CONSTANT (C > 0)';
20 INPUT C
30 PRINT 'WHAT IS THE INITIAL POPULATION (0 < X < 1)';
40 INPUT X
50 PRINT 'GENERATION', 'POPULATION'
60 FOR G = 1 TO 100
70 LET X = C * X * (1 - X)
80 PRINT G, X
90 NEXT G
100 END

```

### Example 2

Using the program above and the seed  $x_0 = 0.2$ , iterate each function.

- a.  $f(x) = 2.9x(1 - x)$
- b.  $f(x) = 3.1x(1 - x)$

### Solution

Although the program gives 100 generations of a population, only the last 6 generations are shown for each function.

a. GENERATION	POPULATION	b. GENERATION	POPULATION
95	.655169176	95	.558014125
96	.655175328	96	.76456652
97	.655169792	97	.558014125
98	.655174774	98	.76456652
99	.65517029	99	.558014125
100	.655174325	100	.76456652

Part (a) of Example 2 shows that the orbit of  $x_0 = 0.2$  for iterations of the logistic function  $f(x) = 2.9x(1 - x)$  seems to have a limit of approximately 0.655 (that is, about 65.5% of capacity). We therefore say that 0.655 is an *equilibrium point* for the population.

Part (b) of Example 2 shows that the orbit of  $x_0 = 0.2$  for iterations of the logistic function  $f(x) = 3.1x(1 - x)$  seems to cycle between approximately 0.558 and 0.765 (that is, about 55.8% and 76.5% of capacity). We therefore say that the population has a *period-2 cycle*.

Although iterating either of the functions in Example 2 using seeds other than  $x_0 = 0.2$  produces essentially the same orbit each time, not all logistic functions behave so well. There are values of  $c$  for which iterating  $f(x) = cx(1 - x)$  using seeds that are nearly identical produces orbits that are wildly different. (For example, see Written Exercise 10.) Such functions are said to behave *chaotically*.

## CLASS EXERCISES

1. A loan of \$100 at 1% monthly interest is paid back at the rate of \$20.60 per month. The loan's monthly outstanding balances form a sequence. Define this sequence (a) by using a recursion formula and (b) by using an iterated function.
2. Using a calculator and the seed  $x_0 = 0.2$ , find the first four iterations of the logistic function  $f(x) = 2x(1 - x)$ . (Note: The "memory in" and "memory recall" keys on your calculator are quite useful in iterating  $f$ .) What do you observe about the resulting sequence of population values?
3. Repeat Exercise 2, this time using the function  $f(x) = 0.5x(1 - x)$ .

## WRITTEN EXERCISES



Many of these Written Exercises require the use of a computer or programmable calculator.

In Exercises 1–4, use (a) a recursion formula and (b) an iterated function to describe each situation. In each case, be sure to state the initial value as well as the formula or function.

**A**

1. **Personal Finance** A savings account with an annual interest rate of 6%, an opening balance of \$100, and no deposits or withdrawals
2. **Personal Finance** An annuity with an annual interest rate of 8%, an initial deposit of \$5000, and annual deposits of \$3000
3. **Personal Finance** A retirement account with an annual interest rate of 6.5%, a balance of \$400,000 at retirement, and annual withdrawals of \$30,000 after retirement
4. **Personal Finance** A loan of \$400 with a monthly interest rate of 1.5% and monthly payments of \$23.30

**Writing** For Exercises 5–10, use the program on page 745 to iterate the logistic function  $f(x) = cx(1 - x)$  for the given value of  $c$  and the three given values of  $x_0$ . Write a sentence or two to describe your findings.

5.  $c = 1.5$ ;  $x_0 = 0.9$ ,  $x_0 = 0.75$ ,  $x_0 = 0.001$
6.  $c = 3.2$ ;  $x_0 = 0.4$ ,  $x_0 = 0.8$ ,  $x_0 = 0.99$
7.  $c = 3.5$ ;  $x_0 = 0.162$ ,  $x_0 = 0.437$ ,  $x_0 = 0.85$
8.  $c = 3.55$ ;  $x_0 = 0.2$ ,  $x_0 = 0.73$ ,  $x_0 = 0.825$
9.  $c = 3.83$ ;  $x_0 = 0.15$ ,  $x_0 = 0.48$ ,  $x_0 = 0.625$
10.  $c = 4$ ;  $x_0 = 0.5$ ,  $x_0 = 0.50001$ ,  $x_0 = 0.49999$

- B** 11. If you win a certain state's million dollar lottery, you are given payments of \$50,000 each year for 20 years. Use a computer or programmable calculator to complete the following.
- a. How much does it cost the state to pay you off if the state finances the payoffs by opening a savings account that pays 10% per year and has a balance of \$0 after 20 annual withdrawals of \$50,000 have been made?
  - b. Explain why you would be much better off at the end of the 20 years if the state had given you \$1,000,000 outright and you had deposited it in an account paying 10% annual interest and then withdrew \$50,000 per year.
12. **Finance** Suppose an account with annual interest rate  $r$  has an opening balance of  $x_0$  dollars, and no deposits or withdrawals are ever made. To obtain the sequence of yearly account balances, we would iterate the function

$$f(x) = (1 + r)x$$

using the seed  $x_0$ . If the function is iterated  $n$  times, then  $f^n(x_0) = \underline{\hspace{1cm}}$ . This shows that iteration leads to what type of growth?

13. The function given in Exercise 12 is based on a *constant* rate of growth,  $r$ . Although iterating the function accurately models the growth of money in a bank account, it is inadequate for modeling the growth of a population (except under ideal circumstances), because the rule for the function does not take into account the room-available-for-growth factor discussed on page 745. Suppose the rate of growth,  $r$ , is proportional to the room available for growth,  $1 - x$ . That is,  $r = c(1 - x)$  for some positive proportionality constant  $c$ . Then:

$$\begin{array}{lcl} \text{population of} & = & \text{population of} + \text{rate of} \times \text{population of} \\ \text{next generation} & = & \text{current generation} \quad \text{growth} \quad \text{current generation} \end{array}$$

$$\begin{aligned} f(x) &= x + rx \\ &= (1 + r)x \\ &= (1 + c(1 - x))x \end{aligned}$$

- a. **Writing** Suppose  $c = 0.8$  in  $f(x) = (1 + c(1 - x))x$ . Using a computer or programmable calculator, iterate the function for various seeds  $x_0$  such that  $0 < x_0 < 1$ . Write a sentence or two to describe the results.
- b. **Investigation** Using other values of  $c$ , repeat part (a). Can you find any  $c$ -values that produce periodic orbits? chaotic orbits?



14. For a given population, if a fraction  $h$  of the capacity (which is defined to be 1) is added or removed each year, then the function in Exercise 13 becomes:

$$f(x) = (1 + c(1 - x))x + h$$

For example, given a population of trees in a forest, if 10% of the forest's capacity is harvested each year, then  $h = -0.1$  in the function above.

- a. **Writing** Suppose  $c = 0.8$  and  $h = -0.1$  in the function above. Using a computer or programmable calculator, iterate the function for various seeds  $x_0$  such that  $0 < x_0 < 1$ . Write a sentence or two to describe the results.

- b. **Investigation** Using other values of  $c$  and  $h$ , repeat part (a). Can you find any  $c$ - and  $h$ -values that produce periodic orbits? chaotic orbits?

15. In Exercise 14, suppose a fraction  $h$  of the *current population* (instead of the capacity) is added or removed each year. Give a rule for the modified function. Then repeat parts (a) and (b) of Exercise 14.



## Chapter Summary


1. The *limit* of a function  $f(x)$  as  $x$  approaches a real number  $c$ , infinity, or negative infinity (denoted  $\lim_{x \rightarrow c} f(x)$ ,  $\lim_{x \rightarrow \infty} f(x)$ , and  $\lim_{x \rightarrow -\infty} f(x)$ , respectively) is discussed in Section 19-1. (Formal definitions are given on pages 722 and 723.) For example,  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$  means that the value of  $\frac{1}{x}$  gets arbitrarily close to 0 as  $x$  becomes arbitrarily large.
2. A function  $f(x)$  is *continuous* at  $x = c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$ .
3. Various techniques for evaluating the limit of the quotient of two functions are given on page 721.
4. A *rational* function has the form  $y = \frac{f(x)}{g(x)}$  where  $f(x)$  and  $g(x)$  are polynomial functions and  $g(x) \neq 0$ . The procedure for graphing a rational function is given on page 726.
5. The area of a region bounded by the graph of the function  $y = f(x)$ , the  $x$ -axis, and the vertical lines  $x = a$  and  $x = b$  can be approximated by dividing the region into rectangles of equal width and summing the rectangles' areas.
6. The *power series* for several important functions are given on page 733.
7. The infinite  $p$ -series converges if  $p > 1$  and diverges if  $p \leq 1$ .

8. One way to create a *dynamical system* is to *iterate* a function  $f(x)$  for a given *seed*  $x_0$ . The *orbit* of  $x_0$  is the sequence  $x_0, f(x_0), f^2(x_0), f^3(x_0), \dots$ , which can be analyzed graphically using a *web diagram*.
9. If  $x$  is a *fixed point* of a function  $f$ , then  $f(x) = x$ . Orbits of nearby seeds move toward an *attracting* fixed point and away from a *repelling* fixed point.
10. Iterated functions model both money and population growth. The *logistic* function  $f(x) = cx(1 - x)$  models population growth. Iterating  $f(x)$  for a given seed may result in an *equilibrium point*, a *periodic cycle*, or a *chaotic orbit*.

### Key vocabulary and ideas

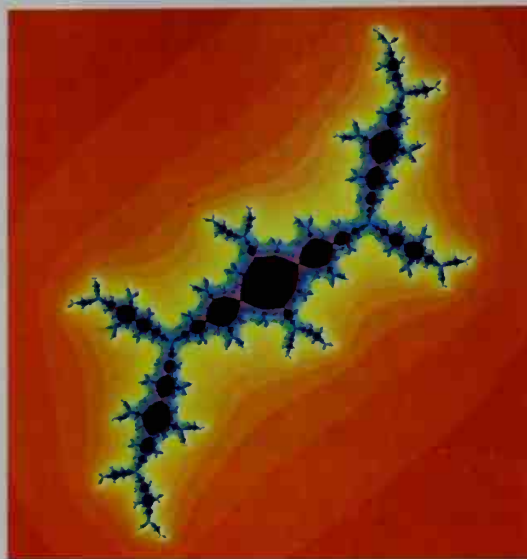
$\lim_{x \rightarrow \infty} f(x), \lim_{x \rightarrow -\infty} f(x)$ (pp. 717–718)	dynamical system (p. 737)
$\lim_{x \rightarrow c} f(x)$ (pp. 718–719)	seed, orbit, iteration (p. 737)
continuous function (p. 720)	web diagram (p. 738)
rational function (p. 726)	fixed point (p. 739)
area under a curve (p. 729)	attracting and repelling fixed points (p. 740)
power series (p. 733)	logistic function (p. 745)
$p$ -series (p. 735)	equilibrium point, periodic cycle (p. 746)

## Chapter Test

1. Evaluate each limit or explain why the limit does not exist. 19-1
    - a.  $\lim_{x \rightarrow \infty} \frac{x^2 - x}{x^2 - 1}$
    - b.  $\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 1}$
    - c.  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$
    - d.  $\lim_{x \rightarrow 0} \frac{|x|}{x}$
  2. Graph each rational function. Show all asymptotes and  $x$ -intercepts. 19-2
    - a.  $y = \frac{x}{x - 4}$
    - b.  $y = \frac{x^2 + 1}{x^2 - 1}$
-  **Part (a) of Exercise 3 requires the use of a computer or programmable calculator.**
3. a. Modify and run the program on page 730 to approximate the area under the curve  $y = 4 - x^2$  from  $x = 0$  to  $x = 2$ . 19-3  
 b. **Visual Thinking** Using part (a) and the symmetry of the curve  $y = 4 - x^2$ , give an approximation of the area under the curve from  $x = -2$  to  $x = 2$ .
  4. Find an infinite series for (a)  $\cos(-1)$  and (b)  $\ln(1 + x^2)$ . 19-4
  5. If  $f(x) = e^x - 2$ , use a calculator to approximate  $\lim_{n \rightarrow \infty} f^n(1)$ . 19-5
  6. Analyze all orbits for iterations of the function  $f(x) = x^3 + x$ .
  7. **Writing** In the logistic function  $f(x) = cx(1 - x)$ , the constant  $c$  reflects the environmental factors that affect a population's growth. If  $0 < c \leq 1$ , find  $\lim_{n \rightarrow \infty} f^n(x_0)$  for any seed  $x_0$ . Describe the significance of this result for any population to which the model is applied. 19-6

## Chaos in the Complex Plane

If you've explored some of the dynamical systems presented in Sections 19-5 and 19-6, then you're probably well aware that iterating a function—no matter how simple its rule—can produce unexpectedly complicated orbits for certain seeds. The results are even more unusual and exciting when a function's domain and range, which until now were sets of real numbers, are allowed to be sets of complex numbers instead. In fact, iterating a function with a complex domain and range can produce exceptionally intricate and beautiful graphic images, like the one shown.



Filled-in Julia set of  $f(z) = z^2 - 0.112 + 0.861i$

### Materials:

a computer or programmable calculator with  
graphics mode (preferably color graphics)

a printer that can print  
a graphics screen

### Complex Orbits

Suppose we have a function  $f(z)$  where  $z$  is a complex number. Iterating  $f(z)$  using some complex seed  $z_0$  gives the orbit of  $z_0$ :

$$\begin{aligned} z_0 & \\ z_1 &= f(z_0) \\ z_2 &= f(z_1) = f^2(z_0) \\ z_3 &= f(z_2) = f^3(z_0) \\ &\vdots \end{aligned}$$

← { Recall from Section 19-5  
that  $f^2(z_0)$  means  $f(f(z_0))$ .

For example, the orbit of  $z_0 = 1 + 2i$  for iterations of the squaring function  $f(z) = z^2$  is:

$$\begin{aligned} z_0 &= 1 + 2i \\ z_1 &= (1 + 2i)^2 = -3 + 4i \\ z_2 &= (-3 + 4i)^2 = -7 - 24i \\ z_3 &= (-7 - 24i)^2 = -527 + 336i \\ &\vdots \end{aligned}$$

← { Recall from Section 1-5 that  
 $(x + yi)^2 = x^2 - y^2 + 2xyi$ .

The calculation of such orbits is best left to a computer or programmable calculator. At the top of the next page is a BASIC program that performs 20 iterations of  $f(z) = z^2$  for any seed  $z_0$ .

```

10 PRINT 'REAL PART OF SEED';
20 INPUT X0
30 PRINT 'IMAGINARY PART OF SEED';
40 INPUT Y0
50 PRINT 'ITERATION', 'ORBIT'
60 FOR I = 0 TO 20
70 PRINT I, X0;'' + ''Y0;''I''
80 LET X1 = X0 * X0 - Y0 * Y0
90 LET Y1 = 2 * X0 * Y0
100 LET X0 = X1
110 LET Y0 = Y1
120 NEXT I
130 END

```

- Run the program for several values of  $z_0 = x_0 + y_0i$  such that

$$|z_0| = \sqrt{x_0^2 + y_0^2} < 1.$$

(Note that in the complex plane, these values of  $z_0$  all lie inside the unit circle.) You should find that all orbits tend to zero. The reason is based on these facts:

1. Since  $z_1 = z_0^2$ ,  $|z_1| = |z_0|^2 = |z_0|^2$ .
  2. Since  $|z_0| < 1$ ,  $|z_0|^2 < |z_0|$ .
  3. Thus,  $|z_1| = |z_0|^2 < |z_0|$ , which means that  $z_1$  is closer to the origin of the complex plane than  $z_0$  is.
  4. Likewise,  $z_2$  is closer to the origin than  $z_1$ ,  $z_3$  is closer than  $z_2$ , and so on.
- Choose several values of  $z_0$  such that  $|z_0| > 1$ . (Where are these values in the complex plane?) This time you should find that all orbits escape to infinity. (How does your computer or calculator indicate this?) Use an argument similar to the one for  $|z_0| < 1$  to explain why.
  - Now let's consider the orbit of  $z_0$  when  $|z_0| = 1$ . In this case, note that

$$|z_1| = |z_0^2| = |z_0|^2 = 1^2 = 1.$$

Likewise,  $|z_2| = 1$ ,  $|z_3| = 1$ , and so on. In other words, the orbit of a point on the unit circle in the complex plane always stays on the unit circle. Test this result by running the program for several values of  $z_0$  such that  $|z_0| = 1$ . (Be sure to try values other than  $z_0 = \pm 1$  and  $z_0 = \pm i$ . For example, try  $z_0 = 0.8 - 0.6i$ .) You should find that some of the orbits fall off the unit circle. This happens because the orbits are very sensitive to slight round-off errors made by the computer or calculator. The orbits exhibit what is called *sensitive dependence on initial conditions*, a hallmark of *chaos*.

## Julia Sets

For a complex polynomial function (such as  $f(z) = z^2$ ), the set of seeds on the boundary between those whose orbits converge or cycle in some way and those whose orbits escape to infinity is called a *Julia set*, named after the French mathematician Gaston Julia. In the case of  $f(z) = z^2$ , the Julia set consists of all points on the unit circle.

So what do the Julia sets of other functions, such as

$$f(z) = z^2 + c$$

where  $c$  is some complex constant, look like? This is by no means an easy question to answer. In fact, Gaston Julia and others first asked this question in the early 1900s, but they lacked the technological tools that are used today to answer it. Not until the 1970s did mathematicians have computers that could quickly and accurately generate and display Julia sets. Thus, your own investigation of Julia sets puts you at the forefront of a very exciting branch of modern mathematics.

To obtain the graph of the Julia set of  $f(z) = z^2 + c$  for a particular complex constant  $c$ , it seems that we would need to look at the orbits of *all* seeds  $z_0$  in the complex plane. Fortunately, the following theorem limits the scope of our work.

### Escaping Theorem

Consider the orbit of  $z_0$  for iterations of the function  $f(z) = z^2 + c$  for some complex constant  $c$ . If  $|f^n(z_0)| > 2$  for some integer  $n \geq 0$ , then the orbit of  $z_0$  escapes to infinity.

The escaping theorem tells us that we don't need to examine the orbit of any seed  $z_0$  whose distance from the origin of the complex plane is more than 2 units. (Why?) Thus, if  $z_0 = x_0 + y_0i$ , we can impose the restrictions  $-2 \leq x_0 \leq 2$  and  $-2 \leq y_0 \leq 2$  on the real and imaginary parts of  $z_0$ . These restrictions give us a square in the complex plane that we can directly associate with some square grid, say 100 dots by 100 dots, on a computer or calculator's graphics screen.

On a typical graphics screen, the point in the upper left corner is  $(0, 0)$  in the computer or calculator's memory, and the point that is 100 dots over and 100 dots down from  $(0, 0)$  is  $(100, 100)$ . We want these two graphics screen points to correspond respectively to the points  $(-2, 2)$  and  $(2, -2)$  in the complex plane. The transformation that converts graphics screen coordinates  $(a, b)$  to complex plane coordinates  $(x, y)$  is given by the pair of equations:

$$x = -2 + \frac{a}{25} \quad \text{and} \quad y = 2 - \frac{b}{25}$$

(You should be able to derive these equations for yourself. Try it.)

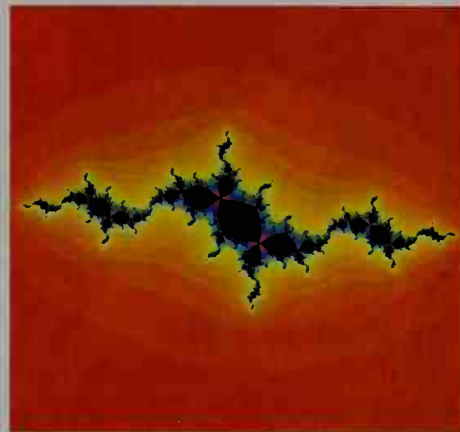
The BASIC program on the next page uses the transformation equations above to convert each graphics screen point  $(a, b)$  to a complex plane point  $(x_0, y_0)$ . This point represents the seed  $z_0 = x_0 + y_0i$  whose orbit for iterations of  $f(z) = z^2 + c$  (for an input value of  $c$ ) is then examined. If the orbit of  $z_0$  escapes to infinity within 20 iterations of  $f$ , then the point  $(a, b)$  is *skipped* (so that its color remains the background color of the screen). If the orbit of  $z_0$  does *not* escape to infinity within 20 iterations of  $f$ , then the point  $(a, b)$  is *plotted* (so that its color is different from the background color of the screen). Notice in line 130 of the program that the test for an orbit escaping to infinity is based on  $|f^n(z_0)|^2 > 4$  rather than  $|f^n(z_0)| > 2$ . (Why is this more efficient?)



- Lines 50 and 170 of the program below have been left blank, because different versions of BASIC differ in their commands for obtaining a blank graphics screen and for plotting points on the screen. Determine the proper commands for your version of BASIC and then complete the program.

<pre> 10 PRINT 'REAL PART OF COMPLEX CONSTANT'; 20 INPUT CX 30 PRINT 'IMAGINARY PART OF COMPLEX CONSTANT'; 40 INPUT CY 50 60 FOR A = 0 TO 100 70 FOR B = 0 TO 100 80 LET XO = -2 + A / 25 90 LET YO = 2 - B / 25 100 FOR I = 1 TO 20 110 LET X1 = XO * XO - YO * YO + CX 120 LET Y1 = 2 * XO * YO + CY 130 IF X1 * X1 + Y1 * Y1 &gt; 4 THEN GOTO 180 140 LET XO = X1 150 LET YO = Y1 160 NEXT I 170 180 NEXT B 190 NEXT A 200 END </pre>	<p>For the version of BASIC that you're using, insert the command(s) for obtaining a blank graphics screen. (Note: If the screen does not have a minimum width and depth of 100 dots each, then you will need to adjust lines 60–90.)</p> <p>←</p> <p>For the version of BASIC that you're using, insert the command(s) for plotting the point (A, B).</p> <p>←</p>
--	---

- Run the program for  $c = 0 + 0i$ . (It may take several minutes before any points appear on the screen, so be patient.) You should get a disk of radius 1. Of course, the *edge* of the disk (that is, the unit circle) is the Julia set of  $f(z) = z^2$ . The disk itself is called the *filled-in* Julia set of  $f$ .
- Run the program for various values of  $c$ , such as  $c = -1$ ,  $c = -0.1 + 0.8i$ , and  $c = 0.3 - 0.5i$ . You should get some fascinating filled-in Julia sets. (Remember that the edge of each graph is the actual Julia set.) If possible, obtain printouts of your graphs.
- If your graphics screen can display multiple colors, you might modify the program so that different colors are used to indicate the “speeds” with which orbits escape to infinity outside the filled-in Julia sets. For example, one color can be given seeds that escape in 10 or fewer iterations, while another color can be given seeds that escape in more than 10 iterations. Using more colors will produce an even more striking work of art, like the one shown.



Filled-in Julia set of  
 $f(z) = z^2 - 1.139 + 0.238i$

## Fractals

One of the most unusual properties of Julia sets is *self-similarity at different levels of scale*. That is, if some portion of a Julia set is magnified, the magnified portion will resemble the original set. (See the pair of graphic images below.) Moreover, magnified portions of portions still resemble the original set. This amazing property is what typifies objects known as *fractals*.

To investigate the fractal quality of Julia sets, we need to alter the transformation equations discussed on page 752. If we want a complex plane point  $(x, y)$  in the square region defined by  $m \leq x \leq m + s$  and  $n \leq y \leq n + s$  to correspond to the graphics screen point  $(a, b)$  in the square region defined by  $0 \leq a \leq 100$  and  $0 \leq b \leq 100$ , then we would use the transformation equations:

$$x = m + \frac{s \cdot a}{100} \quad \text{and} \quad y = n + s - \frac{s \cdot b}{100}$$

For example, if we want the square region defined by  $-1.25 \leq x \leq -0.75$  and  $-1 \leq y \leq -0.5$  to fill the screen grid of 100 dots by 100 dots, then we would modify the program on the preceding page by changing lines 80 and 90 as follows:

```
80 LET XO = -1.25 + A / 200
90 LET YO = -0.5 - B / 200
```

- If you have previously obtained the filled-in Julia set of  $f(z) = z^2 + c$  for  $c = 0.3 - 0.5i$ , then try running the program with the modifications shown above to enlarge a portion of the graph. What do you notice?
- Experiment by enlarging portions of other filled-in Julia sets whose graphs you've already obtained.



Filled-in Julia set of  $f(z) = z^2 + 0.36 + 0.1i$



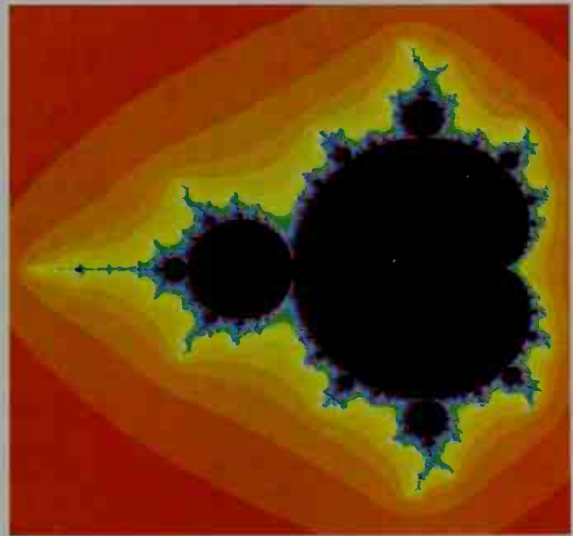
Magnified portion of the Julia set

## The Mandelbrot Set

As we have seen, obtaining the Julia set of  $f(z) = z^2 + c$  involves fixing the value of  $c$  and then iterating  $f(z)$  for various seeds  $z_0$ . Suppose we turn the tables, however. What happens if we fix the value of  $z_0$  and iterate  $f(z)$  for various values of  $c$ ? The result is a famous figure known as the *Mandelbrot set*, first discovered in the 1970s by Benoit Mandelbrot (see page 736).

The program below generates the Mandelbrot set. Note that the seed  $z_0$  is given the value  $0 + 0i$  in lines 60 and 70. (For reasons that we will not discuss here, the orbit of  $z_0 = 0 + 0i$  is called the *critical orbit* for iterations of  $f(z)$ .) The program tests  $c$ -values to determine whether the orbit of  $z_0 = 0 + 0i$  escapes to infinity. If the orbit does *not* escape for a particular  $c$ -value, then the  $c$ -value belongs to the Mandelbrot set (and is therefore plotted).

- Complete lines 10 and 150 of the program below and then run it. You should get a figure like the one shown in black at the right.



Mandelbrot set (in black)

```

10
20  FOR A = 0 TO 100
30  LET CX = - 2 + A / 25
40  FOR B = 0 TO 100
50  LET CY = 2 - B / 25
60  LET XO = 0
70  LET YO = 0
80  FOR I = 1 TO 20
90  LET X1 = XO * XO - YO * YO + CX
100 LET Y1 = 2 * XO * YO + CY
110 IF X1 * X1 + Y1 * Y1 > 4 THEN GOTO 160
120 LET XO = X1
130 LET YO = Y1
140 NEXT I
150
160 NEXT B
170 NEXT A
180 END

```

← For the version of BASIC that you're using, insert the command(s) for obtaining a blank graphics screen. (Note: If the screen does not have a minimum width and depth of 100 dots each, then you will need to adjust lines 20–50.)

← For the version of BASIC that you're using, insert the command(s) for plotting the point (A, B).

- If color graphics are available, try modifying and running the program to produce an image similar to the one shown above, where color is used outside the Mandelbrot set to indicate the “speeds” of escaping orbits.
- Also try modifying and running the program so that one of the small “bulbs” on the Mandelbrot set is magnified. What do you notice?

## Preparing a Report

Now that you've investigated some Julia sets as well as the Mandelbrot set, it is time to compile your work and summarize your findings. If you've obtained print-outs, be sure to indicate on them both the function that was iterated and the portion of the complex plane that was examined. Also point out any unusual features or unexpected results that you observe.

20

*An Introduction to Calculus*





## 20-1 The Slope of a Curve

**Objective** To find derivatives of functions.

The *slope of a curve* at a point  $P$  is a measure of the steepness of the curve. If  $Q$  is a point on the curve near  $P$ , then the slope of the curve at  $P$  is approximately the slope of segment  $PQ$ . The approximation becomes better as  $Q$  moves along the curve toward  $P$ . We therefore define the **slope of the curve** at  $P$  to be the limit of the slope of  $\overline{PQ}$  as  $Q$  approaches  $P$  along the curve:

$$\text{slope of curve at } P = \lim_{Q \rightarrow P} (\text{slope of } \overline{PQ})$$

Suppose we want to find the slope of the curve  $y = x^2$  at  $P(1, 1)$ . We can let  $Q$  be a point on the curve near  $P$  by letting the  $x$ -coordinate of  $Q$  be  $1 + h$ , which differs from the  $x$ -coordinate of  $P$  by  $h$  units. The  $y$ -coordinate of  $Q$  is  $(1 + h)^2$ , as shown in the diagram below. We now calculate the slope of  $\overline{PQ}$ :

$$\begin{aligned} \text{slope of } \overline{PQ} &= \frac{(1 + h)^2 - 1}{h} \\ &= \frac{(1 + 2h + h^2) - 1}{h} \\ &= \frac{h(2 + h)}{h} = 2 + h \end{aligned}$$

As  $Q$  approaches  $P$ ,  $h$  approaches 0. Thus:

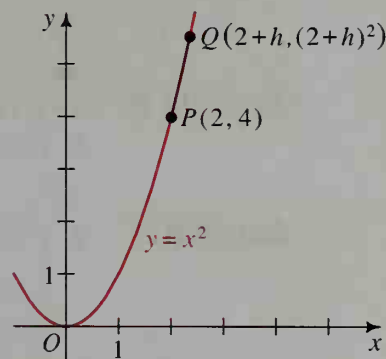
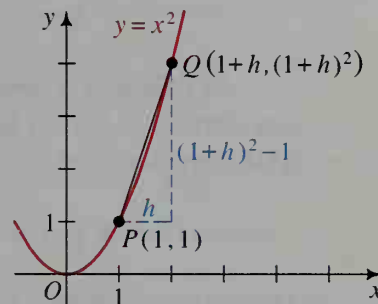
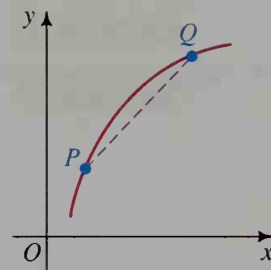
$$\begin{aligned} \text{slope of curve at } (1, 1) &= \lim_{Q \rightarrow P} (\text{slope of } \overline{PQ}) \\ &= \lim_{h \rightarrow 0} (2 + h) = 2 \end{aligned}$$

Therefore, the slope of the curve  $y = x^2$  at  $P(1, 1)$  is 2.

To find the slope of the curve  $y = x^2$  at  $P(2, 4)$ , we proceed similarly:

$$\begin{aligned} \text{slope of } \overline{PQ} &= \frac{(2 + h)^2 - 4}{h} \\ &= \frac{4 + 4h + h^2 - 4}{h} \\ &= \frac{4h + h^2}{h} = 4 + h \\ \lim_{Q \rightarrow P} (\text{slope of } \overline{PQ}) &= \lim_{h \rightarrow 0} (4 + h) = 4 \end{aligned}$$

Therefore, the slope of the curve  $y = x^2$  at  $P(2, 4)$  is 4.



◀ The strobe photo at the left shows how a *continuous* motion can be seen as an infinite sequence of *discrete* steps. Calculus accomplishes this mathematically, allowing us to determine, for example, the *instantaneous velocity* of a drumstick at any given instant.



We have seen that the slope of the curve  $y = x^2$  is 2 when  $x = 1$  and that the slope is 4 when  $x = 2$ . Example 1 generalizes these results by finding the slope of the curve  $y = x^2$  at an arbitrary point  $(x, x^2)$  on the curve.

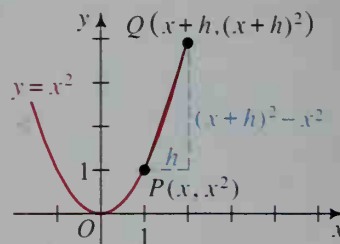
**Example 1** Find the slope of the curve  $y = x^2$  at the point  $P(x, x^2)$ .

**Solution** Let  $Q$  be any other point on the graph, as shown at the right. Then:

$$\begin{aligned}\text{slope of } \overline{PQ} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} = 2x + h\end{aligned}$$

$$\lim_{Q \rightarrow P} (\text{slope of } \overline{PQ}) = \lim_{h \rightarrow 0} (2x + h) = 2x$$

Thus, the slope of the curve  $y = x^2$  at  $P(x, x^2)$  is  $2x$ .

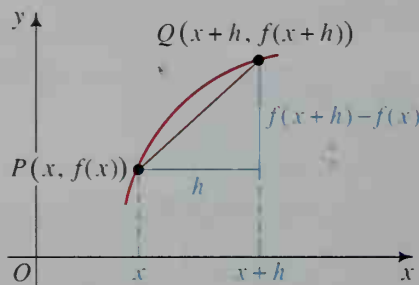


In Example 1, notice that the slope,  $2x$ , is a function of  $x$ . In fact, the slope for any curve  $y = f(x)$  at an arbitrary point  $(x, f(x))$  on the curve is a function of  $x$ . This slope function, called the **derivative of  $f(x)$**  and denoted  $f'(x)$ , is defined as:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

For a given value of  $x$ , the value of  $f'(x)$  is the slope of the curve at  $(x, f(x))$  for the following reason:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{Q \rightarrow P} (\text{slope of } \overline{PQ}) \\ &= \text{slope of curve} \\ &\quad \text{at } P(x, f(x))\end{aligned}$$



**Example 2** a. If  $f(x) = x^3$ , find  $f'(x)$ .

b. Find the slope of the curve  $y = f(x)$  at the point  $(2, 8)$ .

**Solution**

$$\begin{aligned}\text{a. } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \\ &= 3x^2\end{aligned}$$

b. Since  $f'(2) = 3 \cdot 2^2 = 12$ , the slope at  $(2, 8)$  is 12.

The next example shows that the derivative of a function is not always defined at a point where the function itself is defined.

**Example 3** If  $f(x) = |x|$ , show that  $f'(0)$  is undefined.

**Solution** For  $f'(0)$  to be defined,  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$  must exist. Consider the right- and left-hand limits, however:

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{|h| - |0|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h - 0}{h} \quad \leftarrow \begin{cases} \text{Since } h > 0, \\ |h| = h. \end{cases} \\ &= \lim_{h \rightarrow 0^+} 1 = 1 \\ \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{|h| - |0|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h - 0}{h} \quad \leftarrow \begin{cases} \text{Since } h < 0, \\ |h| = -h. \end{cases} \\ &= \lim_{h \rightarrow 0^-} -1 = -1 \end{aligned}$$

Since the right- and left-hand limits are not equal,  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$  does not exist (see page 719), and therefore  $f'(0)$  is undefined.

The following four theorems enable us to find the derivative of every polynomial function, and some other functions as well. Although the proof of Theorem 1 is given, Written Exercises 37–39 ask you to prove Theorems 2–4.

### Theorem 1

If  $f(x)$  is a constant function (that is,  $f(x) = c$  for all  $x$ ), then  $f'(x) = 0$  for all  $x$ .

**Proof**

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c - c}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} \\ &= \lim_{h \rightarrow 0} 0 \\ &= 0 \end{aligned}$$

## Theorem 2

If  $n$  is any nonzero real number and  $f(x) = x^n$ , then  $f'(x) = nx^{n-1}$ .

Note that the results of Examples 1 and 2, where we saw that the derivative of  $f(x) = x^2$  is  $f'(x) = 2x$  and the derivative of  $f(x) = x^3$  is  $f'(x) = 3x^2$ , agree with Theorem 2.

## Theorem 3

If  $c$  and  $n$  are any nonzero real numbers and  $f(x) = cx^n$ , then  $f'(x) = cnx^{n-1}$ .

## Theorem 4

If  $f(x) = p(x) + q(x)$ , then  $f'(x) = p'(x) + q'(x)$ .

### Example 4

Find the derivative of each function.

a.  $f(x) = x^5$

b.  $f(x) = 3x^5$

c.  $f(x) = 12x^2 - 4x + 7$

d.  $f(x) = 8\sqrt{x} - \frac{7}{x^2}$

### Solution

a. To find the derivative of  $f(x) = x^5$ , we use Theorem 2 with  $n = 5$ :

$$f'(x) = 5x^4$$

b. To find the derivative of  $f(x) = 3x^5$ , we use Theorem 3 with  $c = 3$  and  $n = 5$ :

$$f'(x) = 3 \cdot 5x^4 = 15x^4$$

c. We regard  $f(x) = 12x^2 - 4x + 7$  as the sum of three functions defined by its terms and use Theorems 1, 3, and 4:

$$\begin{aligned} f'(x) &= 12 \cdot 2x^1 - 4 \cdot 1x^0 + 0 \\ &= 24x - 4 \end{aligned}$$

d. We rewrite  $f(x) = 8\sqrt{x} - \frac{7}{x^2}$  as  $f(x) = 8x^{1/2} - 7x^{-2}$  and proceed as in part (c) of this example.

$$\begin{aligned} f'(x) &= 8 \cdot \frac{1}{2}x^{(1/2)-1} - 7(-2)x^{-2-1} \\ &= 4x^{-1/2} + 14x^{-3} = \frac{4\sqrt{x}}{x} + \frac{14}{x^3} \end{aligned}$$

The line **tangent** to a curve at a point on the curve is defined to be the line passing through the point and having a slope equal to the slope of the curve at that point.

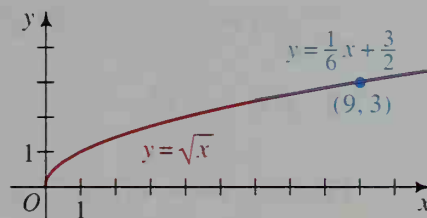
**Example 5** Find an equation of the line that is tangent to the curve  $y = \sqrt{x}$  at the point  $(9, 3)$ .

**Solution** Since  $f(x) = \sqrt{x} = x^{1/2}$ ,  $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ . Therefore:

$$\begin{aligned}\text{slope of tangent at } (9, 3) &= \text{slope of curve at } (9, 3) \\ &= f'(9) \\ &= \frac{1}{2\sqrt{9}} = \frac{1}{6}\end{aligned}$$

The tangent line has slope  $\frac{1}{6}$   
and contains the point  $(9, 3)$ .  
Thus, an equation of the line is

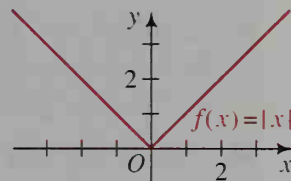
$$\begin{aligned}\frac{y-3}{x-9} &= \frac{1}{6}, \\ \text{or } y &= \frac{1}{6}x + \frac{3}{2}.\end{aligned}$$



## CLASS EXERCISES

In Exercises 1–10, find the derivative of each function. Express the derivative without using fractional or negative exponents.

1.  $f(x) = x^8$
2.  $f(x) = x^{12}$
3.  $f(x) = x^{-4}$
4.  $g(x) = 3x^7$
5.  $h(x) = x^{3/2}$
6.  $f(x) = 4x^{-3/2}$
7.  $g(x) = \sqrt[3]{x}$
8.  $h(x) = \frac{1}{\sqrt[3]{x}}$
9.  $f(x) = 8x^3 - 7x^2 + 3x - 2$
10.  $g(x) = 4x^{-5} - 2x^{-3} + 9x^{-1} + 5$
11. a. If  $f(x) = 4x^2$ , then  $f'(x) = \underline{\hspace{1cm}}$  and  $f'(-1) = \underline{\hspace{1cm}}$ .  
b. What is the slope of  $y = 4x^2$  at the point  $(-1, 4)$ ?
12. a. If  $f(x) = mx + k$  where  $m \neq 0$ , find  $f'(1)$ ,  $f'(2)$ , and  $f'(3)$ .  
b. **Visual Thinking** Give a geometric interpretation of the results of part (a).
13. **Discussion** The graph of  $f(x) = |x|$ , shown at the right, has a “corner” at  $x = 0$ . Using part (b) of Exercise 12, discuss why  $f'(0)$  is undefined. (See Example 3 on page 759.)



## WRITTEN EXERCISES

- A**
- $P(1, 1)$  is a point on the graph of  $y = x^3$ . Find the slope of  $\overline{PQ}$  if  $Q$  is:
    - $(2, 8)$
    - $(1.5, 3.375)$
    - $(1.1, 1.331)$
  - If  $f(x) = x^3$ , find  $f'(x)$  and  $f'(1)$ .
  - $P\left(2, \frac{1}{2}\right)$  is a point on the graph of  $y = \frac{1}{x}$ . Find the slope of  $\overline{PQ}$  if  $Q$  is:
    - $(3, f(3))$
    - $(2.5, f(2.5))$
    - $(2.1, f(2.1))$
  - If  $f(x) = \frac{1}{x}$ , find  $f'(x)$  and  $f'(2)$ .

Find the derivative of the given function. Express the derivative without using fractional or negative exponents.

- $f(x) = 2x^5$
- $f(x) = 3x^6$
- $f(x) = -2x^3$
- $f(x) = 8x^{3/4}$
- $f(x) = \frac{3}{x^2}$
- $g(x) = \frac{4}{\sqrt{x}}$
- $f(x) = \sqrt{5x}$
- $f(x) = \frac{1}{4x^4}$
- $g(x) = \frac{7}{2}x^2 - 5x + 3 - \frac{1}{x}$
- $f(x) = \frac{4}{3}x^3 - \frac{2}{x} - \frac{5}{x^2} + \pi$

Find the slope of each curve at the given point  $P$ .

- $y = x^3$ ;  $P(-1, -1)$
- $y = \frac{x^4}{2}$ ;  $P\left(-1, \frac{1}{2}\right)$
- $y = 3x^2 - 2x + 1$ ;  $P(0, 1)$
- $y = 2\sqrt{x}$ ;  $P(9, 6)$
- $y = 8x^4 - 7x^2 + 5x + 6$ ;  $P(-1, 2)$
- $y = x^3 - 5x^2 + 4x + 2$ ;  $P(2, -2)$
- B**  $y = \frac{2}{x} - \frac{4}{x^2}$ ;  $P(2, 0)$
- $y = \sqrt[3]{x}$ ;  $P(-1, -1)$
- If  $f(x) = ax^2 + bx + c$  where  $a \neq 0$ , find  $f'(x)$ .
  - Visual Thinking** Evaluate  $f'\left(-\frac{b}{2a}\right)$  and explain what the result says about the tangent to the parabola  $y = f(x)$  at the vertex.
- Visual Thinking** Give a geometric argument to explain why two functions whose rules differ only by a constant have the same derivative.

Find an equation of the line tangent to the given curve at the given point  $P$ .

- $y = x^3$ ;  $P(-2, -8)$
- $y = 3x^2 - 5x$ ;  $P(2, 2)$
- $y = \frac{1}{x^2}$ ;  $P(-1, 1)$
- $y = \frac{1}{\sqrt[3]{x}}$ ;  $P\left(8, \frac{1}{2}\right)$
- Sketch the graph of the function  $f(x) = 6x^2 - x^3$ .
  - For what values of  $x$  does  $f'(x) = 0$ ?
  - On your graph of  $f(x)$ , indicate the two points where  $f'(x) = 0$ .
- Repeat Exercise 27 using the function  $f(x) = x^3 - 9x$ .



29. a. If  $f(x) = x^{2/3}$ , find  $f'(x)$ .  
 b. For what values of  $x$  is  $f'(x)$  undefined?  
 c. Sketch the graph of  $f(x)$ . (If you use a computer or graphing calculator, you may need to enter the rule for the function as  $(x^2)^{1/3}$  to obtain the complete graph.) Explain how the graph of  $f(x)$  supports the result of part (b).
30. Repeat Exercise 29 using the function  $f(x) = x^{1/3}$ .

In Exercises 31–36, find a function that has the given derivative.

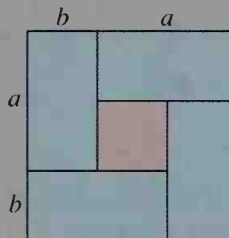
31.  $f'(x) = 4x^3$                       32.  $f'(x) = 5x^4$                       33.  $g'(x) = 3x^5$   
 34.  $g'(x) = x^7$                       35.  $h'(x) = 3\sqrt{x}$                       36.  $h'(x) = \sqrt[3]{x}$

- C** 37. a. Use the binomial theorem (page 591) to write the first three terms and the last term in the expansion of  $(x + h)^n$ .  
 b. Use part (a) and the definition of  $f'(x)$  to prove Theorem 2 on page 760 for positive integral values of  $n$ .  
 (Note: The binomial theorem can be generalized to apply to any nonzero real number  $n$ . This form of the binomial theorem can then be used to give a general proof of Theorem 2.)
38. Prove Theorem 3 on page 760. (Hint: Use the definition of  $f'(x)$ .)
39. Prove Theorem 4 on page 760. (Hint: Use the definition of  $f'(x)$ .)

### COMMUNICATION: Visual Thinking

The saying “a picture is worth a thousand words” is often true in mathematics. One well-drawn diagram may be a very convincing argument on its own, without being a formal proof. Diagrams and sketches can offer evidence that a conjecture may be true, and they can offer clues that lead to interesting discoveries.

For example, consider the diagram at the right, where the lengths  $a$  and  $b$  are unequal positive numbers.



1. Explain how the areas of the squares and rectangles in the diagram allow you to conclude that  $(a + b)^2 = (a - b)^2 + 4ab$ .
2. Explain how the saying “the whole is greater than its parts” allows you to conclude from the diagram that  $(a + b)^2 > 4ab$ .
3. By taking the square root of each side of the inequality above, what relationship do you discover between the arithmetic mean,  $\frac{a + b}{2}$ , and the geometric mean,  $\sqrt{ab}$ , of  $a$  and  $b$ ?

## 20-2 Using Derivatives in Curve Sketching

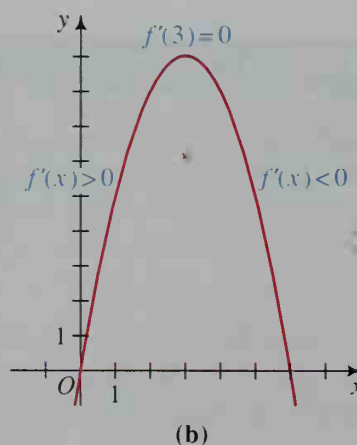
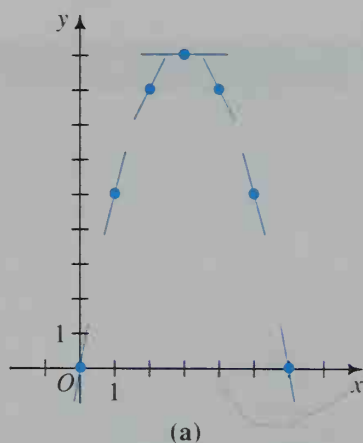
**Objective** To sketch the graphs of functions using derivatives.

In Section 2-3, we discussed techniques for sketching the graph of a polynomial function. In this section, we will see how the derivative of a polynomial function can be used in sketching the graph.

Consider the function  $f(x) = 6x - x^2$ , whose derivative is  $f'(x) = 6 - 2x$ . Values of  $x$ ,  $f(x)$ , and  $f'(x)$  are given in the following table.

$x$	0	1	2	3	4	5	6
$f(x) = 6x - x^2$	0	5	8	9	8	5	0
$f'(x) = 6 - 2x$	6	4	2	0	-2	-4	-6

Much can be learned about the graph of  $f(x)$  by plotting each point  $(x, f(x))$  from the table and then drawing the tangent to the curve at each point plotted. The tangent in each case is the line through  $(x, f(x))$  that has slope  $f'(x)$ . Figure (a) below shows these points and tangent lines. Figure (b) shows a smooth curve drawn through the points and having the corresponding tangent lines. The smooth curve is the graph of  $f(x)$ .



In figure (b), notice that as  $x$  increases, the curve rises whenever  $f'(x) > 0$ , and the curve falls whenever  $f'(x) < 0$ . When  $f'(x) = 0$ , the curve has a horizontal tangent and  $f(x)$  attains its maximum value.

We can perform a similar analysis for the graph of

$$f(x) = x^3 - 9x^2 + 24x - 14.$$

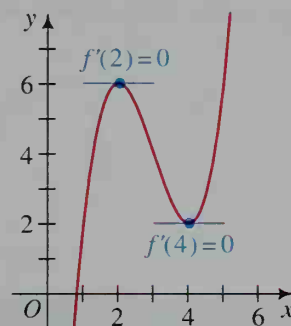
We have:

$$\begin{aligned} f'(x) &= 3x^2 - 18x + 24 \\ &= 3(x^2 - 6x + 8) \\ &= 3(x - 2)(x - 4) \end{aligned}$$

The signs of  $f'(x) = 3(x - 2)(x - 4)$  are given in the following table.

	$x < 2$	$x = 2$	$2 < x < 4$	$x = 4$	$x > 4$
$f'(x)$	+	0	-	0	+

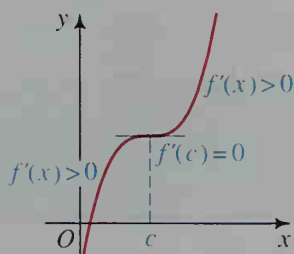
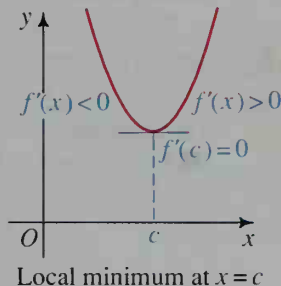
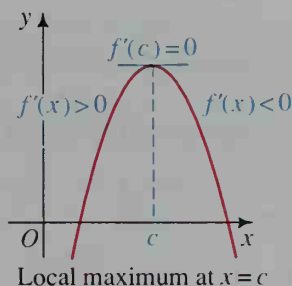
From the graph of  $f(x)$  shown at the right, we see that the curve is rising whenever  $f'(x) > 0$ , the curve is falling whenever  $f'(x) < 0$ , and the curve has a horizontal tangent when  $f'(x) = 0$ . A point such as  $(2, 6)$ , where  $f'(x) = 0$  and the curve changes from rising to falling as  $x$  increases, is called a *local maximum point*, meaning that there is no nearby point for which  $f(x) > 6$ . Similarly, a point such as  $(4, 2)$ , where  $f'(x) = 0$  and the curve changes from falling to rising as  $x$  increases, is called a *local minimum point*, meaning that there is no nearby point for which  $f(x) < 2$ .



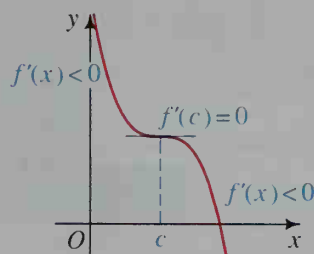
If the graph of a function  $f(x)$  has a local maximum (or minimum) point, then the  $y$ -coordinate of the point is a local maximum (or minimum) *value* of the function. Thus,  $f(x) = x^3 - 9x^2 + 24x - 14$  has a local maximum value of 6 at  $x = 2$  and a local minimum value of 2 at  $x = 4$ .

The observations that we have so far made for polynomial functions are valid for any function  $f(x)$  that has a derivative on an interval  $I$ :

1. If  $f'(x) > 0$  on an interval  $I$ , then the graph of  $f(x)$  rises as  $x$  increases. The function  $f(x)$  is said to be *increasing on the interval I*.
2. If  $f'(x) < 0$  on an interval  $I$ , then the graph of  $f(x)$  falls as  $x$  increases. The function  $f(x)$  is said to be *decreasing on the interval I*.
3. If  $f'(c) = 0$ , then the graph of  $f(x)$  has a horizontal tangent at  $x = c$ . The function may have a local maximum or minimum value, or neither, as shown below.



Neither local maximum nor local minimum at  $x = c$ ;  
 $f'(x)$  has the same sign on both sides of  $x = c$ .



**Example 1**

Graph  $f(x) = x^4 - 4x^3 + 15$ . Identify any local maximum and minimum points.

**Solution**

To determine where local maximum and minimum points may occur, we find  $f'(x)$  and solve  $f'(x) = 0$ .

$$f(x) = x^4 - 4x^3 + 15$$

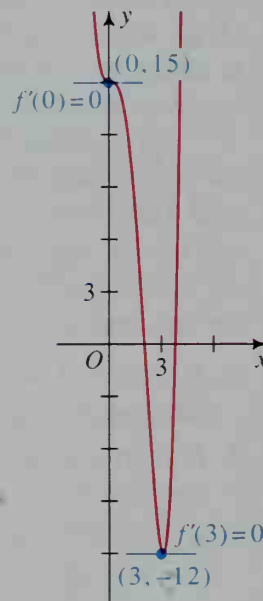
$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x - 3)$$

$$f'(x) = 0 \text{ when } x = 0 \text{ or } x = 3.$$

	$f'(x)$
$x < 0$	-
$x = 0$	0
$0 < x < 3$	-
$x = 3$	0
$x > 3$	+

Note that the graph of  $f(x)$  does *not* have a local maximum or minimum point at  $x = 0$ , since  $f'(x)$  is negative on both sides of  $x = 0$ . The graph of  $f(x)$  *does* have a local minimum point at  $x = 3$ , since  $f'(3) = 0$  and the curve changes from falling to rising. The local minimum point is  $(3, f(3)) = (3, -12)$ .

**Example 2**

Graph  $f(x) = x + x^{-1}$ . Identify any local maximum and minimum points.

**Solution**

To determine where local maximum and minimum points may occur, we find  $f'(x)$  and solve  $f'(x) = 0$ .

$$f(x) = x + x^{-1}$$

$$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$$

$$f'(x) = 0 \text{ when } x = -1 \text{ or } x = 1.$$

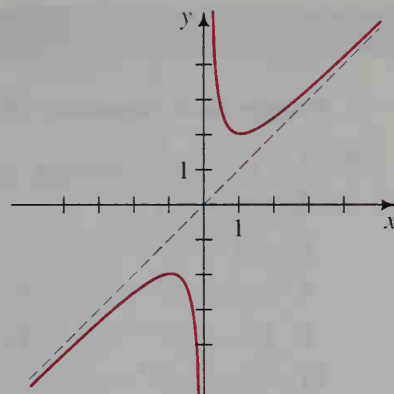
	$x < -1$	$x = -1$	$-1 < x < 1,$ $x \neq 0$	$x = 1$	$x > 1$
$f'(x)$	+	0	-	0	+

Neither  $f(x)$  nor  $f'(x)$  is defined at  $x = 0$ . A local maximum point is  $(-1, -2)$ . A local minimum point is  $(1, 2)$ . Observe that for large  $|x|$ ,  $y = x + x^{-1}$  is approximately the same as  $y = x$ . This indicates that  $y = x$  is an asymptote. Also, since

$$\lim_{x \rightarrow 0^+} f(x) = \infty$$

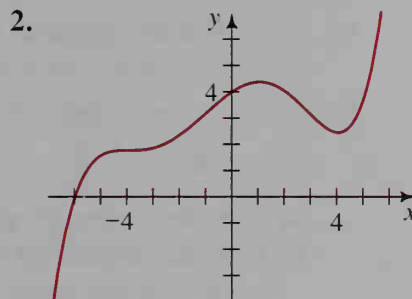
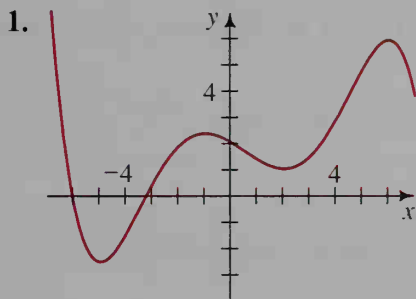
and  $\lim_{x \rightarrow 0^-} f(x) = -\infty$ ,

the  $y$ -axis is an asymptote.



## CLASS EXERCISES

**Visual Thinking** Study the graph of each polynomial function. Then give the values of  $x$  for which  $f'(x)$  is (a) equal to 0, (b) positive, and (c) negative.



- Visual Thinking** Is it possible for a function to have a local maximum value that is less than a local minimum value? Sketch a graph to illustrate.
- Suppose  $f(x)$  is a second-degree polynomial function such that  $f'(1) = 4$ ,  $f'(2) = 0$ , and  $f'(3) = -4$ . At  $x = 2$ , does  $f(x)$  have a local maximum value, a local minimum value, or neither?
- The derivative of a polynomial function  $f(x)$  has values as given in the table below. Give the values of  $x$  for which  $f(x)$  has local maximum and local minimum values.

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$f'(x)$	+	0	-	0	-	0	+

- Give the  $x$ -coordinates of the local maximum and minimum points on the graph of the function  $f(x) = \frac{1}{3}x^3 - 4x$ .



# WRITTEN EXERCISES

Graph each function. Identify any local maximum and minimum points.

- A**
- $f(x) = x^3 - 12x$
  - $f(x) = x^3 - 3x^2$
  - $f(x) = 9x - x^3$
  - $f(x) = x^3 + 4x^2 - 3x - 12$
  - $g(x) = 2x^2 - x^4$
  - $h(x) = 40 + 8x - 5x^2 - x^3$
  - $g(x) = x^4 - 6x^2 + 8$
  - $f(x) = x^4 - 8x^2 - 9$
  - $f(t) = 3t^4 - 4t^3 - 12t^2$
  - $g(t) = 8t^3 - 3t^4$
  - $h(x) = 5x^4 - 4x^5$
  - $f(x) = x^3 - 3x^2 - 9x + 27$

In Exercises 13–16, use the derivative to find the vertex of each parabola.

- $y = x^2 - 6x + 4$
- $y = 5 + 8x - 4x^2$
- $y = (x - a)^2$
- $y = ax^2 + bx + c$
- Refer to Class Exercise 1 and draw an approximate graph of  $f'(x)$ .
- Refer to Class Exercise 2 and draw an approximate graph of  $f'(x)$ .
- Show that the function  $f(x) = x^3 - 3x^2 + 3x - 5$  has neither a local maximum value nor a local minimum value.
- The symmetry point of the graph of the equation  $y = ax^3 + bx^2 + cx + d$  occurs at  $x = -\frac{b}{3a}$ .
  - Find the symmetry point  $S$  of the graph of  $y = x^3 - 6x^2 + 9x - 5$ .
  - Find the coordinates of the local maximum point  $P$  and the local minimum point  $Q$  of the graph of  $y = x^3 - 6x^2 + 9x - 5$ .
  - Show that  $P$ ,  $S$ , and  $Q$  are collinear.
  - Show that  $S$  is the midpoint of  $PQ$ .
- Evaluate  $\lim_{h \rightarrow 0} \frac{\sin(0 + h) - \sin 0}{h}$  to determine the derivative of  $\sin x$  when  $x = 0$ . (*Hint*: See Exercise 43 on page 725.)
  - Visual Thinking** Make a sketch of the curve  $y = \sin x$ , where  $x$  is in radians and the same scale is used on the  $x$ - and  $y$ -axes. Then, using your sketch and your answer to part (a), complete the table below.

At the point on the sine curve having $x$ -coordinate:	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
The slope of the tangent to the sine curve is:	?	?	?	?	?

- Using the table in part (b), try to guess what function is the derivative of  $\sin x$ .

22. Given that the derivative of  $f(x) = \cos x$  is  $f'(x) = -\sin x$ , show that the function  $g(x) = x + 2 \cos x$  has infinitely many local maximum and minimum values.

Graph each function. Identify local maximum and minimum points and asymptotes.

23.  $y = x + 2x^{-1}$

24.  $y = x^2 + x^{-2}$

**C** 25.  $y = x^2 + 3x + \frac{1}{x}$

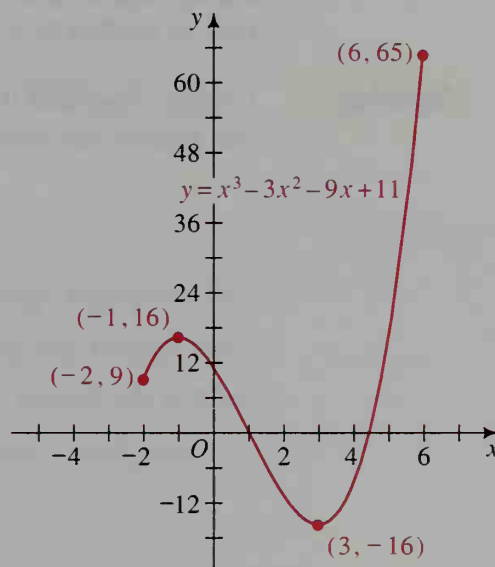
26.  $y = x^2 + \frac{1}{x}$

## 20-3 Extreme Value Problems

**Objective** To solve extreme value problems using derivatives.

Shown at the right is the graph of the function  $f(x) = x^3 - 3x^2 - 9x + 11$  on the interval  $-2 \leq x \leq 6$ . Notice that the function has local *extreme* values (that is, local maximum or minimum values) not only at  $x = -1$  and  $x = 3$ , where  $f'(x) = 0$ , but also at  $x = -2$  and  $x = 6$ , which are the *endpoints* of the interval.

Of the two local minimum values,  $f(-2) = 9$  and  $f(3) = -16$ , we call  $f(3) = -16$  the *global minimum value* of  $f(x)$  on the interval  $-2 \leq x \leq 6$  because there is no other  $x$ -value in the interval for which  $f(x)$  is smaller. Likewise,  $f(6) = 65$  is the *global maximum value* of  $f(x)$  on the interval.



### Activity

Use the graph of  $f(x) = x^3 - 3x^2 - 9x + 11$  shown above to answer the following questions.

- What are the global maximum and minimum values of  $f(x)$  on the interval  $-2 \leq x \leq 0$ ? on the interval  $0 \leq x \leq 4$ ? on the interval  $0 \leq x \leq 1$ ?
- Does  $f(x)$  have either a global maximum value or a global minimum value on the interval  $-2 < x < 6$ ? Explain.
- Consider the function  $g(x) = |f(x)|$ . What are the global maximum and minimum values of  $g(x)$  on the interval  $-2 \leq x \leq 6$ ?
- In part (c), what is true about  $g'(x)$  at those  $x$ -values for which  $g(x)$  has its global minimum value on the given interval?

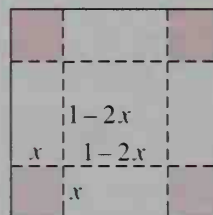
The preceding discussion and activity suggest that a continuous function  $f(x)$  always has a global maximum value and a global minimum value on the interval  $a \leq x \leq b$ . Moreover, these global extreme values will occur:

1. at the endpoints  $x = a$  or  $x = b$  of the interval,
2. at  $x$ -values between  $a$  and  $b$  where  $f'(x) = 0$ , or
3. at  $x$ -values between  $a$  and  $b$  where  $f'(x)$  is undefined.

When we find the global maximum or minimum value of a function on a given interval, we say that we are *maximizing* or *minimizing* the function.

### Example 1

Congruent squares are cut from the corners of a 1 m square piece of tin, and the edges are then turned up to make an open rectangular box. How large should the squares cut from the corners be in order to maximize the volume of the box?



### Solution

Letting  $x$  represent the length of a side of the square cut from a corner, we can express the volume  $V$  of the box as a function of  $x$ :

$$\begin{aligned} V(x) &= (1 - 2x)(1 - 2x)x \\ &= x - 4x^2 + 4x^3 \end{aligned}$$

The domain of  $V(x)$  is  $0 \leq x \leq \frac{1}{2}$ . Since  $V'(x) = 1 - 8x + 12x^2$  is always defined, the global maximum value of  $V(x)$  must occur at an endpoint of the domain or where  $V'(x) = 0$ . Since  $V(0) = V\left(\frac{1}{2}\right) = 0$ , we can eliminate these values. We next solve  $V'(x) = 0$ :

$$\begin{aligned} 1 - 8x + 12x^2 &= 0 \\ (1 - 6x)(1 - 2x) &= 0 \\ x = \frac{1}{6} \quad \text{or} \quad x &= \frac{1}{2} \end{aligned}$$

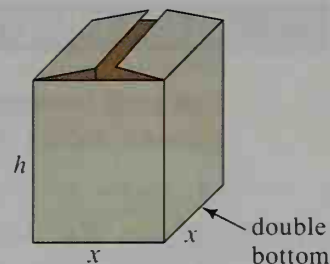
Since we have already eliminated  $x = \frac{1}{2}$ , we check  $x = \frac{1}{6}$ :

	$0 < x < \frac{1}{6}$	$x = \frac{1}{6}$	$\frac{1}{6} < x < \frac{1}{2}$
$V'(x) = (1 - 6x)(1 - 2x)$	+	0	-

Thus, the global maximum value of  $V(x)$  occurs at  $x = \frac{1}{6}$ . The squares cut from each corner should therefore be  $\frac{1}{6}$  m by  $\frac{1}{6}$  m for the box to have its greatest volume. Since  $V\left(\frac{1}{6}\right) = \frac{2}{27}$ , the maximum volume is  $\frac{2}{27} \text{ m}^3$ .

**Example 2**

A manufacturer produces cardboard boxes with square bases. The top of each box is a double flap that opens as shown. The bottom of the box has a double layer of cardboard for strength. If each box must have a volume of  $12 \text{ ft}^3$ , what dimensions will minimize the amount of cardboard used?

**Solution**

**Step 1** Let the dimensions of the box be  $x$ ,  $x$ , and  $h$ , as indicated above. Express the amount of cardboard  $C$  as a function of  $x$  and  $h$ :

$$\begin{aligned} C(x, h) &= \text{double bottom} + \text{top} + 4 \text{ sides} \\ &= 2x^2 + x^2 + 4xh \\ &= 3x^2 + 4xh \end{aligned}$$

**Step 2** Express  $C$  as a function of one variable,  $x$ , by using the fact that the volume is fixed at  $12 \text{ ft}^3$ :

$$\text{Volume} = x^2 h = 12$$

$$h = \frac{12}{x^2}$$

$$\text{Therefore, } C(x) = 3x^2 + 4x \left( \frac{12}{x^2} \right) = 3x^2 + \frac{48}{x}, \text{ where } x > 0.$$

**Step 3** Find  $C'(x)$  and solve  $C'(x) = 0$ :

$$C'(x) = 6x - 48x^{-2} = 0$$

$$6x = \frac{48}{x^2}$$

$$6x^3 = 48$$

$$x = 2$$

$$\text{Note that when } x = 2, h = \frac{12}{x^2} = 3.$$

**Step 4** Check to see whether  $x = 2$  gives a minimum or maximum value of  $C(x)$ :

	$0 < x < 2$	$x = 2$	$x > 2$
$C'(x) = 6x - 48x^{-2}$ $= \frac{6x^3 - 48}{x^2}$ $= \frac{6(x^3 - 8)}{x^2}$	-	0	+

Thus,  $C(x)$  has a minimum value when  $x = 2$ . The dimensions of the box should be 2 ft by 2 ft by 3 ft.

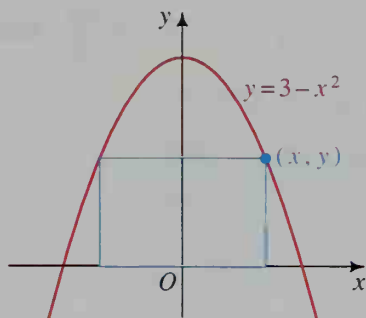
## CLASS EXERCISES

For each function defined on the given interval, determine where the global extreme values occur and what they are.

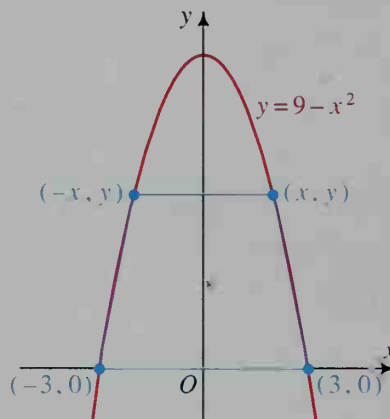
1.  $f(x) = 2x - 1$ ;  $0 \leq x \leq 3$
2.  $f(x) = -x + 3$ ;  $1 \leq x \leq 4$
3.  $g(x) = x^2 + 2$ ;  $-1 \leq x \leq 2$
4.  $g(x) = 2 - x^2$ ;  $-2 \leq x \leq 1$
5.  $h(x) = x^3 - 3x$ ;  $0 \leq x \leq 2$
6.  $h(x) = x^3 + 3x$ ;  $-2 \leq x \leq 2$
7.  $f(x) = 2 \sin x$ ;  $0 \leq x \leq \pi$
8.  $f(x) = \cos \frac{x}{2}$ ;  $0 \leq x \leq \pi$

## WRITTEN EXERCISES

- A** 1. The base of a rectangle is on the  $x$ -axis and the upper two vertices are on the parabola  $y = 3 - x^2$ , as shown at the left below.
- a. Show that the area of the rectangle is  $A(x) = 6x - 2x^3$ ,  $x \geq 0$ .
  - b. Show that the rectangle has the greatest area when  $x = 1$ .



Ex. 1



Ex. 2

2. An isosceles trapezoid is inscribed in the parabola  $y = 9 - x^2$  with the base of the trapezoid on the  $x$ -axis, as shown at the right above.
  - a. Show that the area of the trapezoid is  $A(x) = \frac{1}{2} (9 - x^2)(6 + 2x)$ ,  $x \geq 0$ .
  - b. Show that the trapezoid has its greatest area when  $x = 1$ .
3. A rectangle has one vertex at the origin, another on the positive  $x$ -axis, another on the positive  $y$ -axis, and the fourth vertex on the line  $y = 8 - 2x$ . What is the greatest area the rectangle can have?
4. A rectangle has area  $A$ , where  $A$  is a constant. The length of the rectangle is  $x$ .
  - a. Express the width of the rectangle in terms of  $A$  and  $x$ .
  - b. Express the perimeter in terms of  $A$  and  $x$ .
  - c. Find the length and width that minimize the perimeter.
  - d. Of all rectangles with a given area, which one has the least perimeter?

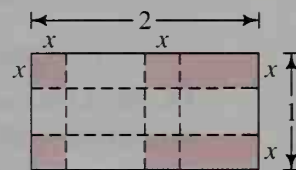


- B** 5. **Manufacturing** Congruent squares are cut from the corners of a rectangular sheet of metal that is 8 cm wide and 15 cm long. The edges are then turned up to make an open box. Let  $x$  be the length of a side of the square cut from each corner.

- Express the volume  $V$  of the box as a function of  $x$ .
- What value of  $x$  maximizes the volume?

6. **Manufacturing** A 1 m by 2 m sheet of metal is cut and folded, as shown, to make a box with a top.

- Express the volume  $V$  of the box as a function of  $x$ .
- What value of  $x$  maximizes the volume?



Ex. 6

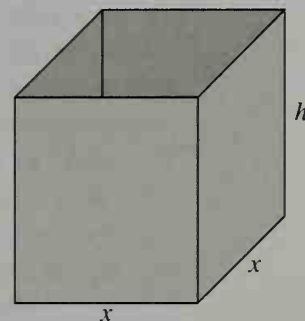
- The graphs of  $y = x$  and  $y = x^3$  intersect three times.
  - Find the coordinates of the points of intersection.
  - For what positive value of  $x$  between the intersection points is the vertical distance between the graphs the greatest?

8. **Manufacturing** A manufacturer produces a metal box with a square base, no top, and a volume of  $4000 \text{ cm}^3$ .

- If the base edges are  $x$  cm long, express the height  $h$  in terms of  $x$ .
- Show that the area of the metal used in the box is given by the function

$$A(x) = x^2 + 16,000x^{-1}.$$

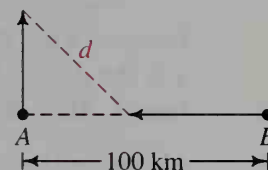
- What values of  $x$  and  $h$  minimize the amount of metal used to make the box?



9. **Business** A toy manufacturer can make  $x$  toy tool sets a day at a total cost of  $(300 + 12x + 0.2x^{3/2})$  dollars. Each set sells for \$18. How many sets should be made each day in order to maximize the profit?

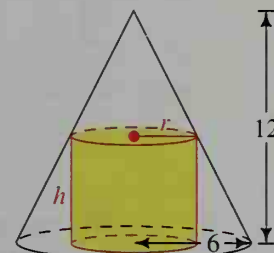
10. Point  $B$  is 100 km east of point  $A$ . At noon, a truck leaves  $A$  and travels north at 60 km/h, while a car leaves  $B$  and travels west at 80 km/h.

- Write an expression for the distance  $d$  between the car and the truck after  $t$  hours of travel.
- When will the distance between the vehicles be a minimum? (*Hint:* Minimize the *square* of the distance.)
- What is the minimum distance between the vehicles?



11. **Manufacturing** A box company produces a box with a square base and no top that has a volume of  $8 \text{ ft}^3$ . Material for the bottom costs  $\$6/\text{ft}^2$  and the material for the sides costs  $\$3/\text{ft}^2$ . Find the dimensions of the box that minimize the cost.

- C** 12. A cone has height 12 and radius 6. A cylinder is inscribed in the cone, as shown.
- Show that  $h = 12 - 2r$  for  $0 \leq r \leq 6$ .
  - Find the maximum volume of the inscribed cylinder.



Ex. 12

13. A cone is inscribed in a sphere with radius 3, as shown at the right.

a. Show that the volume of the cone is

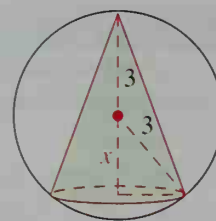
$$V(x) = \frac{1}{3} \pi (9 - x^2)(3 + x) \text{ for } 0 \leq x \leq 3.$$

b. Find the maximum volume of the cone.

14. Find the maximum volume of a cylinder inscribed in a sphere with radius 10.

15. **Business** It costs a construction company  $(6 + 0.004x)$  dollars per mile to operate a truck at  $x$  miles per hour. In addition, it costs \$14.40 per hour to pay the driver.

- a. What is the total cost per mile if the truck is driven at 30 mi/h? at 40 mi/h?
- b. At what speed should the truck be driven to minimize the total cost per mile?
- c. Why might the company decide to *not* require the use of the speed found in part (b)?



Ex. 13

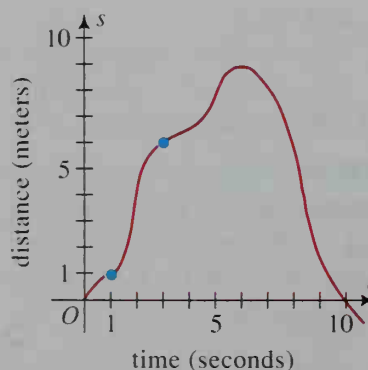
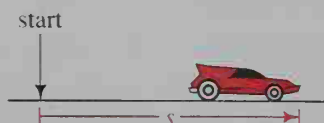


## 20-4 Velocity and Acceleration

### Objective

To find instantaneous velocities and accelerations.

Imagine a toy car that can move forward or backward along a long straight track. If  $s$  is the directed distance of the car from its starting point, then  $s$  is a function of time  $t$ . This *position function* is illustrated in the graph below.



We can tell a great deal about the motion of the car from the graph. For example, the car apparently slowed and nearly stopped at time  $t = 4$  s. At time  $t = 6$  s, the car stopped moving forward and began to move backward. At time

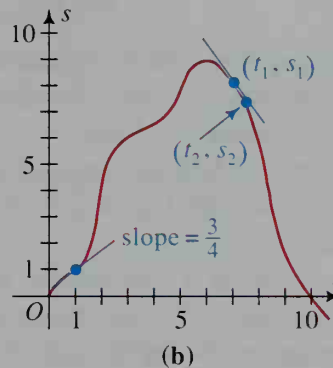
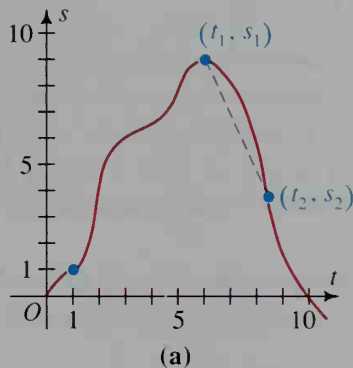
$t = 10$  s, the car returned to its starting point. After 10 s,  $s$  is negative, indicating that the car is to the left of its starting point.

We can also learn about the car's velocity from the graph. For example, if we consider the time interval from  $t = 1$  s to  $t = 3$  s, we see that the car has moved from 1 m to the right of its starting point to 6 m to the right, a distance of 5 m in 2 s. Thus, the *average velocity* of the car in that 2-second interval is  $\frac{5}{2} = 2.5$  m/s.

In general, suppose the positions of the car are  $s_1$  and  $s_2$  at times  $t_1$  and  $t_2$ , respectively. If we use the Greek letter  $\Delta$  (pronounced "delta") to denote "change in," then the car's change in position is  $\Delta s = s_2 - s_1$ , and the change in time is  $\Delta t = t_2 - t_1$ . Thus, the **average velocity** of the car is:

$$\begin{aligned}\text{average velocity} &= \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} \\ &= \text{slope of line joining } (t_1, s_1) \text{ and } (t_2, s_2)\end{aligned}$$

Notice that the average velocity can be interpreted geometrically as the slope of the line joining points  $(t_1, s_1)$  and  $(t_2, s_2)$  on the graph of the position function. When the slope is positive, the velocity is positive and the car is moving forward. When the slope is negative (as shown in figure (a) below), the velocity is negative and the car is moving backward.



When the time interval between  $t_1$  and  $t_2$  is very small, the line joining  $(t_1, s_1)$  and  $(t_2, s_2)$  is almost the same as the line tangent to the graph at  $(t_1, s_1)$ , as shown in figure (b) above. The slope of this tangent line is called the *instantaneous velocity* of the car at the time  $t_1$  to distinguish it from the average velocity of the car over the time interval containing  $t_1$ . For example, figure (b) shows that the instantaneous velocity of the car at time  $t = 1$  s is  $\frac{3}{4}$  m/s.

In general, if the position of an object is given as a function  $s(t)$  of time, then the **instantaneous velocity** at time  $t$  is defined as follows:

$$v(t) = \text{instantaneous velocity} = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t)$$

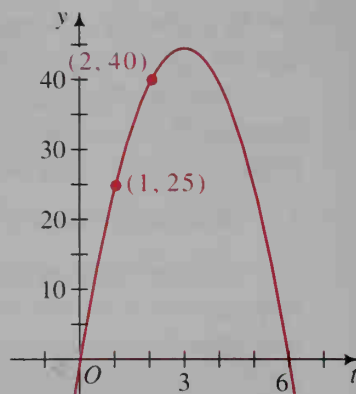
Note that the instantaneous velocity is the derivative of the position function.

Consider now a ball that is thrown vertically upward with an initial speed of 30 m/s. Its height in meters  $t$  seconds later is approximately:

$$h(t) = 30t - 5t^2$$

The average velocity of the ball between times  $t = 1$  s and  $t = 2$  s is:

$$\begin{aligned}\text{average velocity} &= \frac{h(2) - h(1)}{2 - 1} \\ &= \frac{40 - 25}{2 - 1} = 15 \text{ m/s}\end{aligned}$$



In the graph of  $h(t)$  at the right, notice that the average velocity is the slope of the line joining the points  $(1, 25)$  and  $(2, 40)$ .

For contrast, we can find the instantaneous velocity of the ball at times  $t = 1$  s and  $t = 2$  s:

$$\begin{aligned}\text{instantaneous velocity} &= h'(t) \\ h(t) &= 30t - 5t^2 \\ h'(t) &= 30 - 10t \\ h'(1) &= 20 \quad \text{and} \quad h'(2) = 10\end{aligned}$$

The instantaneous velocity of the ball is 20 m/s at  $t = 1$  s and 10 m/s at  $t = 2$  s. These values are, of course, the slopes of the tangents to the graph of the function at the points where  $t = 1$  and  $t = 2$ . The velocity of the ball is 0 m/s when  $t = 3$  s. At this time the ball has reached its maximum height of  $h(3) = 45$  m. The ball “stops” for an instant, changing its direction of motion from upward ( $h'(t) > 0$ ) to downward ( $h'(t) < 0$ ).

*Acceleration* is a measure of the rate of change of the velocity of an object. If the position function of the object is  $s(t)$ , then the velocity function is  $v(t) = s'(t)$ . The **average acceleration** from time  $t_1$  to  $t_2$  is:

$$\text{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{s'(t_2) - s'(t_1)}{t_2 - t_1}$$

The **instantaneous acceleration** at time  $t$  is defined as:

$$a(t) = \text{instantaneous acceleration} = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = v'(t)$$

Acceleration is the derivative of velocity, and velocity is the derivative of the position function. Acceleration is called the *second derivative* of the position function, and we write:

$$a(t) = s''(t)$$

Since acceleration is the rate of change of velocity over time, it is given in units such as meters per second per second, (m/s)/s, or feet per second per second, (ft/s)/s. These are often abbreviated as  $\text{m/s}^2$  and  $\text{ft/s}^2$ , respectively.

**Example 1**

An object moves along a straight line so that the object's distance in meters to the right of its starting point after  $t$  seconds is given by the function  $s(t) = t^3 - 3t^2 + 5t$ . Find the velocity and acceleration of the object at the end of the first two seconds of motion.

**Solution**

Position function:  $s(t) = t^3 - 3t^2 + 5t$

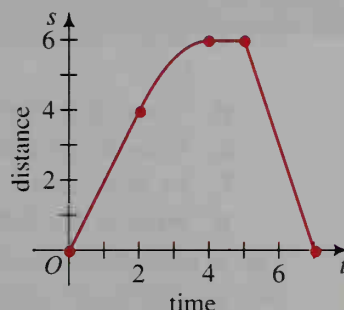
Velocity function:  $v(t) = s'(t) = 3t^2 - 6t + 5$

Acceleration function:  $a(t) = s''(t) = 6t - 6$  ← derivative of  $s'(t)$

Thus, when  $t = 2$  s, the velocity of the object is  $s'(2) = 5$ , or 5 m/s to the right, and the acceleration is  $s''(2) = 6$ , or 6 m/s<sup>2</sup> to the right.

**Example 2**

Suppose a particle travels along a number line. The graph at the right shows the particle's distance  $s$  to the right of the origin as a function of the time  $t$ . Sketch approximate graphs of (a) the velocity function  $v(t)$ , and (b) the acceleration function  $a(t)$ .

**Solution**

- a. The velocity function  $v(t)$  is the derivative of the distance function  $s(t)$ . This means that  $v(t)$  gives the slope of the graph of  $s(t)$ . Thus:

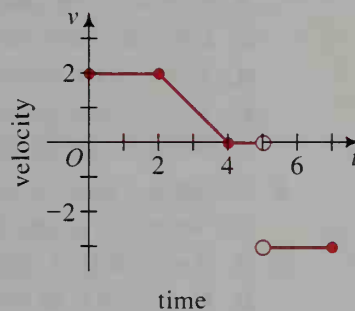
For  $0 \leq t \leq 2$ ,  $v(t) = 2$ .

For  $2 \leq t \leq 4$ ,  $v(t)$  decreases to 0.

For  $4 \leq t < 5$ ,  $v(t) = 0$ .

For  $5 < t \leq 7$ ,  $v(t) = -3$ .

Note that  $v(t)$  is not defined for  $t = 5$  because the graph of  $s(t)$  has a “corner” there.



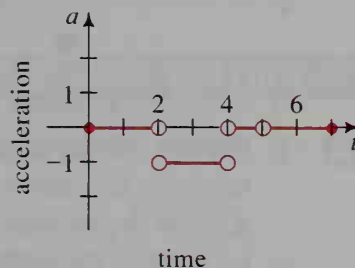
- b. The acceleration function  $a(t)$  is the derivative of the velocity function  $v(t)$ . This means that  $a(t)$  gives the slope of the graph of  $v(t)$ . Thus:

For  $0 \leq t < 2$ ,  $a(t) = 0$ .

For  $2 < t < 4$ ,  $a(t) = -1$ .

For  $4 < t < 5$ ,  $a(t) = 0$ .

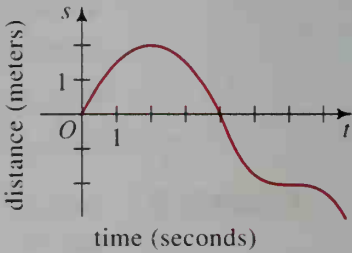
For  $5 < t \leq 7$ ,  $a(t) = 0$ .



Note that  $a(t)$  is not defined for  $t = 2$  and  $t = 4$  because the graph of  $v(t)$  has “corners” there. Also,  $a(t)$  is not defined at  $t = 5$  because  $v(t)$  is not defined there.



## CLASS EXERCISES

- A ball is thrown upward so that its height in feet above the ground after  $t$  seconds is  $h(t) = 48t - 16t^2$ .
  - What is the velocity at time  $t$ ? at time  $t = 1$  s?
  - What is the acceleration at time  $t$ ?
- Visual Thinking** A distance-time graph for a toy car is shown at the right.
 
  - How far does the toy car travel in the first two seconds?
  - What is its average velocity during the first two seconds?
  - When does the car move to the right? to the left?
  - When does the car return to its starting point?
  - What is the instantaneous velocity of the car at  $t = 1$  s?  $t = 2$  s?  $t = 3$  s?
  - At what times does the car move fastest to the right? to the left?
  - What is the farthest the car moves to the right during the first 6 seconds?
- A particle moves along a line so that its distance in meters to the right of its starting point after  $t$  seconds is  $s(t) = t^2 - 5t$ .
  - Does the particle begin moving left or right from the origin?
  - Where is the particle after 1 s?
  - How far does the particle move in the first two seconds?
  - What is its average velocity over the first two seconds?
  - When does the particle return to the origin?
  - What is the acceleration of the particle at time  $t = 1$  s? at  $t = 2$  s?
- Suppose a particle moves along a number line and its position to the right of the origin is given by the function  $s(t)$ , where  $t$  represents time.
  - If  $s(t) < 0$  at some time  $t$ , then the particle is ? at that time.
  - If  $s'(t) > 0$  at some time  $t$ , then the particle is ? at that time.
  - If  $s''(t) < 0$  at some time  $t$ , then the particle is ? at that time.

## WRITTEN EXERCISES

- A** **Physics** A ball is thrown upward from the top of an 80 ft building so that its height in feet above the ground after  $t$  seconds is  $h(t) = 80 + 64t - 16t^2$ .
  - Find the average velocity over the interval from  $t = 0$  s to  $t = 2$  s.
  - What is the instantaneous velocity at time  $t$ ? at time  $t = 1$  s?
  - When is the velocity equal to 0?
  - What is the ball's maximum height above the ground?
  - When does the ball hit the ground?
  - For what values of  $t$  is the ball falling?
  - What is the acceleration at time  $t$ ?
  - Graph the function  $h(t)$ .

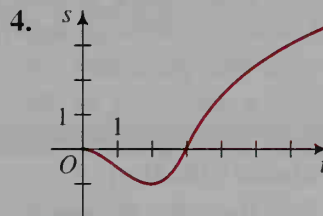
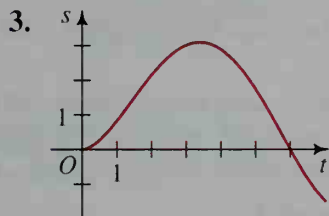
2. **Physics** A helicopter climbs vertically from the top of a 98 m building so that its height in meters above the ground after  $t$  seconds is:

$$h(t) = 98 + 49t - 4.9t^2$$

- Find the average velocity over the interval from  $t = 0$  s to  $t = 2$  s.
- What is the instantaneous velocity at time  $t$ ? at time  $t = 1$  s?
- When is the velocity equal to 0?
- What is the helicopter's maximum height above the ground?
- When does the helicopter reach the ground?
- For what values of  $t$  is the helicopter descending?
- What is the acceleration at time  $t$ ?
- Graph the function  $h(t)$ .

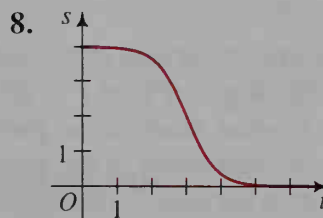
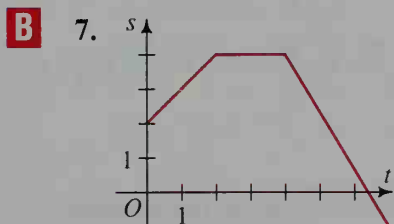


**Visual Thinking** In Exercises 3 and 4, answer the questions of Class Exercise 2 for the graphs shown below. In each graph,  $s$  is the distance (in meters) to the right of the starting point and  $t$  is the time (in seconds).



- Visual Thinking** Study the graph of  $s(t)$  in Exercise 3. Then sketch an approximate graph of the velocity function  $v(t)$ .
- Visual Thinking** Study the graph of  $s(t)$  in Exercise 4. Then sketch an approximate graph of the velocity function  $v(t)$ .

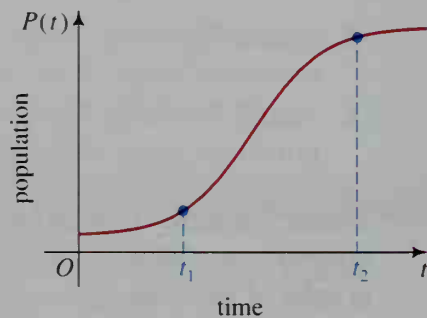
**Visual Thinking** In Exercises 7 and 8, copy the given graph of  $s(t)$ . Then, on separate sets of axes aligned vertically with the copied graph, sketch the graphs of  $v(t)$  and  $a(t)$ .



**Visual Thinking** In Exercises 9 and 10, read the given description of the motion of a particle along a number line. Then make an approximate graph (like those in Exercises 3 and 4) showing the distance of the particle to the right of the origin as a function of time.

9. A particle leaves the origin, traveling at a velocity of 1 m/s. Then it gradually slows until it reverses its direction at time  $t = 3$  s. It returns to the origin at  $t = 5$  s, traveling at a velocity of  $-2$  m/s and then gradually slows until it stops to the left of the origin at time  $t = 10$  s.
10. A particle leaves the origin, traveling right very slowly, and gradually increasing its velocity until it reaches 2 m/s at time  $t = 4$  s. Then the velocity begins to decrease and finally becomes zero at  $t = 8$  s. At this time, the particle begins to move slowly left, and at  $t = 20$  s it finally returns to its starting point and stops.
11. **Economics** If  $P(t)$  represents the general price level of goods and services in an economy at a given time  $t$ , then during an inflationary period,  $P(t)$  increases rapidly. Suppose that our economy is in such an inflationary period, but that the president's economic advisers find that the rate of inflation is slowing. In a subsequent address to the nation, the president announces that "while inflation is still with us, it has been brought under control" and predicts that "prices will soon stabilize."
  - a. Use a derivative to describe why "inflation is still with us."
  - b. Use a derivative to describe why the president believes that inflation "has been brought under control."
  - c. Use a derivative to describe the president's prediction that "prices will soon stabilize."
  - d. **Visual Thinking** Assuming that the president is correct, sketch a graph of  $P(t)$  from the present to the time when prices stabilize.

12. **Biology** At the right is the graph of  $P(t)$ , which shows the growth over time in the population of a protected endangered species.



- a. Compare the signs of  $P'(t_1)$  and  $P''(t_1)$  with  $P'(t_2)$  and  $P''(t_2)$  at times  $t_1$  and  $t_2$ .
- b. **Writing** What conclusions might you expect biologists who are studying the endangered species to make at times  $t_1$  and  $t_2$ ?
13. **Physics** A particle moves along a number line so that its distance to the right of the origin at time  $t$  is  $s(t) = 2t^3 - 6t^2 + 8$ .
  - a. At what times is the particle at the origin?
  - b. At what times is the particle not moving?
  - c. At what time is the velocity of the particle neither increasing nor decreasing; that is, at what time is the particle neither accelerating nor decelerating?

14. **Physics** Suppose a car is traveling so that its distance, in miles, west of Rockford at a time  $t$  hours after noon is  $s(t) = 10t^{3/2} - 15t + 20$ .
- How fast is the car going, and in which direction, at 12:15 P.M.?
  - Where is the car, and what is the time, when its velocity is zero?
- C** 15. **Physics** A ball is thrown horizontally from the top of a 144 ft building with an initial speed of 80 ft/s. Unlike Exercise 1 where the motion of the ball is along a line, here the motion is in a plane and the position  $(x, y)$  of the ball at time  $t$  seconds is a vector  $\mathbf{s}(t) = (x(t), y(t)) = (80t, 144 - 16t^2)$ .
- Make a sketch showing the path of the ball and its position at times  $t = 0$  s,  $t = 1$  s,  $t = 2$  s, and  $t = 3$  s.
  - Show that the distance between the ball's positions at times  $t = 1$  s and  $t = 2$  s is approximately 93.3 ft.
  - From  $t = 1$  s to  $t = 2$  s, the ball actually traveled along a curved path which is a little longer than 93.3 ft. Thus, its average speed over the time interval is somewhat more than 93.3 ft/s. Use similar reasoning to find an approximate average speed of the ball over the time interval from  $t = 1.9$  s to  $t = 2.0$  s.
  - Using calculus, one can show that the instantaneous velocity of the ball is the vector  $\mathbf{v}(t) = (x'(t), y'(t))$ . In other words, the components of the velocity vector are the derivatives of the components of the position vector. Moreover, the speed of the ball at time  $t$  is  $|\mathbf{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ . Use this information to find the velocity vector and speed of the ball when  $t = 2$  s and at the instant the ball hits the ground.

## Chapter Summary

- The *derivative* of a function  $f(x)$  is denoted  $f'(x)$  and gives the slope of the graph of  $f(x)$  at the point  $(x, f(x))$ . By definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

- Theorems 1–4 on pages 759 and 760 offer short cuts for calculating derivatives.
- The graph of a function  $f(x)$  rises where  $f'(x) > 0$  and falls where  $f'(x) < 0$ .
- A point  $(c, f(c))$  is a *local maximum point* or a *local minimum point* on the graph of a function  $f(x)$  if  $f'(c) = 0$  and  $f'(x)$  changes sign at  $x = c$ . The  $y$ -coordinate of a local maximum or minimum point is the *local maximum* or *minimum value* of the function.
- If  $s(t)$  is the position function of an object at time  $t$ , then the *instantaneous velocity* of the object at time  $t$  is:

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t)$$

The velocity function is the derivative of the position function.

6. If  $s(t)$  is the position function and  $v(t)$  is the instantaneous velocity function of an object at time  $t$ , then the object's *instantaneous acceleration* at time  $t$  is:

$$a(t) = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = v'(t) = s''(t)$$

The function  $a(t)$  is the *first* derivative of  $v(t)$  and the *second* derivative of  $s(t)$ .

### Key vocabulary and ideas

derivative (p. 758)

curve sketching (pp. 764–765)

local maximum (or minimum) point (p. 765)

local maximum (or minimum) value (p. 765)

position function (p. 774)

instantaneous velocity (p. 775)

instantaneous acceleration (p. 776)

second derivative (p. 776)

## Chapter Test

1. Find the derivative of each function.

20-1

a.  $f(x) = 5x^3 - 3x^2 + x - 1$

b.  $f(x) = \frac{3}{x^2} - \frac{2}{x} + 4$

c.  $f(x) = \sqrt[3]{3x^2} - \sqrt[3]{\frac{1}{x}}$

d.  $f(x) = \frac{2}{x^6} + 3x^6$

2. **Writing** Describe how to graph a function using its derivative.

20-2

3. Graph the function  $f(x) = 12x - x^3$ . Identify any local maximum and local minimum points.

4. Given a function  $f(x)$ , if  $f'(2) = 0$ ,  $f'(x) > 0$  for  $x < 2$ , and  $f'(x) < 0$  for  $x > 2$ , what can you say about  $f(2)$ ?

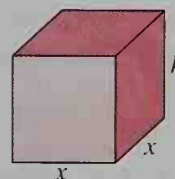
20-3

5. a. A rectangular solid has a square base, as shown. Express its surface area  $A$  as a function of  $x$  and  $h$ .

- b. If the volume of the solid is 1000 cubic units, express  $h$  in terms of  $x$ .

- c. Express  $A$  as a function of  $x$  alone.

- d. Show that the surface area is minimized when the solid is a cube.



6. **Physics** A particle is moving along a number line so that its position to the right of the origin at time  $t$  is  $s(t) = 3t^2 - 5t + 4$ . Find the position, velocity, and acceleration of the particle at time  $t = 5$ .

20-4

7. **Physics** An arrow is shot upward from the bottom of a canyon that is 336 ft below the edge of a cliff. If the initial velocity of the arrow is 160 ft/s, then  $s(t) = 160t - 16t^2$  gives the arrow's height, in feet, above the bottom of the canyon  $t$  seconds after the arrow is shot.

- a. In how many seconds does the arrow first pass the cliff's edge?  
 b. How high does the arrow travel before it starts to descend?  
 c. If it lands on the cliff's edge, how long was the arrow in the air?  
 d. At what velocity does the arrow hit the cliff's edge?





You will need to use a computer or programmable calculator to complete Exercises 3 and 7.

1. Evaluate each limit or state that it does not exist.

a.  $\lim_{x \rightarrow 1} \frac{2x^2 - x - 1}{2x^2 - 3x + 1}$

b.  $\lim_{x \rightarrow \infty} \frac{3x^2 + 2}{2x^2 + x}$

c.  $\lim_{x \rightarrow 2^+} \frac{x + 1}{2 - x}$

2. Sketch the graph of each function. Show vertical and horizontal asymptotes and  $x$ -intercepts.

a.  $y = \frac{x + 2}{1 - x}$

b.  $y = \frac{x}{x^2 - 4}$

c.  $y = \frac{x + 1}{x^2 + 1}$

3. a. Using rectangles of width 0.01, modify and run the program on page 730 to approximate the area under the curve  $y = \log x$  from  $x = 1$  to  $x = 2$ .

- b. **Visual Thinking** Using your answer to part (a), give an approximation of the area under the curve  $y = 1 + \log x$  from  $x = 1$  to  $x = 2$ . Also give a geometric argument to defend your answer.

4. a. Use series (4) on page 733 to find a power series for  $\tan^{-1} \frac{1}{x}$ .

- b. For what values of  $x$  is the series in part (a) convergent?

5. Consider the function  $f(x) = 0.2x - 1$ .

- a. Find the first four terms in the orbit of  $x_0 = 10$  for iterations of  $f(x)$ .

- b. Find  $\lim_{n \rightarrow \infty} f^n(10)$ .

6. Analyze all orbits for iterations of the function  $f(x) = x^2 + \frac{1}{4}$ .

7. **Writing** Modify and run the program on page 745 to iterate the logistic function  $f(x) = 2.5x(1 - x)$  for the seed  $x_0 = 0.3$ . Write a sentence or two to describe your findings.

8. Find the derivative of each function. Express the derivative without using fractional or negative exponents.

a.  $f(x) = 3x^2 - x + \frac{4}{x} - \frac{1}{2x^2}$

b.  $f(x) = \sqrt{2x} - \frac{2}{\sqrt{x}}$

9. Find the slope of the curve  $y = 1 - 2x^2$  at the point  $P(-1, -1)$ .

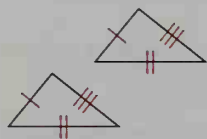
10. Use the derivative of  $f(x) = 3x^4 + 4x^3 - 12x^2 + 6$  to sketch the graph of  $f(x)$ . Identify any local maximum and minimum points.

11. From a square piece of cardboard 10 in. on a side, a box with a square base and an open top is to be constructed by cutting out four squares at the corners and folding up the sides. What should the dimensions of the box be in order to maximize the volume of the box?

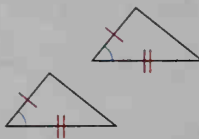
12. A particle moves along a line so that its distance in feet to the right of its starting point after  $t$  seconds is  $s(t) = 8t - t^2$ . Find the times during which the particle (a) moves to the right and (b) moves to the left.

# Geometry Review

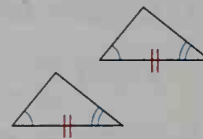
## 1. Conditions for congruent triangles



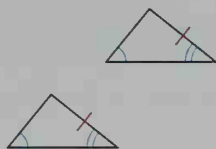
SSS: 3 pairs of corresponding sides congruent



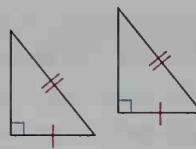
SAS: 2 pairs of corresponding sides and included angles congruent



ASA: 2 pairs of corresponding angles and included sides congruent

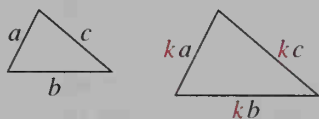


AAS: 2 pairs of corresponding angles and a pair of corresponding non-included sides congruent

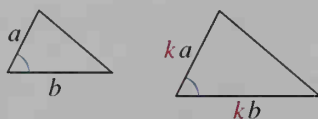


HL: for right triangles, one pair of corresponding legs and hypotenuses congruent

## 2. Conditions for similar triangles



SSS: 3 pairs of corresponding sides proportional

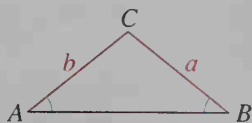


SAS: 2 pairs of corresponding sides proportional, and included angles congruent

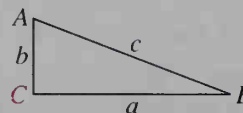


AA: 2 pairs of corresponding angles congruent

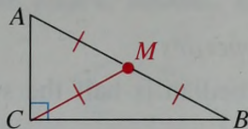
## 3. Special triangle relationships



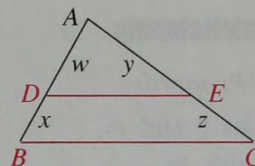
$a = b$  if and only if  $\angle A \cong \angle B$ .  
 $\angle A + \angle B + \angle C = 180^\circ$ .



$c^2 = a^2 + b^2$  if and only if  $\angle C = 90^\circ$ ;  
 otherwise,  $c^2 = a^2 + b^2 - 2ab \cos C$ .



The midpoint of the hypotenuse of a right triangle is equidistant from the 3 vertices.

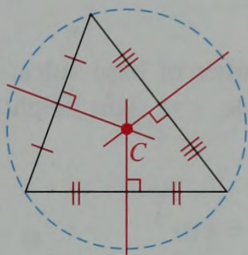


$\overline{DE} \parallel \overline{BC}$  if and only if

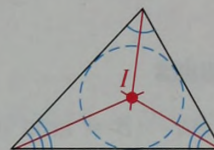
(1)  $\frac{w}{x} = \frac{y}{z}$ , or

(2)  $\frac{w}{w+x} = \frac{y}{y+z} = \frac{DE}{BC}$ .

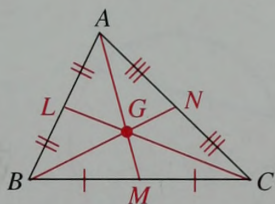
If  $w = x$  and  $y = z$ , then  $DE = \frac{1}{2}BC$ .



Perpendicular bisectors meet at circumcenter  $C$ .



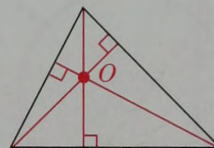
Angle bisectors meet at incenter  $I$ .



Medians meet at centroid  $G$ ;

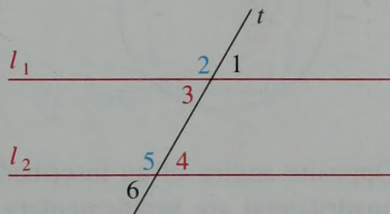
$$AG = \frac{2}{3}AM, \quad BG = \frac{2}{3}BN,$$

$$CG = \frac{2}{3}CL.$$



Altitudes meet at orthocenter  $O$ .

#### 4. Parallel lines



$l_1 \parallel l_2$  if and only if

(1)  $\angle 2 \cong \angle 5$ , or

(2)  $\angle 3 \cong \angle 4$ , or

(3)  $\angle 3$  and  $\angle 5$  are supplements.



## 5. Special quadrilaterals

### Parallelogram Properties

- Opposite sides  $\parallel$  and  $\cong$ .
- Opposite angles  $\cong$ .
- Diagonals bisect each other.
- If a rhombus: Diagonals  $\perp$ .
- If a rectangle: Diagonals  $\cong$ .

### Trapezoid Properties

- Length of median is half the sum of base lengths.
- Median  $\parallel$  bases.
- If isosceles: Base angles  $\cong$ .

## 6. Area formulas

Triangle:  $A = \frac{1}{2}bh$

Parallelogram:  $A = bh$

Trapezoid:  $A = \frac{1}{2}h(b_1 + b_2)$

Cone:  $A = \pi r(r + s)$  ( $s$  = slant height)

Regular Pyramid:  $A = B + \frac{1}{2}nas$  ( $B$  = area of base,  $n$  = number of base sides,  $a$  = length of one base side,  $s$  = slant height)

Circle:  $A = \pi r^2$

Sphere:  $A = 4\pi r^2$

Cylinder:  $A = 2\pi r(r + h)$

## 7. Volume formulas

Prism:  $V = Bh$  ( $B$  = area of base)

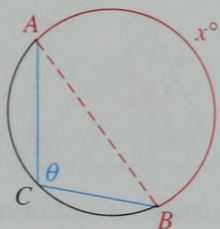
Cone:  $V = \frac{1}{3}\pi r^2 h$

Pyramid:  $V = \frac{1}{3}Bh$  ( $B$  = area of base)

Sphere:  $V = \frac{4}{3}\pi r^3$

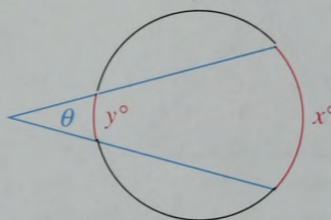
Cylinder:  $V = \pi r^2 h$

## 8. Circles and angle relationships

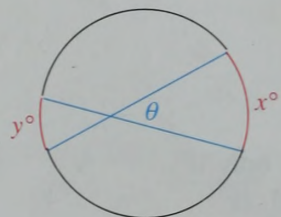


(1)  $\theta = \frac{1}{2}x$

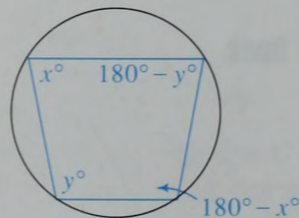
(2)  $\theta = 90^\circ$  if  $\overline{AB}$  is a diameter.



$\theta = \frac{1}{2}(x - y)$

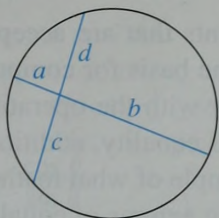


$\theta = \frac{1}{2}(x + y)$

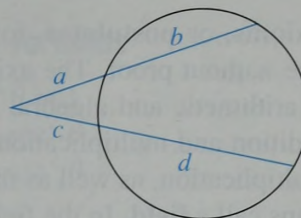


Opposite angles of an inscribed quadrilateral are supplementary.

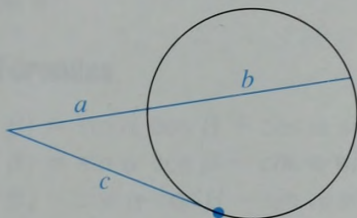
## 9. Circles and chord, secant, and tangent relationships



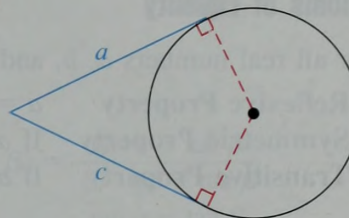
$$a \cdot b = c \cdot d$$



$$a \cdot (a + b) = c \cdot (c + d)$$



$$a \cdot (a + b) = c^2$$



- (1)  $a = c$
- (2) Radius  $\perp$  tangent.



## Properties of the Real Number System

**Axioms**, or **postulates**, for the real numbers are statements that are accepted as true without proof. The axioms for the real numbers are the basis for computation in arithmetic and algebra. A set of real numbers, together with the operations of addition and multiplication, that satisfies all the axioms of equality, addition, and multiplication, as well as the distributive axiom, is an example of what mathematicians call a **field**. In the field of real numbers, the following axioms of equality are assumed to be true.

### Axioms of Equality

For all real numbers  $a$ ,  $b$ , and  $c$ :

**Reflexive Property**  $a = a$

**Symmetric Property** If  $a = b$ , then  $b = a$ .

**Transitive Property** If  $a = b$  and  $b = c$ , then  $a = c$ .

The following *Substitution Property* is also useful.

**Substitution Property** For all real numbers  $a$  and  $b$ , if  $a = b$ , then  $a$  can be replaced by  $b$  in any expression without changing the value of the expression.

Throughout our discussion we will use the symbol  $\mathcal{R}$  to denote the set of real numbers. The symbol  $\in$  means “is a member of” or “belongs to.” Thus,  $a \in \mathcal{R}$  means “ $a$  belongs to the set of real numbers  $\mathcal{R}$ .” The following axioms are true for all  $a$ ,  $b$ , and  $c \in \mathcal{R}$  unless noted otherwise.

### Real Number Axioms

Axiom Name	Addition	Multiplication
<b>Closure</b>	$a + b$ is a unique real number.	$ab$ is a unique real number.
<b>Commutative</b>	$a + b = b + a$	$ab = ba$
<b>Associative</b>	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
<b>Identity</b>	There is a unique real number 0 such that $a + 0 = 0 + a = a$	There is a unique real number 1 such that $a \cdot 1 = 1 \cdot a = a$ .
<b>Inverse</b>	For every $a \in \mathcal{R}$ there is a unique $-a \in \mathcal{R}$ such that $a + (-a) = (-a) + a = 0$ .	For every nonzero $a \in \mathcal{R}$ there is a unique $\frac{1}{a} \in \mathcal{R}$ such that $a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1$ .
<b>Distributive</b>	$a(b + c) = ab + ac$	

# Trigonometry Review

## Definitions

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

## Addition Formulas

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

## Double Angle Formulas

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$= 2 \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

## Pythagorean Formulas

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

## Half-Angle Formulas

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

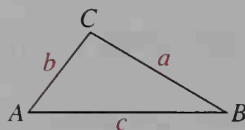
$$\begin{aligned} \tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} \\ &= \frac{1 - \cos \alpha}{\sin \alpha} \end{aligned}$$

## Area of Triangle

$$K = \frac{1}{2}ab \sin C$$

## Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



## Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}| |\vec{CB}|} \quad (\text{Vector form of law})$$

## Sectors

$\theta$  in radians

$\theta$  in degrees

Arc length  $s = r\theta$

$$s = \frac{\theta}{360} \cdot 2\pi r$$

Area  $K = \frac{1}{2}r^2\theta = \frac{1}{2}rs$

$$K = \frac{\theta}{360} \cdot \pi r^2$$

## The Scholastic Aptitude Test (SAT)

There are two 30-minute mathematical sections in the SAT. The questions (a total of 60) in these sections are designed to measure fundamental quantitative abilities closely related to college-level work.

In some questions you will be asked to apply graphic, spatial, numerical, symbolic, and logical techniques to problems that may be similar to exercises in your textbooks. Other questions may require you to do some original thinking.

The mathematical content is restricted to what is typically taught through first-year algebra and some geometry. The *arithmetic* includes simple addition, subtraction, multiplication, and division; percent; average; properties of odd and even numbers; prime numbers; and divisibility of numbers. The *algebra* includes negative numbers; simplifying algebraic expressions; factoring; linear equations; inequalities; simple quadratic equations; positive integer exponents; and roots. The *geometry* includes area, perimeter, circumference, and volume of a box and cube; special properties of isosceles, equilateral, and right triangles;  $30^\circ$ - $60^\circ$ - $90^\circ$  and  $45^\circ$ - $45^\circ$ - $90^\circ$  triangles; properties of parallel and perpendicular lines; and locating points on a coordinate grid.

Two kinds of multiple-choice questions are in the mathematical sections:

- (1) Standard multiple-choice questions (approximately  $\frac{2}{3}$  of the test)
- (2) Quantitative comparison questions (approximately  $\frac{1}{3}$  of the test)

Some sample questions of both types are given on the following pages. Answers to these questions appear at the end of the book.

*Note: The directions given for both types of questions, as well as the examples given for the quantitative comparison questions, are printed as they would appear in an actual test booklet. Please disregard references made to using extra space for scratchwork and filling in answers on answer sheets.*

The questions and accompanying text are reprinted, with permission, from copyrighted publications of the College Entrance Examination Board and Educational Testing Service, the copyright owner of the sample questions. Permission to reprint SAT Test and Achievement Test material does not constitute review or endorsement by Educational Testing Service or the College Board of this publication.

## Standard Multiple-Choice Questions

**Directions:** In this section solve each problem, using any available space on the page for scratchwork. Then decide which is the best of the choices given and fill in the corresponding oval on the answer sheet.

The following information is for your reference in solving some of the problems:

Circle of radius  $r$ :

$$\text{Area} = \pi r^2; \text{Circumference} = 2\pi r$$

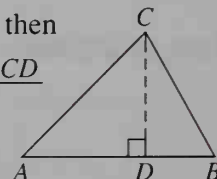
The number of degrees of arc in a circle is 360.  
The measure in degrees of a straight angle is 180.

**Triangle:** The sum of the measures in degrees of the angles of a triangle is 180.

If  $\angle CDA$  is a right angle, then

$$(1) \text{ area of } \triangle ABC = \frac{AB \times CD}{2}$$

$$(2) AC^2 = AD^2 + DC^2$$

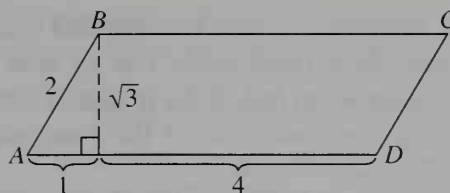


Definitions of symbols:

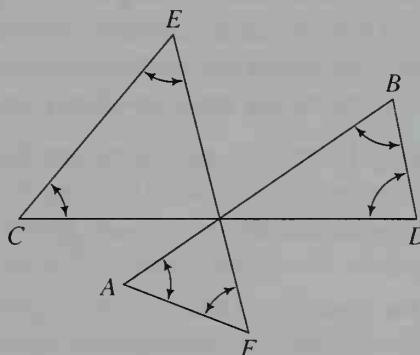
$=$  is equal to       $\neq$  is unequal to  
 $<$  is less than       $\leq$  is less than or equal to  
 $>$  is greater than       $\geq$  is greater than or equal to  
 $\parallel$  is parallel to       $\perp$  is perpendicular to

**Note:** Figures that accompany problems in this test are intended to provide information useful in solving the problems. They are drawn as accurately as possible EXCEPT when it is stated in a specific problem that its figure is not drawn to scale. All figures lie in a plane unless otherwise indicated. All numbers used are real numbers.

- If  $2a + b = 5$ , then  $4a + 2b =$   
(A)  $\frac{5}{4}$  (B)  $\frac{5}{2}$  (C) 10 (D) 20 (E) 25
- If  $16 \cdot 16 \cdot 16 = 8 \cdot 8 \cdot P$ , then  $P =$   
(A) 4 (B) 8 (C) 32 (D) 48 (E) 64
- The town of Mason is located on Eagle Lake. Canton is west of Mason. Sinclair is east of Canton, but west of Mason. Dexter is east of Richmond, but west of Sinclair and Canton. If these towns are in the United States, which town is farthest west?  
(A) Mason (B) Dexter (C) Canton  
(D) Sinclair (E) Richmond
- If the average of seven  $x$ 's is 7, what is the average of fourteen  $x$ 's?  
(A)  $\frac{1}{7}$  (B)  $\frac{1}{2}$  (C) 1 (D) 7 (E) 14
- If an asterisk (\*) between two expressions indicates that the expression on the right exceeds the expression on the left by 1, which of the following is (are) true for all real numbers  $x$ ?  
I.  $x(x + 2) * (x + 1)^2$  II.  $x^2 * (x + 1)^2$   
III.  $\frac{x}{y} * \frac{x + 1}{y + 1}$   
(A) None (B) I only (C) II only  
(D) III only (E) I and III
- If a car travels  $x$  kilometers of a trip in  $h$  hours, in how many hours can it travel the next  $y$  kilometers at this rate?  
(A)  $\frac{xy}{h}$  (B)  $\frac{hy}{x}$  (C)  $\frac{hx}{y}$  (D)  $\frac{h + y}{x}$   
(E)  $\frac{x + y}{h}$
- Which of the following fractions is greater than  $\frac{2}{3}$  and less than  $\frac{3}{4}$ ?  
(A)  $\frac{5}{8}$  (B)  $\frac{4}{5}$  (C)  $\frac{7}{12}$  (D)  $\frac{9}{16}$  (E)  $\frac{7}{10}$



- The figure above shows a piece of paper in the shape of a parallelogram with measurements as indicated. If the paper is tacked at its center to a flat surface and then rotated about its center, the points covered by the paper will be a circular region of diameter  
(A)  $\sqrt{3}$  (B) 2 (C) 5 (D)  $\sqrt{28}$   
(E)  $\sqrt{39}$
- An accurate 12-hour clock shows 3 o'clock at a certain instant. Exactly 11,999,999,995 hours later, what time does the clock show?  
(A) 5 o'clock (B) 7 o'clock  
(C) 8 o'clock (D) 9 o'clock  
(E) 10 o'clock



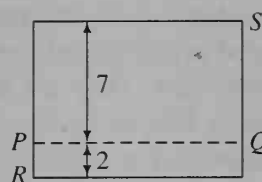
- In the triangles above, if  $AB$ ,  $CD$ , and  $EF$  are line segments, what is the sum of the measures of the six marked angles?  
(A)  $180^\circ$  (B)  $360^\circ$  (C)  $540^\circ$  (D)  $720^\circ$   
(E) It cannot be determined from the information given.
- If the area of a square is 16 and the coordinates of one corner are (3, 2), which of the following could be the coordinates of the diagonally opposite corner?  
(A) (7, 2) (B) (6, 7) (C) (3, 6)  
(D) (-1, 2) (E) (-1, 6)

12. A box with a square base is filled with grass seed. How many cubic feet of seed will it contain when full, if the height of the box is 2 feet and one side of the base measures 18 inches?

(A)  $3 \text{ ft}^3$  (B)  $4\frac{1}{2} \text{ ft}^3$  (C)  $6 \text{ ft}^3$  (D)  $36 \text{ ft}^3$   
(E)  $648 \text{ ft}^3$

13. The houses on the east side of a street are numbered with the consecutive even integers from 256 to 834 inclusive. How many houses are there on the east side of the street?

(A) 287 (B) 288 (C) 289 (D) 290  
(E) 291



14. If the 9-inch by 12-inch piece of paper shown in the figure above were folded flat along the line  $PQ$ , then  $R$  would be how many inches closer to  $S$ ?

(A) 4 (B) 2 (C)  $\sqrt{193} - \sqrt{153}$   
(D)  $15 - \sqrt{193}$  (E)  $\sqrt{193} - 13$

15. If 90 percent of  $p$  is 30 percent of  $q$ , then  $q$  is what percent of  $p$ ?

(A) 3% (B) 27% (C) 30% (D) 270%  
(E) 300%

## Quantitative Comparison Questions

Quantitative comparison questions emphasize the concepts of equalities, inequalities, and estimation. They generally involve less reading, take less time to answer, and require less computation than regular multiple-choice questions.

**Directions:** Each of the following questions consists of two quantities, one in Column A and one in Column B. You are to compare the two quantities and on the answer sheet fill in the oval

- A if the quantity in *Column A* is greater;  
B if the quantity in *Column B* is greater;  
C if the two quantities are equal;  
D if the relationship cannot be determined from the information given.

(Reminder: Since Questions 16–23 which follow are merely samples, no answer spaces are provided in this text.)

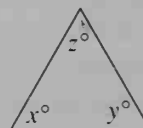
**Notes:** (1) In certain questions, information concerning one or both of the quantities to be compared is centered above the two columns.

(2) A symbol that appears in both columns represents the same thing in *Column A* as it does in *Column B*.

(3) Letters such as  $x$ ,  $n$ , and  $k$  stand for real numbers.

### EXAMPLES

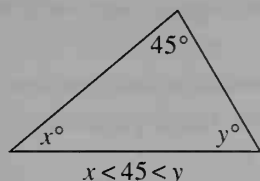
	Column A	Column B	Answers
E1.	$2 \times 6$	$2 + 6$	<input checked="" type="radio"/> A <input type="radio"/> B <input type="radio"/> C <input type="radio"/> D



E2.	$x + y$	$180 - z$	<input type="radio"/> A <input type="radio"/> B <input checked="" type="radio"/> C <input type="radio"/> D
E3.	$p - q$	$q - p$	<input type="radio"/> A <input type="radio"/> B <input type="radio"/> C <input checked="" type="radio"/> D

	Column A	Column B
		$a < 0$ $b < 0$
16.	$a + b$	$a - b$
17.	The average of (30 - 4) and (30 + 4)	The average of 28, 29, 30, 31, and 32
18.	$\sqrt{2} - 1$	$\sqrt{3} - 1$
		$0 < m < 6$ $0 < n < 8$
19.	$m$	$n$

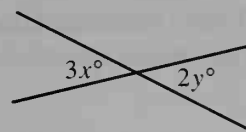




Note: Figure not drawn to scale.

20.	$y$	90
-----	-----	----

21.	$x + 1$	$2x + 1$
-----	---------	----------



22.	$x$	$y$
-----	-----	-----

23.	Area of a triangle with altitude 4	Area of a triangle with base 5
-----	------------------------------------	--------------------------------

## How the Tests Are Scored

The SAT Test, and also the Achievement Tests described next, are scored as follows: You get one point for each right answer, and you lose a fraction of a point for each wrong answer. You neither gain nor lose points for questions you omit. If you mark more than one answer for a question, it is scored like an omitted question.

For questions with 4 answer-choices  
(questions 16–23), your raw score =  
number right  $- \frac{1}{3}(\text{number wrong})$ .

For questions with 5 answer-choices  
(questions 1–15), your raw score =  
number right  $- \frac{1}{4}(\text{number wrong})$ .

## The Achievement Tests

The College Board offers three Achievement Tests in mathematics, Mathematics Level I, Level II, and Level IIC.

**Mathematics Level I** is a broad survey test that covers content typical of three years of college-preparatory mathematics (two years of algebra and a year of geometry). The Level I test contains questions in algebra, geometry, basic trigonometry, algebraic functions, and miscellaneous topics such as elementary statistics, counting problems, data interpretation, logic, elementary number theory, and arithmetic and geometric sequences.

Your total raw score is the sum of these raw scores. The College Board converts your raw score for the entire SAT test or achievement test to a scaled score between 200 and 800.

Because of the way the test is scored, haphazard or random guessing on questions you know nothing about is not recommended. When you know that one or more choices can be eliminated, however, guessing from the remaining choices should be to your advantage.

**Mathematics Level II** is intended for students who have taken college-preparatory mathematics for more than three years—two years of algebra and a year of geometry, elementary functions (precalculus), and/or trigonometry. The Level II test contains questions in algebra, geometry, trigonometry, elementary functions, and miscellaneous topics such as probability, permutations and combinations, statistics, logic and proof, elementary number theory, and sequences and limits.

**Mathematics Level IIC** requires the use of a scientific calculator. The mathematical preparation required and topics tested are the same as for the Level II test. Students taking IIC must use one of the calculators specified by The College Board.

The questions on the Level IIC test fall into three categories with respect to calculator use:

*Category (1)* Calculator inactive—there is no advantage (perhaps even a disadvantage) to using a calculator. See sample questions 1 and 11.

*Category (2)* Calculator neutral—these problems could be solved without a calculator, but a calculator may be useful. See sample questions 12 and 16.

*Category (3)* Calculator active—a calculator is necessary to solve these problems. See sample questions 4 and 9.

Approximately 40% of the questions are in category 1 and 60% are in categories 2 and 3. Skill in knowing both when and how to use a scientific calculator is required. If you have not used a scientific calculator on a regular basis, you should take the Level II test instead.

### Comparisons among the Tests: Level I, Level II, and Level IIC

Both Level I and Level II tests consist of approximately 50 multiple-choice questions with a 1 hour time limit. The content of the Level I test overlaps somewhat with that of the Level II and Level IIC tests, especially in questions on elementary algebra, coordinate geometry, statistics, and basic trigonometry. Consequently, some questions may be appropriate for any of these tests. However, the emphasis in the Level II and Level IIC tests is on more advanced content, with a greater percentage of questions devoted to trigonometry, elementary functions, and precalculus topics. Further, a significant percentage of questions on Level I is devoted to plane geometry, a topic not tested directly in Level II or IIC; the geometry questions on Levels II and IIC cover topics such as coordinate geometry in two or three dimensions, transformations, and solid geometry. The trigonometry questions on Level I are primarily limited to right triangle trigonometry and the fundamental relationships among the trigonometric ratios, whereas the Level II and IIC tests place more emphasis on the properties and graphs of the trigonometric functions, the inverse trigonometric functions, trigonometric equations and identities, and the laws of sines and cosines.

If you have had preparation in trigonometry and elementary functions and have attained grades of B or better in your mathematics

courses, you should select either the Level II or Level IIC test. Students who are sufficiently prepared to take the Level II or IIC test, but who elect to take the Level I test in hopes of receiving high scores, sometimes do not do as well as they expect. No student taking Level I, Level II, or Level IIC is necessarily expected to have studied every topic on the test.

The chart below shows approximately how the questions in each test are distributed among the major curriculum areas. Comparison of the percentages for the levels should help you decide which test you are better prepared to take.

Topics Covered	Approximate Percentage of Test	
	Level I	Levels II and IIC
Algebra	30	18
Geometry		
Plane Geometry	20	—
Solid Geometry	6	8
Coordinate Geometry	12	12
Trigonometry	8	20
Functions	12	24
*Miscellaneous	12	18

\*In Level I, one-half of these questions include elementary statistics, counting problems, and data interpretation. In Level II and Level IIC, one-third of these questions include probability, permutations and combinations, and statistics.

As a general rule, do not try to work a problem by testing each of the answer choices. Such a method is time-consuming and, in many instances, will not work. However, you should look at the answer choices while studying the question. The form of the answer choices

may help to clarify the problem.

The following sample questions are reprinted, with permission, from copyrighted publications of the College Board and Educational Testing Service, the copyright owner of the questions.

## Mathematics Level I

**Directions:** For each of the following problems, decide which is the best of the choices given. Then fill in the corresponding oval on the answer sheet.

**Notes:** (1) Figures that accompany problems in this test are intended to provide information useful in solving the problems. They are drawn

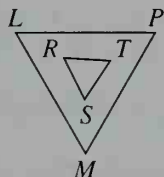
as accurately as possible EXCEPT when it is stated in a specific problem that its figure is not drawn to scale. All figures lie in a plane unless otherwise indicated.

(2) Unless otherwise specified, the domain of a function  $f$  is assumed to be the set of all real numbers  $x$  for which  $f(x)$  is a real number.

### MATHEMATICS LEVEL I TEST-SAMPLE QUESTIONS

1. If  $\frac{3^x}{3^2} = 3^8$ , then  $x =$

- (A) 4 (B) 6 (C) 10 (D) 16 (E) 64



2. The figure above consists of two equilateral triangles,  $LMP$  and  $RST$ . If  $RS = \frac{1}{3}LM$ , then

$$\frac{\text{area } \triangle RST}{\text{area } \triangle LMP} =$$

- (A)  $\frac{1}{9}$  (B)  $\frac{1}{6}$  (C)  $\frac{1}{4}$  (D)  $\frac{1}{3}$  (E)  $\frac{1}{\sqrt{3}}$

3. The positive difference between the mean and the median of the numbers 27, 27, 29, 32, and 35 is

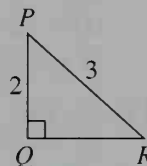
- (A) 0 (B) 1 (C) 2 (D) 3 (E) 8

4. If  $i = \sqrt{-1}$ , which of the following is an expression for  $\sqrt{-49} - \sqrt{-25}$  in the form  $a + bi$ , where  $a$  and  $b$  are real numbers?

- (A)  $0 + (\sqrt{24})i$  (B)  $0 + (\sqrt{74})i$   
(C)  $0 + 24i$  (D)  $0 + 2i$  (E)  $2 + 0i$

5. In the figure,  $\sin \angle P =$

- (A)  $\frac{2}{5}$  (B)  $\frac{2}{3}$  (C)  $\frac{\sqrt{5}}{3}$   
(D)  $\frac{\sqrt{5}}{2}$  (E)  $\frac{3}{2}$



6. If  $x + 2 = y$ , what is the value of  $|x - y| + |y - x|$ ?

- (A) -4 (B) 0 (C) 2 (D) 4  
(E) It cannot be determined.

7. Two rectangular solids have dimensions 4, 6,  $h$ , and 8, 2,  $(2h - 1)$ , respectively. Their volumes are equal when  $h =$

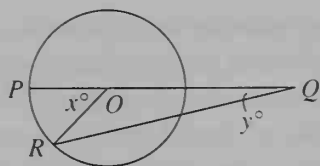
- (A)  $\frac{1}{8}$  (B)  $\frac{4}{5}$  (C) 1 (D) 2 (E) 4

8. If  $f(x) = 5x + 6$ , then  $f(x) < 16$  if and only if

- (A)  $x < 2$  (B)  $x > 2$  (C)  $x = 2$   
(D)  $-2 < x < 2$  (E)  $x$  is any real number

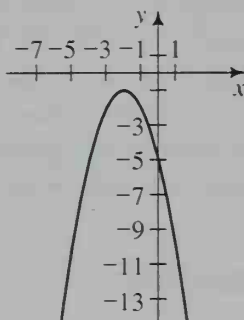
9. If  $x$  is the measure of an acute angle such that  $\tan x = \frac{k}{3}$ , then  $\sin x =$

- (A)  $\frac{k}{3 + k}$  (B)  $\frac{3}{\sqrt{9 - k^2}}$  (C)  $\frac{k}{\sqrt{9 - k^2}}$   
(D)  $\frac{3}{\sqrt{9 + k^2}}$  (E)  $\frac{k}{\sqrt{9 + k^2}}$

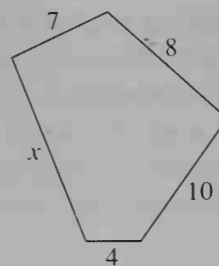


10. The circle in the figure above has center at  $O$ . If  $PQ$  and  $QR$  are secants and if  $x = 40$ , what is  $y$ ?
- (A) 10 (B) 20 (C) 30 (D) 40  
(E) It cannot be determined.

11. On the curve shown in the figure, determine the  $y$ -coordinate of each point at which  $y = 2x$ .



- (A) There is no such point.  
(B)  $-1$   
(C)  $-2$  only  
(D)  $-5$   
(E)  $-2$  and  $-10$
12. In how many points do the graphs of  $x^2 + y^2 = 9$  and  $x^2 = 8y$  intersect?  
(A) One (B) Two (C) Three (D) Four  
(E) More than four
13. If  $f(x) = 2x + 1$  and  $g(x) = 3x - 1$ , then  $f(g(x)) =$   
(A)  $6x - 1$  (B)  $6x + 2$  (C)  $x - 2$   
(D)  $5x$  (E)  $6x^2 + x - 1$
14. If  $a$  and  $b$  are positive integers and if 9 is a factor of their product, which of the following must be true?  
(A) 9 is a factor of both  $a$  and  $b$ .  
(B) 9 is a factor of one number but not of the other.  
(C) 9 is a factor of at least one of the numbers  $a$ ,  $b$ .  
(D) 3 is a factor of both  $a$  and  $b$ .  
(E) 3 is a factor of at least one of the numbers  $a$ ,  $b$ .



Note: Figure not drawn to scale.

15. In the figure above, a pentagon has four of its sides with lengths 4, 7, 8, and 10. If the length  $x$  of the fifth side is also an integer, then the greatest possible value of  $x$  is  
(A) 16 (B) 22 (C) 26 (D) 28 (E) 30
16. If one angle of a rhombus is  $60^\circ$ , then the ratio of the length of its longer diagonal to the length of its shorter diagonal is  
(A) 2 (B)  $\sqrt{3}$  (C)  $\sqrt{2}$  (D)  $\frac{\sqrt{3}}{2}$   
(E)  $\frac{\sqrt{2}}{2}$
17. The solution set of  $2x^2 - 4x = 30$  is  $\{-3, 5\}$ . What is the solution set of  $2(x - 4)^2 - 4(x - 4) = 30$ ?  
(A)  $\{-12, 20\}$  (B)  $\{-7, 1\}$   
(C)  $\{-\frac{3}{4}, \frac{5}{4}\}$  (D)  $\{1, 9\}$  (E)  $\{12, -20\}$
18. Which quadrants of the plane contain points of the graph  $2x - y > 4$ ?  
(A) First, second, and third only  
(B) First, second, and fourth only  
(C) First, third, and fourth only  
(D) Second, third, and fourth only  
(E) First, second, third, and fourth
19. If  $x = bc$  and  $y = bd$ , then  $y - x =$   
(A)  $bc(1 - d)$  (B)  $bd(1 - c)$   
(C)  $d(b - c)$  (D)  $b(d - c)$   
(E)  $c(b - d)$
20. The graph of a certain linear function has a negative slope. If its  $x$ -intercept is negative, then its  $y$ -intercept must be  
(A) positive (B) negative (C) nonnegative  
(D) less than its  $x$ -intercept  
(E) greater than its  $x$ -intercept

## Mathematics Level II

**Directions:** For each of the following problems, decide which is the best of the choices given. Then fill in the corresponding oval on the answer sheet.

**Notes:** (1) Figures that accompany problems in this test are intended to provide information useful in solving the problems. They are drawn

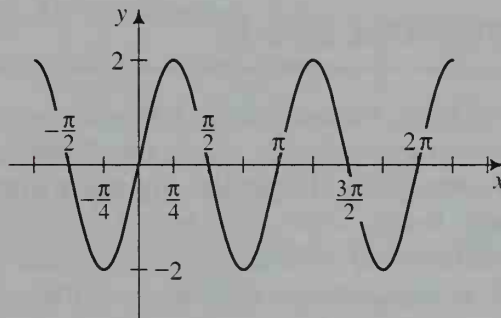
as accurately as possible EXCEPT when it is stated in a specific problem that its figure is not drawn to scale. All figures lie in a plane unless otherwise indicated.

(2) Unless otherwise specified, the domain of a function  $f$  is assumed to be the set of all real numbers  $x$  for which  $f(x)$  is a real number.

### MATHEMATICS LEVEL II TEST – SAMPLE QUESTIONS

- If  $f(x) = x^2 - x^3$ , then  $f(-1) =$   
(A) 2 (B) 1 (C) 0 (D) -1 (E) -2
- The base of an equilateral triangle lies on the  $x$ -axis. What is the sum of the slopes of the three sides?  
(A) -1 (B) 0 (C) 1 (D)  $2\sqrt{3}$   
(E)  $1 + 2\sqrt{3}$
- For what real numbers  $x$  is  $y = 2^{-x}$  a negative number?  
(A) All real  $x$  (B)  $x > 0$  only  
(C)  $x \geq 0$  only (D)  $x < 0$  only  
(E) No real  $x$
- If  $f(x) = 2x$  and  $f(g(x)) = -x$ , then  $g(x) =$   
(A)  $-3x$  (B)  $-\frac{x}{2}$  (C)  $\frac{x}{2}$  (D)  $2 - \frac{x}{2}$   
(E)  $x$
- $\lim_{n \rightarrow \infty} \frac{3n + 12}{2n - 12} =$   
(A) -6 (B)  $-\frac{3}{2}$  (C) -1 (D)  $\frac{2}{3}$  (E)  $\frac{3}{2}$
- If  $0 < y < x < \frac{\pi}{2}$ , which of the following are true?  
I.  $\sin y < \sin x$   
II.  $\cos y < \cos x$   
III.  $\tan y < \tan x$   
(A) None (B) I and II only  
(C) I and III only (D) II and III only  
(E) I, II, and III
- $\sin \left( \arcsin \frac{1}{10} \right) =$   
(A) 0 (B)  $\frac{1}{10}$  (C)  $\frac{1}{9}$  (D)  $\frac{9}{10}$  (E) 1

- If  $f(x) = \log_{10}(x + 3)$  and  $f^{-1}$  denotes the inverse of  $f$ , then  $f^{-1}(2) =$   
(A)  $\frac{1}{10}$  (B)  $\frac{1}{2}$  (C) 97 (D) 103  
(E) 100,000
- The graph of  $\begin{cases} x = 4t - 2 \\ y = 4t^2 \end{cases}$  in the  $xy$ -plane is  
(A) a circle (B) a parabola  
(C) an ellipse that is not a circle  
(D) a hyperbola (E) a straight line



- Which of the following equations has the graph shown in the figure above?  
(A)  $y = \sin \frac{x}{2} + 1$  (B)  $y = \sin 2x$   
(C)  $y = 2 \sin \frac{x}{2}$  (D)  $y = 2 \sin x$   
(E)  $y = 2 \sin 2x$
- All real numbers  $x$  and  $y$  that satisfy the equation  $x + iy = \frac{1}{x - iy}$  also satisfy which of the following equations?  
(A)  $x^2 - y^2 = 0$  (B)  $x^2 - y^2 = 1$   
(C)  $x^2 + y^2 = 0$  (D)  $x^2 + y^2 = 1$   
(E)  $xy = 1$



12. Lines  $l_1$  and  $l_2$  both intersect line  $l_3$  at right angles, but do not necessarily lie in the same plane. Which of the following statements about  $l_1$  and  $l_2$  is correct?

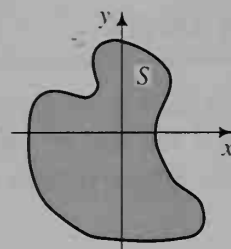
(A) They are necessarily parallel to each other.  
 (B) They cannot be parallel to each other.  
 (C) They are necessarily either parallel or perpendicular to each other.  
 (D) They are necessarily perpendicular to each other.  
 (E) They are not necessarily parallel or perpendicular to each other.

13. The probability that  $R$  hits a certain target is  $\frac{3}{5}$  and the probability that  $T$  hits it is  $\frac{5}{7}$ . What is the probability that  $R$  hits the target and  $T$  misses it?

(A)  $\frac{4}{35}$  (B)  $\frac{6}{35}$  (C)  $\frac{3}{7}$  (D)  $\frac{21}{25}$  (E)  $\frac{31}{35}$

14.  $\frac{(n+1)!}{n!} - n =$

(A) 0 (B) 1 (C)  $n$  (D)  $n+1$  (E)  $n!$



15. In the figure, shaded region  $S$  has area 10. What is the area of region  $T$ , which consists of all points  $(x-1, y+4)$  where the point  $(x, y)$  is in  $S$ ?

(A) 40 (B) 30 (C) 20 (D)  $10\sqrt{2}$   
 (E) 10

16. The graph of  $\begin{cases} x = 2 \cos \theta \\ y = 4 \sin \theta \end{cases}$  in the  $xy$ -plane is

(A) a line (B) a circle of radius 2  
 (C) a circle of radius 4  
 (D) an ellipse with  $x$ -intercepts  $\pm 4$   
 (E) an ellipse with  $y$ -intercepts  $\pm 4$

## Mathematics Level IIC

**Directions:** For each of the following problems, look for your solution among the choices given. **In some cases, the choices given are approximate.** Decide which is the best of the choices by rounding your solution when appropriate. Then fill in the corresponding oval on the answer sheet.

**Notes:** (1) A calculator may be necessary for answering some (but not all) of the questions in this test. For each question, you will have to decide whether you should use a calculator. The calculator you use should be a scientific calculator; programmable calculators and calculators that can display graphs are NOT permitted.

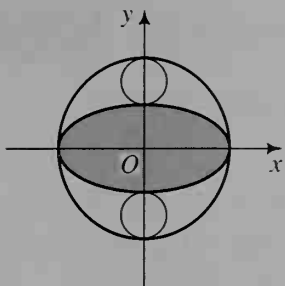
(2) For some questions in this test you may have to decide whether your calculator should be in the radian mode or the degree mode.

(3) Figures that accompany problems in this test are intended to provide information useful in solving the problems. They are drawn as accurately as possible EXCEPT when it is stated in a specific problem that its figure is not drawn to scale. All figures lie in a plane unless otherwise indicated.

(4) Unless otherwise specified, the domain of a function  $f$  is assumed to be the set of all real numbers  $x$  for which  $f(x)$  is a real number.

## MATHEMATICS LEVEL IIC TEST – SAMPLE QUESTIONS

1. If  $f(x) = 2x$  and  $f(g(x)) = -x$ , then  $g(x) =$   
 (A)  $-3x$  (B)  $-\frac{x}{2}$  (C)  $\frac{x}{2}$   
 (D)  $2 - \frac{x}{2}$  (E)  $x$
2. If  $x - 2$  is a factor of  $x^3 + kx^2 + 12x - 8$ , then  $k =$   
 (A)  $-6$  (B)  $-3$  (C) 2 (D) 3 (E) 6



3. The formula for the area enclosed by an ellipse is  $A = \pi ab$ , where  $a$  and  $b$  are one-half the respective lengths of the major and minor axes. In the figure, both small circles have diameter 2, and the equation of the larger circle is  $x^2 + y^2 = 15$ . What is the area enclosed by the ellipse?

(A) 20 (B) 23 (C) 26 (D) 29 (E) 32

4. If  $2x^2 + 4x = 3$ , then, to the nearest tenth, what is the positive value of  $x$ ?

(A) 0.6 (B) 0.8 (C) 1.2 (D) 1.6  
(E) There is no positive value of  $x$  that satisfies the equation.

5. If  $f(x) = (x + 3)^2 + 1$ , what is the minimum value of the function  $f$ ?

(A) -3 (B) 0 (C) 1 (D) 3 (E) 4

6. What is the area of a right triangle with an angle of  $28^\circ$  and with longer leg of length 13?

(A) 40 (B) 45 (C) 75 (D) 90 (E) 159

7. If  $\log_2 3 = x$ , then  $x =$

(A) 0.4 (B) 0.6 (C) 1.6 (D) 2.5  
(E) 3.0

8.  $\lim_{n \rightarrow \infty} \frac{3n + 12}{2n - 12} =$

(A) -6 (B)  $-\frac{3}{2}$  (C) -1

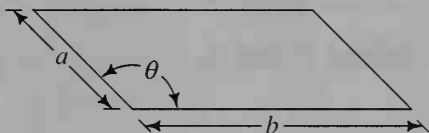
(D)  $\frac{2}{3}$  (E)  $\frac{3}{2}$

9. The diameter and height of a right circular cylinder are equal. If the volume of the cylinder is 2, what is the height of the cylinder?

(A) 1.37 (B) 1.08 (C) 0.86 (D) 0.80  
(E) 0.68

10. If  $f(x) = \sqrt[3]{x^3 + 1}$ , what is  $f^{-1}(1.5)$ ?

(A) 3.4 (B) 2.4 (C) 1.6 (D) 1.5 (E) 1.3



11. The area of the parallelogram in the figure is

(A)  $ab$  (B)  $ab \cos \theta$  (C)  $ab \sin \theta$   
(D)  $ab \tan \theta$  (E)  $a^2 + b^2 - 2ab \cos \theta$

12. A club has 14 members, consisting of 6 men and 8 women. How many slates of 3 officers—president, vice-president, and secretary—can be formed if the president must be a woman and the vice-president must be a man?

(A) 2744 (B) 2184 (C) 672 (D) 576  
(E) 336

13. What is the range of the function defined by

$$f(x) = \frac{1}{x} + 2?$$

(A) All real numbers  
(B) All real numbers except  $-\frac{1}{2}$   
(C) All real numbers except 0  
(D) All real numbers except 2  
(E) All real numbers between 2 and 3

14. The probability that  $R$  hits a certain target is  $\frac{3}{5}$  and the probability that  $T$  hits it is  $\frac{5}{7}$ .

What is the probability that  $R$  hits the target and  $T$  misses it?

(A)  $\frac{4}{35}$  (B)  $\frac{6}{35}$  (C)  $\frac{3}{7}$  (D)  $\frac{21}{25}$  (E)  $\frac{31}{35}$

15. If  $x$  is a real number between 0 and  $2\pi$ , and  $\sin x = 5 \cos x$ , what is the value of  $x$ ?

(A) 0.197 (B) 0.201 (C) 1.373  
(D) 11.310 (E) 78.690

16.  $\sin(\operatorname{Arccos}(-0.6)) =$

(A) 2.2 (B) 0.8 (C) 0.4 (D) -0.8  
(E) -0.4

## Using the Tables

**Table 1.** Example:  $\sqrt{830} = 10\sqrt{8.3} \approx 28.81$

**Tables 2 and 3.** Example a.  $\log 4.72 = 0.6739$

Example b. If  $\log N = 0.4843$ , then  $N = 3.05$

Example c.  $\log 718 = \log (100 \cdot 7.18) = \log 10^2 + \log 7.18$   
 $= 2 + 0.8561 = 2.8561$

We call 2 the characteristic and 0.8561 the mantissa.

**Tables 4 and 5.** To evaluate a trigonometric function of  $\theta$  when  $0^\circ \leq \theta \leq 45^\circ$ , locate  $\theta$  in the left-hand column and use the trigonometric-function headings at the top of the table. When  $45^\circ < \theta \leq 90^\circ$ , locate  $\theta$  in the right-hand column and use the headings at the bottom of the table.

Examples: a.  $\cos 14^\circ 10' = 0.9696$       b.  $\tan 51.9^\circ = 1.275$

**Table 6.** Examples: a.  $\sin 0.95 = 0.8134$       b.  $\csc 1.51 = 1.002$

**Table 7.** Example a.  $e^{2.6} = e^{2.5} \cdot e^{0.1} \approx (12.182)(1.1052) \approx 13.464$

Example b.  $e^{-3.21} = e^{-3} \cdot e^{-0.21} \approx (0.0498)(0.8106) \approx 0.0404$

**Interpolation.** Examples: Use interpolation to find:

a.  $\log 8.217$

b.  $x$  if  $\log x = 0.5725$

c.  $\tan 68^\circ 34'$

d.  $\theta$  if  $\sin \theta = 0.5065$

a. From Table 3:

	$N$	$\log N$	
	8.220	.9149	
.10	8.217	$x$	.0006
	.07	$x - .9143$	
	8.210	.9143	
	$\frac{.07}{.10} = \frac{x - .9143}{.0006}$		
	$.10x = .000042 + .09143$		
	$x \approx 0.91472$		

b. From Table 3:

	$N$	$\log N$	
	3.740	.5729	
.010	$x$	.5725	.0012
	$x - 3.730$	$x$	.0008
	3.730	.5717	
	$\frac{x - 3.730}{.010} = \frac{.0008}{.0012}$		
	$x = 3.730 + \frac{8}{12}(.010)$		
	$\approx 3.737$		

c. From Table 4:

	$\theta$	$\tan \theta$	
	68°40'	2.560	
10'	68°34'	$x$	.021
	4'	$x - 2.539$	
	68°30'	2.539	
	$\frac{4}{10} = \frac{x - 2.539}{.021}$		
	$x = 2.539 + \frac{4}{10}(.021)$		
	$\approx 2.547$		

d. From Table 4:

	$\theta$	$\sin \theta$	
	30°30'	.5075	
10'	30°x'	.5065	.0025
	$x - 20'$	$x$	.0015
	30°20'	.5050	
	$\frac{x - 20}{10} = \frac{.0015}{.0025}$		
	$x = 20 + \frac{3}{5}(10) = 26$		
	$\theta \approx 30^\circ 26'$		

**Table 1****Squares and Square Roots**

$N$	$N^2$	$\sqrt{N}$	$\sqrt{10N}$	$N$	$N^2$	$\sqrt{N}$	$\sqrt{10N}$
1.0	1.00	1.000	3.162	5.5	30.25	2.345	7.416
1.1	1.21	1.049	3.317	5.6	31.36	2.366	7.483
1.2	1.44	1.095	3.464	5.7	32.49	2.387	7.550
1.3	1.69	1.140	3.606	5.8	33.64	2.408	7.616
1.4	1.96	1.183	3.742	5.9	34.81	2.429	7.681
1.5	2.25	1.225	3.873	6.0	36.00	2.449	7.746
1.6	2.56	1.265	4.000	6.1	37.21	2.470	7.810
1.7	2.89	1.304	4.123	6.2	38.44	2.490	7.874
1.8	3.24	1.342	4.243	6.3	39.69	2.510	7.937
1.9	3.61	1.378	4.359	6.4	40.96	2.530	8.000
2.0	4.00	1.414	4.472	6.5	42.25	2.550	8.062
2.1	4.41	1.449	4.583	6.6	43.56	2.569	8.124
2.2	4.84	1.483	4.690	6.7	44.89	2.588	8.185
2.3	5.29	1.517	4.796	6.8	46.24	2.608	8.246
2.4	5.76	1.549	4.899	6.9	47.61	2.627	8.307
2.5	6.25	1.581	5.000	7.0	49.00	2.646	8.367
2.6	6.76	1.612	5.099	7.1	50.41	2.665	8.426
2.7	7.29	1.643	5.196	7.2	51.84	2.683	8.485
2.8	7.84	1.673	5.292	7.3	53.29	2.702	8.544
2.9	8.41	1.703	5.385	7.4	54.76	2.720	8.602
3.0	9.00	1.732	5.477	7.5	56.25	2.739	8.660
3.1	9.61	1.761	5.568	7.6	57.76	2.757	8.718
3.2	10.24	1.789	5.657	7.7	59.29	2.775	8.775
3.3	10.89	1.817	5.745	7.8	60.84	2.793	8.832
3.4	11.56	1.844	5.831	7.9	62.41	2.811	8.888
3.5	12.25	1.871	5.916	8.0	64.00	2.828	8.944
3.6	12.96	1.897	6.000	8.1	65.61	2.846	9.000
3.7	13.69	1.924	6.083	8.2	67.24	2.864	9.055
3.8	14.44	1.949	6.164	8.3	68.89	2.881	9.110
3.9	15.21	1.975	6.245	8.4	70.56	2.898	9.165
4.0	16.00	2.000	6.325	8.5	72.25	2.915	9.220
4.1	16.81	2.025	6.403	8.6	73.96	2.933	9.274
4.2	17.64	2.049	6.481	8.7	75.69	2.950	9.327
4.3	18.49	2.074	6.557	8.8	77.44	2.966	9.381
4.4	19.36	2.098	6.633	8.9	79.21	2.983	9.434
4.5	20.25	2.121	6.708	9.0	81.00	3.000	9.487
4.6	21.16	2.145	6.782	9.1	82.81	3.017	9.539
4.7	22.09	2.168	6.856	9.2	84.64	3.033	9.592
4.8	23.04	2.191	6.928	9.3	86.49	3.050	9.644
4.9	24.01	2.214	7.000	9.4	88.36	3.066	9.695
5.0	25.00	2.236	7.071	9.5	90.25	3.082	9.747
5.1	26.01	2.258	7.141	9.6	92.16	3.098	9.798
5.2	27.04	2.280	7.211	9.7	94.09	3.114	9.849
5.3	28.09	2.302	7.280	9.8	96.04	3.130	9.899
5.4	29.16	2.324	7.348	9.9	98.01	3.146	9.950
5.5	30.25	2.345	7.416	10	100.00	3.162	10.000

**Table 2****Natural Logarithms**

$x$	$\ln x$	$x$	$\ln x$	$x$	$\ln x$
0.1	-2.3026	4.5	1.5041	9.0	2.1972
0.2	-1.6094	4.6	1.5261	9.1	2.2083
0.3	-1.2040	4.7	1.5476	9.2	2.2192
0.4	-0.9163	4.8	1.5686	9.3	2.2300
		4.9	1.5892	9.4	2.2407
0.5	-0.6931	5.0	1.6094	9.5	2.2513
0.6	-0.5108	5.1	1.6292	9.6	2.2618
0.7	-0.3567	5.2	1.6487	9.7	2.2721
0.8	-0.2231	5.3	1.6677	9.8	2.2824
0.9	-0.1054	5.4	1.6864	9.9	2.2925
1.0	0.0000	5.5	1.7047	10	2.3026
1.1	0.0953	5.6	1.7228	11	2.3979
1.2	0.1823	5.7	1.7405	12	2.4849
1.3	0.2624	5.8	1.7579	13	2.5649
1.4	0.3365	5.9	1.7750	14	2.6391
1.5	0.4055	6.0	1.7918	15	2.7081
1.6	0.4700	6.1	1.8083	16	2.7726
1.7	0.5306	6.2	1.8245	17	2.8332
1.8	0.5878	6.3	1.8405	18	2.8904
1.9	0.6419	6.4	1.8563	19	2.9444
2.0	0.6931	6.5	1.8718	20	2.9957
2.1	0.7419	6.6	1.8871	25	3.2189
2.2	0.7885	6.7	1.9021	30	3.4012
2.3	0.8329	6.8	1.9169	35	3.5553
2.4	0.8755	6.9	1.9315	40	3.6889
2.5	0.9163	7.0	1.9459	45	3.8067
2.6	0.9555	7.1	1.9601	50	3.9120
2.7	0.9933	7.2	1.9741	55	4.0073
2.8	1.0296	7.3	1.9879	60	4.0943
2.9	1.0647	7.4	2.0015	65	4.1744
3.0	1.0986	7.5	2.0149	70	4.2485
3.1	1.1314	7.6	2.0281	75	4.3175
3.2	1.1632	7.7	2.0412	80	4.3820
3.3	1.1939	7.8	2.0541	85	4.4427
3.4	1.2238	7.9	2.0669	90	4.4998
3.5	1.2528	8.0	2.0794	100	4.6052
3.6	1.2809	8.1	2.0919	110	4.7005
3.7	1.3083	8.2	2.1041	120	4.7875
3.8	1.3350	8.3	2.1163	130	4.8676
3.9	1.3610	8.4	2.1282	140	4.9416
4.0	1.3863	8.5	2.1401	150	5.0106
4.1	1.4110	8.6	2.1518	160	5.0752
4.2	1.4351	8.7	2.1633	170	5.1358
4.3	1.4586	8.8	2.1748	180	5.1930
4.4	1.4816	8.9	2.1861	190	5.2470



Table 3

## Common Logarithms of Numbers\*

N	0	1	2	3	4	5	6	7	8	9
1.0	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
5.0	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

\*Decimal points omitted.

**Table 3****Common Logarithms of Numbers**

N	0	1	2	3	4	5	6	7	8	9
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Table 4

Trigonometric Functions of  $\theta$  ( $\theta$  in degrees and minutes)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$	
0° 00'	.0000	.0000	1.0000	.0000	Undefined	1.000	Undefined	1.5708
10'	.0029	.0029	1.0000	.0029	343.8	1.000	343.8	1.5679
20'	.0058	.0058	1.0000	.0058	171.9	1.000	171.9	1.5650
30'	.0087	.0087	1.0000	.0087	114.6	1.000	114.6	1.5621
40'	.0116	.0116	.9999	.0116	85.94	1.000	85.95	1.5592
50'	.0145	.0145	.9999	.0145	68.75	1.000	68.76	1.5563
1° 00'	.0175	.0175	.9998	.0175	57.29	1.000	57.30	1.5533
10'	.0204	.0204	.9998	.0204	49.10	1.000	49.11	1.5504
20'	.0233	.0233	.9997	.0233	42.96	1.000	42.98	1.5475
30'	.0262	.0262	.9997	.0262	38.19	1.000	38.20	1.5446
40'	.0291	.0291	.9996	.0291	34.37	1.000	34.38	1.5417
50'	.0320	.0320	.9995	.0320	31.24	1.001	31.26	1.5388
2° 00'	.0349	.0349	.9994	.0349	28.64	1.001	28.65	1.5359
10'	.0378	.0378	.9993	.0378	26.43	1.001	26.45	1.5330
20'	.0407	.0407	.9992	.0407	24.54	1.001	24.56	1.5301
30'	.0436	.0436	.9990	.0437	22.90	1.001	22.93	1.5272
40'	.0465	.0465	.9989	.0466	21.47	1.001	21.49	1.5243
50'	.0495	.0494	.9988	.0495	20.21	1.001	20.23	1.5213
3° 00'	.0524	.0523	.9986	.0524	19.08	1.001	19.11	1.5184
10'	.0553	.0552	.9985	.0553	18.07	1.002	18.10	1.5155
20'	.0582	.0581	.9983	.0582	17.17	1.002	17.20	1.5126
30'	.0611	.0610	.9981	.0612	16.35	1.002	16.38	1.5097
40'	.0640	.0640	.9980	.0641	15.60	1.002	15.64	1.5068
50'	.0669	.0669	.9978	.0670	14.92	1.002	14.96	1.5039
4° 00'	.0698	.0698	.9976	.0699	14.30	1.002	14.34	1.5010
10'	.0727	.0727	.9974	.0729	13.73	1.003	13.76	1.4981
20'	.0756	.0756	.9971	.0758	13.20	1.003	13.23	1.4952
30'	.0785	.0785	.9969	.0787	12.71	1.003	12.75	1.4923
40'	.0814	.0814	.9967	.0816	12.25	1.003	12.29	1.4893
50'	.0844	.0843	.9964	.0846	11.83	1.004	11.87	1.4864
5° 00'	.0873	.0872	.9962	.0875	11.43	1.004	11.47	1.4835
10'	.0902	.0901	.9959	.0904	11.06	1.004	11.10	1.4806
20'	.0931	.0929	.9957	.0934	10.71	1.004	10.76	1.4777
30'	.0960	.0958	.9954	.0963	10.39	1.005	10.43	1.4748
40'	.0989	.0987	.9951	.0992	10.08	1.005	10.13	1.4719
50'	.1018	.1016	.9948	.1022	9.788	1.005	9.839	1.4690
6° 00'	.1047	.1045	.9945	.1051	9.514	1.006	9.567	1.4661
10'	.1076	.1074	.9942	.1080	9.255	1.006	9.309	1.4632
20'	.1105	.1103	.9939	.1110	9.010	1.006	9.065	1.4603
30'	.1134	.1132	.9936	.1139	8.777	1.006	8.834	1.4573
40'	.1164	.1161	.9932	.1169	8.556	1.007	8.614	1.4544
50'	.1193	.1190	.9929	.1198	8.345	1.007	8.405	1.4515
7° 00'	.1222	.1219	.9925	.1228	8.144	1.008	8.206	1.4486
10'	.1251	.1248	.9922	.1257	7.953	1.008	8.016	1.4457
20'	.1280	.1276	.9918	.1287	7.770	1.008	7.834	1.4428
30'	.1309	.1305	.9914	.1317	7.596	1.009	7.661	1.4399
40'	.1338	.1334	.9911	.1346	7.429	1.009	7.496	1.4370
50'	.1367	.1363	.9907	.1376	7.269	1.009	7.337	1.4341
8° 00'	.1396	.1392	.9903	.1405	7.115	1.010	7.185	1.4312
10'	.1425	.1421	.9899	.1435	6.968	1.010	7.040	1.4283
20'	.1454	.1449	.9894	.1465	6.827	1.011	6.900	1.4254
30'	.1484	.1478	.9890	.1495	6.691	1.011	6.765	1.4224
40'	.1513	.1507	.9886	.1524	6.561	1.012	6.636	1.4195
50'	.1542	.1536	.9881	.1554	6.435	1.012	6.512	1.4166
9° 00'	.1571	.1564	.9877	.1584	6.314	1.012	6.392	1.4137
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$
								Radians
								Degrees

Table 4

Trigonometric Functions of  $\theta$  ( $\theta$  in degrees and minutes)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
9° 00'	.1571	.1564	.9877	.1584	6.314	1.012	6.392	1.4137	81° 00'
10'	.1600	.1593	.9872	.1614	6.197	1.013	6.277	1.4108	50'
20'	.1629	.1622	.9868	.1644	6.084	1.013	6.166	1.4079	40'
30'	.1658	.1650	.9863	.1673	5.976	1.014	6.059	1.4050	30'
40'	.1687	.1679	.9858	.1703	5.871	1.014	5.955	1.4021	20'
50'	.1716	.1708	.9853	.1733	5.769	1.015	5.855	1.3992	10'
10° 00'	.1745	.1736	.9848	.1763	5.671	1.015	5.759	1.3963	80° 00'
10'	.1774	.1765	.9843	.1793	5.576	1.016	5.665	1.3934	50'
20'	.1804	.1794	.9838	.1823	5.485	1.016	5.575	1.3904	40'
30'	.1833	.1822	.9833	.1853	5.396	1.017	5.487	1.3875	30'
40'	.1862	.1851	.9827	.1883	5.309	1.018	5.403	1.3846	20'
50'	.1891	.1880	.9822	.1914	5.226	1.018	5.320	1.3817	10'
11° 00'	.1920	.1908	.9816	.1944	5.145	1.019	5.241	1.3788	79° 00'
10'	.1949	.1937	.9811	.1974	5.066	1.019	5.164	1.3759	50'
20'	.1978	.1965	.9805	.2004	4.989	1.020	5.089	1.3730	40'
30'	.2007	.1994	.9799	.2035	4.915	1.020	5.016	1.3701	30'
40'	.2036	.2022	.9793	.2065	4.843	1.021	4.945	1.3672	20'
50'	.2065	.2051	.9787	.2095	4.773	1.022	4.876	1.3643	10'
12° 00'	.2094	.2079	.9781	.2126	4.705	1.022	4.810	1.3614	78° 00'
10'	.2123	.2108	.9775	.2156	4.638	1.023	4.745	1.3584	50'
20'	.2153	.2136	.9769	.2186	4.574	1.024	4.682	1.3555	40'
30'	.2182	.2164	.9763	.2217	4.511	1.024	4.620	1.3526	30'
40'	.2211	.2193	.9757	.2247	4.449	1.025	4.560	1.3497	20'
50'	.2240	.2221	.9750	.2278	4.390	1.026	4.502	1.3468	10'
13° 00'	.2269	.2250	.9744	.2309	4.331	1.026	4.445	1.3439	77° 00'
10'	.2298	.2278	.9737	.2339	4.275	1.027	4.390	1.3410	50'
20'	.2327	.2306	.9730	.2370	4.219	1.028	4.336	1.3381	40'
30'	.2356	.2334	.9724	.2401	4.165	1.028	4.284	1.3352	30'
40'	.2385	.2363	.9717	.2432	4.113	1.029	4.232	1.3323	20'
50'	.2414	.2391	.9710	.2462	4.061	1.030	4.182	1.3294	10'
14° 00'	.2443	.2419	.9703	.2493	4.011	1.031	4.134	1.3265	76° 00'
10'	.2473	.2447	.9696	.2524	3.962	1.031	4.086	1.3235	50'
20'	.2502	.2476	.9689	.2555	3.914	1.032	4.039	1.3206	40'
30'	.2531	.2504	.9681	.2586	3.867	1.033	3.994	1.3177	30'
40'	.2560	.2532	.9674	.2617	3.821	1.034	3.950	1.3148	20'
50'	.2589	.2560	.9667	.2648	3.776	1.034	3.906	1.3119	10'
15° 00'	.2618	.2588	.9659	.2679	3.732	1.035	3.864	1.3090	75° 00'
10'	.2647	.2616	.9652	.2711	3.689	1.036	3.822	1.3061	50'
20'	.2676	.2644	.9644	.2742	3.647	1.037	3.782	1.3032	40'
30'	.2705	.2672	.9636	.2773	3.606	1.038	3.742	1.3003	30'
40'	.2734	.2700	.9628	.2805	3.566	1.039	3.703	1.2974	20'
50'	.2763	.2728	.9621	.2836	3.526	1.039	3.665	1.2945	10'
16° 00'	.2793	.2756	.9613	.2867	3.487	1.040	3.628	1.2915	74° 00'
10'	.2822	.2784	.9605	.2899	3.450	1.041	3.592	1.2886	50'
20'	.2851	.2812	.9596	.2931	3.412	1.042	3.556	1.2857	40'
30'	.2880	.2840	.9588	.2962	3.376	1.043	3.521	1.2828	30'
40'	.2909	.2868	.9580	.2994	3.340	1.044	3.487	1.2799	20'
50'	.2938	.2896	.9572	.3026	3.305	1.045	3.453	1.2770	10'
17° 00'	.2967	.2924	.9563	.3057	3.271	1.046	3.420	1.2741	73° 00'
10'	.2996	.2952	.9555	.3089	3.237	1.047	3.388	1.2712	50'
20'	.3025	.2979	.9546	.3121	3.204	1.048	3.357	1.2683	40'
30'	.3054	.3007	.9537	.3153	3.172	1.049	3.326	1.2654	30'
40'	.3083	.3035	.9528	.3185	3.140	1.049	3.295	1.2625	20'
50'	.3113	.3062	.9520	.3217	3.108	1.050	3.265	1.2595	10'
18° 00'	.3142	.3090	.9511	.3249	3.078	1.051	3.236	1.2566	72° 00'
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees



Table 4

Trigonometric Functions of  $\theta$  ( $\theta$  in degrees and minutes)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
18° 00'	.3142	.3090	.9511	.3249	3.078	1.051	3.236	1.2566	72° 00'
10'	.3171	.3118	.9502	.3281	3.047	1.052	3.207	1.2537	50'
20'	.3200	.3145	.9492	.3314	3.018	1.053	3.179	1.2508	40'
30'	.3229	.3173	.9483	.3346	2.989	1.054	3.152	1.2479	30'
40'	.3258	.3201	.9474	.3378	2.960	1.056	3.124	1.2450	20'
50'	.3287	.3228	.9465	.3411	2.932	1.057	3.098	1.2421	10'
19° 00'	.3316	.3256	.9455	.3443	2.904	1.058	3.072	1.2392	71° 00'
10'	.3345	.3283	.9446	.3476	2.877	1.059	3.046	1.2363	50'
20'	.3374	.3311	.9436	.3508	2.850	1.060	3.021	1.2334	40'
30'	.3403	.3338	.9426	.3541	2.824	1.061	2.996	1.2305	30'
40'	.3432	.3365	.9417	.3574	2.798	1.062	2.971	1.2275	20'
50'	.3462	.3393	.9407	.3607	2.773	1.063	2.947	1.2246	10'
20° 00'	.3491	.3420	.9397	.3640	2.747	1.064	2.924	1.2217	70° 00'
10'	.3520	.3448	.9387	.3673	2.723	1.065	2.901	1.2188	50'
20'	.3549	.3475	.9377	.3706	2.699	1.066	2.878	1.2159	40'
30'	.3578	.3502	.9367	.3739	2.675	1.068	2.855	1.2130	30'
40'	.3607	.3529	.9356	.3772	2.651	1.069	2.833	1.2101	20'
50'	.3636	.3557	.9346	.3805	2.628	1.070	2.812	1.2072	10'
21° 00'	.3665	.3584	.9336	.3839	2.605	1.071	2.790	1.2043	69° 00'
10'	.3694	.3611	.9325	.3872	2.583	1.072	2.769	1.2014	50'
20'	.3723	.3638	.9315	.3906	2.560	1.074	2.749	1.1985	40'
30'	.3752	.3665	.9304	.3939	2.539	1.075	2.729	1.1956	30'
40'	.3782	.3692	.9293	.3973	2.517	1.076	2.709	1.1926	20'
50'	.3811	.3719	.9283	.4006	2.496	1.077	2.689	1.1897	10'
22° 00'	.3840	.3746	.9272	.4040	2.475	1.079	2.669	1.1868	68° 00'
10'	.3869	.3773	.9261	.4074	2.455	1.080	2.650	1.1839	50'
20'	.3898	.3800	.9250	.4108	2.434	1.081	2.632	1.1810	40'
30'	.3927	.3827	.9239	.4142	2.414	1.082	2.613	1.1781	30'
40'	.3956	.3854	.9228	.4176	2.394	1.084	2.595	1.1752	20'
50'	.3985	.3881	.9216	.4210	2.375	1.085	2.577	1.1723	10'
23° 00'	.4014	.3907	.9205	.4245	2.356	1.086	2.559	1.1694	67° 00'
10'	.4043	.3934	.9194	.4279	2.337	1.088	2.542	1.1665	50'
20'	.4072	.3961	.9182	.4314	2.318	1.089	2.525	1.1636	40'
30'	.4102	.3987	.9171	.4348	2.300	1.090	2.508	1.1606	30'
40'	.4131	.4014	.9159	.4383	2.282	1.092	2.491	1.1577	20'
50'	.4160	.4041	.9147	.4417	2.264	1.093	2.475	1.1548	10'
24° 00'	.4189	.4067	.9135	.4452	2.246	1.095	2.459	1.1519	66° 00'
10'	.4218	.4094	.9124	.4487	2.229	1.096	2.443	1.1490	50'
20'	.4247	.4120	.9112	.4522	2.211	1.097	2.427	1.1461	40'
30'	.4276	.4147	.9100	.4557	2.194	1.099	2.411	1.1432	30'
40'	.4305	.4173	.9088	.4592	2.177	1.100	2.396	1.1403	20'
50'	.4334	.4200	.9075	.4628	2.161	1.102	2.381	1.1374	10'
25° 00'	.4363	.4226	.9063	.4663	2.145	1.103	2.366	1.1345	65° 00'
10'	.4392	.4253	.9051	.4699	2.128	1.105	2.352	1.1316	50'
20'	.4422	.4279	.9038	.4734	2.112	1.106	2.337	1.1286	40'
30'	.4451	.4305	.9026	.4770	2.097	1.108	2.323	1.1257	30'
40'	.4480	.4331	.9013	.4806	2.081	1.109	2.309	1.1228	20'
50'	.4509	.4358	.9001	.4841	2.066	1.111	2.295	1.1199	10'
26° 00'	.4538	.4384	.8988	.4877	2.050	1.113	2.281	1.1170	64° 00'
10'	.4567	.4410	.8975	.4913	2.035	1.114	2.268	1.1141	50'
20'	.4596	.4436	.8962	.4950	2.020	1.116	2.254	1.1112	40'
30'	.4625	.4462	.8949	.4986	2.006	1.117	2.241	1.1083	30'
40'	.4654	.4488	.8936	.5022	1.991	1.119	2.228	1.1054	20'
50'	.4683	.4514	.8923	.5059	1.977	1.121	2.215	1.1025	10'
27° 00'	.4712	.4540	.8910	.5095	1.963	1.122	2.203	1.0996	63° 00'
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees



Table 4

Trigonometric Functions of  $\theta$  ( $\theta$  in degrees and minutes)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
27° 00'	.4712	.4540	.8910	.5095	1.963	1.122	2.203	1.0996	63° 00'
10'	.4741	.4566	.8897	.5132	1.949	1.124	2.190	1.0966	50'
20'	.4771	.4592	.8884	.5169	1.935	1.126	2.178	1.0937	40'
30'	.4800	.4617	.8870	.5206	1.921	1.127	2.166	1.0908	30'
40'	.4829	.4643	.8857	.5243	1.907	1.129	2.154	1.0879	20'
50'	.4858	.4669	.8843	.5280	1.894	1.131	2.142	1.0850	10'
28° 00'	.4887	.4695	.8829	.5317	1.881	1.133	2.130	1.0821	62° 00'
10'	.4916	.4720	.8816	.5354	1.868	1.134	2.118	1.0792	50'
20'	.4945	.4746	.8802	.5392	1.855	1.136	2.107	1.0763	40'
30'	.4974	.4772	.8788	.5430	1.842	1.138	2.096	1.0734	30'
40'	.5003	.4797	.8774	.5467	1.829	1.140	2.085	1.0705	20'
50'	.5032	.4823	.8760	.5505	1.816	1.142	2.074	1.0676	10'
29° 00'	.5061	.4848	.8746	.5543	1.804	1.143	2.063	1.0647	61° 00'
10'	.5091	.4874	.8732	.5581	1.792	1.145	2.052	1.0617	50'
20'	.5120	.4899	.8718	.5619	1.780	1.147	2.041	1.0588	40'
30'	.5149	.4924	.8704	.5658	1.767	1.149	2.031	1.0559	30'
40'	.5178	.4950	.8689	.5696	1.756	1.151	2.020	1.0530	20'
50'	.5207	.4975	.8675	.5735	1.744	1.153	2.010	1.0501	10'
30° 00'	.5236	.5000	.8660	.5774	1.732	1.155	2.000	1.0472	60° 00'
10'	.5265	.5025	.8646	.5812	1.720	1.157	1.990	1.0443	50'
20'	.5294	.5050	.8631	.5851	1.709	1.159	1.980	1.0414	40'
30'	.5323	.5075	.8616	.5890	1.698	1.161	1.970	1.0385	30'
40'	.5352	.5100	.8601	.5930	1.686	1.163	1.961	1.0356	20'
50'	.5381	.5125	.8587	.5969	1.675	1.165	1.951	1.0327	10'
31° 00'	.5411	.5150	.8572	.6009	1.664	1.167	1.942	1.0297	59° 00'
10'	.5440	.5175	.8557	.6048	1.653	1.169	1.932	1.0268	50'
20'	.5469	.5200	.8542	.6088	1.643	1.171	1.923	1.0239	40'
30'	.5498	.5225	.8526	.6128	1.632	1.173	1.914	1.0210	30'
40'	.5527	.5250	.8511	.6168	1.621	1.175	1.905	1.0181	20'
50'	.5556	.5275	.8496	.6208	1.611	1.177	1.896	1.0152	10'
32° 00'	.5585	.5299	.8480	.6249	1.600	1.179	1.887	1.0123	58° 00'
10'	.5614	.5324	.8465	.6289	1.590	1.181	1.878	1.0094	50'
20'	.5643	.5348	.8450	.6330	1.580	1.184	1.870	1.0065	40'
30'	.5672	.5373	.8434	.6371	1.570	1.186	1.861	1.0036	30'
40'	.5701	.5398	.8418	.6412	1.560	1.188	1.853	1.0007	20'
50'	.5730	.5422	.8403	.6453	1.550	1.190	1.844	.9977	10'
33° 00'	.5760	.5446	.8387	.6494	1.540	1.192	1.836	.9948	57° 00'
10'	.5789	.5471	.8371	.6536	1.530	1.195	1.828	.9919	50'
20'	.5818	.5495	.8355	.6577	1.520	1.197	1.820	.9890	40'
30'	.5847	.5519	.8339	.6619	1.511	1.199	1.812	.9861	30'
40'	.5876	.5544	.8323	.6661	1.501	1.202	1.804	.9832	20'
50'	.5905	.5568	.8307	.6703	1.492	1.204	1.796	.9803	10'
34° 00'	.5934	.5592	.8290	.6745	1.483	1.206	1.788	.9774	56° 00'
10'	.5963	.5616	.8274	.6787	1.473	1.209	1.781	.9745	50'
20'	.5992	.5640	.8258	.6830	1.464	1.211	1.773	.9716	40'
30'	.6021	.5664	.8241	.6873	1.455	1.213	1.766	.9687	30'
40'	.6050	.5688	.8225	.6916	1.446	1.216	1.758	.9657	20'
50'	.6080	.5712	.8208	.6959	1.437	1.218	1.751	.9628	10'
35° 00'	.6109	.5736	.8192	.7002	1.428	1.221	1.743	.9599	55° 00'
10'	.6138	.5760	.8175	.7046	1.419	1.223	1.736	.9570	50'
20'	.6167	.5783	.8158	.7089	1.411	1.226	1.729	.9541	40'
30'	.6196	.5807	.8141	.7133	1.402	1.228	1.722	.9512	30'
40'	.6225	.5831	.8124	.7177	1.393	1.231	1.715	.9483	20'
50'	.6254	.5854	.8107	.7221	1.385	1.233	1.708	.9454	10'
36° 00'	.6283	.5878	.8090	.7265	1.376	1.236	1.701	.9425	54° 00'
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees

Table 4

Trigonometric Functions of  $\theta$  ( $\theta$  in degrees and minutes)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
36° 00'	.6283	.5878	.8090	.7265	1.376	1.236	1.701	.9425	54° 00'
10'	.6312	.5901	.8073	.7310	1.368	1.239	1.695	.9396	50'
20'	.6341	.5925	.8056	.7355	1.360	1.241	1.688	.9367	40'
30'	.6370	.5948	.8039	.7400	1.351	1.244	1.681	.9338	30'
40'	.6400	.5972	.8021	.7445	1.343	1.247	1.675	.9308	20'
50'	.6429	.5995	.8004	.7490	1.335	1.249	1.668	.9279	10'
37° 00'	.6458	.6018	.7986	.7536	1.327	1.252	1.662	.9250	53° 00'
10'	.6487	.6041	.7969	.7581	1.319	1.255	1.655	.9221	50'
20'	.6516	.6065	.7951	.7627	1.311	1.258	1.649	.9192	40'
30'	.6545	.6088	.7934	.7673	1.303	1.260	1.643	.9163	30'
40'	.6574	.6111	.7916	.7720	1.295	1.263	1.636	.9134	20'
50'	.6603	.6134	.7898	.7766	1.288	1.266	1.630	.9105	10'
38° 00'	.6632	.6157	.7880	.7813	1.280	1.269	1.624	.9076	52° 00'
10'	.6661	.6180	.7862	.7860	1.272	1.272	1.618	.9047	50'
20'	.6690	.6202	.7844	.7907	1.265	1.275	1.612	.9018	40'
30'	.6720	.6225	.7826	.7954	1.257	1.278	1.606	.8988	30'
40'	.6749	.6248	.7808	.8002	1.250	1.281	1.601	.8959	20'
50'	.6778	.6271	.7790	.8050	1.242	1.284	1.595	.8930	10'
39° 00'	.6807	.6293	.7771	.8098	1.235	1.287	1.589	.8901	51° 00'
10'	.6836	.6316	.7753	.8146	1.228	1.290	1.583	.8872	50'
20'	.6865	.6338	.7735	.8195	1.220	1.293	1.578	.8843	40'
30'	.6894	.6361	.7716	.8243	1.213	1.296	1.572	.8814	30'
40'	.6923	.6383	.7698	.8292	1.206	1.299	1.567	.8785	20'
50'	.6952	.6406	.7679	.8342	1.199	1.302	1.561	.8756	10'
40° 00'	.6981	.6428	.7660	.8391	1.192	1.305	1.556	.8727	50° 00'
10'	.7010	.6450	.7642	.8441	1.185	1.309	1.550	.8698	50'
20'	.7039	.6472	.7623	.8491	1.178	1.312	1.545	.8668	40'
30'	.7069	.6494	.7604	.8541	1.171	1.315	1.540	.8639	30'
40'	.7098	.6517	.7585	.8591	1.164	1.318	1.535	.8610	20'
50'	.7127	.6539	.7566	.8642	1.157	1.322	1.529	.8581	10'
41° 00'	.7156	.6561	.7547	.8693	1.150	1.325	1.524	.8552	49° 00'
10'	.7185	.6583	.7528	.8744	1.144	1.328	1.519	.8523	50'
20'	.7214	.6604	.7509	.8796	1.137	1.332	1.514	.8494	40'
30'	.7243	.6626	.7490	.8847	1.130	1.335	1.509	.8465	30'
40'	.7272	.6648	.7470	.8899	1.124	1.339	1.504	.8436	20'
50'	.7301	.6670	.7451	.8952	1.117	1.342	1.499	.8407	10'
42° 00'	.7330	.6691	.7431	.9004	1.111	1.346	1.494	.8378	48° 00'
10'	.7359	.6713	.7412	.9057	1.104	1.349	1.490	.8348	50'
20'	.7389	.6734	.7392	.9110	1.098	1.353	1.485	.8319	40'
30'	.7418	.6756	.7373	.9163	1.091	1.356	1.480	.8290	30'
40'	.7447	.6777	.7353	.9217	1.085	1.360	1.476	.8261	20'
50'	.7476	.6799	.7333	.9271	1.079	1.364	1.471	.8232	10'
43° 00'	.7505	.6820	.7314	.9325	1.072	1.367	1.466	.8203	47° 00'
10'	.7534	.6841	.7294	.9380	1.066	1.371	1.462	.8174	50'
20'	.7563	.6862	.7274	.9435	1.060	1.375	1.457	.8145	40'
30'	.7592	.6884	.7254	.9490	1.054	1.379	1.453	.8116	30'
40'	.7621	.6905	.7234	.9545	1.048	1.382	1.448	.8087	20'
50'	.7650	.6926	.7214	.9601	1.042	1.386	1.444	.8058	10'
44° 00'	.7679	.6947	.7193	.9657	1.036	1.390	1.440	.8029	46° 00'
10'	.7709	.6967	.7173	.9713	1.030	1.394	1.435	.7999	50'
20'	.7738	.6988	.7153	.9770	1.024	1.398	1.431	.7970	40'
30'	.7767	.7009	.7133	.9827	1.018	1.402	1.427	.7941	30'
40'	.7796	.7030	.7112	.9884	1.012	1.406	1.423	.7912	20'
50'	.7825	.7050	.7092	.9942	1.006	1.410	1.418	.7883	10'
45° 00'	.7854	.7071	.7071	1.000	1.000	1.414	1.414	.7854	45° 00'
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees

Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
0.0	.0000	.0000	1.0000	.0000	undefined	1.000	undefined	1.5708	<b>90.0</b>
0.1	.0017	.0017	1.0000	.0017	573.0	1.000	573.0	1.5691	89.9
0.2	.0035	.0035	1.0000	.0035	286.5	1.000	286.5	1.5673	89.8
0.3	.0052	.0052	1.0000	.0052	191.0	1.000	191.0	1.5656	89.7
0.4	.0070	.0070	1.0000	.0070	143.2	1.000	143.2	1.5638	89.6
0.5	.0087	.0087	1.0000	.0087	114.6	1.000	114.6	1.5621	89.5
0.6	.0105	.0105	.9999	.0105	95.49	1.000	95.49	1.5603	89.4
0.7	.0122	.0122	.9999	.0122	81.85	1.000	81.85	1.5586	89.3
0.8	.0140	.0140	.9999	.0140	71.62	1.000	71.62	1.5568	89.2
0.9	.0157	.0157	.9999	.0157	63.66	1.000	63.66	1.5551	89.1
1.0	.0175	.0175	.9998	.0175	57.29	1.000	57.30	1.5533	<b>89.0</b>
1.1	.0192	.0192	.9998	.0192	52.08	1.000	52.09	1.5516	88.9
1.2	.0209	.0209	.9998	.0209	47.74	1.000	47.75	1.5499	88.8
1.3	.0227	.0227	.9997	.0227	44.07	1.000	44.08	1.5481	88.7
1.4	.0244	.0244	.9997	.0244	40.92	1.000	40.93	1.5464	88.6
1.5	.0262	.0262	.9997	.0262	38.19	1.000	38.20	1.5446	88.5
1.6	.0279	.0279	.9996	.0279	35.80	1.000	35.81	1.5429	88.4
1.7	.0297	.0297	.9996	.0297	33.69	1.000	33.71	1.5411	88.3
1.8	.0314	.0314	.9995	.0314	31.82	1.000	31.84	1.5394	88.2
1.9	.0332	.0332	.9995	.0332	30.14	1.001	30.16	1.5376	88.1
2.0	.0349	.0349	.9994	.0349	28.64	1.001	28.65	1.5359	<b>88.0</b>
2.1	.0367	.0366	.9993	.0367	27.27	1.001	27.29	1.5341	87.9
2.2	.0384	.0384	.9993	.0384	26.03	1.001	26.05	1.5324	87.8
2.3	.0401	.0401	.9992	.0402	24.90	1.001	24.92	1.5307	87.7
2.4	.0419	.0419	.9991	.0419	23.86	1.001	23.88	1.5289	87.6
2.5	.0436	.0436	.9990	.0437	22.90	1.001	22.93	1.5272	87.5
2.6	.0454	.0454	.9990	.0454	22.02	1.001	22.04	1.5254	87.4
2.7	.0471	.0471	.9989	.0472	21.20	1.001	21.23	1.5237	87.3
2.8	.0489	.0488	.9988	.0489	20.45	1.001	20.47	1.5219	87.2
2.9	.0506	.0506	.9987	.0507	19.74	1.001	19.77	1.5202	87.1
3.0	.0524	.0523	.9986	.0524	19.08	1.001	19.11	1.5184	<b>87.0</b>
3.1	.0541	.0541	.9985	.0542	18.46	1.001	18.49	1.5167	86.9
3.2	.0559	.0558	.9984	.0559	17.89	1.002	17.91	1.5149	86.8
3.3	.0576	.0576	.9983	.0577	17.34	1.002	17.37	1.5132	86.7
3.4	.0593	.0593	.9982	.0594	16.83	1.002	16.86	1.5115	86.6
3.5	.0611	.0610	.9981	.0612	16.35	1.002	16.38	1.5097	86.5
3.6	.0628	.0628	.9980	.0629	15.89	1.002	15.93	1.5080	86.4
3.7	.0646	.0645	.9979	.0647	15.46	1.002	15.50	1.5062	86.3
3.8	.0663	.0663	.9978	.0664	15.06	1.002	15.09	1.5045	86.2
3.9	.0681	.0680	.9977	.0682	14.67	1.002	14.70	1.5027	86.1
4.0	.0698	.0698	.9976	.0699	14.30	1.002	14.34	1.5010	<b>86.0</b>
4.1	.0716	.0715	.9974	.0717	13.95	1.003	13.99	1.4992	85.9
4.2	.0733	.0732	.9973	.0734	13.62	1.003	13.65	1.4975	85.8
4.3	.0750	.0750	.9972	.0752	13.30	1.003	13.34	1.4957	85.7
4.4	.0768	.0767	.9971	.0769	13.00	1.003	13.03	1.4940	85.6
4.5	.0785	.0785	.9969	.0787	12.71	1.003	12.75	1.4923	85.5
4.6	.0803	.0802	.9968	.0805	12.43	1.003	12.47	1.4905	85.4
4.7	.0820	.0819	.9966	.0822	12.16	1.003	12.20	1.4888	85.3
4.8	.0838	.0837	.9965	.0840	11.91	1.004	11.95	1.4870	85.2
4.9	.0855	.0854	.9963	.0857	11.66	1.004	11.71	1.4853	85.1
5.0	.0873	.0872	.9962	.0875	11.43	1.004	11.47	1.4835	<b>85.0</b>
5.1	.0890	.0889	.9960	.0892	11.20	1.004	11.25	1.4818	84.9
5.2	.0908	.0906	.9959	.0910	10.99	1.004	11.03	1.4800	84.8
5.3	.0925	.0924	.9957	.0928	10.78	1.004	10.83	1.4783	84.7
5.4	.0942	.0941	.9956	.0945	10.58	1.004	10.63	1.4765	84.6
5.5	.0960	.0958	.9954	.0963	10.39	1.005	10.43	1.4748	84.5
5.6	.0977	.0976	.9952	.0981	10.20	1.005	10.25	1.4731	84.4
5.7	.0995	.0993	.9951	.0998	10.02	1.005	10.07	1.4713	84.3
5.8	.1012	.1011	.9949	.1016	9.845	1.005	9.895	1.4696	84.2
5.9	.1030	.1028	.9947	.1033	9.677	1.005	9.728	1.4678	84.1
6.0	.1047	.1045	.9945	.1051	9.514	1.006	9.567	1.4661	<b>84.0</b>
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees



Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
6.0	.1047	.1045	.9945	.1051	9.514	1.006	9.567	1.4661	81.0
6.1	.1065	.1063	.9943	.1069	9.357	1.006	9.411	1.4643	83.9
6.2	.1082	.1080	.9942	.1086	9.205	1.006	9.259	1.4626	83.8
6.3	.1100	.1097	.9940	.1104	9.058	1.006	9.113	1.4608	83.7
6.4	.1117	.1115	.9938	.1122	8.915	1.006	8.971	1.4591	83.6
6.5	.1134	.1132	.9936	.1139	8.777	1.006	8.834	1.4574	83.5
6.6	.1152	.1149	.9934	.1157	8.643	1.007	8.700	1.4556	83.4
6.7	.1169	.1167	.9932	.1175	8.513	1.007	8.571	1.4539	83.3
6.8	.1187	.1184	.9930	.1192	8.386	1.007	8.446	1.4521	83.2
6.9	.1204	.1201	.9928	.1210	8.264	1.007	8.324	1.4504	83.1
7.0	.1222	.1219	.9925	.1228	8.144	1.008	8.206	1.4486	83.0
7.1	.1239	.1236	.9923	.1246	8.028	1.008	8.091	1.4469	82.9
7.2	.1257	.1253	.9921	.1263	7.916	1.008	7.979	1.4451	82.8
7.3	.1274	.1271	.9919	.1281	7.806	1.008	7.870	1.4434	82.7
7.4	.1292	.1288	.9917	.1299	7.700	1.008	7.764	1.4416	82.6
7.5	.1309	.1305	.9914	.1317	7.596	1.009	7.661	1.4399	82.5
7.6	.1326	.1323	.9912	.1334	7.495	1.009	7.561	1.4382	82.4
7.7	.1344	.1340	.9910	.1352	7.396	1.009	7.463	1.4364	82.3
7.8	.1361	.1357	.9907	.1370	7.300	1.009	7.368	1.4347	82.2
7.9	.1379	.1374	.9905	.1388	7.207	1.010	7.276	1.4329	82.1
8.0	.1396	.1392	.9903	.1405	7.115	1.010	7.185	1.4312	82.0
8.1	.1414	.1409	.9900	.1423	7.026	1.010	7.097	1.4294	81.9
8.2	.1431	.1426	.9898	.1441	6.940	1.010	7.011	1.4277	81.8
8.3	.1449	.1444	.9895	.1459	6.855	1.011	6.927	1.4259	81.7
8.4	.1466	.1461	.9893	.1477	6.772	1.011	6.845	1.4242	81.6
8.5	.1484	.1478	.9890	.1495	6.691	1.011	6.765	1.4224	81.5
8.6	.1501	.1495	.9888	.1512	6.612	1.011	6.687	1.4207	81.4
8.7	.1518	.1513	.9885	.1530	6.535	1.012	6.611	1.4190	81.3
8.8	.1536	.1530	.9882	.1548	6.460	1.012	6.537	1.4172	81.2
8.9	.1553	.1547	.9880	.1566	6.386	1.012	6.464	1.4155	81.1
9.0	.1571	.1564	.9877	.1584	6.314	1.012	6.392	1.4137	81.0
9.1	.1588	.1582	.9874	.1602	6.243	1.013	6.323	1.4120	80.9
9.2	.1606	.1599	.9871	.1620	6.174	1.013	6.255	1.4102	80.8
9.3	.1623	.1616	.9869	.1638	6.107	1.013	6.188	1.4085	80.7
9.4	.1641	.1633	.9866	.1655	6.041	1.014	6.123	1.4067	80.6
9.5	.1658	.1650	.9863	.1673	5.976	1.014	6.059	1.4050	80.5
9.6	.1676	.1668	.9860	.1691	5.912	1.014	5.996	1.4032	80.4
9.7	.1693	.1685	.9857	.1709	5.850	1.015	5.935	1.4015	80.3
9.8	.1710	.1702	.9854	.1727	5.789	1.015	5.875	1.3998	80.2
9.9	.1728	.1719	.9851	.1745	5.730	1.015	5.816	1.3980	80.1
10.0	.1745	.1736	.9848	.1763	5.671	1.015	5.759	1.3963	80.0
10.1	.1763	.1754	.9845	.1781	5.614	1.016	5.702	1.3945	79.9
10.2	.1780	.1771	.9842	.1799	5.558	1.016	5.647	1.3928	79.8
10.3	.1798	.1788	.9839	.1817	5.503	1.016	5.593	1.3910	79.7
10.4	.1815	.1805	.9836	.1835	5.449	1.017	5.540	1.3893	79.6
10.5	.1833	.1822	.9833	.1853	5.396	1.017	5.487	1.3875	79.5
10.6	.1850	.1840	.9829	.1871	5.343	1.017	5.436	1.3858	79.4
10.7	.1868	.1857	.9826	.1890	5.292	1.018	5.386	1.3840	79.3
10.8	.1885	.1874	.9823	.1908	5.242	1.018	5.337	1.3823	79.2
10.9	.1902	.1891	.9820	.1926	5.193	1.018	5.288	1.3806	79.1
11.0	.1920	.1908	.9816	.1944	5.145	1.019	5.241	1.3788	79.0
11.1	.1937	.1925	.9813	.1962	5.097	1.019	5.194	1.3771	78.9
11.2	.1955	.1942	.9810	.1980	5.050	1.019	5.148	1.3753	78.8
11.3	.1972	.1959	.9806	.1998	5.005	1.020	5.103	1.3736	78.7
11.4	.1990	.1977	.9803	.2016	4.959	1.020	5.059	1.3718	78.6
11.5	.2007	.1994	.9799	.2035	4.915	1.020	5.016	1.3701	78.5
11.6	.2025	.2011	.9796	.2053	4.872	1.021	4.973	1.3683	78.4
11.7	.2042	.2028	.9792	.2071	4.829	1.021	4.931	1.3666	78.3
11.8	.2059	.2045	.9789	.2089	4.787	1.022	4.890	1.3648	78.2
11.9	.2077	.2062	.9785	.2107	4.745	1.022	4.850	1.3631	78.1
12.0	.2094	.2079	.9781	.2126	4.705	1.022	4.810	1.3614	78.0
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees

Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
12.0	.2094	.2079	.9781	.2126	4.705	1.022	4.810	1.3614	<b>78.0</b>
12.1	.2112	.2096	.9778	.2144	4.665	1.023	4.771	1.3596	77.9
12.2	.2129	.2113	.9774	.2162	4.625	1.023	4.732	1.3579	77.8
12.3	.2147	.2130	.9770	.2180	4.586	1.023	4.694	1.3561	77.7
12.4	.2164	.2147	.9767	.2199	4.548	1.024	4.657	1.3544	77.6
12.5	.2182	.2164	.9763	.2217	4.511	1.024	4.620	1.3526	77.5
12.6	.2199	.2181	.9759	.2235	4.474	1.025	4.584	1.3509	77.4
12.7	.2217	.2198	.9755	.2254	4.437	1.025	4.549	1.3491	77.3
12.8	.2234	.2215	.9751	.2272	4.402	1.025	4.514	1.3474	77.2
12.9	.2251	.2233	.9748	.2290	4.366	1.026	4.479	1.3456	77.1
13.0	.2269	.2250	.9744	.2309	4.331	1.026	4.445	1.3439	<b>77.0</b>
13.1	.2286	.2267	.9740	.2327	4.297	1.027	4.412	1.3422	76.9
13.2	.2304	.2284	.9736	.2345	4.264	1.027	4.379	1.3404	76.8
13.3	.2321	.2300	.9732	.2364	4.230	1.028	4.347	1.3387	76.7
13.4	.2339	.2317	.9728	.2382	4.198	1.028	4.315	1.3369	76.6
13.5	.2356	.2334	.9724	.2401	4.165	1.028	4.284	1.3352	76.5
13.6	.2374	.2351	.9720	.2419	4.134	1.029	4.253	1.3334	76.4
13.7	.2391	.2368	.9715	.2438	4.102	1.029	4.222	1.3317	76.3
13.8	.2409	.2385	.9711	.2456	4.071	1.030	4.192	1.3299	76.2
13.9	.2426	.2402	.9707	.2475	4.041	1.030	4.163	1.3282	76.1
14.0	.2443	.2419	.9703	.2493	4.011	1.031	4.134	1.3265	<b>76.0</b>
14.1	.2461	.2436	.9699	.2512	3.981	1.031	4.105	1.3247	75.9
14.2	.2478	.2453	.9694	.2530	3.952	1.032	4.077	1.3230	75.8
14.3	.2496	.2470	.9690	.2549	3.923	1.032	4.049	1.3212	75.7
14.4	.2513	.2487	.9686	.2568	3.895	1.032	4.021	1.3195	75.6
14.5	.2531	.2504	.9681	.2586	3.867	1.033	3.994	1.3177	75.5
14.6	.2548	.2521	.9677	.2605	3.839	1.033	3.967	1.3160	75.4
14.7	.2566	.2538	.9673	.2623	3.812	1.034	3.941	1.3142	75.3
14.8	.2583	.2554	.9668	.2642	3.785	1.034	3.915	1.3125	75.2
14.9	.2601	.2571	.9664	.2661	3.758	1.035	3.889	1.3107	75.1
15.0	.2618	.2588	.9659	.2679	3.732	1.035	3.864	1.3090	<b>75.0</b>
15.1	.2635	.2605	.9655	.2698	3.706	1.036	3.839	1.3073	74.9
15.2	.2653	.2622	.9650	.2717	3.681	1.036	3.814	1.3055	74.8
15.3	.2670	.2639	.9646	.2736	3.655	1.037	3.790	1.3038	74.7
15.4	.2688	.2656	.9641	.2754	3.630	1.037	3.766	1.3020	74.6
15.5	.2705	.2672	.9636	.2773	3.606	1.038	3.742	1.3003	74.5
15.6	.2723	.2689	.9632	.2792	3.582	1.038	3.719	1.2985	74.4
15.7	.2740	.2706	.9627	.2811	3.558	1.039	3.695	1.2968	74.3
15.8	.2758	.2723	.9622	.2830	3.534	1.039	3.673	1.2950	74.2
15.9	.2775	.2740	.9617	.2849	3.511	1.040	3.650	1.2933	74.1
16.0	.2793	.2756	.9613	.2867	3.487	1.040	3.628	1.2915	<b>74.0</b>
16.1	.2810	.2773	.9608	.2886	3.465	1.041	3.606	1.2898	73.9
16.2	.2827	.2790	.9603	.2905	3.442	1.041	3.584	1.2881	73.8
16.3	.2845	.2807	.9598	.2924	3.420	1.042	3.563	1.2863	73.7
16.4	.2862	.2823	.9593	.2943	3.398	1.042	3.542	1.2846	73.6
16.5	.2880	.2840	.9588	.2962	3.376	1.043	3.521	1.2828	73.5
16.6	.2897	.2857	.9583	.2981	3.354	1.043	3.500	1.2811	73.4
16.7	.2915	.2874	.9578	.3000	3.333	1.044	3.480	1.2793	73.3
16.8	.2932	.2890	.9573	.3019	3.312	1.045	3.460	1.2776	73.2
16.9	.2950	.2907	.9568	.3038	3.291	1.045	3.440	1.2758	73.1
17.0	.2967	.2924	.9563	.3057	3.271	1.046	3.420	1.2741	<b>73.0</b>
17.1	.2985	.2940	.9558	.3076	3.251	1.046	3.401	1.2723	72.9
17.2	.3002	.2957	.9553	.3096	3.230	1.047	3.382	1.2706	72.8
17.3	.3019	.2974	.9548	.3115	3.211	1.047	3.363	1.2689	72.7
17.4	.3037	.2990	.9542	.3134	3.191	1.048	3.344	1.2671	72.6
17.5	.3054	.3007	.9537	.3153	3.172	1.049	3.326	1.2654	72.5
17.6	.3072	.3024	.9532	.3172	3.152	1.049	3.307	1.2636	72.4
17.7	.3089	.3040	.9527	.3191	3.133	1.050	3.289	1.2619	72.3
17.8	.3107	.3057	.9521	.3211	3.115	1.050	3.271	1.2601	72.2
17.9	.3124	.3074	.9516	.3230	3.096	1.051	3.254	1.2584	72.1
18.0	.3142	.3090	.9511	.3249	3.078	1.051	3.236	1.2566	<b>72.0</b>
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees



Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
18.0	.3142	.3090	.9511	.3249	3.078	1.051	3.236	1.2566	72.0
18.1	.3159	.3107	.9505	.3269	3.060	1.052	3.219	1.2549	71.9
18.2	.3177	.3123	.9500	.3288	3.042	1.053	3.202	1.2531	71.8
18.3	.3194	.3140	.9494	.3307	3.024	1.053	3.185	1.2514	71.7
18.4	.3211	.3156	.9489	.3327	3.006	1.054	3.168	1.2497	71.6
18.5	.3229	.3173	.9483	.3346	2.989	1.054	3.152	1.2479	71.5
18.6	.3246	.3190	.9478	.3365	2.971	1.055	3.135	1.2462	71.4
18.7	.3264	.3206	.9472	.3385	2.954	1.056	3.119	1.2444	71.3
18.8	.3281	.3223	.9466	.3404	2.937	1.056	3.103	1.2427	71.2
18.9	.3299	.3239	.9461	.3424	2.921	1.057	3.087	1.2409	71.1
19.0	.3316	.3256	.9455	.3443	2.904	1.058	3.072	1.2392	71.0
19.1	.3334	.3272	.9449	.3463	2.888	1.058	3.056	1.2374	70.9
19.2	.3351	.3289	.9444	.3482	2.872	1.059	3.041	1.2357	70.8
19.3	.3368	.3305	.9438	.3502	2.856	1.060	3.026	1.2339	70.7
19.4	.3386	.3322	.9432	.3522	2.840	1.060	3.011	1.2322	70.6
19.5	.3403	.3338	.9426	.3541	2.824	1.061	2.996	1.2305	70.5
19.6	.3421	.3355	.9421	.3561	2.808	1.062	2.981	1.2287	70.4
19.7	.3438	.3371	.9415	.3581	2.793	1.062	2.967	1.2270	70.3
19.8	.3456	.3387	.9409	.3600	2.778	1.063	2.952	1.2252	70.2
19.9	.3473	.3404	.9403	.3620	2.762	1.064	2.938	1.2235	70.1
20.0	.3491	.3420	.9397	.3640	2.747	1.064	2.924	1.2217	70.0
20.1	.3508	.3437	.9391	.3659	2.733	1.065	2.910	1.2200	69.9
20.2	.3526	.3453	.9385	.3679	2.718	1.066	2.896	1.2182	69.8
20.3	.3543	.3469	.9379	.3699	2.703	1.066	2.882	1.2165	69.7
20.4	.3560	.3486	.9373	.3719	2.689	1.067	2.869	1.2147	69.6
20.5	.3578	.3502	.9367	.3739	2.675	1.068	2.855	1.2130	69.5
20.6	.3595	.3518	.9361	.3759	2.660	1.068	2.842	1.2113	69.4
20.7	.3613	.3535	.9354	.3779	2.646	1.069	2.829	1.2095	69.3
20.8	.3630	.3551	.9348	.3799	2.633	1.070	2.816	1.2078	69.2
20.9	.3648	.3567	.9342	.3819	2.619	1.070	2.803	1.2060	69.1
21.0	.3665	.3584	.9336	.3839	2.605	1.071	2.790	1.2043	69.0
21.1	.3683	.3600	.9330	.3859	2.592	1.072	2.778	1.2025	68.9
21.2	.3700	.3616	.9323	.3879	2.578	1.073	2.765	1.2008	68.8
21.3	.3718	.3633	.9317	.3899	2.565	1.073	2.753	1.1991	68.7
21.4	.3735	.3649	.9311	.3919	2.552	1.074	2.741	1.1973	68.6
21.5	.3752	.3665	.9304	.3939	2.539	1.075	2.729	1.1956	68.5
21.6	.3770	.3681	.9298	.3959	2.526	1.076	2.716	1.1938	68.4
21.7	.3787	.3697	.9291	.3979	2.513	1.076	2.705	1.1921	68.3
21.8	.3805	.3714	.9285	.4000	2.500	1.077	2.693	1.1903	68.2
21.9	.3822	.3730	.9278	.4020	2.488	1.078	2.681	1.1886	68.1
22.0	.3840	.3746	.9272	.4040	2.475	1.079	2.669	1.1868	68.0
22.1	.3857	.3762	.9265	.4061	2.463	1.079	2.658	1.1851	67.9
22.2	.3875	.3778	.9259	.4081	2.450	1.080	2.647	1.1833	67.8
22.3	.3892	.3795	.9252	.4101	2.438	1.081	2.635	1.1816	67.7
22.4	.3910	.3811	.9245	.4122	2.426	1.082	2.624	1.1798	67.6
22.5	.3927	.3827	.9239	.4142	2.414	1.082	2.613	1.1781	67.5
22.6	.3944	.3843	.9232	.4163	2.402	1.083	2.602	1.1764	67.4
22.7	.3962	.3859	.9225	.4183	2.391	1.084	2.591	1.1746	67.3
22.8	.3979	.3875	.9219	.4204	2.379	1.085	2.581	1.1729	67.2
22.9	.3997	.3891	.9212	.4224	2.367	1.086	2.570	1.1711	67.1
23.0	.4014	.3907	.9205	.4245	2.356	1.086	2.559	1.1694	67.0
23.1	.4032	.3923	.9198	.4265	2.344	1.087	2.549	1.1676	66.9
23.2	.4049	.3939	.9191	.4286	2.333	1.088	2.538	1.1659	66.8
23.3	.4067	.3955	.9184	.4307	2.322	1.089	2.528	1.1641	66.7
23.4	.4084	.3971	.9178	.4327	2.311	1.090	2.518	1.1624	66.6
23.5	.4102	.3987	.9171	.4348	2.300	1.090	2.508	1.1606	66.5
23.6	.4119	.4003	.9164	.4369	2.289	1.091	2.498	1.1589	66.4
23.7	.4136	.4019	.9157	.4390	2.278	1.092	2.488	1.1572	66.3
23.8	.4154	.4035	.9150	.4411	2.267	1.093	2.478	1.1554	66.2
23.9	.4171	.4051	.9143	.4431	2.257	1.094	2.468	1.1537	66.1
24.0	.4189	.4067	.9135	.4452	2.246	1.095	2.459	1.1519	66.0
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees

Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
24.0	.4189	.4067	.9135	.4452	2.246	1.095	2.459	1.1519	<b>66.0</b>
24.1	.4206	.4083	.9128	.4473	2.236	1.095	2.449	1.1502	65.9
24.2	.4224	.4099	.9121	.4494	2.225	1.096	2.439	1.1484	65.8
24.3	.4241	.4115	.9114	.4515	2.215	1.097	2.430	1.1467	65.7
24.4	.4259	.4131	.9107	.4536	2.204	1.098	2.421	1.1449	65.6
24.5	.4276	.4147	.9100	.4557	2.194	1.099	2.411	1.1432	65.5
24.6	.4294	.4163	.9092	.4578	2.184	1.100	2.402	1.1414	65.4
24.7	.4311	.4179	.9085	.4599	2.174	1.101	2.393	1.1397	65.3
24.8	.4328	.4195	.9078	.4621	2.164	1.102	2.384	1.1380	65.2
24.9	.4346	.4210	.9070	.4642	2.154	1.102	2.375	1.1362	65.1
25.0	.4363	.4226	.9063	.4663	2.145	1.103	2.366	1.1345	<b>65.0</b>
25.1	.4381	.4242	.9056	.4684	2.135	1.104	2.357	1.1327	64.9
25.2	.4398	.4258	.9048	.4706	2.125	1.105	2.349	1.1310	64.8
25.3	.4416	.4274	.9041	.4727	2.116	1.106	2.340	1.1292	64.7
25.4	.4433	.4289	.9033	.4748	2.106	1.107	2.331	1.1275	64.6
25.5	.4451	.4305	.9026	.4770	2.097	1.108	2.323	1.1257	64.5
25.6	.4468	.4321	.9018	.4791	2.087	1.109	2.314	1.1240	64.4
25.7	.4485	.4337	.9011	.4813	2.078	1.110	2.306	1.1222	64.3
25.8	.4503	.4352	.9003	.4834	2.069	1.111	2.298	1.1205	64.2
25.9	.4520	.4368	.8996	.4856	2.059	1.112	2.289	1.1188	64.1
26.0	.4538	.4384	.8988	.4877	2.050	1.113	2.281	1.1170	<b>64.0</b>
26.1	.4555	.4399	.8980	.4899	2.041	1.114	2.273	1.1153	63.9
26.2	.4573	.4415	.8973	.4921	2.032	1.115	2.265	1.1135	63.8
26.3	.4590	.4431	.8965	.4942	2.023	1.115	2.257	1.1118	63.7
26.4	.4608	.4446	.8957	.4964	2.014	1.116	2.249	1.1100	63.6
26.5	.4625	.4462	.8949	.4986	2.006	1.117	2.241	1.1083	63.5
26.6	.4643	.4478	.8942	.5008	1.997	1.118	2.233	1.1065	63.4
26.7	.4660	.4493	.8934	.5029	1.988	1.119	2.226	1.1048	63.3
26.8	.4677	.4509	.8926	.5051	1.980	1.120	2.218	1.1030	63.2
26.9	.4695	.4524	.8918	.5073	1.971	1.121	2.210	1.1013	63.1
27.0	.4712	.4540	.8910	.5095	1.963	1.122	2.203	1.0996	<b>63.0</b>
27.1	.4730	.4555	.8902	.5117	1.954	1.123	2.195	1.0978	62.9
27.2	.4747	.4571	.8894	.5139	1.946	1.124	2.188	1.0961	62.8
27.3	.4765	.4586	.8886	.5161	1.937	1.125	2.180	1.0943	62.7
27.4	.4782	.4602	.8878	.5184	1.929	1.126	2.173	1.0926	62.6
27.5	.4800	.4617	.8870	.5206	1.921	1.127	2.166	1.0908	62.5
27.6	.4817	.4633	.8862	.5228	1.913	1.128	2.158	1.0891	62.4
27.7	.4835	.4648	.8854	.5250	1.905	1.129	2.151	1.0873	62.3
27.8	.4852	.4664	.8846	.5272	1.897	1.130	2.144	1.0856	62.2
27.9	.4869	.4679	.8838	.5295	1.889	1.132	2.137	1.0838	62.1
28.0	.4887	.4695	.8829	.5317	1.881	1.133	2.130	1.0821	<b>62.0</b>
28.1	.4904	.4710	.8821	.5340	1.873	1.134	2.123	1.0804	61.9
28.2	.4922	.4726	.8813	.5362	1.865	1.135	2.116	1.0786	61.8
28.3	.4939	.4741	.8805	.5384	1.857	1.136	2.109	1.0769	61.7
28.4	.4957	.4756	.8796	.5407	1.849	1.137	2.103	1.0751	61.6
28.5	.4974	.4772	.8788	.5430	1.842	1.138	2.096	1.0734	61.5
28.6	.4992	.4787	.8780	.5452	1.834	1.139	2.089	1.0716	61.4
28.7	.5009	.4802	.8771	.5475	1.827	1.140	2.082	1.0699	61.3
28.8	.5027	.4818	.8763	.5498	1.819	1.141	2.076	1.0681	61.2
28.9	.5044	.4833	.8755	.5520	1.811	1.142	2.069	1.0664	61.1
29.0	.5061	.4848	.8746	.5543	1.804	1.143	2.063	1.0647	<b>61.0</b>
29.1	.5079	.4863	.8738	.5566	1.797	1.144	2.056	1.0629	60.9
29.2	.5096	.4879	.8729	.5589	1.789	1.146	2.050	1.0612	60.8
29.3	.5114	.4894	.8721	.5612	1.782	1.147	2.043	1.0594	60.7
29.4	.5131	.4909	.8712	.5635	1.775	1.148	2.037	1.0577	60.6
29.5	.5149	.4924	.8704	.5658	1.767	1.149	2.031	1.0559	60.5
29.6	.5166	.4939	.8695	.5681	1.760	1.150	2.025	1.0542	60.4
29.7	.5184	.4955	.8686	.5704	1.753	1.151	2.018	1.0524	60.3
29.8	.5201	.4970	.8678	.5727	1.746	1.152	2.012	1.0507	60.2
29.9	.5219	.4985	.8669	.5750	1.739	1.154	2.006	1.0489	60.1
30.0	.5236	.5000	.8660	.5774	1.732	1.155	2.000	1.0472	<b>60.0</b>
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees

Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
30.0	.5236	.5000	.8660	.5774	1.732	1.155	2.000	1.0472	60.0
30.1	.5253	.5015	.8652	.5797	1.725	1.156	1.994	1.0455	59.9
30.2	.5271	.5030	.8643	.5820	1.718	1.157	1.988	1.0437	59.8
30.3	.5288	.5045	.8634	.5844	1.711	1.158	1.982	1.0420	59.7
30.4	.5306	.5060	.8625	.5867	1.704	1.159	1.976	1.0402	59.6
30.5	.5323	.5075	.8616	.5890	1.698	1.161	1.970	1.0385	59.5
30.6	.5341	.5090	.8607	.5914	1.691	1.162	1.964	1.0367	59.4
30.7	.5358	.5105	.8599	.5938	1.684	1.163	1.959	1.0350	59.3
30.8	.5376	.5120	.8590	.5961	1.678	1.164	1.953	1.0332	59.2
30.9	.5393	.5135	.8581	.5985	1.671	1.165	1.947	1.0315	59.1
31.0	.5411	.5150	.8572	.6009	1.664	1.167	1.942	1.0297	59.0
31.1	.5428	.5165	.8563	.6032	1.658	1.168	1.936	1.0280	58.9
31.2	.5445	.5180	.8554	.6056	1.651	1.169	1.930	1.0263	58.8
31.3	.5463	.5195	.8545	.6080	1.645	1.170	1.925	1.0245	58.7
31.4	.5480	.5210	.8535	.6104	1.638	1.172	1.919	1.0228	58.6
31.5	.5498	.5225	.8526	.6128	1.632	1.173	1.914	1.0210	58.5
31.6	.5515	.5240	.8517	.6152	1.625	1.174	1.908	1.0193	58.4
31.7	.5533	.5255	.8508	.6176	1.619	1.175	1.903	1.0175	58.3
31.8	.5550	.5270	.8499	.6200	1.613	1.177	1.898	1.0158	58.2
31.9	.5568	.5284	.8490	.6224	1.607	1.178	1.892	1.0140	58.1
32.0	.5585	.5299	.8480	.6249	1.600	1.179	1.887	1.0123	58.0
32.1	.5603	.5314	.8471	.6273	1.594	1.180	1.882	1.0105	57.9
32.2	.5620	.5329	.8462	.6297	1.588	1.182	1.877	1.0088	57.8
32.3	.5637	.5344	.8453	.6322	1.582	1.183	1.871	1.0071	57.7
32.4	.5655	.5358	.8443	.6346	1.576	1.184	1.866	1.0053	57.6
32.5	.5672	.5373	.8434	.6371	1.570	1.186	1.861	1.0036	57.5
32.6	.5690	.5388	.8425	.6395	1.564	1.187	1.856	1.0018	57.4
32.7	.5707	.5402	.8415	.6420	1.558	1.188	1.851	1.0001	57.3
32.8	.5725	.5417	.8406	.6445	1.552	1.190	1.846	.9983	57.2
32.9	.5742	.5432	.8396	.6469	1.546	1.191	1.841	.9966	57.1
33.0	.5760	.5446	.8387	.6494	1.540	1.192	1.836	.9948	57.0
33.1	.5777	.5461	.8377	.6519	1.534	1.194	1.831	.9931	56.9
33.2	.5794	.5476	.8368	.6544	1.528	1.195	1.826	.9913	56.8
33.3	.5812	.5490	.8358	.6569	1.522	1.196	1.821	.9896	56.7
33.4	.5829	.5505	.8348	.6594	1.517	1.198	1.817	.9879	56.6
33.5	.5847	.5519	.8339	.6619	1.511	1.199	1.812	.9861	56.5
33.6	.5864	.5534	.8329	.6644	1.505	1.201	1.807	.9844	56.4
33.7	.5882	.5548	.8320	.6669	1.499	1.202	1.802	.9826	56.3
33.8	.5899	.5563	.8310	.6694	1.494	1.203	1.798	.9809	56.2
33.9	.5917	.5577	.8300	.6720	1.488	1.205	1.793	.9791	56.1
34.0	.5934	.5592	.8290	.6745	1.483	1.206	1.788	.9774	56.0
34.1	.5952	.5606	.8281	.6771	1.477	1.208	1.784	.9756	55.9
34.2	.5969	.5621	.8271	.6796	1.471	1.209	1.779	.9739	55.8
34.3	.5986	.5635	.8261	.6822	1.466	1.211	1.775	.9721	55.7
34.4	.6004	.5650	.8251	.6847	1.460	1.212	1.770	.9704	55.6
34.5	.6021	.5664	.8241	.6873	1.455	1.213	1.766	.9687	55.5
34.6	.6039	.5678	.8231	.6899	1.450	1.215	1.761	.9669	55.4
34.7	.6056	.5693	.8221	.6924	1.444	1.216	1.757	.9652	55.3
34.8	.6074	.5707	.8211	.6950	1.439	1.218	1.752	.9634	55.2
34.9	.6091	.5721	.8202	.6976	1.433	1.219	1.748	.9617	55.1
35.0	.6109	.5736	.8192	.7002	1.428	1.221	1.743	.9599	55.0
35.1	.6126	.5750	.8181	.7028	1.423	1.222	1.739	.9582	54.9
35.2	.6144	.5764	.8171	.7054	1.418	1.224	1.735	.9564	54.8
35.3	.6161	.5779	.8161	.7080	1.412	1.225	1.731	.9547	54.7
35.4	.6178	.5793	.8151	.7107	1.407	1.227	1.726	.9529	54.6
35.5	.6196	.5807	.8141	.7133	1.402	1.228	1.722	.9512	54.5
35.6	.6213	.5821	.8131	.7159	1.397	1.230	1.718	.9495	54.4
35.7	.6231	.5835	.8121	.7186	1.392	1.231	1.714	.9477	54.3
35.8	.6248	.5850	.8111	.7212	1.387	1.233	1.710	.9460	54.2
35.9	.6266	.5864	.8100	.7239	1.381	1.235	1.705	.9442	54.1
36.0	.6283	.5878	.8090	.7265	1.376	1.236	1.701	.9425	54.0
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees



Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
36.0	.6283	.5878	.8090	.7265	1.376	1.236	1.701	.9425	<b>54.0</b>
36.1	.6301	.5892	.8080	.7292	1.371	1.238	1.697	.9407	53.9
36.2	.6318	.5906	.8070	.7319	1.366	1.239	1.693	.9390	53.8
36.3	.6336	.5920	.8059	.7346	1.361	1.241	1.689	.9372	53.7
36.4	.6353	.5934	.8049	.7373	1.356	1.242	1.685	.9355	53.6
36.5	.6370	.5948	.8039	.7400	1.351	1.244	1.681	.9338	53.5
36.6	.6388	.5962	.8028	.7427	1.347	1.246	1.677	.9320	53.4
36.7	.6405	.5976	.8018	.7454	1.342	1.247	1.673	.9303	53.3
36.8	.6423	.5990	.8007	.7481	1.337	1.249	1.669	.9285	53.2
36.9	.6440	.6004	.7997	.7508	1.332	1.250	1.666	.9268	53.1
37.0	.6458	.6018	.7986	.7536	1.327	1.252	1.662	.9250	<b>53.0</b>
37.1	.6475	.6032	.7976	.7563	1.322	1.254	1.658	.9233	52.9
37.2	.6493	.6046	.7965	.7590	1.317	1.255	1.654	.9215	52.8
37.3	.6510	.6060	.7955	.7618	1.313	1.257	1.650	.9198	52.7
37.4	.6528	.6074	.7944	.7646	1.308	1.259	1.646	.9180	52.6
37.5	.6545	.6088	.7934	.7673	1.303	1.260	1.643	.9163	52.5
37.6	.6562	.6101	.7923	.7701	1.299	1.262	1.639	.9146	52.4
37.7	.6580	.6115	.7912	.7729	1.294	1.264	1.635	.9128	52.3
37.8	.6597	.6129	.7902	.7757	1.289	1.266	1.632	.9111	52.2
37.9	.6615	.6143	.7891	.7785	1.285	1.267	1.628	.9093	52.1
38.0	.6632	.6157	.7880	.7813	1.280	1.269	1.624	.9076	<b>52.0</b>
38.1	.6650	.6170	.7869	.7841	1.275	1.271	1.621	.9058	51.9
38.2	.6667	.6184	.7859	.7869	1.271	1.272	1.617	.9041	51.8
38.3	.6685	.6198	.7848	.7898	1.266	1.274	1.613	.9023	51.7
38.4	.6702	.6211	.7837	.7926	1.262	1.276	1.610	.9006	51.6
38.5	.6720	.6225	.7826	.7954	1.257	1.278	1.606	.8988	51.5
38.6	.6737	.6239	.7815	.7983	1.253	1.280	1.603	.8971	51.4
38.7	.6754	.6252	.7804	.8012	1.248	1.281	1.599	.8954	51.3
38.8	.6772	.6266	.7793	.8040	1.244	1.283	1.596	.8936	51.2
38.9	.6789	.6280	.7782	.8069	1.239	1.285	1.592	.8919	51.1
39.0	.6807	.6293	.7771	.8098	1.235	1.287	1.589	.8901	<b>51.0</b>
39.1	.6824	.6307	.7760	.8127	1.230	1.289	1.586	.8884	50.9
39.2	.6842	.6320	.7749	.8156	1.226	1.290	1.582	.8866	50.8
39.3	.6859	.6334	.7738	.8185	1.222	1.292	1.579	.8849	50.7
39.4	.6877	.6347	.7727	.8214	1.217	1.294	1.575	.8831	50.6
39.5	.6894	.6361	.7716	.8243	1.213	1.296	1.572	.8814	50.5
39.6	.6912	.6374	.7705	.8273	1.209	1.298	1.569	.8796	50.4
39.7	.6929	.6388	.7694	.8302	1.205	1.300	1.566	.8779	50.3
39.8	.6946	.6401	.7683	.8332	1.200	1.302	1.562	.8762	50.2
39.9	.6964	.6414	.7672	.8361	1.196	1.304	1.559	.8744	50.1
40.0	.6981	.6428	.7660	.8391	1.192	1.305	1.556	.8727	<b>50.0</b>
40.1	.6999	.6441	.7649	.8421	1.188	1.307	1.552	.8709	49.9
40.2	.7016	.6455	.7638	.8451	1.183	1.309	1.549	.8692	49.8
40.3	.7034	.6468	.7627	.8481	1.179	1.311	1.546	.8674	49.7
40.4	.7051	.6481	.7615	.8511	1.175	1.313	1.543	.8657	49.6
40.5	.7069	.6494	.7604	.8541	1.171	1.315	1.540	.8639	49.5
40.6	.7086	.6508	.7593	.8571	1.167	1.317	1.537	.8622	49.4
40.7	.7103	.6521	.7581	.8601	1.163	1.319	1.534	.8604	49.3
40.8	.7121	.6534	.7570	.8632	1.159	1.321	1.530	.8587	49.2
40.9	.7138	.6547	.7559	.8662	1.154	1.323	1.527	.8570	49.1
41.0	.7156	.6561	.7547	.8693	1.150	1.325	1.524	.8552	<b>49.0</b>
41.1	.7173	.6574	.7536	.8724	1.146	1.327	1.521	.8535	48.9
41.2	.7191	.6587	.7524	.8754	1.142	1.329	1.518	.8517	48.8
41.3	.7208	.6600	.7513	.8785	1.138	1.331	1.515	.8500	48.7
41.4	.7226	.6613	.7501	.8816	1.134	1.333	1.512	.8482	48.6
41.5	.7243	.6626	.7490	.8847	1.130	1.335	1.509	.8465	48.5
41.6	.7261	.6639	.7478	.8878	1.126	1.337	1.506	.8447	48.4
41.7	.7278	.6652	.7466	.8910	1.122	1.339	1.503	.8430	48.3
41.8	.7295	.6665	.7455	.8941	1.118	1.341	1.500	.8412	48.2
41.9	.7313	.6678	.7443	.8972	1.115	1.344	1.497	.8395	48.1
42.0	.7330	.6691	.7431	.9004	1.111	1.346	1.494	.8378	<b>48.0</b>
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees

Table 5

Trigonometric Functions of  $\theta$  ( $\theta$  in decimal degrees)

$\theta$ Degrees	$\theta$ Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$		
42.0	.7330	.6691	.7431	.9004	1.111	1.346	1.494	.8378	48.0
42.1	.7348	.6704	.7420	.9036	1.107	1.348	1.492	.8360	47.9
42.2	.7365	.6717	.7408	.9067	1.103	1.350	1.489	.8343	47.8
42.3	.7383	.6730	.7396	.9099	1.099	1.352	1.486	.8325	47.7
42.4	.7400	.6743	.7385	.9131	1.095	1.354	1.483	.8308	47.6
42.5	.7418	.6756	.7373	.9163	1.091	1.356	1.480	.8290	47.5
42.6	.7435	.6769	.7361	.9195	1.087	1.359	1.477	.8273	47.4
42.7	.7453	.6782	.7349	.9228	1.084	1.361	1.475	.8255	47.3
42.8	.7470	.6794	.7337	.9260	1.080	1.363	1.472	.8238	47.2
42.9	.7487	.6807	.7325	.9293	1.076	1.365	1.469	.8221	47.1
43.0	.7505	.6820	.7314	.9325	1.072	1.367	1.466	.8203	47.0
43.1	.7522	.6833	.7302	.9358	1.069	1.370	1.464	.8186	46.9
43.2	.7540	.6845	.7290	.9391	1.065	1.372	1.461	.8168	46.8
43.3	.7557	.6858	.7278	.9424	1.061	1.374	1.458	.8151	46.7
43.4	.7575	.6871	.7266	.9457	1.057	1.376	1.455	.8133	46.6
43.5	.7592	.6884	.7254	.9490	1.054	1.379	1.453	.8116	46.5
43.6	.7610	.6896	.7242	.9523	1.050	1.381	1.450	.8098	46.4
43.7	.7627	.6909	.7230	.9556	1.046	1.383	1.447	.8081	46.3
43.8	.7645	.6921	.7218	.9590	1.043	1.386	1.445	.8063	46.2
43.9	.7662	.6934	.7206	.9623	1.039	1.388	1.442	.8046	46.1
44.0	.7679	.6947	.7193	.9657	1.036	1.390	1.440	.8029	46.0
44.1	.7697	.6959	.7181	.9691	1.032	1.393	1.437	.8011	45.9
44.2	.7714	.6972	.7169	.9725	1.028	1.395	1.434	.7994	45.8
44.3	.7732	.6984	.7157	.9759	1.025	1.397	1.432	.7976	45.7
44.4	.7749	.6997	.7145	.9793	1.021	1.400	1.429	.7959	45.6
44.5	.7767	.7009	.7133	.9827	1.018	1.402	1.427	.7941	45.5
44.6	.7784	.7022	.7120	.9861	1.014	1.404	1.424	.7924	45.4
44.7	.7802	.7034	.7108	.9896	1.011	1.407	1.422	.7906	45.3
44.8	.7819	.7046	.7096	.9930	1.007	1.409	1.419	.7889	45.2
44.9	.7837	.7059	.7083	.9965	1.003	1.412	1.417	.7871	45.1
45.0	.7854	.7071	.7071	1.0000	1.000	1.414	1.414	.7854	45.0
		$\cos \theta$	$\sin \theta$	$\cot \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\theta$ Radians	$\theta$ Degrees



Table 6

Trigonometric Functions of  $\theta$  ( $\theta$  in radians)

$\theta$ Radians	$\theta$ Degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
0.00	0° 00'	0.0000	1.000	0.0000	Undefined	1.000	Undefined
.01	0° 34'	.0100	1.000	.0100	100.0	1.000	100.0
.02	1° 09'	.0200	0.9998	.0200	49.99	1.000	50.00
.03	1° 43'	.0300	0.9996	.0300	33.32	1.000	33.34
.04	2° 18'	.0400	0.9992	.0400	24.99	1.001	25.01
0.05	2° 52'	0.0500	0.9988	0.0500	19.98	1.001	20.01
.06	3° 26'	.0600	.9982	.0601	16.65	1.002	16.68
.07	4° 01'	.0699	.9976	.0701	14.26	1.002	14.30
.08	4° 35'	.0799	.9968	.0802	12.47	1.003	12.51
.09	5° 09'	.0899	.9960	.0902	11.08	1.004	11.13
0.10	5° 44'	0.0998	0.9950	0.1003	9.967	1.005	10.02
.11	6° 18'	.1098	.9940	.1104	9.054	1.006	9.109
.12	6° 53'	.1197	.9928	.1206	8.293	1.007	8.353
.13	7° 27'	.1296	.9916	.1307	7.649	1.009	7.714
.14	8° 01'	.1395	.9902	.1409	7.096	1.010	7.166
0.15	8° 36'	0.1494	0.9888	0.1511	6.617	1.011	6.692
.16	9° 10'	.1593	.9872	.1614	6.197	1.013	6.277
.17	9° 44'	.1692	.9856	.1717	5.826	1.015	5.911
.18	10° 19'	.1790	.9838	.1820	5.495	1.016	5.586
.19	10° 53'	.1889	.9820	.1923	5.200	1.018	5.295
0.20	11° 28'	0.1987	0.9801	0.2027	4.933	1.020	5.033
.21	12° 02'	.2085	.9780	.2131	4.692	1.022	4.797
.22	12° 36'	.2182	.9759	.2236	4.472	1.025	4.582
.23	13° 11'	.2280	.9737	.2341	4.271	1.027	4.386
.24	13° 45'	.2377	.9713	.2447	4.086	1.030	4.207
0.25	14° 19'	0.2474	0.9689	0.2553	3.916	1.032	4.042
.26	14° 54'	.2571	.9664	.2660	3.759	1.035	3.890
.27	15° 28'	.2667	.9638	.2768	3.613	1.038	3.749
.28	16° 03'	.2764	.9611	.2876	3.478	1.041	3.619
.29	16° 37'	.2860	.9582	.2984	3.351	1.044	3.497
0.30	17° 11'	0.2955	0.9553	0.3093	3.233	1.047	3.384
.31	17° 46'	.3051	.9523	.3203	3.122	1.050	3.278
.32	18° 20'	.3146	.9492	.3314	3.018	1.053	3.179
.33	18° 55'	.3240	.9460	.3425	2.920	1.057	3.086
.34	19° 29'	.3335	.9428	.3537	2.827	1.061	2.999
0.35	20° 03'	0.3429	0.9394	0.3650	2.740	1.065	2.916
.36	20° 38'	.3523	.9359	.3764	2.657	1.068	2.839
.37	21° 12'	.3616	.9323	.3879	2.578	1.073	2.765
.38	21° 46'	.3709	.9287	.3994	2.504	1.077	2.696
.39	22° 21'	.3802	.9249	.4111	2.433	1.081	2.630
0.40	22° 55'	0.3894	0.9211	0.4228	2.365	1.086	2.568
.41	23° 30'	.3986	.9171	.4346	2.301	1.090	2.509
.42	24° 04'	.4078	.9131	.4466	2.239	1.095	2.452
.43	24° 38'	.4169	.9090	.4586	2.180	1.100	2.399
.44	25° 13'	.4259	.9048	.4708	2.124	1.105	2.348
0.45	25° 47'	0.4350	0.9004	0.4831	2.070	1.111	2.299
.46	26° 21'	.4439	.8961	.4954	2.018	1.116	2.253
.47	26° 56'	.4529	.8916	.5080	1.969	1.122	2.208
.48	27° 30'	.4618	.8870	.5206	1.921	1.127	2.166
.49	28° 05'	.4706	.8823	.5334	1.875	1.133	2.125

Table 6

Trigonometric Functions of  $\theta$  ( $\theta$  in radians)

$\theta$ Radians	$\theta$ Degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
$\frac{\pi}{6} \rightarrow$							
.50	28° 39'	.4794	0.8776	0.5463	1.830	1.139	2.086
.51	29° 13'	.4882	.8727	.5594	1.788	1.146	2.048
.52	29° 48'	.4969	.8678	.5726	1.747	1.152	2.013
.53	30° 22'	.5055	.8628	.5859	1.707	1.159	1.978
.54	30° 56'	.5141	.8577	.5994	1.668	1.166	1.945
.55	31° 31'	.5227	0.8525	0.6131	1.631	1.173	1.913
.56	32° 05'	.5312	.8473	.6269	1.595	1.180	1.883
.57	32° 40'	.5396	.8419	.6410	1.560	1.188	1.853
.58	33° 14'	.5480	.8365	.6552	1.526	1.196	1.825
.59	33° 48'	.5564	.8309	.6696	1.494	1.203	1.797
.60	34° 23'	.5646	0.8253	0.6841	1.462	1.212	1.771
.61	34° 57'	.5729	.8196	.6989	1.431	1.220	1.746
.62	35° 31'	.5810	.8139	.7139	1.401	1.229	1.721
.63	36° 06'	.5891	.8080	.7291	1.372	1.238	1.697
.64	36° 40'	.5972	.8021	.7445	1.343	1.247	1.674
.65	37° 15'	0.6052	0.7961	0.7602	1.315	1.256	1.652
.66	37° 49'	.6131	.7900	.7761	1.288	1.266	1.631
.67	38° 23'	.6210	.7838	.7923	1.262	1.276	1.610
.68	38° 58'	.6288	.7776	.8087	1.237	1.286	1.590
.69	39° 32'	.6365	.7712	.8253	1.212	1.297	1.571
.70	40° 06'	0.6442	0.7648	0.8423	1.187	1.307	1.552
.71	40° 41'	.6518	.7584	.8595	1.163	1.319	1.534
.72	41° 15'	.6594	.7518	.8771	1.140	1.330	1.517
.73	41° 50'	.6669	.7452	.8949	1.117	1.342	1.500
.74	42° 24'	.6743	.7385	.9131	1.095	1.354	1.483
.75	42° 58'	0.6816	0.7317	0.9316	1.073	1.367	1.467
.76	43° 33'	.6889	.7248	.9505	1.052	1.380	1.452
.77	44° 07'	.6961	.7179	.9697	1.031	1.393	1.437
$\frac{\pi}{4} \rightarrow$							
.78	44° 41'	.7033	.7109	.9893	1.011	1.407	1.422
.79	45° 16'	.7104	.7038	1.009	.9908	1.421	1.408
.80	45° 50'	0.7174	0.6967	1.030	0.9712	1.435	1.394
.81	46° 25'	.7243	.6895	1.050	.9520	1.450	1.381
.82	46° 59'	.7311	.6822	1.072	.9331	1.466	1.368
.83	47° 33'	.7379	.6749	1.093	.9146	1.482	1.355
.84	48° 08'	.7446	.6675	1.116	.8964	1.498	1.343
.85	48° 42'	0.7513	0.6600	1.138	0.8785	1.515	1.331
.86	49° 17'	.7578	.6524	1.162	.8609	1.533	1.320
.87	49° 51'	.7643	.6448	1.185	.8437	1.551	1.308
.88	50° 25'	.7707	.6372	1.210	.8267	1.569	1.297
.89	51° 00'	.7771	.6294	1.235	.8100	1.589	1.287
.90	51° 34'	0.7833	0.6216	1.260	0.7936	1.609	1.277
.91	52° 08'	.7895	.6137	1.286	.7774	1.629	1.267
.92	52° 43'	.7956	.6058	1.313	.7615	1.651	1.257
.93	53° 17'	.8016	.5978	1.341	.7458	1.673	1.247
.94	53° 52'	.8076	.5898	1.369	.7303	1.696	1.238
.95	54° 26'	0.8134	0.5817	1.398	0.7151	1.719	1.229
.96	55° 00'	.8192	.5735	1.428	.7001	1.744	1.221
.97	55° 35'	.8249	.5653	1.459	.6853	1.769	1.212
.98	56° 09'	.8305	.5570	1.491	.6707	1.795	1.204
.99	56° 43'	.8360	.5487	1.524	.6563	1.823	1.196

Table 6

Trigonometric Functions of  $\theta$  ( $\theta$  in radians)

$\theta$ Radians	$\theta$ Degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
1.00	57° 18'	0.8415	0.5403	1.557	0.6421	1.851	1.188
1.01	57° 52'	.8468	.5319	1.592	.6281	1.880	1.181
1.02	58° 27'	.8521	.5234	1.628	.6142	1.911	1.174
1.03	59° 01'	.8573	.5148	1.665	.6005	1.942	1.166
1.04	59° 35'	.8624	.5062	1.704	.5870	1.975	1.160
$\frac{\pi}{3}$ → 1.05	60° 10'	0.8674	0.4976	1.743	0.5736	2.010	1.153
1.06	60° 44'	.8724	.4889	1.784	.5604	2.046	1.146
1.07	61° 18'	.8772	.4801	1.827	.5473	2.083	1.140
1.08	61° 53'	.8820	.4713	1.871	.5344	2.122	1.134
1.09	62° 27'	.8866	.4625	1.917	.5216	2.162	1.128
1.10	63° 02'	0.8912	0.4536	1.965	0.5090	2.205	1.122
1.11	63° 36'	.8957	.4447	2.014	.4964	2.249	1.116
1.12	64° 10'	.9001	.4357	2.066	.4840	2.295	1.111
1.13	64° 45'	.9044	.4267	2.120	.4718	2.344	1.106
1.14	65° 19'	.9086	.4176	2.176	.4596	2.395	1.101
1.15	65° 53'	0.9128	0.4085	2.234	0.4475	2.448	1.096
1.16	66° 28'	.9168	.3993	2.296	.4356	2.504	1.091
1.17	67° 02'	.9208	.3902	2.360	.4237	2.563	1.086
1.18	67° 37'	.9246	.3809	2.428	.4120	2.625	1.082
1.19	68° 11'	.9284	.3717	2.498	.4003	2.691	1.077
1.20	68° 45'	0.9320	0.3624	2.572	0.3888	2.760	1.073
1.21	69° 20'	.9356	.3530	2.650	.3773	2.833	1.069
1.22	69° 54'	.9391	.3436	2.733	.3659	2.910	1.065
1.23	70° 28'	.9425	.3342	2.820	.3546	2.992	1.061
1.24	71° 03'	.9458	.3248	2.912	.3434	3.079	1.057
1.25	71° 37'	0.9490	0.3153	3.010	0.3323	3.171	1.054
1.26	72° 12'	.9521	.3058	3.113	.3212	3.270	1.050
1.27	72° 46'	.9551	.2963	3.224	.3102	3.375	1.047
1.28	73° 20'	.9580	.2867	3.341	.2993	3.488	1.044
1.29	73° 55'	.9608	.2771	3.467	.2884	3.609	1.041
1.30	74° 29'	0.9636	0.2675	3.602	0.2776	3.738	1.038
1.31	75° 03'	.9662	.2579	3.747	.2669	3.878	1.035
1.32	75° 38'	.9687	.2482	3.903	.2562	4.029	1.032
1.33	76° 12'	.9711	.2385	4.072	.2456	4.193	1.030
1.34	76° 47'	.9735	.2288	4.256	.2350	4.372	1.027
1.35	77° 21'	0.9757	0.2190	4.455	0.2245	4.566	1.025
1.36	77° 55'	.9779	.2092	4.673	.2140	4.779	1.023
1.37	78° 30'	.9799	.1994	4.913	.2035	5.014	1.021
1.38	79° 04'	.9819	.1896	5.177	.1931	5.273	1.018
1.39	79° 39'	.9837	.1798	5.471	.1828	5.561	1.017
1.40	80° 13'	0.9854	0.1700	5.798	0.1725	5.883	1.015
1.41	80° 47'	.9871	.1601	6.165	.1622	6.246	1.013
1.42	81° 22'	.9887	.1502	6.581	.1519	6.657	1.011
1.43	81° 56'	.9901	.1403	7.055	.1417	7.126	1.010
1.44	82° 30'	.9915	.1304	7.602	.1315	7.667	1.009
1.45	83° 05'	0.9927	0.1205	8.238	0.1214	8.299	1.007
1.46	83° 39'	.9939	.1106	8.989	.1113	9.044	1.006
1.47	84° 14'	.9949	.1006	9.887	.1011	9.938	1.005
1.48	84° 48'	.9959	.0907	10.98	.0911	11.03	1.004
1.49	85° 22'	.9967	.0807	12.35	.0810	12.39	1.003

Table 6

Trigonometric Functions of  $\theta$  ( $\theta$  in radians)

$\theta$ Radians	$\theta$ Degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
1.50	85° 57'	0.9975	0.0707	14.10	0.0709	14.14	1.003
1.51	86° 31'	.9982	.0608	16.43	.0609	16.46	1.002
1.52	87° 05'	.9987	.0508	19.67	.0508	19.70	1.001
1.53	87° 40'	.9992	.0408	24.50	.0408	24.52	1.001
1.54	88° 14'	.9995	.0308	32.46	.0308	32.48	1.000
1.55	88° 49'	0.9998	0.0208	48.08	0.0208	48.09	1.000
1.56	89° 23'	.9999	.0108	92.62	.0108	92.63	1.000
1.57	89° 57'	1.000	.0008	1256	.0008	1256	1.000

 $\frac{\pi}{2}$  →

Table 7

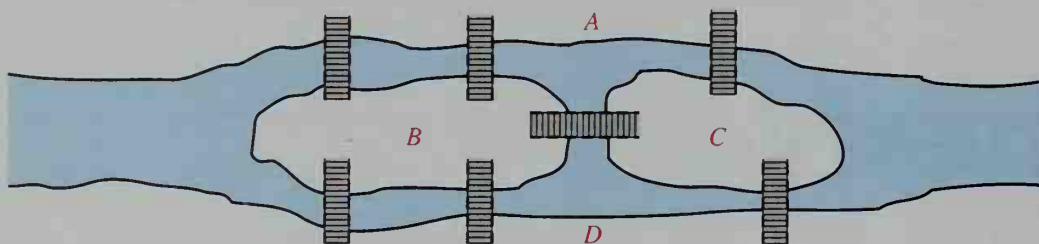
 $e^x$  and  $e^{-x}$ 

$x$	$e^x$	$e^{-x}$	$x$	$e^x$	$e^{-x}$
0.00	1.0000	1.0000	0.36	1.4333	0.6977
0.01	1.0101	0.9900	0.37	1.4477	0.6907
0.02	1.0202	0.9802	0.38	1.4623	0.6839
0.03	1.0305	0.9704	0.39	1.4770	0.6771
0.04	1.0408	0.9608	0.40	1.4918	0.6703
0.05	1.0513	0.9512	0.41	1.5068	0.6637
0.06	1.0618	0.9418	0.42	1.5220	0.6570
0.07	1.0725	0.9324	0.43	1.5373	0.6505
0.08	1.0833	0.9231	0.44	1.5527	0.6440
0.09	1.0942	0.9139	0.45	1.5683	0.6376
0.10	1.1052	0.9048	0.46	1.5841	0.6313
0.11	1.1163	0.8958	0.47	1.6000	0.6250
0.12	1.1275	0.8869	0.48	1.6161	0.6188
0.13	1.1388	0.8781	0.49	1.6323	0.6126
0.14	1.1503	0.8694	0.50	1.6487	0.6065
0.15	1.1618	0.8607	1.0	2.7183	0.3679
0.16	1.1735	0.8521	1.5	4.4817	0.2231
0.17	1.1853	0.8437	2.0	7.3891	0.1353
0.18	1.1972	0.8353	2.5	12.182	0.0821
0.19	1.2092	0.8270	3.0	20.086	0.0498
0.20	1.2214	0.8187	3.5	33.115	0.0302
0.21	1.2337	0.8106	4.0	54.598	0.0183
0.22	1.2461	0.8025	4.5	90.017	0.0111
0.23	1.2586	0.7945	5.0	148.41	0.0067
0.24	1.2712	0.7866	5.5	244.69	0.0041
0.25	1.2840	0.7788	6.0	403.43	0.0025
0.26	1.2969	0.7711	6.5	665.14	0.0015
0.27	1.3100	0.7634	7.0	1096.6	0.0009
0.28	1.3231	0.7558	7.5	1808.0	0.0006
0.29	1.3364	0.7483	8.0	2981.0	0.0003
0.30	1.3499	0.7408	8.5	4914.8	0.0002
0.31	1.3634	0.7334	9.0	8103.1	0.0001
0.32	1.3771	0.7261	9.5	13359	0.0001
0.33	1.3910	0.7189	10.0	22026	.00005
0.34	1.4049	0.7118			
0.35	1.4191	0.7047			

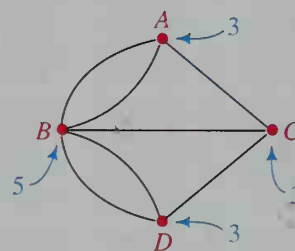
## Graph Theory

### A Famous Bridge Problem

In the 1700s, the European city of Königsberg (now Kaliningrad) had seven bridges connecting both sides of the Pregel River to two islands in the river, as shown below. As villagers strolled about the city, a problem was suggested: Is it possible to start at some point in the city, travel over each of the seven bridges exactly once, and return to the starting point? What do you think?

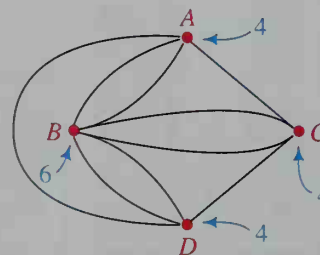


To analyze this problem, the Swiss mathematician Leonhard Euler (1707–1783) drew a *graph* like the one at the right. This graph, which is not like the graphs that you studied in algebra, is a diagram in which the four land masses of the city (A, B, C, and D) are pictured as points, or *vertices*, and the seven bridges are pictured as arcs, or *edges*. Next to each vertex is a number indicating how many edges are attached to the vertex. This number is called the *valence* of the vertex.



Euler reasoned that in order for a person to travel over every bridge once and return to the starting point, every vertex must have an even valence. This is because a person traveling *into* a vertex must also *leave* it; thus the edges must be paired: one “in” with one “out.”

In the 7-bridge problem, none of the vertices has an even valence, so a circuit over all 7 bridges is impossible. However, if two more bridges are added (giving the 9 bridges shown at the right), then every vertex will have an even valence and a circuit over all 9 bridges will be possible. Such a circuit, which traces every edge of a graph exactly once and returns to its starting point, is called an *Euler circuit*. Can you find an Euler circuit for the graph at the right?

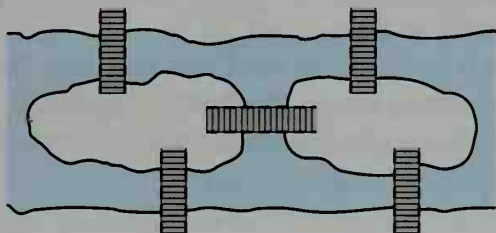




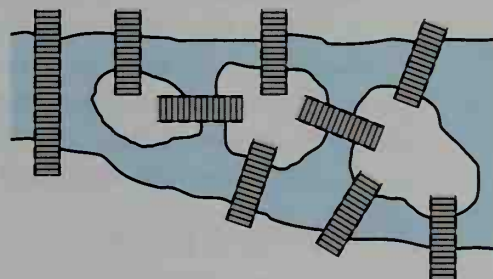
## EXERCISES

Draw a graph for each bridge problem pictured. Find the valence of each vertex and decide whether an Euler circuit is possible.

1.



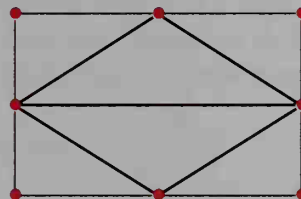
2.



3. Although an Euler circuit is not possible in Exercise 1, it is possible to start at one point of the city, travel over each bridge once, and end at a part of the city different from where you started. Describe how to do this.

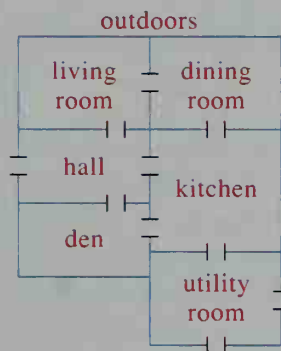
4. A network of city streets is shown at the right.

- If a member of the street department wants to inspect each street without traveling over any portion of a street twice, can the inspector do so by starting and ending at the same point?
- Can part (a) be done if the inspector starts and ends at different points?



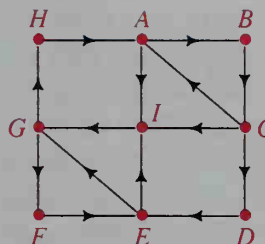
5. Consider a house having a floor plan as shown at the right.

- Can you enter the house, travel through each doorway once and only once, and return to your starting point? (*Hint*: Make a graph whose vertices represent the rooms and the outdoors.)
- Can you travel through each door exactly once and end at a different place from your starting point?

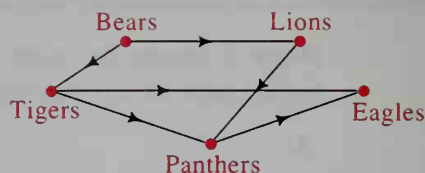


6. The graph at the right represents a network of one-way streets with the arrowheads indicating the direction of travel. Because 2 streets lead into vertex  $A$ , we say that the *invalence* of  $A$  is 2; likewise, because 2 streets lead out of  $A$ , we say that the *outvalence* of  $A$  is 2.

- Find the invalence and outvalence of each of the other vertices.
- Tell why an Euler circuit is not possible.
- Change the direction of just one arrowhead so that an Euler circuit is possible. Name the vertices in the order that they would be visited.



7. Five teams are involved in a tournament in which each team plays each of the other four exactly once. After a few games have been played, the results are shown as a graph in which an arrow from one team to another team indicates that the first team defeated the second.



- Which two teams have the most wins so far?
- Copy the graph and add arrows to it to show that the Lions beat both the Eagles and the Tigers, while the Panthers lost to the Bears.
- At the end of the competition, which team do you think should be ranked first and which team last? Explain.

## A Traveler's Puzzle

A business traveler located in Denver must travel to each of the cities shown in the graph below. The airline on which the businessperson likes to travel connects certain pairs of cities, but not all pairs. Is it possible to make a circuit starting and ending in Denver and visiting each of the cities exactly once? This kind of circuit is called a *Hamilton circuit* in honor of William Rowan Hamilton (1805–1865), an Irish mathematician who worked on such circuits. Note that in a Hamilton circuit, each vertex is visited once, while in an Euler circuit, each edge is traveled once.

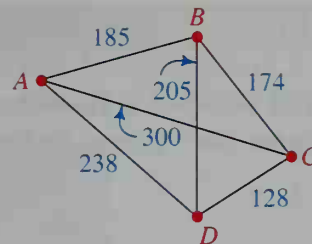


In the above graph, note that there are only 10 vertices (representing the 10 cities). Any other points where two edges intersect are not considered vertices. The graph *does* have a Hamilton circuit. Can you find it?

It is easy to tell whether a graph has an Euler circuit by investigating the valences of the vertices, but there is no known general rule for deciding when a Hamilton circuit is possible. For a small number of cities, you can use a trial-and-error method. For a larger number of cities, a computer can be programmed to use trial and error. But there are surprising limitations to what a computer can do, as the example at the top of the next page shows. The example presents a specific case of what is commonly called the *traveling salesman problem*.

### Example

The graph at the right shows the distances between the 4 cities A, B, C, and D. A traveler wishes to leave A, visit the 3 other cities exactly once, and return to A. What is the shortest circuit the traveler can take?



### Solution

The traveler can visit any of the 3 cities first. From the first stop the traveler can go to either of the 2 remaining cities, and then the traveler must visit the 1 remaining city. Thus, there are  $3 \times 2 \times 1 = 6$  possible circuits. The total distance for each circuit is shown below.

Circuit	Total distance
<i>ABCD</i> A	$185 + 174 + 128 + 238 = 725$
<i>ABDC</i> A	$185 + 205 + 128 + 300 = 818$
<i>ACBD</i> A	$300 + 174 + 205 + 238 = 917$
<i>ACDB</i> A	$238 + 128 + 174 + 185 = 725$
<i>ADCB</i> A	$300 + 128 + 205 + 185 = 818$
<i>ADBC</i> A	$238 + 205 + 174 + 300 = 917$

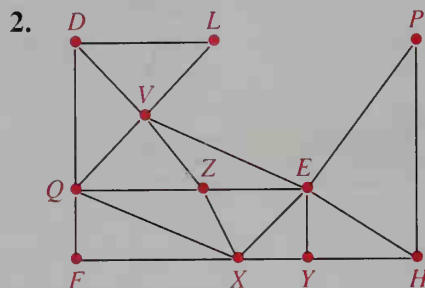
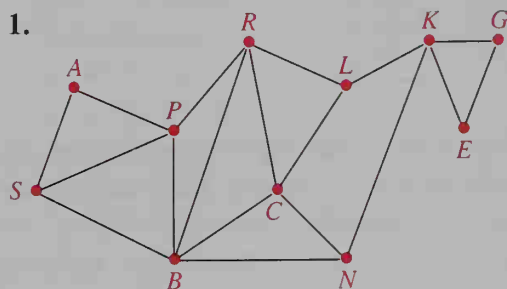
Thus, the shortest total distance is 725 mi for circuit *ABCD*A or its reverse circuit *ADCBA*.

The analysis used in the example for 4 cities can also be used for 10 cities, but now there are many more Hamilton circuits to analyze. In fact, the number of circuits is  $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880$ . A reasonably fast computer can check all of these circuits and find the shortest one in less than a second. However, with 15 cities the computation time becomes about  $1\frac{1}{2}$  minutes, and with 20 cities it becomes almost 4 years! Exercises 6–8 on pages 826 and 827 will help you understand why it takes even a fast computer so long to check all possible circuits.

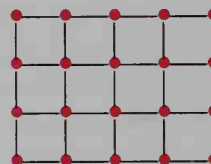
Although the shortest circuit can be found only by testing all circuits, it is possible to get a very good circuit, although it might not be the shortest, by using the *nearest-neighbor algorithm*. With this method, you go from each city to the nearest city you haven't yet visited. Thus, for the graph shown above, you would go from A to B to C to D and back to A. This gives a total distance of 725 mi, which in this case is the shortest distance. Although the nearest-neighbor algorithm doesn't *always* give the shortest distance, it *is* easy and quick to use.

## EXERCISES

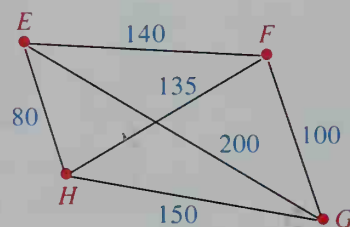
Tell whether a Hamilton circuit is possible for each graph. If it is possible, name the vertices in the order visited. (More than one answer is possible.)



3. a. Is a Hamilton circuit possible for the 3 by 4 grid shown at the right?
- b. Is a Hamilton circuit possible for a 4 by 6 grid?
- c. Given an  $x$  by  $y$  grid, what must be true about  $x$  and  $y$  if a Hamilton circuit is possible?



4. Use the procedure of the example on page 825 to solve the traveler problem for the four cities shown in the graph at the right. Begin and end at vertex  $F$ .
5. Use the nearest-neighbor algorithm for the graph in Exercise 4. Begin and end at vertex  $F$ . Does the algorithm give you the shortest circuit?



6. a. Suppose a traveler must leave city  $A$  and visit 14 other cities before returning to  $A$ . How many possible Hamilton circuits are there? (Give your answer as a factorial and then evaluate this factorial with a calculator.)
- b. Since each circuit involves adding 15 numbers, there are 14 additions per circuit. Thus, the total number of additions to check all  $14!$  circuits is  $14 \times 14!$ . If a computer can do 1 addition per nanosecond (one billionth of a second), how long will it take a computer to compute the distances for all circuits?
7. Suppose a traveler must leave city  $A$  and visit each of 20 cities before returning to  $A$ .
  - a. How many circuits are possible?
  - b. How many additions per circuit are there to compute the distance traveled?
  - c. What is the total number of additions required to find the distance traveled for each circuit?
  - d. How long would it take a computer to do all of the additions given in part (c)? Assume the computer can do 1 billion additions per second.

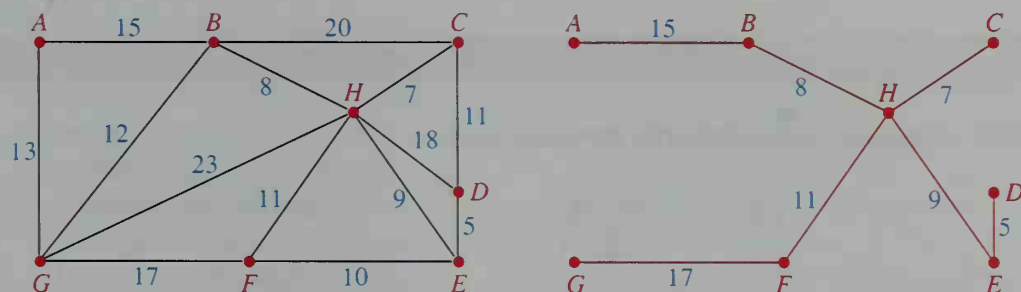
8. Repeat Exercise 7 if 25 cities are to be visited before returning to A.
9. The table below gives the cost of transportation between the 5 cities A, B, C, D, and E.

From \ To	A	B	C	D	E
A	—	\$220	\$150	\$100	\$130
B	\$220	—	\$160	\$200	\$240
C	\$150	\$160	—	\$180	\$110
D	\$100	\$200	\$180	—	\$190
E	\$130	\$240	\$110	\$190	—

- a. Make a graph showing these costs.
  - b. Use the nearest-neighbor algorithm to find the circuit beginning and ending at A. (Note that “nearest neighbor” in this problem means “the neighbor it costs the least to reach.”)
10. Many real-life problems can be modeled with Hamilton circuits. For example, a telephone company must decide on the most efficient route for a worker to collect money from pay phones. Give another example.

## Minimizing the Cost of a Network

The graph at the left below gives the costs, in thousands of dollars, of joining several locations with roads. The total cost of all fourteen roads is \$179,000. For far less money, just the roads colored in red at the right below could be built and people could still get from any location to any other. The cost of the red road network is \$72,000, the sum of the costs for the 7 roads. Can you find another network of roads that costs less than this?



A network of least possible cost that allows one to travel from any vertex to any other vertex is called a *minimal spanning tree*. It is called “minimal” because the cost is least, it is called “spanning” because the network spans out to touch every vertex, and it is called a “tree” because it roughly resembles a tree with branches.



## Algorithm for Finding a Minimal Spanning Tree

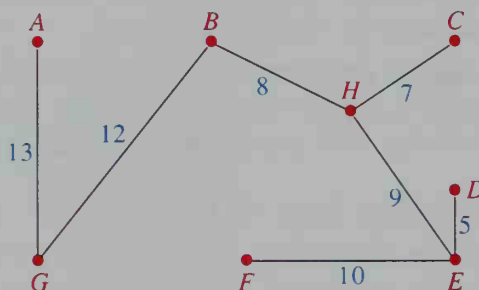
1. Build the least expensive road (edge) first.
2. Then build the road next lowest in cost.
3. At each stage, build the road that is next lowest in cost and *does not* form a circuit. Stop when all vertices have been spanned.

### Example

Use the algorithm above to find the minimal spanning tree for the road network on the left at the bottom of the preceding page.

### Solution

1. The least expensive road is  $DE$ .
2. The roads next lowest in cost are  $HC$ ,  $HB$ ,  $HE$ , and  $EF$ .
3. The road next lowest in cost is  $HF$ , but this road is not built because it forms the circuit  $HEFH$ . We therefore go to the road next lowest in cost,  $BG$ , and finally to  $GA$ .

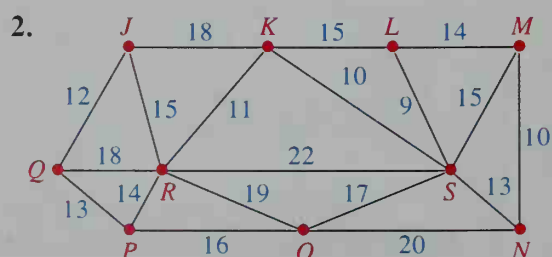
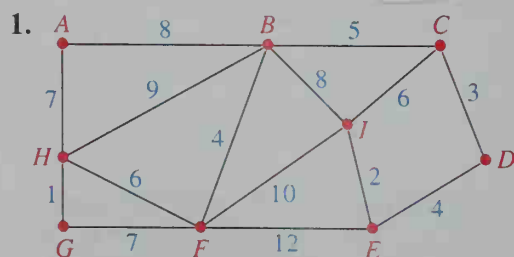


Thus, the total cost of the minimal spanning tree is

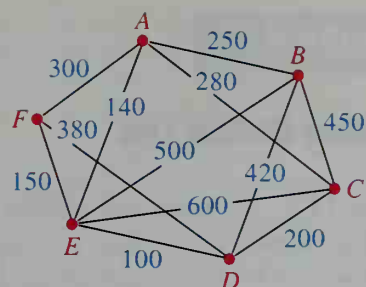
\$5000 + \$7000 + \$8000 + \$9000 + \$10,000 + \$12,000 + \$13,000, or \$64,000.

## EXERCISES

Find a minimal spanning tree for each graph.



3. If a graph has  $n$  vertices, how many edges does a minimal spanning tree have?
4. The graph at the right shows the dollar costs required to connect various computers to form a computer network. What is the minimal cost of a network that links the six computers?

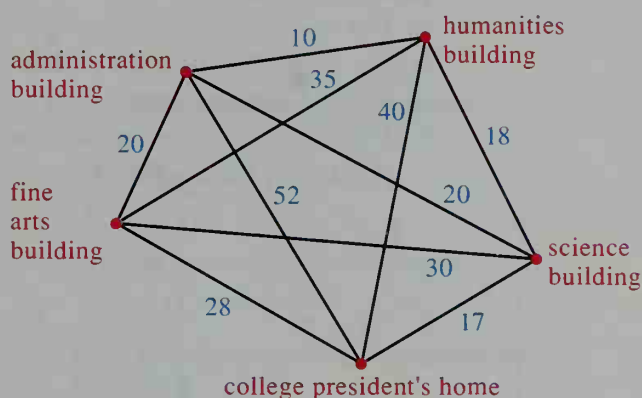


Ex. 4

5. a. Give an example of a situation in which a person or company might want to find a maximal spanning tree.  
b. Copy the graph in Exercise 1 and find a *maximal* spanning tree. Remember that a tree has no circuits.
6. A real estate developer wants to connect locations  $A, B, C, D, E$ , and  $F$  with roads while keeping costs to a minimum. The table below gives the estimated costs (in thousands of dollars) for building roads between pairs of locations. Make a graph and find the minimum cost of connecting the six locations.

	A	B	C	D	E	F
A	—	18	21	30	15	9
B		—	10	18	13	22
C			—	8	12	18
D				—	16	24
E					—	11
F						—

7. The graph at the right shows the costs (in hundreds of dollars) of building sidewalks between various locations on a college campus. Although a sidewalk between the administration building and the college president's home is the most expensive, the college president insists that it be built. What then is the minimum cost of joining the locations?



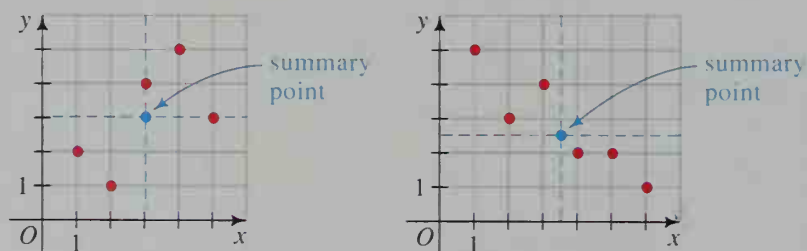
## The Median-Median Line

In Section 18-1, we discussed how to fit a line, called the least-squares line, to a scatter plot. Another line that we can fit to a scatter plot is called the *median-median line*, and it has two advantages over the least-squares line. First, it is easier to use if you are not using a computer or calculator. Second, an outlying data point does not affect the median-median line very much, whereas it can have a great effect on the least-squares line.

When finding a median-median line, you must know how to find a *summary point* for a set of points  $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$ .

summary point = (median of  $x$ -coordinates, median of  $y$ -coordinates)

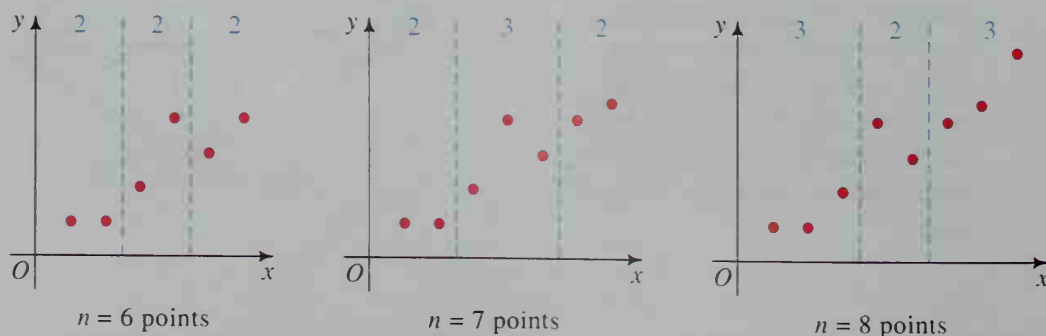
To find the median of the  $x$ -coordinates of the points in a scatter plot, look at the points from left to right and draw a vertical line at the median  $x$ -position. To find the median of the  $y$ -coordinates, look at the points from the bottom to the top and draw a horizontal line at the median  $y$ -position. Two illustrations are shown below.



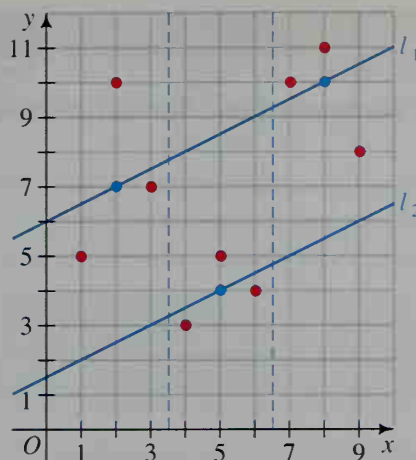
## How to Find the Median-Median Line

**Step 1** Draw two vertical lines that divide a scatter plot with  $n$  points into three parts as follows:

- If  $n$  is divisible by 3, say  $n = 3k$ , then each part will have  $k$  points.
- If  $n = 3k + 1$ , then the middle part will have one more data point than the two outside parts.
- If  $n = 3k + 2$ , then the two outside parts will each have one more data point than the middle part.



- Step 2**
- Plot three summary points, one for each of the three subdivisions of the scatter plot. (See the graph at the right.)
  - Draw a line  $l_1$  through the left and right summary points.
  - Draw a parallel line  $l_2$  through the middle summary point.



- Step 3** The median-median line is drawn parallel to the two lines in Step 2 and such that its distance to line  $l_1$  is half its distance to line  $l_2$ . This line is shown at the right and its equation is computed as follows:

Equation of  $l_1$ :

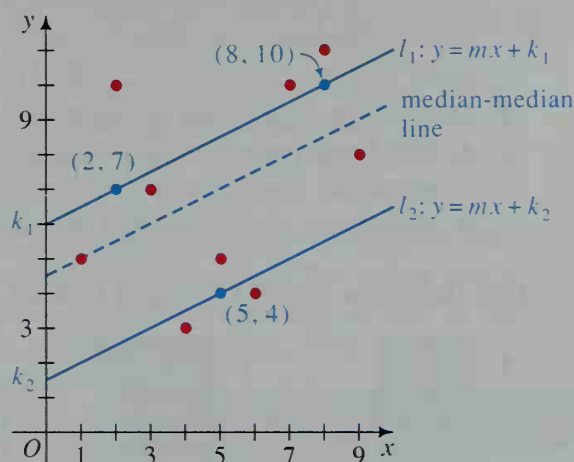
$$y = mx + k_1$$

Equation of  $l_2$ :

$$y = mx + k_2$$

Equation of the median-median line:

$$y = mx + k, \text{ where } k = \frac{2}{3}k_1 + \frac{1}{3}k_2$$



### Example

Find an equation of the median-median line shown above.

### Solution

1. Slope of  $l_1$ :  $m = \frac{10 - 7}{8 - 2} = \frac{3}{6} = 0.5$

2. Since  $(2, 7)$  and  $(8, 10)$  lie on  $l_1$ , an equation of  $l_1$  is:  
 $y = 0.5x + 6$

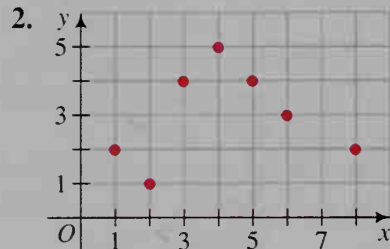
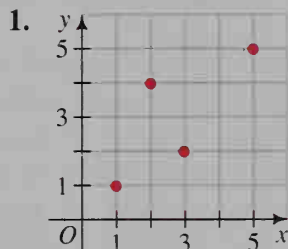
3. Since  $l_2$  has slope 0.5 and  $(5, 4)$  lies on  $l_2$ , an equation of  $l_2$  is:  
 $y = 0.5x + 1.5$

4. Value of  $k$ :  $k = \frac{2}{3}k_1 + \frac{1}{3}k_2 = \frac{2}{3}(6) + \frac{1}{3}(1.5) = 4.5$

An equation of the median-median line is  $y = mx + k = 0.5x + 4.5$ .

## EXERCISES

Find the summary point for each set of points.



3. Refer to Step 1 in the instructions for finding the median-median line. How many points would you put into each of the three subdivisions of the data if the scatter plot has:
  - a. 24 points?
  - b. 17 points?
  - c. 61 points?
4. Suppose the three summary points of a scatter plot are  $(2, 3)$ ,  $(5, 6)$ , and  $(8, 6)$ .
  - a. Plot these points and draw the lines  $l_1$  and  $l_2$  referred to in Step 2 on the preceding page. Find equations of these lines.
  - b. Sketch the median-median line and find its equation.

Plot the given points  $(x, y)$  and find an equation of the median-median line.

5.

$x$	$y$
0	1
2	3
3	3
5	3
6	4
8	6

6.

$x$	$y$
2	9
6	7
7	8
8	5
9	6
10	3
14	1

7.

$x$	$y$
-3	5
-1	4
0	4
2	3
3	0
4	3
5	2
6	1
9	-1
11	-1

8.

$x$	$y$
1	0
2	4
3	4
4	5
6	5
6	7
8	7
8	9

9. Refer to the data in Exercise 7 and suppose that the point  $(-3, 5)$  is replaced by  $(-3, 25)$ . Describe the effect, if any, that this replacement has on:
  - a. the summary points for the three subdivisions of the data.
  - b. the equation of the median-median line.
  - c. the point  $(\bar{x}, \bar{y})$ , which lies on the least-squares line (see page 684).



10. **Writing** Fit a median-median line to the age-versus-shoe-size data whose scatter plot is given for Class Exercises 1 and 2 on page 687. Write a sentence or two comparing the median-median line with the least-squares line, which has the equation  $y = 0.47x + 1.82$ .
11. Parents of small children often wonder how tall their children will grow to be. In a study of 9 boys and 9 girls, their heights (in centimeters) were recorded on their second birthday and again on their eighteenth birthday, as shown below.

$x$ = boy's height (age 2)	77	79	80	81	83	84	85	87	91
$y$ = boy's height (age 18)	157	168	180	163	179	175	181	182	189

$x$ = girl's height (age 2)	76	79	80	81	82	83	84	86	89
$y$ = girl's height (age 18)	155	150	152	159	161	169	165	183	178

- Using the data for either the boys or the girls, make a scatter plot of the set of points  $(x, y)$  and then fit a median-median line to the data.
  - Use the line in part (a) to predict how tall a two-year-old will be on his or her eighteenth birthday if he or she is now 78 cm tall.
12. During a promotion, an airline offers special fares to people flying from Boston, Massachusetts, to various cities in the United States. The tables below give the air distance (in miles) between Boston and each city as well as the one-way cost (based on a round-trip ticket) of each flight.

City	Distance	Cost
Cleveland, Ohio	551	\$120.75
Denver, Colorado	1769	\$194.25
Houston, Texas	1605	\$183.25
Indianapolis, Indiana	807	\$106.25
Kansas City, Missouri	1251	\$119.25
Louisville, Kentucky	826	\$141.25

City	Distance	Cost
New Orleans, Louisiana	1359	\$119.25
Philadelphia, Pennsylvania	271	\$72.75
Phoenix, Arizona	2300	\$182.75
San Francisco, California	2699	\$214.25
Seattle, Washington	2493	\$214.75
Washington, D.C.	393	\$89.00

- Make a scatter plot of the set of points  $(x, y) = (\text{distance}, \text{cost})$  and then fit a median-median line to the data.
- If the air distance between Boston and Miami, Florida, is 1255 mi, what would you predict the one-way cost (based on a round-trip ticket) to be?

13. The table below gives the number of telephones and the number of television sets per 100 people for several countries.

Country	Phones	Televisions
Argentina	11.6	21.7
Australia	55.0	48.3
Canada	78.0	57.7
Costa Rica	14.7	7.9
Denmark	86.4	38.6
Egypt	2.8	8.3
Finland	61.7	37.4
France	60.8	33.3
India	0.6	0.7

Country	Phones	Televisions
Japan	55.5	58.7
Kenya	1.4	0.6
Malaysia	9.1	14.0
Mexico	9.6	12.0
Spain	39.6	36.8
Switzerland	85.6	40.5
U.K.	52.4	43.4
U.S.S.R.	11.3	31.4
U.S.A.	76.0	81.1

- Make a scatter plot of the set of points  $(x, y) = (\text{telephones}, \text{televisions})$  and then fit a median-median line to the data.
  - What nation's point has the greatest vertical distance to the line?
  - If you are told that Sweden has 89.0 telephones per 100 people, how many televisions per 100 people would you predict? (The actual number is 39.5 televisions per 100 people.)
14. The table below gives the life expectancy at birth (in years) for male and female United States citizens born in various years.

$x$ = year of birth	1920	1930	1940	1950	1960	1970	1980	1990
$y_m$ = life expectancy of males at birth	53.6	58.1	60.8	65.6	66.6	67.1	70.0	72.0
$y_f$ = life expectancy of females at birth	54.6	61.6	65.2	71.1	73.1	74.7	77.4	78.8

- Make separate scatter plots of the sets of points  $(x, y_m)$  and  $(x, y_f)$  and then fit a median-median line to each set of data.
  - Writing** Write a paragraph in which you compare the two median-median lines in part (a).
  - What is the predicted life expectancy of a baby boy born in 2000? of a baby girl born in 2000?
15. **Research** Use an almanac or other reference book to research two data sets that you think might be linearly related. Make a scatter plot and fit a median-median line to the data. Then use the line to explore a question that the data might suggest.

## Descartes' Rule of Signs and Bounds for Real Roots

The problem of determining the roots of polynomial equations occurs frequently in mathematics. Information that narrows down the set of *possible* roots of a polynomial equation greatly reduces the work involved in finding the roots. **Descartes' rule of signs** gives us information about the number of real roots of polynomial equations with real coefficients.

First consider a simplified polynomial  $P(x)$  written in order of decreasing powers of  $x$ . Whenever the coefficients of two adjacent terms have opposite signs, we say that  $P(x)$  has a *variation in sign*. Thus, the polynomial

$$x^6 - 3x^4 - 2x^2 + 4x - 5$$

has three variations in sign. (Notice that you ignore terms having coefficient 0.)

### Descartes' Rule of Signs

Let  $P(x)$  be a polynomial with real coefficients; then

1. the number of positive roots of  $P(x) = 0$  is either equal to the number of variations in sign of  $P(x)$  or is less than this number by a positive even integer;
2. the number of negative roots of  $P(x) = 0$  is either equal to the number of variations in sign of  $P(-x)$  or is less than this number by a positive even integer.

Sometimes Descartes' rule gives complete information about the numbers of different types of roots. For example,  $P(x) = x^4 + 11x - 6$  has *one* variation in sign and thus  $P(x) = 0$  has *one positive root*. Since  $P(-x) = (-x)^4 + 11(-x) - 6 = x^4 - 11x - 6$  has *one* variation in sign,  $P(x) = 0$  has *one negative root*. Since  $P(x)$  is of degree four,  $P(x) = 0$  has exactly four roots. (See the fundamental theorem of algebra, page 85.) Since the equation has one positive real root and one negative real root, the remaining two roots must be imaginary.

More frequently, Descartes' rule leaves us with several possibilities.

### Example 1

List the possibilities for the nature of the roots (positive, negative, and imaginary) of  $P(x) = 0$  for the polynomial

$$P(x) = -2x^4 + x^3 - 7x^2 + 4x + 1.$$

### Solution

The given polynomial has three variations in sign, so the number of positive roots of  $P(x) = 0$  is 3 or 1.

(Solution continues on the next page.)

Examine  $P(-x)$  to find the possible numbers of negative roots.

$$P(-x) = -2x^4 - x^3 - 7x^2 - 4x + 1$$

Since  $P(-x)$  has one variation in sign, there is one negative root.

Since there are exactly four roots, we can make a chart to show the possible combinations of roots.

Number of positive real roots	Number of negative real roots	Number of imaginary roots
3	1	0
1	1	2

Descartes' rule of signs is used to determine the *number* of positive and negative roots of a polynomial equation  $P(x) = 0$ . The following rule can be used to narrow the region of possible real roots down to an interval on the number line, that is, to find *bounds* for the real roots of  $P(x) = 0$ .

### Upper and Lower Bounds for Real Roots

Let  $P(x)$  be a polynomial with real coefficients and a positive leading coefficient.

1. Let  $M$  be a nonnegative real number. If the coefficients of the quotient and remainder obtained by dividing  $P(x)$  by  $x - M$  are all nonnegative, then  $P(x) = 0$  has no roots greater than  $M$ .
2. Let  $L$  be a nonpositive real number. If the coefficients of the quotient and remainder obtained by dividing  $P(x)$  by  $x - L$  are alternately nonnegative and nonpositive, then  $P(x) = 0$  has no roots less than  $L$ .

The numbers  $M$  and  $L$  are called **upper** and **lower bounds**, respectively, for the real roots of  $P(x) = 0$ . To apply this rule, use synthetic division for different values of  $x$  until the last line of numbers satisfies the conditions for  $M$  or  $L$ .

**Example 2** Find the least nonnegative integral upper bound and the greatest nonpositive integral lower bound for the roots of  $3x^3 - 4x^2 - 4x + 4 = 0$ .

**Solution** Use synthetic substitution with  $x = \pm 1, \pm 2, \pm 3, \dots$  until numbers that satisfy the respective conditions for upper and lower bounds are found. The results are shown in the table at the top of the next page.

$x$	coefficients of quotient and remainder				
$\begin{array}{r} 1 \end{array}$	3	-1	-5	-1	
$\begin{array}{r} 2 \end{array}$	3	2	0	4	$\leftarrow$ all $\geq 0$
$\begin{array}{r} -1 \end{array}$	3	-7	3	1	
$\begin{array}{r} -2 \end{array}$	3	-10	16	-28	$\leftarrow$ alternately $\geq 0$ and $\leq 0$

Thus,  $M = 2$  is the upper bound and  $L = -2$  is the lower bound. If  $3x^3 - 4x^2 - 4x + 4 = 0$  has any real roots, they must lie in the interval  $-2 < x < 2$ .

## EXERCISES

Use Descartes' rule to make a chart summarizing the possible combinations of positive, negative, and imaginary roots of each polynomial equation.

- $y^2 - y - 4 = 0$
- $-y^2 + y - 3 = 0$
- $y^3 - y^2 + 2y + 3 = 0$
- $x^3 + 6x^2 - x - 12 = 0$
- $2x^4 - 3x^3 - x^2 + x + 1 = 0$
- $3y^4 - y^3 + 4y^2 + 8 = 0$
- $5y^4 - y^3 + y^2 + 2y - 4 = 0$
- $-4x^4 - 3x^3 + x^2 - x + 5 = 0$

For each polynomial equation (a) find the least nonnegative integral upper bound and the greatest nonpositive integral lower bound of the real roots, and (b) write an interval in which the real roots lie.

- $x^3 - 2x^2 + 3x + 5 = 0$
- $x^3 - 4x^2 + 8x - 5 = 0$
- $2x^3 - 3x^2 - 3x + 3 = 0$
- $2x^3 - 2x^2 + x + 8 = 0$
- $y^4 + y^3 + 16 = 0$
- $y^4 - 2y - 3 = 0$
- $y^4 - 2y^3 - 3y^2 + 4y + 8 = 0$
- $y^4 + 2y^3 - y^2 + 7y - 5 = 0$



## Partial Fractions

In your algebra courses, you learned to simplify sums of rational expressions by using the least common denominator. For example,

$$\frac{-3}{x-2} + \frac{5}{x-1} = \frac{-3(x-1) + 5(x-2)}{(x-2)(x-1)} = \frac{2x-7}{(x-2)(x-1)}.$$

Some calculus problems involving rational expressions can be solved by reversing this process, that is, by rewriting a single rational expression as a sum of two or more rational expressions, called *partial fractions*:

$$\frac{2x-7}{(x-2)(x-1)} = \frac{-3}{x-2} + \frac{5}{x-1}$$

There may be several ways of rewriting a rational expression as the sum of other rational expressions. For example, both

$$\frac{2x}{(x-2)(x-1)} + \frac{-7}{(x-2)(x-1)} \quad \text{and} \quad \frac{x-5}{x-2} + \frac{6-x}{x-1}$$

are also equivalent to  $\frac{2x-7}{(x-2)(x-1)}$ . When writing a rational expression as the sum of partial fractions, however, the goal is to use partial fractions with different denominators and with numerators of the lowest degree possible. In the examples and in most of the exercises, the numerators of the partial fractions are constants.

### Example 1

Write  $\frac{3x+17}{(x-5)(x+3)}$  as a sum of partial fractions.

### Solution

Using the factors of the denominator as the denominators of the partial fractions, we want to find constants  $A$  and  $B$  such that

$$\frac{3x+17}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}.$$

Multiply both sides of the equation above by  $(x-5)(x+3)$ :

$$3x+17 = A(x+3) + B(x-5) \quad (1)$$

Expand the right side of equation (1) and combine like terms:

$$3x+17 = (A+B)x + (3A-5B) \quad (2)$$

We can use either equation (1) or equation (2) to find  $A$  and  $B$ .

**Method 1** In equation (2),  $A$  and  $B$  must satisfy the following system:

$$\begin{aligned} A+B &= 3 && \leftarrow \text{The coefficients of } x \text{ must be equal.} \\ 3A-5B &= 17 && \leftarrow \text{The constant terms must be equal.} \end{aligned}$$

Solving this system of equations simultaneously gives the solution  $A = 4$  and  $B = -1$ .

The values  $A = 4$  and  $B = -1$  make equation (2) true for *all* values of  $x$ . Then equation (1) must also be true for all values of  $x$ . Thus, even though the original rational expression is undefined for  $x = 5$  and  $x = -3$ , we can legitimately substitute  $x = 5$  and  $x = -3$  into equation (1). This gives us a second method of finding  $A$  and  $B$ .

**Method 2** We use equation (1):

$$3x + 17 = A(x + 3) + B(x - 5)$$

Substituting  $x = 5$  causes the term  $B(x - 5)$  to become 0:

$$3(5) + 17 = A(5 + 3) + 0$$

$$32 = 8A$$

$$A = 4$$

Substituting  $x = -3$  causes the term  $A(x + 3)$  to become 0:

$$3(-3) + 17 = 0 + B(-3 - 5)$$

$$8 = -8B$$

$$B = -1$$

Both methods give the solution  $A = 4$  and  $B = -1$ . Thus,

$$\frac{3x + 17}{(x - 5)(x + 3)} = \frac{4}{x - 5} + \frac{-1}{x + 3}, \text{ or } \frac{4}{x - 5} - \frac{1}{x + 3}.$$

You can easily verify this answer by combining the partial fractions using the common denominator  $(x - 5)(x + 3)$ .

Example 2 shows how to deal with a rational expression in which the denominator, when factored, contains a repeated factor.

### Example 2

Write  $\frac{5x^2 - 27x + 21}{x^3 - 5x^2 - 8x + 48}$  as a sum of partial fractions.

### Solution

Using the methods presented in Section 2-2 and Section 2-6, factor the denominator:

$$x^3 - 5x^2 - 8x + 48 = (x - 4)^2(x + 3)$$

Because the factor  $(x - 4)$  is squared, the original rational expression *must* have a partial fraction with denominator  $(x - 4)^2$ , and it *may also* have a partial fraction with denominator  $(x - 4)$ . The factor  $(x + 3)$  will be the denominator of a third partial fraction. Thus, we must find constants  $A$ ,  $B$ , and  $C$  such that

$$\frac{5x^2 - 27x + 21}{(x - 4)^2(x + 3)} = \frac{A}{x - 4} + \frac{B}{(x - 4)^2} + \frac{C}{x + 3}.$$

Multiplying each side of the equation above by  $(x - 4)^2(x + 3)$  produces equation (3), shown at the top of the next page.

(Solution continues on the next page.)

$$5x^2 - 27x + 21 = A(x - 4)(x + 3) + B(x + 3) + C(x - 4)^2 \quad (3)$$

Expanding the right side of equation (3) would be time-consuming, so in this case we will use Method 2 presented in Example 1. Although we cannot find  $A$  directly by substituting a value for  $x$  in equation (3), we can find  $B$  and  $C$  by substituting  $x = 4$  and  $x = -3$ , respectively.

When  $x = 4$ ,

$$\begin{aligned} 5(4)^2 - 27(4) + 21 &= 0 + B(4 + 3) + 0 \\ -7 &= 7B \\ B &= -1 \end{aligned}$$

When  $x = -3$ ,

$$\begin{aligned} 5(-3)^2 - 27(-3) + 21 &= 0 + 0 + C(-3 - 4)^2 \\ 147 &= 49C \\ C &= 3 \end{aligned}$$

To find  $A$ , return to equation (3) and substitute  $-1$  for  $B$ ,  $3$  for  $C$ , and any convenient value, such as  $0$ , for  $x$ :

$$\begin{aligned} 5(0)^2 - 27(0) + 21 &= A(0 - 4)(0 + 3) + (-1)(0 + 3) + (3)(0 - 4)^2 \\ -24 &= A(-12) \\ A &= 2 \end{aligned}$$

$$\text{Thus, } \frac{5x^2 - 27x + 21}{x^3 - 5x^2 - 8x + 48} = \frac{2}{x - 4} - \frac{1}{(x - 4)^2} + \frac{3}{x + 3}.$$

In Example 2, the denominator's repeated factor is squared. If the denominator of the original rational expression had contained a cubed factor,  $(x - a)^3$ , then we would have used each of  $\frac{A}{x - a}$ ,  $\frac{B}{(x - a)^2}$ , and  $\frac{C}{(x - a)^3}$  as partial fractions.

In both Example 1 and Example 2, the original rational expression has the following characteristics: (1) the degree of the numerator is less than the degree of the denominator, and (2) all factors of the denominator are linear. In the exercises, you will consider some other cases.

## Exercises

In Exercises 1–15, write each rational expression as a sum of partial fractions.

1.  $\frac{6x}{(x - 1)(x + 5)}$

2.  $\frac{2x + 1}{(x - 4)(x - 1)}$

3.  $\frac{7x + 6}{4x^2 + 9x - 9}$

4.  $\frac{4x + 1}{6x^2 - 7x + 2}$

5.  $\frac{x^2 + 8}{x^3 + 3x^2 + 2x}$

6.  $\frac{x - 6}{x^3 + x^2 - 6x}$

7.  $\frac{x + 6}{x^2 - 4}$

8.  $\frac{x^2 + x + 6}{x^3 + 2x^2 - 5x - 6}$

9.  $\frac{3x - 1}{(x + 4)^2}$

10.  $\frac{x^2 + x + 2}{(x - 2)^3}$

11.  $\frac{x^2 + 1}{x^3 + 3x^2 + 3x + 1}$

12.  $\frac{x + 2}{x^2 - 2x + 1}$

13.  $\frac{2x^2}{x^3 + x^2 - x - 1}$

14.  $\frac{6}{x^4 - x^2}$

15.  $\frac{x^3 - x^2}{(x + 3)^4}$

16. Use Method 1 of Example 1 on page 838 to find  $A$ ,  $B$ , and  $C$  in Example 2 on page 839. First show that  $A$ ,  $B$ , and  $C$  satisfy the system of equations shown at the right. Then solve the system using substitution, determinants (see Section 12-8), or matrices (see Section 14-3).

$$\begin{aligned} A + C &= 5 \\ -A + B - 8C &= -27 \\ -12A + 3B + 16C &= 21 \end{aligned}$$

17. a. Before trying to write the rational expression  $\frac{2x^2 - 28}{x^2 - 6x - 16}$  as a sum of partial fractions, notice that its numerator and denominator are of the same degree. Explain why you cannot find constants  $A$  and  $B$  such that

$$\frac{2x^2 - 28}{x^2 - 6x - 16} = \frac{2x^2 - 28}{(x - 8)(x + 2)} = \frac{A}{x - 8} + \frac{B}{x + 2}.$$

- b. To write  $\frac{2x^2 - 28}{x^2 - 6x - 16}$  as a sum of partial fractions, first use polynomial long division to divide the numerator by the denominator. Verify that

$$\frac{2x^2 - 28}{x^2 - 6x - 16} = 2 + \frac{12x + 4}{x^2 - 6x - 16}.$$

- c. Now write  $\frac{12x + 4}{x^2 - 6x - 16}$  as a sum of partial fractions. Verify that

$$\frac{12x + 4}{x^2 - 6x - 16} = \frac{10}{x - 8} + \frac{2}{x + 2}.$$

$$\text{Thus, } \frac{2x^2 - 28}{x^2 - 6x - 16} = 2 + \frac{10}{x - 8} + \frac{2}{x + 2}.$$

**In Exercises 18–20, use the method presented in parts (b) and (c) of Exercise 17 to write the given rational expression as a sum of partial fractions.**

18.  $\frac{4x^2 - x}{2x^2 + x - 3}$

19.  $\frac{4x^3 - 3x^2 + 2x + 1}{x^2 + x}$

20.  $\frac{4x^3 - x + 2}{4x^2 + 4x + 1}$

21. In the rational expression  $\frac{5x^2 - 2x - 1}{(x - 2)(x^2 + x - 1)}$ , one factor of the denominator is a nonfactorable quadratic. Write the rational expression as a sum of partial fractions by finding constants  $A$ ,  $B$ , and  $C$  such that

$$\frac{5x^2 - 2x - 1}{(x - 2)(x^2 + x - 1)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + x - 1}.$$

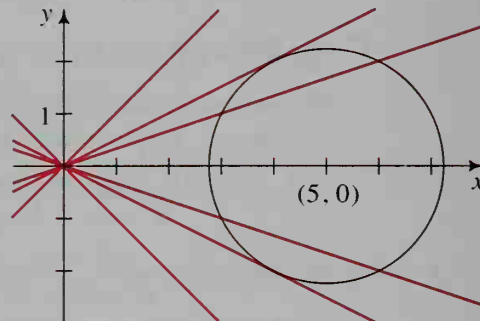
22. Write  $\frac{x^4 + 8}{x^3 - 15x - 4}$  as a sum of partial fractions.

## Tangents and Normals to Conic Sections

### Families of Lines

A set of lines satisfying a given condition is called a *family of lines*. For example, the set of all nonvertical lines that contain the point  $(0, 0)$  is the family of nonvertical lines through the origin. Each nonvertical line has an equation in the form  $y = mx$ . The slope  $m$  is called the *parameter* of the family. Specifying a value for  $m$  determines the equation of a line in the family.

The diagram at the right shows a circle with center  $(5, 0)$  and radius  $\sqrt{5}$ , and also several lines from the family of nonvertical lines through the origin. Some of these lines intersect the circle in two points, some are tangent to the circle, and some do not intersect the circle. To classify such lines by the number of points of intersection, you can use your knowledge of the discriminant of a quadratic equation (see pages 31–32). The following four steps show how.



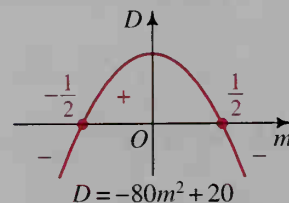
1. The equation  $y = mx$  represents a nonvertical line through the origin. We wish to determine which values of  $m$  give equations of lines that intersect the circle.
2. We start as if we were solving the equations of the line and the circle simultaneously under the assumption that  $m$  is given.

$$\begin{aligned} \text{Line: } & y = mx \\ \text{Circle: } & (x - 5)^2 + y^2 = 5 \\ \text{Substituting: } & (x - 5)^2 + (mx)^2 = 5 \\ & x^2 - 10x + 25 + m^2x^2 = 5 \\ & (1 + m^2)x^2 - 10x + 20 = 0 \quad (1) \end{aligned}$$

3. Instead of solving equation (1) we determine how many solutions it has. Equation (1) is a quadratic equation whose basic form is  $ax^2 + bx + c = 0$ . In this case  $a = 1 + m^2$ ,  $b = -10$ , and  $c = 20$ . The discriminant of this quadratic equation is:

$$\begin{aligned} D &= b^2 - 4ac = (-10)^2 - 4(1 + m^2)(20) = 100 - 80 - 80m^2 \\ &= -80m^2 + 20 \end{aligned}$$

4. The graph of  $D = -80m^2 + 20$  is a parabola that opens downward, crossing the horizontal axis when  $m = -\frac{1}{2}$  and  $m = \frac{1}{2}$ . Using the graph to determine which values of  $m$  cause  $D$  to be positive, to be zero, and to be negative, we draw three conclusions.

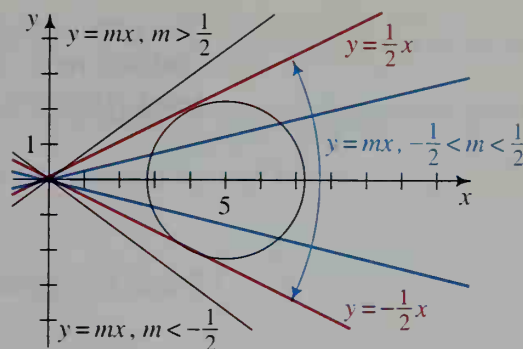




a.  $D > 0$  when  $-\frac{1}{2} < m < \frac{1}{2}$ . In this case, equation (1) has two distinct roots, indicating that the line and the circle intersect twice. Thus, when  $-\frac{1}{2} < m < \frac{1}{2}$ , the line  $y = mx$  intersects the circle in two distinct points.

b.  $D = 0$  when  $m = \pm \frac{1}{2}$ . In this case, equation (1) has one (double) root, indicating that the line and the circle intersect in exactly one point. Thus, when  $m = -\frac{1}{2}$  or  $m = \frac{1}{2}$ , the line  $y = mx$  is tangent to the circle.

c.  $D < 0$  when  $m < -\frac{1}{2}$  or  $m > \frac{1}{2}$ . In this case, equation (1) has no real roots, indicating that the line and the circle have no intersection points. Thus, when  $m < -\frac{1}{2}$  or  $m > \frac{1}{2}$ , the line  $y = mx$  does not intersect the circle.



## Tangent and Normal Lines

The following example applies the method discussed above to finding the equation of a line tangent to a parabola. Notice that the lines of the family discussed in the example do not all pass through a common point; instead, these lines all have the same slope. The parameter for this family of lines is the y-intercept  $k$ .

### Example 1

The diagram shows the parabola  $y = x^2$  and several lines from the family of lines with slope 3. Find an equation of the line with slope 3 that is tangent to the parabola.

### Solution

The equation  $y = 3x + k$  represents the family of lines with slope 3. We proceed as if we were solving the equation of the line and the parabola simultaneously under the assumption that  $k$  is given.

$$\text{Line: } y = 3x + k$$

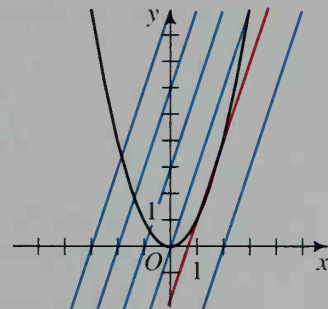
$$\text{Parabola: } y = x^2$$

$$\begin{aligned} \text{Substituting: } 3x + k &= x^2 \\ x^2 - 3x - k &= 0 \quad (2) \end{aligned}$$

Equation (2) is a quadratic equation with discriminant

$$D = b^2 - 4ac = (-3)^2 - 4(1)(-k) = 9 + 4k.$$

(Solution continues on the next page.)



We wish to find the value of  $k$  for which equation (2) has exactly one (double) root, indicating that the line intersects the parabola in a single point. This will happen when  $D = 0$ :

$$9 + 4k = 0$$

$$k = -\frac{9}{4}$$

Thus, the equation of the tangent line is  $y = 3x - \frac{9}{4}$ .

When a line is tangent to a curve, the point of intersection is called the point of tangency. A line that passes through the point of tangency and is perpendicular to the tangent line is called a **normal line**. Such a line is said to be *normal to the curve* at the point of tangency.

**Example 2** As shown in Example 1, the line  $y = 3x - \frac{9}{4}$  is tangent to the parabola  $y = x^2$ . Find (a) the coordinates of the point of tangency, and (b) the equation of the line normal to the parabola at the point of tangency.

**Solution**

a. We can take advantage of the fact that in Example 1 we began to solve the equations  $y = 3x + k$  and  $y = x^2$  simultaneously. We stopped when we reached equation (2):

$$x^2 - 3x - k = 0 \quad (2)$$

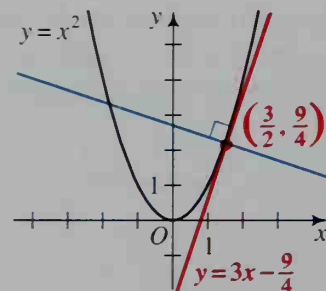
We then determined that when  $k = -\frac{9}{4}$ , equation (2) has exactly one root. This root is the  $x$ -coordinate of the only intersection point of the line  $y = 3x - \frac{9}{4}$  and the parabola  $y = x^2$ . Thus, to find the  $x$ -coordinate of the point of tangency, substitute  $k = -\frac{9}{4}$  in equation (2) and solve for  $x$ :

$$x^2 - 3x - \left(-\frac{9}{4}\right) = 0$$

$$x^2 - 3x + \frac{9}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = 0$$

$$x = \frac{3}{2}$$



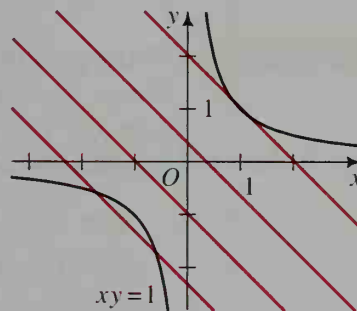
Substituting  $x = \frac{3}{2}$  in the equation  $y = x^2$  gives  $y = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$ . Thus, the line and the parabola are tangent at the point  $\left(\frac{3}{2}, \frac{9}{4}\right)$ .

- b. The slope of the tangent line  $y = 3x - \frac{9}{4}$  is 3; therefore, the slope of the normal line is  $-\frac{1}{3}$ . Since the normal line passes through the point of tangency  $(\frac{3}{2}, \frac{9}{4})$ , the equation of the normal line is

$$\frac{y - \frac{9}{4}}{x - \frac{3}{2}} = -\frac{1}{3}, \text{ or } y = -\frac{1}{3}x + \frac{11}{4}.$$

## Exercises

- Using a parameter, write an equation to represent each family of lines.
  - Nonvertical lines through the point  $(0, 5)$
  - Lines with slope  $-2$
- From the family of lines with slope 3, which lines intersect the circle  $x^2 + y^2 = 10$  in (a) 2 points? (b) 1 point? (c) 0 points?
- From the family of nonvertical lines through the origin, find an equation of each line that is tangent to the circle  $x^2 + (y - 4)^2 = 8$ .
- Repeat Exercise 3 for the parabola  $y = (x + 3)^2 - 5$ .
- The diagram at the right shows the hyperbola  $xy = 1$  and several lines from the family of lines with slope  $-1$ . Find an equation of each line with slope  $-1$  that is tangent to the hyperbola  $xy = 1$ .
- From the family of lines with slope  $-3$ , find an equation of each line that is tangent to the ellipse  $9x^2 + 4y^2 = 36$ . Illustrate your solution with a sketch.
- From the family of lines that contain the point  $(0, -1)$ , which lines are tangent to the ellipse  $(x + 1)^2 + 4y^2 = 4$ ? Illustrate your solution with a sketch.
- From the family of lines that contain the point  $(-2, 0)$ , which lines do not intersect the parabola  $y = 6x - x^2$ ?



In Exercises 9–12, do each of the following:

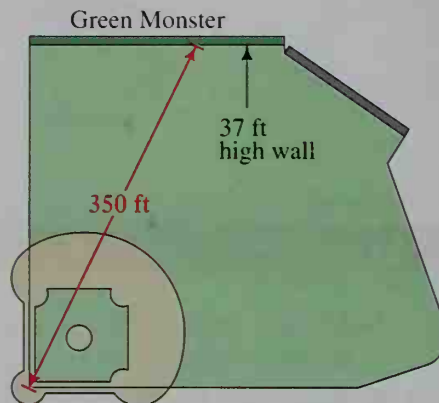
- From the family of lines with the given condition, find an equation of each line that is tangent to the given curve.
  - For each tangent line in part (a), find the coordinates of the point of tangency.
  - Find an equation of the line normal to the given curve at each point of tangency in part (b). Illustrate your solution with a sketch.
- $m = -2$ ;  $y = x^2 + 4x$
  - $m = 2$ ;  $xy = -2$
  - Contains  $(0, 4)$ ;  $x^2 + y^2 = 8$
  - Contains  $(0, 0)$ ;  $y = x^2 + 4$

## Graphing Calculator Applications

### Baseball

The left field wall at Boston's Fenway Park is the tallest in baseball. The imposing 37 ft high wall is known as the "Green Monster."

Boston Red Sox player Mo Vaughn was the American League's Most Valuable Player in 1995. Many of his home runs were hit over the Green Monster at about the 350 ft mark.



### Example

A hitter's bat meets a ball about 3 ft above home plate. Suppose the horizontal component of the ball's initial velocity is 100 ft/s and the vertical component is 60 ft/s. If the ball is hit toward the Green Monster at the 350 ft mark, will it go over the wall for a home run?

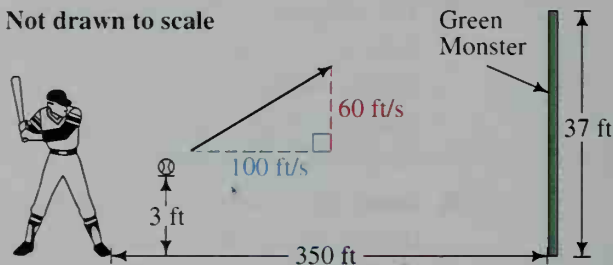
### Solution

Write parametric equations for the position  $(x, y)$  of the ball  $t$  seconds after it is hit.

$$x = 100t$$

$$y = 3 + 60t - 16t^2$$

Not drawn to scale



Set your graphing calculator's mode to Degree, Par (parametric), and Sequential. Press **Y=** and enter the equations shown below. Use the viewing window described.

$$X_{1T} = 350$$

$$Y_{1T} = 5T$$

(This draws the  
Green Monster)

$$X_{2T} = 100T$$

$$Y_{2T} = 3 + 60T - 16T^2$$

(This draws the path of  
the ball)

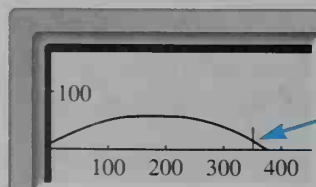
**viewing window:**

$$0 \leq t \leq 7.4, \text{ Tstep} = 0.1$$

$$0 \leq x \leq 450, \text{ Xscl} = 100$$

$$0 \leq y \leq 300, \text{ Yscl} = 100$$

Press **GRAPH** to view the graphs.



The ball hits the  
Green Monster.

The ball does not go over the Green Monster.

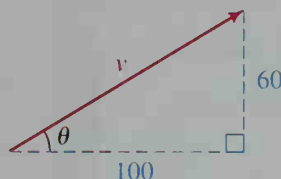
## EXERCISES

Exercises 1–4 refer to the situation described in the example on page 846.

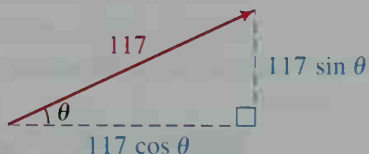
1. Explain why the equations for  $X_{1T}$  and  $Y_{1T}$  draw the Green Monster.  
(Hint:  $5 \cdot 7.4 = 37$  ft.)
2. Use your graphing calculator's TRACE feature on the graph of the ball's path. Would a good outfielder be able to leap and catch the ball just before it hits the Green Monster? Explain why or why not.
3. a. If the vertical velocity component of the batted baseball is changed from 60 ft/s to 65 ft/s, what is the new equation for  $Y_{2T}$ ?  
b. Enter the new equation into your graphing calculator. Does the ball go over the wall for a home run? Explain your reasoning.
4. If the vertical velocity component is increased from 65 ft/s to 66 ft/s, does the ball go over the wall for a home run? Explain your reasoning.

In Exercises 5 and 6, a ball is hit from the same height and in the same direction as in the example on page 846.

5. Find the initial speed  $v$  and angle of elevation  $\theta$  of a ball that has initial velocity  $(100, 60)$ . Round  $v$  to the nearest foot per second and round  $\theta$  to the nearest tenth of a degree.



6. If you keep the ball's initial speed  $v$  constant at 117 ft/s and let the angle  $\theta$  vary, the equations of the ball's path are:



$$X_{2T} = (117 \cos \theta) T$$

$$Y_{2T} = 3 + (117 \sin \theta) T - 16T^2$$

- a. Follow the steps below to graph the ball's path.

**Step 1** Enter the equations for  $X_{2T}$  and  $Y_{2T}$  into your graphing calculator.

**Step 2** Choose an angle  $(0 \leq \theta \leq 90^\circ)$  at which the ball is hit. Store the angle measure as the value of  $\theta$ . For example, to try  $\theta = 30^\circ$ , press 30 **STO>**  $\theta$ .

**Step 3** After you have stored an angle, press **GRAPH** to see the ball's path. Record the angle measure and whether or not the ball goes over the wall.

- b. Repeat steps 2 and 3 of part (a) using several different angles. What values of  $\theta$  result in a home run?



A few hours after a person takes some medicine, the medicine reaches its peak concentration in the person's blood. After reaching its peak, the concentration of the medicine decays exponentially.

After a person takes a single dose of an allergy medicine, its concentration in the person's blood peaks at 263 nanograms per milliliter (ng/mL) and then begins to decrease. Each day, the concentration is about 14% of what it was the previous day. Find the medicine's concentration:

**a.** after 1 day                                      **b.** after 6 days

### Method 1: Repeated Calculation

Enter the initial concentration and press **ENTER**. Press **×** 0.14 **ENTER**. Repeatedly press **ENTER** until you reach the desired number of days.

263

Ans\*0.14

36.82

5.1548

.721672

.10103408

.0141447712

.001980268

This is the concentration after one day.

This is the concentration after six days.

- The concentration after one day is about 36.8 ng/mL.
- The concentration after six days is about 0.00198 ng/mL.

Set your graphing calculator's mode to Seq (sequence). Enter the following information into the calculator.

**Y=**  $U_n = 0.14U_{n-1}$  **WINDOW**  $U_n\text{Start} = 263$   
**2nd TblSet**  $\text{TblMin} = 0$   $n\text{Start} = 0$   
 $\Delta\text{Tbl} = 1$   $n\text{Min} = 0$   
 $n\text{Max} = 6$

Press **2nd** **TABLE** to view the results.

$n$	$U_n$
0	263
1	36.82
2	5.1548
3	.72167
4	.10103
5	.01414
6	.00198

- The concentration after one day is about 36.8 ng/mL.
- The concentration after six days is about 0.00198 ng/mL.

## EXERCISES

After a medicine for hypertension reaches its peak concentration in a person's blood, the concentration each day is about 30% of what it was the previous day.

1. When a patient takes one dose of the medicine, its concentration in the blood peaks at 115 ng/mL.
  - a. Use Method 1 to find the medicine's concentration in the patient's blood after 2 days.
  - b. Use Method 2 to find the medicine's concentration in the patient's blood after 7 days.
2. A doctor prescribes the same hypertension medicine in a dose that is twice as strong as the dose in Exercise 1. This double-strength dose results in a peak concentration of 230 ng/mL.
  - a. Use Method 1 to find the medicine's concentration after 2 days. How is the information you entered into your graphing calculator different from the information that you entered in Exercise 1(a)?
  - b. Use Method 2 to find the concentration after 7 days. How is the information you entered into your graphing calculator different from the information that you entered in Exercise 1(b)?

For Exercises 3–5, use the information below.

After a patient takes a single dose of a medicine for high blood pressure, its concentration in the blood decreases by about 50% each day. Suppose that the patient takes one dose of the medicine *every day*. With each dose, the concentration of the medicine in the patient's blood increases by 40 ng/mL. To model the situation using Methods 1 and 2, make the following changes:

### Changes to Method 1

Enter the initial concentration (40).

Press **ENTER**.

Press **×** 0.5 **+** 40 **ENTER**.

Repeatedly press **ENTER** until you reach the desired number of days.

### Changes to Method 2

**Y=**  $U_n = 0.5U_{n-1} + 40$

**WINDOW**  $U_n \text{Start} = 40$

3. Use Method 1 to find the medicine's concentration after 4 days.
4. Use Method 2 to find the medicine's concentration after 8 days.
5. While viewing the table from Exercise 4, press **▼** repeatedly and examine how the concentration of medicine in the patient's blood changes as the number of days increases. Describe what happens to the concentration over time.

6. A doctor prescribes the same high blood pressure medicine in a daily dose that is three times as strong as the dose in Exercises 3–5. With each of these triple-strength doses, the concentration of the medicine in the patient's blood increases by 120 ng/mL.
- What is the concentration in the patient's blood after 1 day? After 10 days?
  - As the number of days increases, what limiting value does the concentration approach?
7. A doctor prescribing the medicine from Exercises 3–6 wants the concentration of the medicine in the patient's blood to approach a limiting value of about 400 units. What size dose should the patient take every day?

## Environment

The table below shows the amount of municipal waste (trash) generated in the United States and the population of the United States from 1960 to 1995.

Year	60	65	70	75	80	85	90	95
Trash (millions of tons per year)	88	103	122	128	151	164	196	208
Population (millions of people)	180	194	204	215	227	238	249	263

### Example

Make a scatter plot of the amount of trash generated in each year.

### Solution

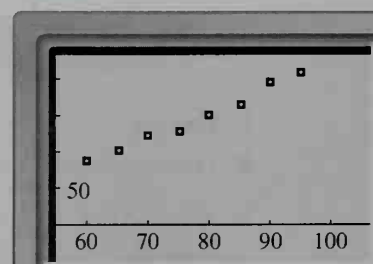
**Step 1** Press **STAT** and select “1:Edit...” Enter the years into list  $L_1$  and enter the trash data into list  $L_2$ .

**Step 2** Set your calculator's mode to Func (function).

**Step 3** Press **Y=** and make sure all of the functions are either cleared or deselected.

**Step 4** Press **2nd STAT PLOT**.  
Select “1:Plot1...” and  
choose the options below.

**Step 5** Enter an appropriate  
viewing window and  
press **GRAPH**.



## EXERCISES

On your graphing calculator, make the scatter plot from the example on page 850. Then answer Exercises 1–3.


1. What viewing window did you use?
2. Does the scatter plot appear to be linear? Explain.
3. Suggest some possible reasons for the increase in the amount of trash that is thrown away in the United States every year.

For Exercises 4–6, use the data from the table on page 850.

4. Follow the steps below to create a list of the amount of trash thrown away by the average person in each year.

**Step 1** If you have not already done so, enter the years into list  $L_1$  and the trash data into list  $L_2$  as in Step 1 of the solution to the example on page 850.

**Step 2** Press **STAT** and select “1:Edit...”. Enter the population data into list  $L_3$ .

**Step 3** Press  until you see a column for list  $L_4$ . Position the cursor on the “ $L_4$ ” at the top of the column and press **ENTER**. Then enter the formula  $L_4 = L_2 \div L_3$ .

$L_2$	$L_3$	$L_4$
88	180	-----
103	194	
122	204	
128	215	
151	227	
164	238	
196	249	
$L_4 = L_2 / L_3$		

5. Since  $L_4 = \frac{L_2}{L_3}$ , the units of  $L_4$  are  $\frac{\text{millions of tons per year}}{\text{millions of people}}$ , or tons per year per person.
  - a. How many tons of trash did the average person throw away in 1965?
  - b. How many tons of trash did the average person throw away in 1980?
  - c. How did the amount of trash thrown away by the average person change between 1960 and 1995?

6. **Writing** Consider this statement:

*The increase in the amount of trash disposed of yearly in the United States is entirely due to the increase in the population.*

Based on your results from Exercises 4 and 5, do you agree or disagree with the statement above? Write a short paragraph explaining your reasoning.

7. Follow the instructions below to create a list giving the average amount of trash in pounds per day per person.

Position the cursor on the “ $L_5$ ” at the top of the column for list  $L_5$  and press **ENTER**. Then enter the formula  $L_5 = L_4 \times 2000 \div 365$ .

8. Explain why the formula for  $L_5$  that you entered in Exercise 7 converts from tons per year to pounds per day.

9. Choose one of the materials from the table at right to use in this exercise.

- Use the percent value from the table and the 1995 value in list  $L_5$  from Exercise 7. On average, how many pounds of the material does a person dispose of in one day?
- On average, how many pounds of the material do you think you and your family dispose of each day? How do you compare to the 1995 national average?

Composition of Municipal Waste	
Material	Percent
Food wastes	15
Paper	38
Cardboard	5
Plastics	5
Leather and textiles	2
Yard wastes	10
Wood	3
Metals and glass	18
Dirt, ashes, etc.	4

## Investment

Tim and Tom are twins who are identical except in the way each handles his money. On their 21st birthday, Tim decides to invest \$1000 on that birthday and on every succeeding birthday. His brother Tom spends all of his money on clothes and recreation, never saving a penny.

Sometime between their 30th and 31st birthdays, both twins have a change of heart. Tim decides that instead of saving his money, he will spend every extra dollar he has on exotic vacations. Tom, on the other hand, realizes that he had better start saving for retirement, so he decides to invest \$1000 on each birthday from age 31 to age 65.

	Birthdays 21–30	Birthdays 31–65
Tim	Saves \$1000	Saves \$0
Tom	Saves \$0	Saves \$1000

Tim saves for only 10 years and Tom saves for 35 years, but Tim starts earlier than Tom. Do you think there is an advantage to starting earlier? The example and the exercises below will help you to reach a conclusion.

### Example

Suppose the money Tim saves earns 8% interest every year. How much money, to the nearest dollar, does he have on his 30th birthday?

### Solution

The amount of money Tim has on each birthday between age 21 and age 30 is represented by the following sequence.

$$U_{21} = 1000; U_n = 1.08U_{n-1} + 1000$$

Set your calculator's mode to Seq (sequence) and enter the following information.

**Y=**  $U_n = 1.08U_{n-1} + 1000$

**2nd TblSet** TblMin = 0  
 $\Delta Tbl = 1$

**WINDOW**  $U_n$ Start = 1000  
 $n$ Start = 21  
 $n$ Min = 21  
 $n$ Max = 30



Press **2nd** **TABLE** to view the results. Use the **▼** key to scroll down to  $n = 30$ .

$n$	$U_n$
24	4506.1
25	5866.6
26	7335.9
27	8922.8
28	10637
29	12488
<b>30</b>	14487
$n = 30$	

On his 30th birthday, Tim has \$14,487.

## EXERCISES

In Exercises 1–4, assume that all of the money Tim and Tom save earns 8% interest every year. Round your answers to the nearest dollar.

- The amounts of money that Tim and Tom each have on each birthday between age 30 and age 65 are represented by the following sequences.

$$\text{Tim: } U_{30} = 14487; U_n = 1.08U_{n-1}$$

$$\text{Tom: } V_{30} = 0; V_n = 1.08V_{n-1} + 1000$$

Explain how the recursive definition for  $U_n$  above is different from the recursive definition for  $U_n$  in the solution to the example on pages 852–853.

- Enter the following information into your graphing calculator to view the sequences from Exercise 1.

**Y=**  $U_n = 1.08U_{n-1}$

$V_n = 1.08V_{n-1} + 1000$

**WINDOW**  $U_n\text{Start} = 14487$

$V_n\text{Start} = 0$

$n\text{Start} = 30$

$n\text{Min} = 30$

$n\text{Max} = 65$

**2nd** **TblSet**  $\text{TblMin} = 30$

$\Delta\text{Tbl} = 1$

Press **2nd** **TABLE**. How much money do Tim and Tom each have:

a. on their 31st birthday?

b. on their 32nd birthday?

- How much money do Tim and Tom each have on their 65th birthday?
- Explain why Tim, who saved \$1000 per year for just 10 years, has so much more money than Tom, who saved \$1000 per year for 35 years.

In Exercises 5 and 6, assume that all of the money Tim and Tom save earns 6% interest every year. Round your answers to the nearest dollar.

- How much money does Tim have on his 30th birthday?
- How much money do Tim and Tom each have on their 65th birthday?

## Refresher Exercises

Each set of Refresher Exercises provides review of prerequisite skills for the indicated chapter, as well as ongoing cumulative review of material covered in previous chapters.

### Refresher Exercises

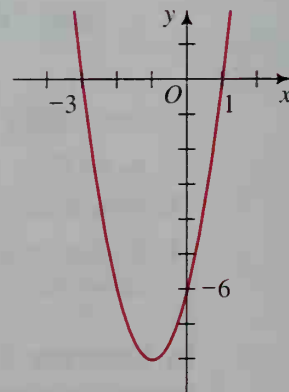
(for use before Chapter 1)

1. In what quadrant does the point  $(-2, 5)$  lie?
2. Is  $(4, -1)$  a point on the graph of the equation  $2x - 5y = 7$ ?
3. Simplify:      a.  $3\sqrt{63}$       b.  $\sqrt{10}(\sqrt{2} + 2)$
4. The lengths of the legs of a right triangle are 3 cm and 6 cm. Find the length of the hypotenuse.
5. Let  $3x + 4y = -9$ . Find  $y$  when  $x = -2$ .
6. Solve for  $y$ :      a.  $5x - 8y = 16$       b.  $\frac{y-4}{x+3} = \frac{3}{4}$
7. Multiply:      a.  $(2x-5)^2$       b.  $(\sqrt{3}+2)(\sqrt{3}-2)$
8. Factor:      a.  $x^2 - 5x + 6$       b.  $x^2 - 25$
9. Let  $y = x^2 + 2x - 3$ . Find  $x$  when  $y = 0$ .
10. Solve for  $x$  by taking the square root of both sides:  $(x-2)^2 = 9$

### Refresher Exercises

(for use before Chapter 2)

1. Classify each function as linear or quadratic.  
a.  $f(x) = 5$       b.  $g(x) = 7x - 3x^2$       c.  $h(x) = 2x - 15$
2. If  $f(x) = 3x^2 - x - 1$ , find  $f(0)$ ,  $f(-2)$ , and  $f(i)$ .
3. Is  $x = -1$  a zero of the function  $P(x) = x^3 - 3x^2 - x + 3$ ?
4. Find all real and imaginary roots of each equation.  
a.  $x^2 - 2x + 6 = 0$       b.  $6x - 4x^2 = 0$       c.  $(x-1)(x^2 + 8) = 0$
5. Factor completely:  $5x^3 - 20x$
6. Find an equation of the parabola graphed at the right.
7. a. Sketch the graph of  $f(x) = -x^2 + 6x - 9$ . Label the vertex, the axis of symmetry, and the intercepts.  
b. What is the maximum value of  $f$ ?
8. The sides of a rectangle are  $x$  and  $3 - 2x$ .  
a. Express the rectangle's area as a function of  $x$ .  
b. Express the rectangle's perimeter as a function of  $x$ .  
c. Explain why  $x$  cannot equal 2.
9. The height and the diameter of a cylinder are equal. Express the volume of the cylinder as a function of its radius.
10. Find the sum and the product of  $1 - 3i$  and  $1 + 3i$ .



Ex. 6

**Refresher Exercises***(for use before Chapter 3)*

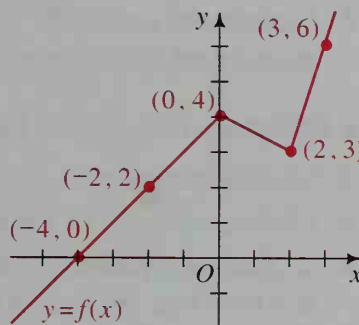
1. Graph the inequality  $-3 \leq x < 2$  on a number line.
2. Solve the inequality  $5 - 4x \leq -7$  for  $x$ .
3. Give the values of  $x$  that make each statement true.
  - a.  $|x| = 5$
  - b.  $|x| < 5$
  - c.  $|x| > 5$
4. Graph  $P(x) = x(x - 3)(x + 1)$ . Then tell if the graph of  $y = P(x)$  is above or below the  $x$ -axis for each given set of  $x$ -values.
  - a.  $x < -1$
  - b.  $-1 < x < 0$
  - c.  $0 < x < 3$
  - d.  $x > 3$
5. Sketch the graphs of  $y = x^2 - 4x + 3$  and  $x - 2y = -6$  on the same set of axes. Find the coordinates of each intersection point.
6. Let  $P = 10x + 8y$ . In parts (a)–(e), evaluate  $P$  for the ordered pair  $(x, y)$ .
  - a.  $(0, 0)$
  - b.  $(0, 5)$
  - c.  $(3, 3)$
  - d.  $(5, 2)$
  - e.  $(6, 0)$
  - f. Which ordered pair in parts (a)–(e) gives the largest value for  $P$ ?
7. Factor:  $a^3 - 2a^2 - a + 2 = 0$
8. Solve for  $x$ :  $4x^4 - 21x^2 + 27 = 0$

**Refresher Exercises***(for use before Chapter 4)*

1. For what value of  $x$  is the function  $g(x) = \frac{2x + 1}{x + 7}$  undefined?
2. Write an inequality that describes the set of  $y$ -values on the graph of the function  $f(x) = (x - 2)^2 - 1$ .
3. Let  $k(x) = 3x + 2$ . Find  $k(a)$ ,  $k(2a)$ , and  $k(a + 1)$ .

In Exercises 4–6, refer to the graph of  $y = f(x)$  shown at the right below. Draw the graph obtained by changing the graph of  $f$  in the way specified.

4. Shift each point of the graph of  $f$  to the right 3 units. For example, instead of plotting the point  $(2, 3)$ , plot the point  $(5, 3)$ .
5. Divide the  $y$ -coordinate of each point of the graph of  $f$  by 2. For example, instead of plotting the point  $(2, 3)$ , plot the point  $(2, 1.5)$ .
6. Reverse the  $x$ - and  $y$ -coordinates of each point of the graph of  $f$ . For example, instead of plotting the point  $(2, 3)$ , plot the point  $(3, 2)$ .



7. Solve  $x = y^3 - 4$  for  $y$  in terms of  $x$ .
8. Find the vertex and the axis of symmetry for the parabola  $y = 2x^2 + 8x + 5$ .
9. Give the dimensions of three different rectangles with area  $6 \text{ cm}^2$ .
10. Each leg of an isosceles triangle is twice as long as its base. Express the perimeter of the triangle in terms of the length  $b$  of the base.

- Simplify: a.  $x^3 \cdot x^2(4x^3)^2$       b.  $\frac{(3x)^3 y^4}{(6xy^2)^2}$
- Use a calculator to evaluate each expression to the nearest thousandth.  
a.  $2(1.23)^3$       b.  $\left(1 + \frac{1}{n}\right)^n$  when  $n = 9999$
- Use a calculator to evaluate  $9^{0.5}$ ,  $100^{0.5}$ , and  $16^{0.5}$ . Use your answers to give the common meaning of  $x^{0.5}$ .

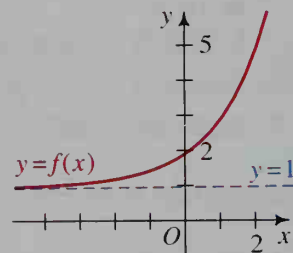
In Exercises 4–7, solve for  $x$ .

- $2^x = 64$
- $10^x = 1000$
- $\left(\frac{x}{2}\right)^3 = 125$
- $x^5 = 0.00001$

- This year a radio costs \$89. The cost is expected to increase by 7% next year. Find the expected cost of the radio next year.

- Sketch the graph of the inverse of the function  $y = f(x)$  whose graph is shown at the right.

- Let  $f$  be a linear function such that  $f(2) = 5$  and  $f(6) = -1$ . Find an equation for  $f(x)$ .


**Refresher Exercises**

(for use before Chapter 6)

For Exercises 1–3, let  $f(x) = 2x^2 + 8x + 5$ .

- a. Rewrite the equation for  $f(x)$  in the form  $f(x) = a(x - h)^2 + k$ .  
b. The graph of  $y = f(x)$  is a parabola. Give the coordinates of its vertex.
- Sketch the graph of  $y = f(x)$ . Label the  $x$ - and  $y$ -intercepts.
- Suppose the graph of  $y = f(x)$  is shifted 2 units to the left and 1 unit up. Write an equation for the new graph.

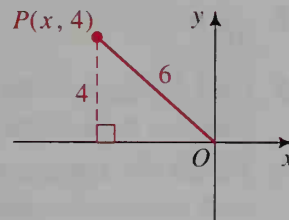
In Exercises 4–7,  $\triangle ABC$  has vertices  $A(0, 0)$ ,  $B(4, 8)$ , and  $C(10, 0)$ .

- Find the coordinates of  $M$ , the midpoint of  $\overline{AB}$ .
- Find an equation of the line that contains  $M$  and is parallel to  $\overline{BC}$ .
- a. Find an equation of the line through points  $C$  and  $M$ .  
b. Show that the line in part (a) is the perpendicular bisector of  $\overline{AB}$ .
- What type of triangle is  $\triangle ABC$ ? Justify your answer.
- Find the domain and range of each function.  
a.  $f(x) = \sqrt{9 - x^2}$       b.  $g(x) = -\sqrt{x - 3}$
- Find the coordinates of any points of intersection of the graphs of  $2x - y = 1$  and  $y = 5 - 5x - 3x^2$ .
- Test the equation  $x^2 + 4y^2 = 1$  for symmetry in (a) the  $x$ -axis, (b) the  $y$ -axis, (c) the line  $y = x$ , and (d) the origin.

## Refresher Exercises

(for use before Chapter 7)

- Sketch the graph of the circle  $x^2 + (y - 2)^2 = 25$ .
  - Find the circumference and the area of the circle in part (a).
- In the diagram at the right, point  $P(x, 4)$  is located 6 units from the origin. Find the value of  $x$ .
- Find the height of an equilateral triangle with sides of length 10.
- The legs of a right isosceles triangle are 5 cm long. Find the length of the triangle's hypotenuse.
- Give the equations of all asymptotes of the graph of the given equation.
  - $y = \log(x + 2)$
  - $y = 2^x - 3$
  - $xy = 1$
  - $36 = x^2 - 9y^2$



In Exercises 6–8, let  $f(x) = x^2 + 2$ .

- Explain why the function  $f$  does not have an inverse.
- Restrict the domain of  $f$  so that  $f$  has an inverse. Then find a rule for  $f^{-1}(x)$ .
- Show that  $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ .

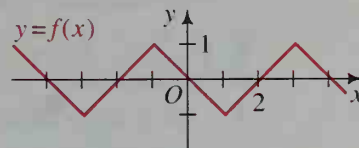
## Refresher Exercises

(for use before Chapter 8)

- Graph  $y = \sin x$  for  $-2\pi \leq x \leq 2\pi$ .
- If  $\tan \theta = \frac{5}{2}$  and  $180^\circ < \theta < 270^\circ$ , find each of the following.
  - $\csc \theta$
  - $\sec \theta$
  - $\cot \theta$
- Find the exact value of each expression in radians.
  - $\cos^{-1} 0$
  - $\sin^{-1}\left(-\frac{1}{2}\right)$
  - $\tan^{-1} 1$

For Exercises 4 and 5, refer to the graph of the function  $f$  shown below.

- Find the fundamental period and the amplitude of  $f$ .
- Sketch the graph of each function.
  - $y = f(2x)$
  - $y = f(x) + 2$
  - $y = f(-x)$
  - $y = 2f(x)$
  - $y = f(x - 1)$
  - $y = -f(x)$



In Exercises 6–8, solve for the variable.

- $2y = 2y(y - 1)$
- $2t^2 - t = 2 - \frac{1}{t}$
- $2\sqrt{a} = a - 3$
- Simplify the complex fraction:  $\frac{\frac{a}{b} - \frac{b}{a}}{\frac{a}{a-b}}$
- Solve the system:  $y^2 = 1 - x^2$   
 $y^2 = x^2 - 3x + 2$



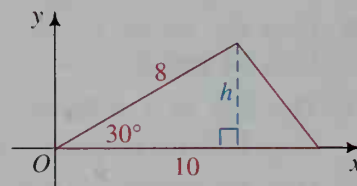
In Exercises 1 and 2, find the value of  $x$  to three significant digits.

1.  $\frac{x}{10} = \sin 40^\circ$

2.  $\frac{3}{x} = \tan 27^\circ$

3. Find the area of the triangle shown in the diagram at the right.

4. Sketch two noncongruent triangles such that each triangle has one side of length 12, one side of length 7, and one angle that measures  $30^\circ$ .



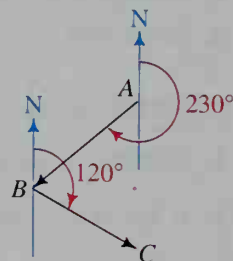
In Exercises 5 and 6, solve for  $0^\circ < \theta < 180^\circ$ . Give your answers to the nearest tenth of a degree.

5.  $9 = 16 + 36 - 2(4)(6)\cos \theta$

6.  $\frac{\sin \theta}{4} = \frac{\sin 27.2^\circ}{5}$

7. In the diagram at the right, the blue arrows indicate the direction of true north. A ship sails from point A to point B along a course of  $230^\circ$ , as shown. The ship then changes its direction, sailing to point C along a course of  $120^\circ$ . Find the measure of  $\angle ABC$ .

8. Graph  $y = \sin x$  and  $y = \ln x$  on the same set of axes. Between what two integers does the equation  $\sin x = \ln x$  have a solution?



Refresher Exercises

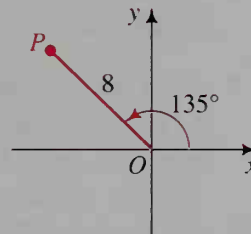
(for use before Chapter 10)

- If  $\sin x = -\frac{15}{17}$  and  $\frac{\pi}{2} < x < \frac{3\pi}{2}$ , find the exact values of the other five trigonometric functions.
- Find the exact value of  $\sin \left( \cos^{-1} \left( -\frac{1}{3} \right) \right)$ .
- Use a calculator to evaluate  $\sin 149^\circ \cos 121^\circ + \cos 149^\circ \sin 121^\circ$ .
- Without using a calculator, evaluate  $\frac{\tan 210^\circ + \tan 120^\circ}{1 - (\tan 210^\circ)(\tan 120^\circ)}$ .
- Prove  $\tan(-\theta) = -\tan \theta$ .
- Find the slope and the inclination of the line with equation  $3x - 5y = 15$ .
- Simplify  $\sin \theta \cdot \sec^2(90^\circ - \theta)$ .
- Solve  $2 + \cos^2 \theta = 3 \sin^2 \theta$  for  $0^\circ \leq \theta < 360^\circ$ .
- Sketch the graph of  $y = \tan x$  for  $-2\pi \leq x \leq 2\pi$ . Then give the domain and range of the tangent function.
- Graph  $y = -2 \cos x$  and  $y = \sqrt{x}$  on the same set of axes. How many solutions does the equation  $-2 \cos x = \sqrt{x}$  have?

## Refresher Exercises

(for use before Chapter 11)

- Express  $\cos 216^\circ$  in terms of a reference angle.
- As shown in the diagram at the right, point  $P$  is 8 units from the origin on the terminal ray of a  $135^\circ$  angle in standard position. Find the  $x$ - and  $y$ -coordinates of  $P$ .
- $\triangle ABC$  has vertices  $A(0, 0)$ ,  $B(4, 0)$ , and  $C(4, 7)$ .
  - Find  $AC$  to the nearest tenth.
  - Find  $\angle BAC$  to the nearest tenth of a degree.



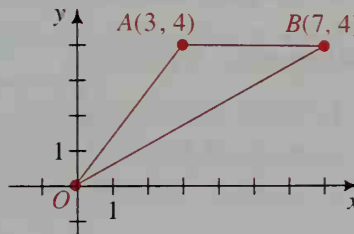
In Exercises 4–7, express each complex number in the form  $a + bi$ .

- $(1 + i)^3$
- $(4 - i\sqrt{3})(2 + i\sqrt{3})$
- $(4 - i\sqrt{3}) + (2 + i\sqrt{3})$
- $\frac{2 - 4i}{1 + i}$
- Show that  $-\sqrt{3} + i$  is a square root of  $2 - 2i\sqrt{3}$ .
- Express  $\cos 4\theta$  in terms of  $\cos \theta$ .
- Solve  $2x^2 + 2x + 1 = 0$ .

## Refresher Exercises

(for use before Chapter 12)

- The diagram at the right shows  $\triangle AOB$  with vertices  $A(3, 4)$ ,  $O(0, 0)$ , and  $B(7, 4)$ .
  - Find the length of each side of  $\triangle AOB$ .
  - Find  $\angle OAB$  to the nearest degree.
- A plane flies 200 km north and then 250 km on a course of  $110^\circ$ . How far is the plane from its starting point? Give your answer to the nearest kilometer.
- Write an equation of the line that contains the points  $(1, -5)$  and  $(-2, 4)$ .
- Sketch the graph of the ellipse with equation  $9(x - 1)^2 + 4(y + 2)^2 = 36$ .
- Given  $A = (2, -3)$  and  $B = (8, -11)$ , find an equation of the circle with diameter  $AB$ .
- Sketch the graphs of  $2x + 3y = 6$  and  $y = 2x^2 - 5x - 3$  to determine the number of intersection points. Then solve the equations simultaneously.
- If  $x = 2 \cos^2 \theta$  and  $y = \sin 2\theta$ , show that  $(x - 1)^2 + y^2 = 1$ .
- Write  $8 \text{ cis } 240^\circ$  in rectangular form.
- Let  $2x + 3y - 4z = 10$ . Find  $z$  when  $x = 0$  and  $y = 0$ .
  - Find two more *ordered triples*  $(x, y, z)$  that are solutions of the equation given in part (a).
- How many solutions does the system shown at the right have? What is the geometric relationship of the graphs of the two lines?
 
$$\begin{aligned} 9x - 6y &= -12 \\ y &= 1.5x - 2 \end{aligned}$$

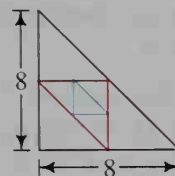


1. Plot the points  $(x, f(x))$  for  $x = 1, 2, 3, 4$ , and  $5$ .

a.  $f(x) = 2x - 3$

b.  $f(x) = \left(-\frac{1}{2}\right)^{x-1}$

2. Each leg of a right isosceles triangle has length 8. The midpoints of the sides of the triangle are joined to form another right isosceles triangle, and the midpoints of the sides of this triangle are joined to form still another right isosceles triangle, and so on.



- a. Find the area of each of the first three triangles in this sequence of triangles.  
b. What is the area of the seventh triangle in the sequence?

In Exercises 3–6, solve for  $n$ .

3.  $4 + 3(n - 1) = 511$

4.  $5 \cdot 2^{n-1} = 20,480$

5.  $\frac{n(5 + 3n)}{2} = 175$

6.  $\frac{1}{1 - 4n} = 2$

7. Evaluate each expression to the nearest thousandth if  $n = 1000$ .

a.  $\frac{4n^2 + 3n - 5}{2n^2 - n}$

b.  $\frac{2 \sin n}{n}$

8. Simplify and factor the expression  $k^2 + k + 2(k + 1)$ .

**Refresher Exercises**

(for use before Chapter 14)

1. Solve each system.

a.  $\begin{cases} 2x + 5y = -11 \\ 3x - 7y = 27 \end{cases}$

b.  $\begin{cases} x + 2y - z = -3 \\ 2x - y + 2z = 7 \\ 3y + 4z = 5 \end{cases}$

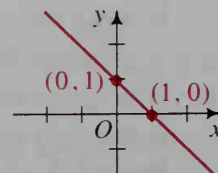
2. Evaluate each determinant.

a.  $\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix}$

b.  $\begin{vmatrix} 1 & 2 & -3 \\ 4 & -4 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

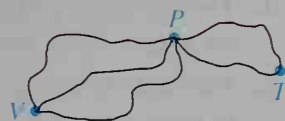
3. Let  $\mathbf{u} = (2, -7, 0)$  and  $\mathbf{v} = (-3, 1, -6)$ . Evaluate  $\mathbf{u} - \mathbf{v}$ ,  $2\mathbf{u} + \mathbf{v}$ , and  $\mathbf{u} \cdot \mathbf{v}$ .  
4. Refer to the vectors  $\mathbf{u}$  and  $\mathbf{v}$  in Exercise 3. Show that  $\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$ .

In Exercises 5–8, suppose that the graph of  $y = -x + 1$ , shown below, is changed in the way specified. (a) Tell what happens to points  $(1, 0)$  and  $(0, 1)$ . (b) Give an equation of the new graph.



5. The graph is reflected in the  $x$ -axis.  
6. The graph is reflected in the  $y$ -axis.  
7. The graph is reflected in the line  $y = x$ .  
8. The graph is shifted 2 units to the left and 1 unit up.

1. From the State Park Visitors' Center, located at point  $V$ , you can choose one of three paths to follow to Pretty Pond, located at point  $P$ . There are 2 paths from the pond to an observation tower at point  $O$ . If you hike from the Visitors' Center to Pretty Pond and then go on to the tower, how many different routes can you follow?
2. Make an organized list to determine the number of four-digit integers that can be formed using each of the digits 1, 2, 3, and 4 exactly once.
3. Make an organized list to determine the number of different ways you can choose two of the four digits 1, 2, 3, and 4.
4. Of the first ten counting numbers, how many are (a) prime? (b) odd? (c) both prime and odd? (d) neither prime nor odd?



**In Exercises 5 and 6, expand the given binomial.**

5.  $(x - y)^3$
6.  $(2x + 1)^3$
7. a. Express  $2 + 2i$  in polar form.  
b. Use De Moivre's Theorem to find  $(2 + 2i)^4$ .
8. Let  $\mathbf{u} = (1, 4, -6)$  and  $\mathbf{v} = (0.5, 0.25, 0.25)$ .  
a. Find  $\mathbf{u} \cdot \mathbf{v}$ .  
b. Based on your answer to part (a), what can you conclude about  $\mathbf{u}$  and  $\mathbf{v}$ ?

**Refresher Exercises**

(for use before Chapter 16)

1. When one card is chosen at random from a standard deck of playing cards, there are 52 possible outcomes. What fraction of the possible outcomes are (a) clubs? (b) face cards? (c) aces?
2. How many different 4-letter "words" can be formed using the letters of the given word?  
a. SINE  
b. NEED
3. From a jar containing 3 green marbles and 2 yellow marbles, 3 marbles are randomly picked, one at a time and without being replaced. Draw a tree diagram showing all possible outcomes.

**In Exercises 4–6, five cards are drawn at random from a well-shuffled standard deck of 52 cards. Find the number of different 5-card hands that contain the specified cards.**

4. No diamonds
5. At least one diamond
6. All diamonds
7. Evaluate the expression  $\frac{{}_4C_1 \cdot {}_5C_2}{{}_9C_3}$ .
8. In the expansion of  $(3x - 2y)^{10}$ , find and simplify the middle term.

- Twenty-eight percent of the 25 students in a precalculus class scored below 70 on a test. How many students in the class scored 70 or above on the test?
- The table below shows the number of students whose English exam scores fell in each grade range. What percent of the students scored between 70 and 89?

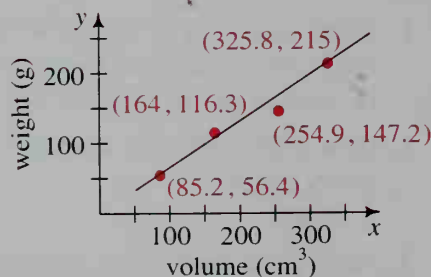
Grade range	Below 60	60–69	70–79	80–89	90–100
Number of students	2	6	12	9	3

- Let  $x_1 = 6$ ,  $x_2 = 8$ ,  $x_3 = 9$ , and  $x_4 = 13$ . Evaluate  $m = \frac{\sum_{i=1}^4 x_i}{4}$ .
  - Use the values of  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ , and  $m$  in part (a) to evaluate  $\frac{\sum_{i=1}^4 (x_i - m)^2}{4}$ .
- Graph the function  $f(x) = e^{-x^2}$ .
- Let  $f(x) = x^2 - 4$  and  $g(x) = \sqrt{x + 3}$ . Find a rule for  $(f \circ g)(x)$  and give the domain of the composite function.
- On a single set of axes, graph  $y = x^3$  and  $y = 2^x$ .
  - Between what two integers is a solution of the equation  $x^3 = 2^x$  located?

**Refresher Exercises**

(for use before Chapter 18)

- Four blocks of different sizes are cut from one piece of wood. The volume and weight of each block are found and then plotted as an ordered pair (volume, weight), as shown at the right. The four points lie on or near the line containing the points that represent the lightest and heaviest blocks. Find an equation of the line.



- Find the standard deviation of the data 2.8, 3.7, 8.4, 9.6, 12.5.
- Sketch the graph of each function.
  - $y = 0.5x^2$
  - $y = 2^{-x} + 2$
  - $y = \sqrt{x}$
- Find the value of  $y$  to three significant digits if  $x = 6$ .
  - $y = 10 - 0.31x$
  - $y = 50(1.06)^x$
  - $y = 25.8x^{0.4}$
- Solve each equation. Give your answers to the nearest thousandth.
  - $\log x = 0.72$
  - $4^x = 3$
  - $\ln x = 1.09$
- Express  $y$  in terms of  $x$ .
  - $\log y = x + 2$
  - $\ln y = 2 \ln x$
  - $\log y = 4 \log x + 3$



## Refresher Exercises

(for use before Chapter 19)

1. Evaluate each limit: a.  $\lim_{n \rightarrow +\infty} \sqrt{n}$  b.  $\lim_{n \rightarrow \infty} \frac{2n^3 - n}{7n^3 - n^2}$

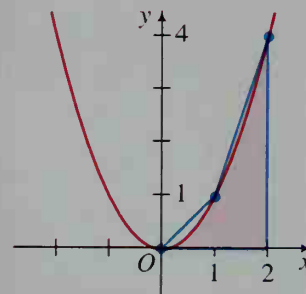
In Exercises 2 and 3, simplify the given expression.

2.  $\frac{x^2 - 9}{x^2 + 4x + 3}$  3.  $\frac{x - 4}{\sqrt{x} - 4}$

In Exercises 4 and 5, solve for  $x$ .

4.  $(x - 4)(x + 2)(x - 1)^2 \geq 0$  5.  $\frac{(x - 5)(x + 3)}{(x - 1)(x + 1)} < 0$

6. In the diagram at the right, the shaded region is bounded by the curve  $y = x^2$ , the  $x$ - and  $y$ -axes, and the vertical line  $x = 2$ . Approximate the area of the shaded region by finding the area of the convex quadrilateral shown in blue.



Ex. 6

7. Use your calculator to find the sum of the first six terms of the series  $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ . What number is approximated by the sum?
8. Let  $f(x) = \sqrt{x} + 1$ . Find  $f(1)$ ,  $f(f(1))$ , and  $f(f(f(1)))$  to the nearest thousandth.

## Refresher Exercises

(for use before Chapter 20)

1. Find an equation of the line with slope  $-\frac{2}{3}$  that contains the point  $(-1, -4)$ .

2. Refer to the diagram at the right.

- a. Write an expression to represent the slope of a line that passes through the point  $(0, 1)$  and through a second point  $(x, 2^x)$  that lies on the graph of the function  $f(x) = 2^x$ .

- b. Evaluate your expression in part (a) to the nearest thousandth for  $x = 2$ ,  $x = 1$ ,  $x = 0.5$ , and  $x = 0.1$ .

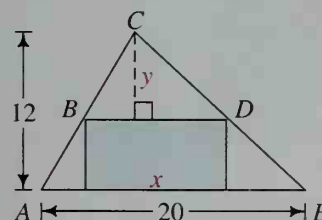
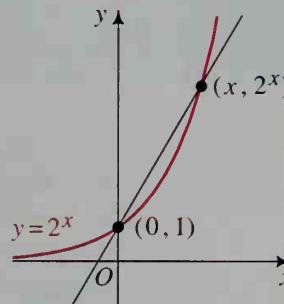
3. Simplify the expression  $\frac{2(x + h)^2 + 1 - (2x^2 + 1)}{h}$ .

4. Find the vertex of the parabola whose equation is  $y = x^2 - 6x + 2$ .

5. Sketch the graph of the equation  $y = 4x^3 - 12x^2$ .

6. The height of  $\triangle ACE$  is 12, and base  $AE = 20$ . A rectangle is inscribed in  $\triangle ACE$  as shown at the right.

- a. Express the rectangle's area as a function of  $x$  and  $y$ .  
b. Use the fact that  $\triangle ACE \sim \triangle BCD$  to write  $y$  as a function of  $x$ .  
c. Express the rectangle's area as a function of  $x$  alone.  
d. Find the maximum possible area of the rectangle.



# List of Symbols

Symbol	Meaning	Page
$P(x, y)$	point $P$ with coordinates $x$ and $y$	1
$\sqrt{\quad}$	square root	4
$\overline{AB}$	line segment with endpoints $A$ and $B$	4
$AB$	the distance from point $A$ to point $B$	4
$\triangle$	triangle	5
$\angle$	angle	6
$m$	slope of a line	7
$\sim$	is similar to	12
$\cong$	is congruent to	12
$\parallel$	is parallel to	12
$\perp$	is perpendicular to	13
$f(x)$	$f$ of $x$ , or function $f$ of $x$	19
$i$	$\sqrt{-1}$	26
$\bar{z}$	conjugate of the complex number $z$	27
$\overleftrightarrow{PB}$	the line through $P$ and $B$	36
$ a $	absolute value of $a$	37
$x^n$	the $n$ th power of $x$	53
$\{(x, y)   y = f(x)\}$	the set of all ordered pairs $(x, y)$ such that $x$ is an element of the domain of $f$ and $y$ is the corresponding element of the range	119
$\lfloor x \rfloor$	the greatest integer less than or equal to $x$	124
$\lceil x \rceil$	the least integer greater than or equal to $x$	124
$(f \circ g)(x)$	$f(g(x))$ , or $f$ of $g$ of $x$	126
$f^{-1}(x)$	$f$ inverse of $x$	147
$P(l, w)$	$P$ as a function of $l$ and $w$	151
$\lim_{n \rightarrow \infty}$	limit as $n$ approaches infinity	186
$e$	irrational number that is approximately 2.718 . . .	187
$\log x$	common logarithm of $x$	191
$\log_b x$	logarithm to base $b$ of $x$	193
$\ln x$	natural logarithm of $x$ , or $\log_e x$	193
$25^\circ 20' 6''$	25 degrees, 20 minutes, and 6 seconds	257
$P(r, \theta)$	point $P$ with polar coordinates $r$ and $\theta$	395
$r \operatorname{cis} \theta$	$r(\cos \theta + i \sin \theta)$	403
$\vec{v}$	vector quantity	419
$\overrightarrow{AB}$	vector depicted by an arrow from point $A$ to point $B$	419
$ \overrightarrow{AB} $	magnitude of $\overrightarrow{AB}$	426
$\mathbf{v}_1 \cdot \mathbf{v}_2$	dot product of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$	441
$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$	$2 \times 2$ determinant	458
$\mathbf{v}_1 \times \mathbf{v}_2$	cross product of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$	465

Symbol	Meaning	Page
$\sum_{k=1}^{100} k^2$	sum of $k^2$ for values of $k$ from 1 to 100	506
$E^t$	transpose of matrix $E$	517
$A^{-1}$	multiplicative inverse of matrix $A$	531
$ A $	determinant of matrix $A$	531
$T: (x, y) \rightarrow (x + 2y, 3x + y)$	transformation $T$ that maps each point $P(x, y)$ to its image point $P'(x + 2y, 3x + y)$	551
$A \cap B$	intersection of sets $A$ and $B$	565
$A \cup B$	union of sets $A$ and $B$	565
$\overline{A}$	complement of set $A$	566
$n(A)$	number of elements in set $A$	566
$n!$	$n$ factorial	572
${}_nP_r$	number of permutations of $r$ things chosen from $n$ things	578
${}_nC_r$	number of combinations of $r$ things chosen from $n$ things	578
$P(A)$	probability of event $A$	598
$P(B A)$	probability of event $B$ , given event $A$	607
$\bar{x}$	the mean of data values $x_1, x_2, x_3, \dots, x_n$	641
$s^2$	variance	654
$s$	standard deviation	654
$z$	standard value	656
$r$	correlation coefficient	685
$f'(x)$	derivative of $f(x)$	758
$\Delta s$	the change in $s$ values, or $s_2 - s_1$	775
$f''(x)$	second derivative of $f(x)$ , or derivative of $f'(x)$	776

## Metric Units

*Time:* second (s), minute (min)

*Temperature:*

degree Celsius ( $^{\circ}\text{C}$ )

*Length:* meter (m)

*Capacity:* liter (L)

*Mass:* kilogram (kg)\*

*Electrical charge:* coulomb (C)

*Force:* newton (N)

*Pressure:* pascal (P)

*Frequency:* hertz (Hz)

*Luminous intensity:*

candela (cd)

A prefix multiplies a unit by the factor given in the table. For example,

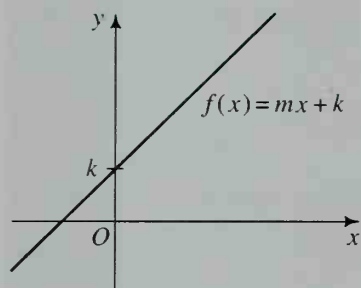
$$1 \text{ MHz} = 10^6 \text{ Hz} = 1,000,000 \text{ Hz}$$

Compound units may also be formed by multiplication or division. Examples are kilometers per hour (km/h) and cubic meters ( $\text{m}^3$ ).

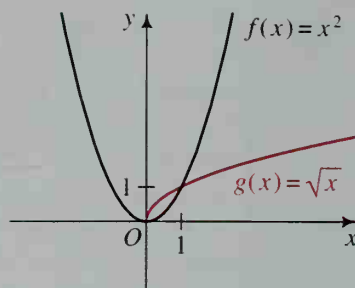
\*Although the kilogram is defined as the base unit, the gram (g) is used with prefixes to name other units of mass.

Factor	Prefix	Symbol
$10^6$	mega	M
$10^3$	kilo	k
$10^{-2}$	centi	c
$10^{-3}$	milli	m

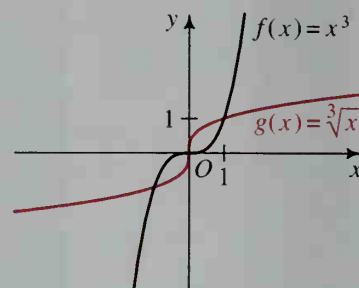
# Visual Glossary of Functions



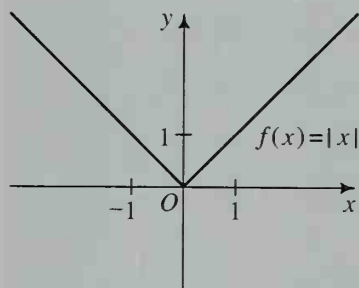
Linear



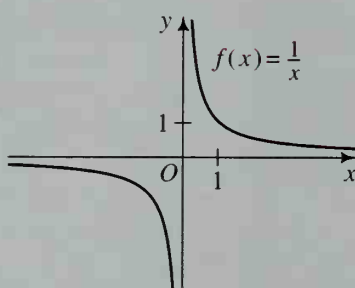
Quadratic; Square root



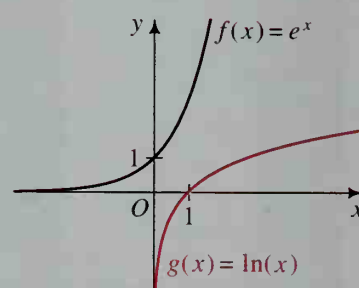
Cubic; Cube root



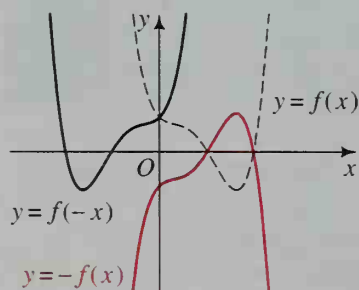
Absolute value



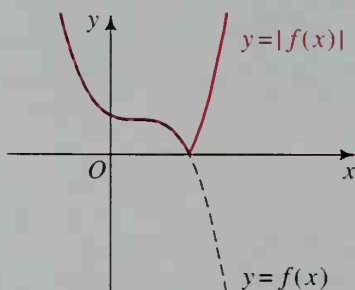
Reciprocal



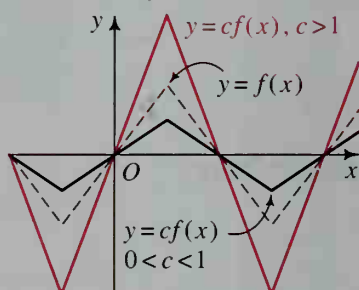
Exponential; Logarithmic



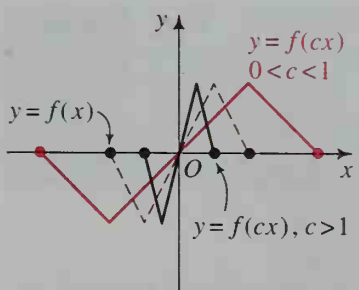
Reflection in the x- or y-axis



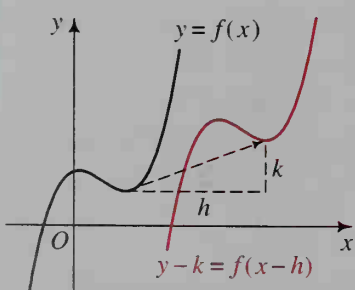
Absolute value reflection



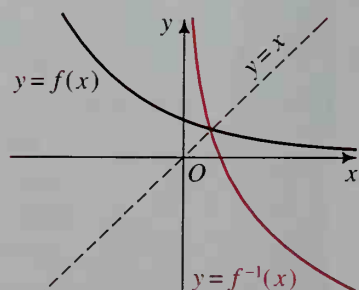
Vertical stretch or shrink



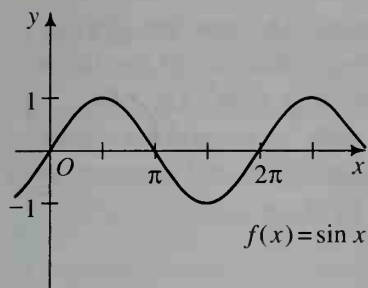
Horizontal stretch or shrink



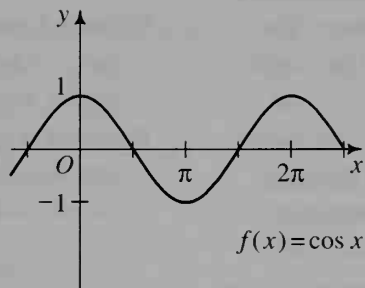
Translation



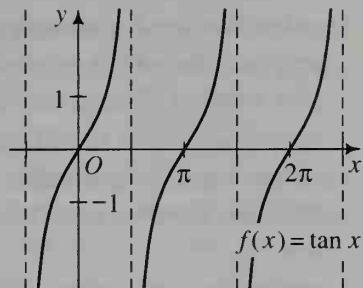
Inverse



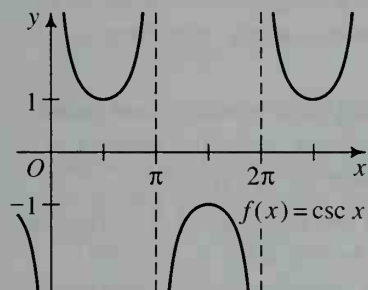
Sine



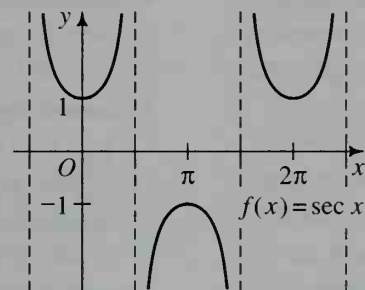
Cosine



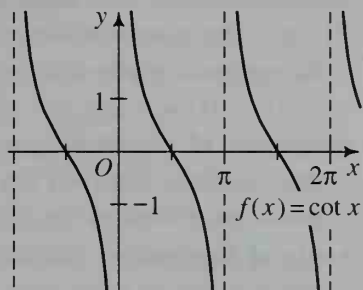
Tangent



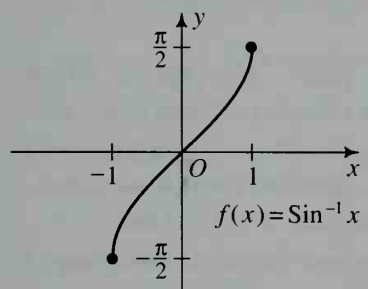
Cosecant



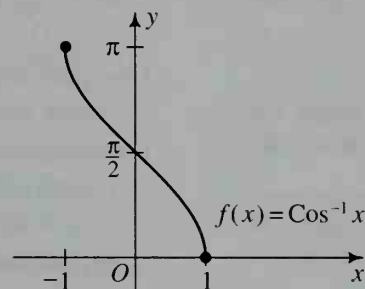
Secant



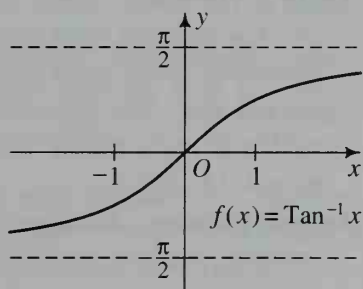
Cotangent



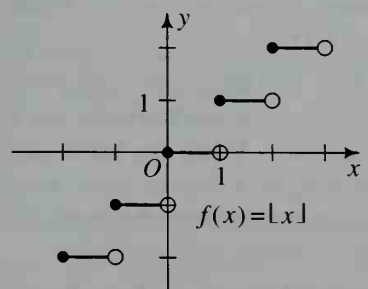
Inverse sine



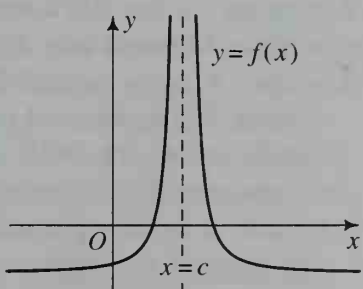
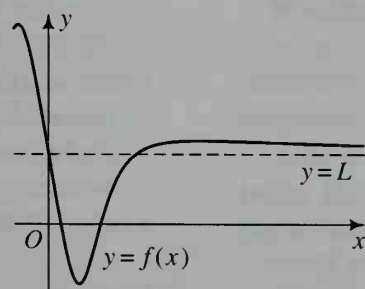
Inverse cosine



Inverse tangent



Greatest integer





**Absolute value of a complex number** The length of the arrow representing the complex number. If  $z = a + bi$ , then

$$|z| = \sqrt{a^2 + b^2}. \text{ (p. 403)}$$

**Absolute value of a number** The distance from the number to zero on the number line. (p. 96)

**Acceleration** A measure of the rate of change of the velocity of an object over time. (p. 776)

**Additive inverse of a matrix  $A$**  (denoted  $-A$ ) The matrix in which each element is the opposite of its corresponding element in  $A$ . (p. 531)

**Amplitude of a periodic function** One half the difference between the maximum and minimum values of the function. (p. 139)

**Angle of depression** Angle by which an observer's line of sight must be depressed from the horizontal to the point observed. (p. 335)

**Angle of elevation** Angle by which an observer's line of sight must be elevated from the horizontal to the point observed. (p. 332)

**Apparent size** The measure of an angle subtended at one's eye by an object being observed. (p. 264)

**Argand diagram** A diagram that represents complex numbers geometrically. (p. 403)

**Arithmetic sequence** A sequence of numbers such that the difference of any two consecutive terms is constant. (p. 473)

**Asymptote** A line that a curve approaches more and more closely. (p. 232)

**Average** A single number, such as the mean, median, and mode, used to represent an entire set of data. (p. 641)

**Axis of symmetry** A graph has a line called an axis of symmetry if it is possible to pair the points of the graph so that the line is the perpendicular bisector of the segment joining each pair. (p. 133)

**Binomial coefficients** The numbers in Pascal's triangle; the coefficients of the terms in the expansion of  $(a + b)^n$ . (p. 590)

**Box-and-whisker plot** A method for displaying the median, quartiles, and extremes of a set of data. (p. 649)

**Cofunctions** Sine, cosine; tangent, cotangent; secant, cosecant. (p. 318)

**Combination** An arrangement of a set of objects in which order is not important. (p. 578)

**Common logarithm of a positive real number  $x$**  The exponent when the number is written as a power of 10. (p. 191)

**Complement of a set  $A$**  (denoted  $\bar{A}$ ) The set of all elements not in a set  $A$ . (p. 566)

**Complex number** Any number of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit.  $a$  is the *real part* and  $b$  is the *imaginary part* of  $a + bi$ . (p. 27)

**Complex number plane** The plane in which points represent complex numbers. (p. 403)

**Composite of two functions**  $(f \circ g)(x) = f(g(x))$  where  $x$  is in the domain of  $g$  and  $g(x)$  is in the domain of  $f$ . (p. 126)

**Composition** The operation that combines two functions to produce their composite. (p. 126)

**Conditional probability** The probability that an event will occur given that another event has occurred. (p. 607)

**Conic sections** Cross sections (circle, ellipse, hyperbola, parabola) resulting from slicing a double cone by a plane. When the result is a single point, a line, or a pair of lines, the cross section is called a *degenerate conic section*. (p. 213)

**Constant** A polynomial (or term of a polynomial) with degree 0. (p. 53)

**Continuous function** A function  $f(x)$  is continuous at a real number  $c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$ . (p. 720)

**Coordinates** An ordered pair of numbers associated with a point on the plane. (p. 1)

**Correlation coefficient** A number between  $-1$  and  $1$ , inclusive, that measures how closely plotted points tend to cluster about the least-squares line. (p. 685)

**Cosecant**  $\csc \theta = \frac{r}{y}$ ,  $y \neq 0$ , where  $P(x, y)$  is a point on circle  $O$  with equation  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ . Also,  $\csc \theta = \frac{1}{\sin \theta}$ ,  $\sin \theta \neq 0$ . (p. 282)

**Cosine**  $\cos \theta = \frac{x}{r}$  where  $P(x, y)$  is a point on circle  $O$  with equation  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ . (p. 268)

**Cotangent**  $\cot \theta = \frac{x}{y}$ ,  $y \neq 0$ , where  $P(x, y)$  is a point on circle  $O$  with equation  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ . Also,  $\cot \theta = \frac{\cos \theta}{\sin \theta}$  and  $\cot \theta = \frac{1}{\tan \theta}$ ,  $\sin \theta \neq 0$  and  $\tan \theta \neq 0$ . (p. 282)

**Coterminal angles** Two angles in standard position having the same terminal ray. (p. 260)

**Cross product of two vectors** If  $\mathbf{v}_1 = (a_1, b_1, c_1)$  and  $\mathbf{v}_2 = (a_2, b_2, c_2)$ , then

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}. \quad (\text{p. 465})$$

**Cubic** A polynomial (or term of a polynomial) with degree 3. (p. 53)

**Degree** A unit for measuring angles,  $\frac{1}{360}$  of a revolution. (p. 257)

**Degree of a polynomial in  $x$**  The power of  $x$  contained in the leading term. (p. 53)

**Derivative of  $f(x)$**  The slope function  $f'(x)$ . (p. 758)

**Determinant** A value associated with a square array of numbers. (pp. 458–459)

**Dimensions of a matrix** The number of rows and columns of the matrix. (p. 517)

**Directrix of a parabola** A fixed line such that all points of the parabola are equidistant from that line and a fixed point called the *focus*. (p. 238)

**Discriminant** The quantity  $b^2 - 4ac$  that appears beneath the radical sign in the quadratic formula. (p. 31)

**Domain of a function** The set of values for which the function is defined. (pp. 20, 119)

**Dot product of two vectors** (also called *scalar product*) In two dimensions,  $\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2$ , where  $\mathbf{v}_1 = (x_1, y_1)$  and  $\mathbf{v}_2 = (x_2, y_2)$ . (p. 441) In three dimensions,  $\mathbf{u} \cdot \mathbf{v} = x_1x_2 + y_1y_2 + z_1z_2$ , where  $\mathbf{u} = (x_1, y_1, z_1)$  and  $\mathbf{v} = (x_2, y_2, z_2)$ . (p. 448)

**Doubling time** Time required for a quantity growing exponentially to double. (p. 182)

$e$  The irrational number, approximately equal to  $2.718 \dots$ , which is the limit as  $n$  approaches infinity of  $\left(1 + \frac{1}{n}\right)^n$ . (p. 186)

**Eccentricity of a conic section** The positive constant  $e$  such that  $PF:PD = e$ , where  $F$  is a fixed point (*focus*) not on a fixed line  $d$  (*directrix*), and  $PD$  is the perpendicular distance from  $P$  to  $d$ . The curve is an ellipse if  $0 < e < 1$ , a parabola if  $e = 1$ , and a hyperbola if  $e > 1$ . (p. 247)

**Element (or entry) of a matrix** A number in a matrix. (p. 517)

**Ellipse** (1) The set of all points  $P$  in the plane such that  $PF_1 + PF_2 = 2a$ , where  $F_1$  and  $F_2$  are fixed points called *foci* and  $a$  is a positive constant. (p. 226)

(2)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $a$  and  $b$  are positive constants. (p. 227)

**Even function**  $f$  is an even function if  $f(-x) = f(x)$ . (p. 137)

**Event** Any subset of a sample space. (p. 598)

**Expected value** If a given situation has payoff  $x_1$ , with probability  $P(x_1)$ , payoff  $x_2$ , with probability  $P(x_2)$ , and so on, then the expected value is  $x_1 \cdot P(x_1) + x_2 \cdot P(x_2) + \cdots + x_n \cdot P(x_n)$ . (p. 630)

**Exponential equation** An equation that contains a variable in the exponent. (p. 203)

**Exponential function with base  $b$**  Any function of the form  $f(x) = ab^x$ ,  $a > 0$ ,  $b > 0$ , and  $b \neq 1$ . (p. 181)

**Extreme values of a function** Minimum and maximum values of the function. (p. 157)

**Feasible solution** Any point  $(x, y)$  that satisfies all the requirements of a problem. The set of these points is called the *feasible region*. (p. 109)

**Frequency polygon** A line graph used to display a set of data. (p. 641)

**Frequency table** A table that is used to organize and display data. (p. 640)

**Function** (1) A dependent relationship between quantities. (p. 19) (2) A correspondence or rule that assigns to every element in a set  $D$  (the *domain*) exactly one element in a set  $R$  (the *range*). (p. 119)

**Geometric sequence** A sequence such that the ratio (*common ratio*) of any two consecutive terms is constant. (p. 473)

**Graph of an equation** The set of all points in the coordinate plane corresponding to solutions of an equation. (p. 1)

**Half-life** The time required for half a quantity decaying exponentially to decay. (p. 182)

**Histogram** A bar graph used to display a set of data. (p. 640)

**Hyperbola** The set of all points  $P$  in the plane such that  $|PF_1 - PF_2| = 2a$ , where  $F_1$  and  $F_2$  are fixed points called *foci* and  $a$  is a positive constant. (p. 231)

**Identity matrix** (denoted  $I_{n \times n}$ ) An  $n \times n$  matrix whose main diagonal elements are 1 and whose other elements are 0. (p. 530)

**Imaginary number** A complex number  $a + bi$  for which  $b \neq 0$ . (p. 27)

**Imaginary unit  $i$**  The number  $\sqrt{-1}$  with the property  $i^2 = -1$ . (p. 26)

**Inclination of a line** The angle  $\alpha$ , where  $0^\circ \leq \alpha < 180^\circ$ , that is measured from the positive  $x$ -axis to the line. (p. 296)

**Independent events** Two events  $A$  and  $B$  are independent if and only if the occurrence of  $A$  does not affect the probability of the occurrence of  $B$ . (p. 607)

**Infinite sequence** A sequence that does not have a last term. (p. 493)

**Inverse functions** Two functions  $f$  and  $g$  are inverse functions if  $g(f(x)) = x$  for all  $x$  in the domain of  $f$  and  $f(g(x)) = x$  for all  $x$  in the domain of  $g$ .  $f^{-1}$  represents the inverse of  $f$ . (p. 146)

**Leading term of a polynomial in  $x$**  The term containing the highest power of  $x$ . The coefficient of the leading term is called the *leading coefficient*. (p. 53)

**Least-squares line** The line that minimizes the sum of the squares of the distances of data points from the line. (p. 684)

**Limit** See pp. 722–723 for a formal definition.

**Line of reflection** A line that acts like a mirror and is located halfway between a point and its reflection. (p. 131)

**Linear equation** Any equation of the form  $Ax + By = C$ , where  $A$  and  $B$  are not both 0. (p. 1)

**Linear function** A function of the form  $f(x) = mx + k$ . (p. 19)

**Linear programming** A branch of mathematics that uses linear inequalities called *constraints* to solve certain decision-making problems where a linear expression is to be maximized or minimized. (p. 108)

**Logarithm to base  $b$  of a positive number  $x$** 

The exponent when the number is written as a power of  $b$ , where  $b > 0$  and  $b \neq 1$ .

(p. 193)

**Logarithmic function with base  $b$**  Inverse of exponential function with base  $b$ . (p. 193)

**Markov chain** A sequence of observations or predictions, each of which depends on the immediately preceding one, and only on it. (p. 545)

**Mathematical induction** A method of proof used to show that a statement is true for all positive integers. (p. 511)

**Mathematical model** One or more functions, graphs, tables, equations, or inequalities that describe a real-world situation. (p. 20)

**Matrix** (plural, *matrices*) A rectangular array of numbers. Each number is an *element* or *entry* of the matrix. (p. 517)

**Mean** (denoted  $\bar{x}$ ) A statistical average found by dividing the sum of a set of data by the number of items of data. (p. 641)

**Median** A statistical average found by arranging a set of data in increasing or decreasing order and finding the middle number or the mean of the two middle numbers. (p. 642)

**Mode** A statistical average that is the one item of data that occurs most often. (p. 642)

**Multiplicative inverse of a matrix  $A$**  (denoted  $A^{-1}$ ) The matrix  $A^{-1}$  such that  $A \cdot A^{-1} = A^{-1} \cdot A = I$ , where  $|A| \neq 0$ . (pp. 531–532)

**Mutually exclusive events** Events that cannot occur simultaneously. (p. 599)

**Natural logarithm of a positive number  $x$**   $\log_e x$ , or  $\ln x$ . (p. 193)

**Normal distribution** A distribution of data along a bell-shaped curve that reaches its maximum height at the mean. (p. 662)

**Odd function**  $f$  is an odd function if  $f(-x) = -f(x)$ . (p. 137)

**One-to-one function** A function that has an inverse. Each  $x$ -value corresponds to exactly one  $y$ -value, and each  $y$ -value corresponds to exactly one  $x$ -value. (p. 148)

**Origin** Point of intersection of  $x$ - and  $y$ -axes. (p. 1)

**Parabola** (1) The graph of a quadratic function. (p. 37) (2) The set of all points  $P$  in the plane that are equidistant from a fixed point  $F$ , called the *focus*, and a fixed line  $d$ , not containing  $F$ , called the *directrix*. (p. 238)

**Pascal's triangle** An array of the coefficients of the terms in the expansion of  $(a + b)^n$ . (p. 590)

**Percentiles** The 99 points that divide a set of data into 100 equal parts when the data are arranged in ascending order. (p. 665)

**Periodic function** A function  $f$  is periodic if there is a positive number  $p$ , called the *period* of  $f$ , such that  $f(x + p) = f(x)$  for all  $x$  in the domain of  $f$ . (p. 138)

**Permutation** An arrangement of a set of objects in which order is important. (p. 578)

**Point of symmetry** A graph has a point of symmetry if it is possible to pair the points of the graph so that the point is the midpoint of the segment joining each pair. (p. 133)

**Polar axis** The reference ray, usually the nonnegative  $x$ -axis, in the polar coordinate system. (p. 395)

**Polar coordinates** The ordered pair  $(r, \theta)$  that describes the position of a point  $P$  in the plane.  $r$  is the distance from the pole to  $P$ , and  $\theta$  is the measure of an angle formed by ray  $OP$  and the polar axis. (p. 395)

**Polar equation** An equation of a curve given in terms of  $r$  and  $\theta$ . (p. 396)

**Pole** The origin in the polar coordinate system. (p. 395)



**Polynomial in  $x$**  An expression that can be written in the form  $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$  where  $n$  is a non-negative integer. The numbers  $a_n, a_{n-1}, \dots, a_2, a_1$ , and  $a_0$  are called the *coefficients* of the polynomial. (p. 53)

**Polynomial inequalities**  $P(x) < 0$  and  $P(x) > 0$  where  $P(x)$  is a polynomial. (p. 100)

**Population** In statistics, the entire set of individuals or objects being studied. (p. 669)

**Power series** An infinite series of the form  $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ . (p. 733)

**Probability** (1) A number between 0 and 1, inclusive, that indicates the likelihood of the occurrence of an event. (p. 597) (2) If a sample space of an experiment contains  $n$  equally likely outcomes and if  $m$  of the  $n$  outcomes correspond to some event  $A$ , then  $P(A) = \frac{m}{n}$ . (p. 598)

**Pure imaginary number** An imaginary number  $a + bi$  for which  $a = 0$  and  $b \neq 0$ . (p. 27)

**Quadrantal angle** An angle in standard position with the terminal ray along an axis. (p. 259)

**Quadrants** The four regions into which the coordinate plane is divided by the  $x$ - and  $y$ -axes. (p. 1)

**Quadratic** A polynomial (or term of a polynomial) with degree 2. (p. 53)

**Quadratic equation** Any equation that can be written in the form  $ax^2 + bx + c = 0$ , where  $a \neq 0$ . (p. 30)

**Quartic** A polynomial (or term of a polynomial) with degree 4. (p. 53)

**Quintic** A polynomial (or term of a polynomial) with degree 5. (p. 53)

**Radian** A unit for measuring angles. (p. 258)

**Range of a function** (1) The set of output values of the function. (p. 20) (2) The set of elements to which the elements of the domain are assigned by the function. (p. 119)

**Rational numbers** Numbers that are the ratio of two integers. (p. 25)

**Real numbers** Zero and all positive and negative integers, rational numbers, and irrational numbers. (p. 26) All complex numbers  $a + bi$  for which  $b = 0$ . (p. 27)

**Reference angle** The acute positive angle formed by the terminal ray of an angle and the  $x$ -axis. (p. 275)

**Resultant** Vector sum of two vectors. (p. 420)

**Revolution** A complete circular motion. (p. 257)

**Sample** A subset of the population. (p. 669)

**Sample space of an experiment** A set  $S$  such that each outcome of the experiment corresponds to exactly one element of  $S$ . (p. 598)

**Sampling** The process of selecting a sample representative of a total population. (p. 669)

**Scalar** In work with vectors and matrices, any real number. (pp. 422, 518)

**Secant**  $\sec \theta = \frac{r}{x}$ ,  $x \neq 0$ , where  $P(x, y)$  is a point on a circle  $O$  with equation  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ . Also,  $\sec \theta = \frac{1}{\cos \theta}$ ,  $\cos \theta \neq 0$ . (p. 282)

**Sector of a circle** A region bounded by a central angle and the intercepted arc. (p. 263)

**Segment of a circle** A region bounded by an arc of the circle and the chord connecting the endpoints of the arc. (p. 340)

**Sequence** (1) A set of numbers, called *terms*, arranged in a particular order. (p. 473) (2) A function whose domain is the set of positive integers. (p. 474)



**Series** An indicated sum of the terms of a sequence. (p. 486)

**Sign graph** A number-line graph that shows when the value of a polynomial function is positive, negative, or zero. (p. 100)

**Sine**  $\sin \theta = \frac{y}{r}$  where  $P(x, y)$  is a point on circle  $O$  with equation  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ . (p. 268)

**Slope of a curve at a point  $P$**  The limit of the slope of  $\overline{PQ}$  as  $Q$  approaches  $P$  along the curve, where  $Q$  is a point on the curve near  $P$ . (p. 758)

**Slope of a nonvertical line** (1) A number measuring the steepness of the line relative to the  $x$ -axis. (p. 7) (2) The tangent of the inclination of the line. (p. 296)

**Slope function**  $f'(x)$ , the derivative of  $f(x)$ . (p. 758)

**Square matrix** A matrix having the same number of rows as of columns. (p. 530)

**Standard deviation** The positive square root of the variance. (p. 654)

**Standard position of an angle** An angle shown in a coordinate plane with its vertex at the origin and its initial ray along the positive  $x$ -axis. (p. 259)

**Stem-and-leaf plot** A method for organizing and displaying data. (p. 639)

**Synthetic substitution** A method for evaluating a polynomial function  $P(x)$  for any value of  $x$ . (p. 55)

**Synthetic division** A short-cut method of dividing a polynomial by a divisor of the form  $x - a$ . (p. 59)

**Tangent**  $\tan \theta = \frac{y}{x}$ ,  $x \neq 0$ , where  $P(x, y)$  is a point on circle  $O$  with equation  $x^2 + y^2 = r^2$  and  $\theta$  is an angle in standard position with terminal ray  $OP$ . Also  $\tan \theta = \frac{\sin \theta}{\cos \theta}$ ,  $\cos \theta \neq 0$ . (p. 282)

**Tangent to a curve at a point on the curve** The line passing through the point and having a slope equal to the slope of the curve at that point. (p. 761)

**Transformation of the plane** A function whose domain and range are sets of points. (p. 551)

**Transition matrix** A matrix that specifies a transition from one observation to the next. (p. 543)

**Transpose of a matrix** The matrix obtained by interchanging the rows and columns of the matrix. (p. 517)

**Trigonometric identity** An equation that is true for all values of the variable for which each side of the equation is defined. (p. 318)

**Unit circle** The circle  $x^2 + y^2 = 1$ . (p. 269)

**Variance** A statistic used to describe the spread of data about the mean. (p. 654)

**Vector** (1) A quantity described by both direction and magnitude (p. 419). (2) A quantity named by an ordered pair (p. 426) or an ordered triple (p. 446) of numbers.

**Venn diagram** A diagram used to represent a relationship between sets. (p. 565)

**$x$ -axis and  $y$ -axis** Perpendicular lines, usually horizontal and vertical, respectively, used in setting up a coordinate system. (p. 1)

**$x$ -intercept** The  $x$ -coordinate of a point at which a graph intersects the  $x$ -axis. (p. 1)

**$y$ -intercept** The  $y$ -coordinate of a point at which a graph intersects the  $y$ -axis. (p. 1)

**Zero of a function** If  $f(a) = 0$ , then  $a$  is a zero of the function  $f$ . (p. 19)

**Zero of a polynomial function  $P(x)$**  Any value of  $x$  for which  $P(x) = 0$ . (p. 53)

**Zero matrix** (denoted  $O_{m \times n}$ ) An  $m \times n$  matrix whose elements are all zero. (p. 530)

## A

**ABEL, NIELS HENRIK**, 88

**Absolute value**, 96

- of complex number, 403
- equations involving, 97–98
- inequalities involving, 97–98
- of vector, 419

**Acceleration**

- average, 776
- instantaneous, 776

**Activities**

- Ch. 1, 10, 19; Ch. 2, 62, 63, 64; Ch. 3, 101; Ch. 4, 131; Ch. 5, 180, 187; Ch. 6, 226; Ch. 7, 269; Ch. 8, 318, 319; Ch. 9, 346; Ch. 10, 380; Ch. 11, 409; Ch. 12, 434, 459; Ch. 13, 474, 480; Ch. 14, 553; Ch. 15, 583; Ch. 16, 597, 605; Ch. 17, 662; Ch. 18, 700; Ch. 19, 726, 730, 737, 739; Ch. 20, 769

**Addition principle**, 573

**Analytic geometry**, 213–255

- coordinate proofs, 214–217

**Angle(s)**

- between line and plane, 457
- between nonzero vectors, 442
- between planes, 456
- between two intersecting lines, 376
- central, 258
- coterminal, 260
- of depression, 335 (Ex. 14)
- of descent, 335 (Ex. 13), 338 (Ex. 37)
- direction, for conic section, 297
- of elevation, 332
- measurement, 257–259
- quadrantal, 259
- radian measure, 258
- reference, 275
- as rotation, 257
- standard position, 259
- trisection, 88

**APOLLONIUS**, 237

**Apparent size**, 264

**Applications**, xvi–xvii

- Accounting
  - straight-line depreciation, 24

**Agriculture**

- maximizing profit, 113
- maximizing yield, 46

**Animal science**

- feeding-mixture matrix, 527

**Archaeology**

- carbon dating, 207

**Astronomy**

- amount of daylight, 314
- apparent size, 265
- brightness of stars, 196, 695
- circumference of Earth, 266
- classifying planets, 569
- distance between celestial bodies, 267
- Jupiter's satellites, 703
- orbit, 250
- planets' distances from sun and year-length, 702
- time of sunset, 315

**Aviation**

- angle of descent, 335, 338
- angle of elevation, 353
- distance from horizon, 357
- shock waves and conic sections, 249
- wind and resultant velocity, 424, 426, 430

**Biology**

- bacterial growth, 182–183, 184
- population growth, 189, 780
- population transition matrix, 549, 550
- weight of salmon, 705

**Business**

- customer transition matrix, 546
- depreciation, 174, 178, 184
- expected value, 635
- market-share matrix, 548
- maximizing profit, 773
- maximizing revenue, 47–48, 72, 110

- minimizing cost, 774
- parts-production matrix, 527
- price matrix, 529
- product-order transition matrix, 547
- profit matrix, 527
- revenue, 23
- sales matrix, 521

**Carpentry**, 246

**Chemistry**

- pH, 196

**Civil engineering**

- ramp elevation, 356

**Communications**

- Morse Code, 589

**Computer**. *See* Computer

exercises and uses.

**Conservation**

- oil consumption, 201

**Consumer economics**

- average price, 646
- compound interest, 154, 187–188, 478
- cost increase, 177, 178, 179, 184
- expected value, 635
- food-sample bacterial count, 659
- fuel economy, 130, 154
- gas mileage, 46
- market value model, 694
- minimizing cost, 111
- modeling cost, 704
- quadratic price model, 46, 47
- rate of price increase, 174

**Ecology**

- estimating wildlife population, 673, 674
- food-chain matrix, 542

**Economics**

- baseball players' salaries, 706
- consumer price index, 179, 707
- gross national product, 707
- inflation, 780
- salary distribution, 648

**Education**

- college expenses, 179

- Electronics
  - AM radio waves, 306
  - FM radio waves, 307
  - modeling voltage, 696
  - power, 47
  - voltage equations, 303–304
- Engineering
  - minimizing cost, 164
  - volume of storage tank, 163, 344
- Environment, 850–852
- Farming
  - expected gain, 634
  - fencing maximum area, 71
  - grazing area, 266
- Finance
  - appreciation, 179
  - compound interest, 189–190, 206
  - doubling investment, 182, 184, 206
  - effective annual yield, 189
  - tripling investment, 205–206
  - value of investment, 483, 747, 852–853
- Forestry
  - lumber production, 703
- Geography
  - classifying states, 569
  - distance on the globe, 281
  - latitude and
    - circumference, 277
  - population distribution, 546, 647
  - population growth, 184, 185, 205, 695
- Geology
  - geyser eruption, 107
  - oil-well drilling, 353
  - Richter scale, 202–203
- Horticulture
  - temperature range, 99
  - tree heights, 667
- Industrial design
  - project-scoring matrix, 528
- Insurance
  - expected gain, 634
  - probability, 627
- Landscaping
  - minimizing cost, 164
- Linear programming, 108–111, 112, 113–114
- Manufacturing
  - ball bearings, 667
  - extreme value, 773
  - filling bottles, 668
  - inventory matrix, 522
  - labor-schedule matrix, 536
  - light-bulb life, 667
  - maximizing profit, 113–114
  - maximizing volume, 72–73, 164
  - minimizing cost, 162
  - probability, 623, 627
  - production costs, 156, 708
  - sampling for defects, 673
  - washer diameter, 108
- Mechanics
  - pistons, 282
- Medicine
  - drug concentration, 848–850
  - drug half-life, 184
  - probability, 628
  - pulse rate, 707
  - Rh factor, 570
- Meteorology
  - average temperatures, 644
  - probability, 602
  - temperature sine waves, 315
  - weather distribution
    - pattern, 586
  - wind chill, 155
- Music
  - note frequencies, 180
  - vibration of piano
    - string, 308
- Navigation, 348, 357
  - angle of depression, 335
  - angle of elevation, 336
  - great circles, 301
  - setting a course, 424, 426, 430
  - submarine dive, 356
- Nutrition
  - menu-cost matrix, 528
  - menu information, 652–653
  - recommended energy
    - intake, 647
  - vitamin-content matrix, 536
- Occupational safety
  - probability, 602
- Oceanography
  - intensity of light in
    - water, 207
  - tides, 314
  - water pressure, 706
- Operations research
  - employee arrival time, 667
  - mail handling, 667
- Optics
  - apparent size, 266
  - Snell's law, 300
- Personal finance, 746, 852–853
- Physics
  - boiling point of water, 706
  - decibel levels, 192, 195
  - displacement of spring, 307
  - Earth's rotational speed, 281
  - half-life, 182, 183, 206, 478
  - height of rotating point, 316
  - height of thrown object, 47, 71–72
  - illumination, 131
  - path of projectile, 438, 439, 846–847
  - pendulum motion, 308
  - resultant force, 424, 429, 430
  - rotational speed, 265
  - speed of sound, 130
  - speed on an incline, 336
  - stopping distance, 46
  - velocity and
    - acceleration, 778, 779, 780, 781, 782
  - work, 445
- Physiology
  - calories burned, 155
  - oxygen consumption, 129
  - reaction time, 645, 668
  - weight distribution, 668
- Plumbing
  - capacity of pipe, 48

- Polling**  
 confidence intervals, 680  
 determining sample size, 679  
 margin of error, 678, 679
- Psychology**  
 learning trials, 547
- Recreation**  
 bicycle wheel rate of turn, 262
- Retailing**  
 constant profit line, 156
- Space science**  
 satellite orbit, 338  
 space shuttle speed, 336
- Sports**  
 basketball shooting, 704  
 batting averages, 644  
 batting order, 575  
 choosing a team, 576  
 distance-time (baseball), 165  
 game-sequence possibilities, 589  
 home-run records, 644  
 marathon qualifying times, 668  
 path of a ball, 42, 846–847  
 probability, 612, 617  
 runners' speeds, 708  
 swimming, 424  
 winning Olympic times, 697, 703  
 yacht rating, 157
- Statistics**  
 weather transition matrix, 547  
 wellness transition matrix, 548
- Surveying**, 348
- Telecommunications**  
 parabolic satellite dish, 240
- Test taking**  
 expected gain, 634
- Thermodynamics**  
 cooling-time model, 697  
 temperature-change model, 316  
 warming-time model, 708
- Transportation**  
 airline no-shows, 667  
 commuter transition matrix, 546  
 parabolic arch, 43  
 Trigonometry in navigation and surveying, 359–364
- Approximation**  
 area of ellipse, 231  
 area under curve, 729–733, 749 (Ex. 3), 783 (Ex. 3)  
 length of and area under one arch of a cycloid, 440  
 of roots, 75–80, 93 (Ex. 8b), 117 (Ex. 20), 129 (Exs. 14–16)  
 value of infinite series, 274
- Arc length**, 258, 263
- Area(s)**  
 regular inscribed and circumscribed polygons, 338 (Exs. 41, 42)  
 sector of circle, 263  
 triangle, 339
- Argand diagram**, 403
- Arithmetic sequence**, 473
- Asymptotes**, 232
- Average**, 641–643  
 mean, 641  
 median, 642  
 mode, 642  
 weighted, 523
- Axis (axes)**  
 of parabola, 37  
 polar, 395  
 of symmetry, 37, 133
- B**
- BERNOULLI, JACQUES**, 401 (Ex. 36)
- Binomial coefficients**, 590
- Binomial probability theorem**, 614
- Binomial theorem**, 591
- BLACKWELL, DAVID**, 375
- BLUFORD, GUION, JR.**, xxxv
- Box-and-whisker plot**, 648–651
- BRAHMAGUPTA**, 88
- C**
- Calculator**  
 graphing. *See* Graphing calculator.  
 programmable. *See* Programmable calculator.  
 scientific. *See* Scientific calculator.
- Calculus, introduction**, 757–783  
 applications  
 biology, 780  
 business, 773, 774  
 economics, 780  
 manufacturing, 773  
 physics, 778–781
- CARDANO, GIROLAMO**, 88
- Cardioid**, 400 (Exs. 17, 18), 401 (Ex. 35)
- Careers**  
 architecture, 117  
 computer graphic arts, xxxv  
 computer programming, xxxvi  
 engineering, xxxvi  
 genetics, 713  
 immunology, xxxiv  
 physics, xxxvii  
 psychology, xxxvii  
 space exploration, xxxv  
 statistics, 713  
 urban planning, 117
- Cartesian coordinates**, 396, 411
- Center-radius form**, 220
- Centroid**, 17 (Exs. 21, 24), 219 (Exs. 15, 16)
- Chaos**, 751
- Chapter Summaries**  
 Ch. 1, 48; Ch. 2, 91; Ch. 3, 114; Ch. 4, 165; Ch. 5, 208; Ch. 6, 252; Ch. 7, 291; Ch. 8, 328; Ch. 9, 364; Ch. 10, 391; Ch. 11, 414; Ch. 12, 468; Ch. 13, 514; Ch. 14, 560; Ch. 15, 594; Ch. 16, 636; Ch. 17, 680; Ch. 18, 709; Ch. 19, 748; Ch. 20, 781
- Chapter Tests**  
 Ch. 1, 50; Ch. 2, 93; Ch. 3, 115; Ch. 4, 166; Ch. 5, 209; Ch. 6, 253; Ch. 7, 293; Ch. 8, 328; Ch. 9, 365; Ch. 10, 393; Ch. 11, 415; Ch. 12, 469; Ch. 13, 515; Ch. 14, 560; Ch. 15, 595; Ch. 16, 636; Ch. 17, 681; Ch. 18, 710; Ch. 19, 749; Ch. 20, 782
- Circle(s)**  
 equation, 219–221  
 center-radius form, 220  
 great, 301 (Ex. 46)  
 intersection with line, 221  
 sector, 263  
 unit, 269



- Circular functions**, 269
- Circumcenter**, 17 (Exs. 22, 24), 219 (Ex. 17)
- Cofunctions**, 282, 318
- College Entrance Examinations**, 790–799
- Combinations**, 578–579
- Combinatorics**, 565–595
- combinations, 578–579
  - permutations, 578–579, 583–585
  - principles, 566, 571, 573, 574
- Communication**, xxii–xxiii
- Discussion, 379
  - Reading, 246, 623
  - Visual thinking, 763
  - See also under* Discussion, Reading, Writing, and Visual thinking.
- Communication matrices**, 537–539
- Complement principle**, 574
- Completing the square**, 30
- Complex conjugates**, 27
- Complex conjugates theorem**, 85
- Complex number plane**, 403
- real number line, 403
- Complex numbers**, 26–27
- absolute value, 403
  - converting between rectangular and polar form, 403
  - equality of, 27
  - geometric representation, 403
  - multiplication in polar form, 404
  - operations with, 27
  - powers, 407–408
  - roots, 412–413
  - sequences and series, 504
- Components of vectors**, 426
- Composition of functions**, 126
- Computer exercises and uses**, xxiv–xxv
- approximating roots, 75–80, 93 (Ex. 8b), 116 (Ex. 13), 129 (Exs. 14–16)
  - bracket-and-halving method, 79
  - area under a curve, 231, 440, 729–733, 749 (Ex. 3), 783 (Ex. 3)
  - area of polygon, 338
  - Cartesian equation of a plane, 467
  - complex orbits, 750–755
  - compound interest, 190
  - curve fitting, 684, 700
  - drawing geometric figures, 334 (Exs. 11, 12)
  - evaluating limits, 499, 722, 724 (Exs. 1–22)
  - evaluating polynomials, 58
  - evaluating trigonometric functions, 322, 391
  - finding maximums and minimums, 67 (Exs. 25b, 26b), 69–70, 72–74 (Exs. 1–9), 158, 162 (Exs. 15, 16), 164–165 (Exs. 25–29), 167 (Ex. 12c), 343 (Ex. 18)
  - graphing, 41 (Exs. 25–26), 42 (Ex. 41), 62, 63, 64, 65 (Cl. Exs. 4, 5), 127, 130 (Exs. 33–36), 131, 138 (Ex. 38), 145 (Ex. 15), 150 (Exs. 24–26), 195 (Exs. 29, 30), 220, 223 (Exs. 21–23), 285 (Ex. 10), 290 (Exs. 15–18), 298, 300 (Exs. 37–45), 319, 321 (Exs. 27, 28), 338 (Ex. 38c), 384 (Exs. 27–30), 434, 437–439 (Exs. 23b, 24b, 26–28, 34–40), 726, 728–729 (Cl. Exs. 1–6, Exs. 1–15)
  - inequalities, 102, 104 (Exs. 31–34, 37–42), 207 (Exs. 41–42)
  - involving  $(a + bi)$ , 29
  - iterated functions, 745–748, 783 (Ex. 7)
  - least-squares line, 690 (Exs. 18–20), 693
  - matrix operations, 536 (Exs. 23–27), 541–542 (Exs. 7, 8, 12, 13), 547–550 (Exs. 5–7, 12–15, 17)
  - Monte Carlo simulation of the binomial probability theorem, 618
  - permutations and combinations, 583
  - polar graphs, 400–401 (Exs. 13–22, 25–34), 402, 411 (Ex. 18)
  - probability, 618 (Exs. 19, 20)
  - quadratic equations, 36
  - sampling, 674
  - series, 274, 486–487, 489–490 (Exs. 21–26), 510
  - solving triangles, 355
  - systems of equations, 3, 5 (Cl. Ex. 10), 40, 41 (Exs. 21–24), 245 (Exs. 9–22)
  - trigonometric equations, 280 (Exs. 21–23), 303, 306 (Exs. 19–26), 311, 314–315 (Exs. 17c, 19b, 21c), 326–327 (Exs. 17–34), 340–341, 343–344 (Exs. 23, 24, 32–36), 386–388, 389–391 (Exs. 11–32, 36–39)
  - recursion equations, 480–481, 483–485 (Exs. 29–31, 33–35), 486–487, 489–490 (Exs. 21–26)
- Confidence intervals**, 674
- 95%, 675
  - 99%, 677
- Conic sections**, 213, 219–241, 247–248
- applications
    - elliptical orbits, 250
    - shock waves, 249
    - telecommunications, 240
  - degenerate, 213
  - direction angle, 297
  - directrix, 247
  - eccentricity, 247
  - equations, 213, 248
  - focus, 247
  - parametric equations of, 434–435, 437–438 (Exs. 26–31)
  - tangent and normal lines, 842–845
- Connections, mathematical**
- algorithms, computer implementation. *See* Computer exercises and uses.
  - circular functions and series, 733–736
  - complex numbers, ways of representing, 403–404
  - coordinates, rectangular and polar, 396–397
  - equations, explicit and parametric representation, 433–435
  - finite graphs and matrices, 551–559
  - function equations,
    - standardized form and geometric transformations, 131–146
  - functions and inverses, 146–148, 193, 286–288



- recursive and closed-form definitions of a sequence, 479–481
- right-triangle ratios/trigonometric functions/circular functions, 331–333
- systems of equations and matrices, 530–537
- See also under Applications, Data analysis, Probability, and Statistics.*
- Constant**, 53
- Continuous function**, 62, 720
- Conversion**
- bases of logarithms, 205
  - between Celsius and Fahrenheit, 99 (Ex. 25)
  - between degree measure and radians, 258
  - between rectangular and polar coordinates, 396
- Convex polygonal region**, 110
- Coordinate geometry proofs**, 214–217
- Coordinates**
- Cartesian, 396, 411
  - converting between rectangular and polar, 396
  - of a point, 1
  - polar, 395
  - rectangular, 396
- Corner-point principle**, 110
- Correlation coefficient**, 685–687
- Cosecant**, 282
- Cosine(s)**, 268
- angle between two vectors, 442
  - applications, 310–312
  - domain and range, 270
  - evaluating, 275–276
  - fundamental period, 271
  - graph, 278
  - law of, 350
  - special angles, 276–277
- Cotangent**, 282
- COWINGS, PATRICIA**, xxxvii
- COX, GERTRUDE MARY**, 660
- Cramer's rule**, 461
- Cross product**, 465
- Cubic functions**
- graph, 62–65
  - local maximum, minimum, 69
- Cubic polynomial**, 53
- Cumulative frequency polygon**, 641
- Cumulative reviews**
- Ch. 1–3, 116; Ch. 4–6, 254; Ch. 7–11, 416; Ch. 12–14, 562; Ch. 15–18, 711; Ch. 19–20, 783
  - See also Refresher Exercises.*
- Curve fitting and models**, xxxiii, 43, 210–211, 683–710
- applications
    - astronomy, 695, 702, 703
    - biology, 705
    - consumer economics, 694, 704
    - economics, 706, 707
    - electronics, 696
    - forestry, 703
    - geography, 695
    - manufacturing, 708
    - medicine, 707
    - oceanography, 706
    - physics, 706
    - sports, 697, 703, 704, 708
    - thermodynamics, 697, 708
- Cycloid**, 440
- D**
- DANDELIN, GERMINAL PIERRE**, 251
- Dandelin spheres**, 251 (Ex. 21b)
- DANTZIG, GEORGE**, 108
- Data, organizing**, 639–650
- Data analysis**, Ch. 16
- (Probability), 597–637; Ch. 17 (Statistics), 639–681; Ch. 18 (Curve Fitting and Models), 683–710; Project (Newton's Law of Cooling), 210
- Decibel**, 191
- Decreasing function**, 138 (Exs. 37–39), 182 (Ex. 3), 765
- Degree**
- in angle measure, 257
  - of polynomial, 53
- De Moivre's theorem**, 408, 593 (Exs. 29, 30)
- De Morgan's laws**, 571 (Ex. 25)
- Derivative**, 758
- of position function, 775, 776
  - second, 776
  - used in curve sketching, 764–767
- DESCARTES, RENÉ**, 246, 411
- Descartes' rule of signs**, 835–837
- Determinant(s)**, 458–466
- applications, 461–463
  - Cramer's rule, 461
  - of a matrix, 531
  - minor, 458
  - value of, 458
- Dimensions of a matrix**, 517
- Directrix**
- conic section, 247
  - parabola, 238
- Discrete mathematics**
- algorithm theory, 29, 36, 75–80, 190, 231, 322, 338, 391, 440, 467, 499, 510, 583, 618, 674, 729–733
  - chaos theory, 750–755
  - combinations, 578–583
  - combinatorics, 565–595
  - determinants, 458–467
  - dynamical systems, 737–743
  - fractal geometry, 736, 754
  - graph theory, 822–829
  - iteration, 737–748
  - linear programming, 108–114
  - mathematical induction, 510–514
  - mathematics of finance, 179, 182, 184, 189–190, 205–206, 483, 746, 747
  - matrices, 517–561
  - permutations, 578–589
  - probability, 597–637
  - recursion, 479–492
  - sequences, 473–485, 493–499
  - series, 486–492, 500–514
  - sets, 565–571
  - statistics, 639–681
  - Venn diagrams, 565–571
- Discriminant**, 31, 842
- Discussion**, 379
- exercises focused on, 65, 154, 175, 177, 199, 205, 206, 222, 224, 228, 240, 249, 299, 320, 377, 389, 399, 409, 423, 438, 449, 461, 464, 466, 476, 496, 502, 520, 525, 534, 540, 545, 602, 617, 633, 658, 666, 687, 688, 701, 704, 707, 728, 761
- Distance**
- between point and plane, 457
  - between points, 4
- Distribution**
- normal, 661, 662

skewed, 661  
 standard normal, 663  
 uniform, 660  
**Division**, 58  
   synthetic, 58–60  
**Domain of a function**, 20, 119  
**Dot product**, 441, 442  
**Double-angle formulas**, 381, 383  
**Dynamical system**, 737

## E

*e*, 186  
**Eccentricity**, 247  
**Element of a matrix**, 517  
**Ellipse**, 225–227  
   algebraic definition, 227  
   axes of symmetry, 227  
   center, 227  
   focus, 225  
   geometric definition, 226  
   graphs, 229  
   vertices, 227  
**Ellipsoid**, 230 (Ex. 29)  
**Equation(s)**  
   classifying second-degree, 248  
   depressed, 82  
   exponential, 203–205  
   general second-degree, 213  
   linear, 1  
   of lines, finding, 14–15  
   parametric, 397, 433–435, 448  
   of plane, 453  
   polar, 396  
   polynomial, 75–91  
   quadratic, 30  
   solution, 1  
   solving higher-degree, 75–77  
   of sphere, 448  
   trigonometric, 295–298, 323–326  
   vector, 432–433, 448  
*See also* Linear equations,  
   Polynomial equations,  
   Quadratic equations,  
   Systems of second-degree  
   equations.  
**ERATOSTHENES**, 266 (Ex. 16)  
**EULER, LEONHARD**, 187, 736 (Ex. 15)  
**Euler's line**, 17 (Ex. 24)  
**Event(s)**, 597–599  
   independent, 607  
   mutually exclusive, 599  
   probability of, 599, 608  
**Expected value**, 630–632

**Experiment**, 598  
**Exploratory activities**. *See*  
   Activities.  
**Exponential decay**, 170  
**Exponential equations**, 203–205  
   change-of-base formula, 205  
**Exponential function(s)**, 180–182  
   applications  
     compound interest, 187–189  
     depreciation, 184  
     doubling investment,  
       182, 184  
     half-life, 182–184  
     population increase,  
       182–185, 189  
   base, 181  
   domain, 181, 193  
   natural, 187  
   range, 181, 193  
**Exponential growth**, 170  
**Exponents**, 169–190  
   applications  
     cost increase, 174, 177,  
       178  
     depreciation, 174, 178  
   equations containing, 176  
   integral, 169  
   irrational, 180  
   laws of, 170  
   negative integral, 170  
   rational, 175  
   zero, 171  
**Extreme values of a function**, 157

## F

**Factor theorem**, 60  
**Factorial  $n$** , 572  
**Fair game**, 630  
**Family of lines**, 842  
**FERRARI, LODOVICO**, 88  
**FIBONACCI**, 484 (Ex. 32)  
**Finite sums, properties of**, 507  
**Fixed point**, 739–740  
**Focus (foci)**  
   conic section, 247  
   ellipse, 225, 226  
   hyperbola, 231  
   parabola, 238  
**Formulas**  
   area of triangle, 6 (Ex. 32),  
     337 (Ex. 32), 339  
   change of base of a logarithm,  
     205  
   circumference of circle, 258  
   conversion of angle measures,  
     258  
   conversion of coordinates, 396  
   distance, 4, 447  
   double-angle, 381, 383  
   doubling time, 182  
   half-angle, 382, 383  
   half-life, 182  
   Hero's (Heron's), 6 (Ex. 32),  
     354 (Ex. 26), 467 (Ex. 15)  
   law of cosines, 350  
   law of sines, 345  
   midpoint, 4, 447  
   quadratic, 31  
   sector of circle: area and arc  
     length, 263  
   speed of sound, 130 (Ex. 31)  
   surface area and volume of  
     sphere, 130 (Ex. 32)  
   trigonometric addition, 369–393  
   trigonometric, summary of, 789  
**Fractals**, 736, 754  
**Frequency polygon**, 641  
**Frequency table**, 640  
**Function(s)**, 1–51, 118–167  
   amplitude, 139  
   applications, 151–160  
   composite, 126  
   constant, 19  
   continuous, 720  
   cosine, 268  
   cubic, 62–63, 69–70  
   decreasing, 138 (Exs. 37–39),  
     765  
   definition, 19, 119  
   derivative of, 758  
   discontinuous, 720  
   domain, 20, 119  
   even, 137  
   exponential, 180–182. *See*  
     *also* Exponential functions.  
   extreme values, 157  
   fixed point of, 739–740  
   forming from verbal  
     descriptions, 157–160  
   fundamental period, 138  
   graphs, 37–40, 62–65, 120–  
     121  
   greatest integer, 124 (Ex. 21)  
   increasing, 138 (Exs. 37–39),  
     765  
   inverse, 146–148  
   inverse trigonometric, 286–288  
   iterated, 737–746  
   iterations of, 737  
   limits of, 717–722

linear, 1–25  
 logarithmic, 191–193  
 logistic, 745  
 maximizing or minimizing, 770  
 maximum or minimum values, 68–70, 157, 765, 769  
 natural exponential, 187  
 natural logarithmic, 193  
 odd, 137  
 one-to-one, 148  
 operations on, 124–127  
 period of, 138  
 periodic, 138–139  
 polynomial, 52–93  
 position, 774  
 quadratic, 37–51, 62  
 quartic, 63–64  
 quintic, 64  
 range, 20, 119  
 rational, 726  
 sine, 268  
 step, 21  
 trigonometric, 268–288  
 of two variables, 151–153  
 vertical-line test, 121  
 visual glossary of, 858–859  
 zero of, 19

**Fundamental period of function, 138**  
**Fundamental theorem of algebra, 85**

## G

**GALILEO, 703 (Ex. 17)**  
**GAUSS, CARL FRIEDRICH, 88, 414 (Ex. 16b)**  
**Geometric sequence, 473**  
**Geometry review, 784–787**  
**Global maximum and minimum values, 769**  
**Golden ratio, 499**  
**GOLDSBERRY, YVONNE, 117**  
**GONZALEZ, JOE FRED, JR., 713**  
**Graph theory, 822–829**  
**Graphing calculator, xxiv–xxv, xxvi–xxxiii**  
 composite functions, 127, 130 (Exs. 33–36), 195 (Exs. 29, 30)  
 conic sections, 220, 223 (Exs. 21–23), 230 (Ex. 37b), 241 (Exs. 19, 20, 26–31), 242, 298, 300 (Exs. 37–44), 437–438 (Exs. 26–28)

connected mode, xxvii  
 curve fitting, xxxiii, 700  
 entering functions, xxvi  
 evaluating limits, 187, 722, 724 (Exs. 1–22)  
 exponential equations and inequalities, 185 (Ex. 22), 207 (Exs. 41, 42)  
 exponential functions, 185 (Exs. 19–22), 190 (Exs. 13–16)  
 finding an intersection point, xxx  
 finding maximums and minimums, 67 (Exs. 25b, 26b), 69–70, 72–74 (Exs. 1–9), 158, 162 (Exs. 15, 16), 164–165 (Exs. 25–29), 167 (Ex. 12c), 224–225 (Exs. 47–49), 229 (Ex. 28b), 343 (Ex. 18b)  
 finding a point of interest, xxix  
 how to use, xxvi–xxxiii  
 increasing and decreasing functions, 138 (Ex. 38)  
 inverse function relationships, 150 (Exs. 24–26), 290 (Exs. 15–18)  
 inverse trigonometric functions, 290 (Exs. 15–18)  
 least-squares line, 689 (Exs. 9–12), 693  
 logarithmic functions, 195 (Exs. 31, 32), 201 (Exs. 35, 36)  
 logarithmic inequalities, 207 (Exs. 41, 42)  
 matrix operations, xxxiii, 536 (Exs. 23–27), 541–542 (Exs. 7, 8, 12, 13), 547–550 (Exs. 5–7, 12–15, 17)  
 parametric equations, xxxi–xxxii, 434, 437–439 (Exs. 23b, 24b, 26–28, 34–40)  
 polar graphs, 397, 400–401 (Exs. 13–22, 25–34), 402, 411 (Ex. 18)  
 polynomial equations and inequalities, 75, 78, 93 (Ex. 8b), 102, 104 (Exs. 31–38), 129 (Ex. 14)  
 polynomial functions, 62–64, 65 (Cl. Exs. 4, 5)

radical equations and inequalities, 36 (Ex. 43), 104 (Exs. 39–42), 129 (Exs. 15, 16)  
 radical functions, 228 (Exs. 8, 9), 235 (Exs. 9, 10), 241 (Exs. 19, 20)  
 rational functions, 726, 728–729 (Cl. Exs. 1–6, Exs. 1–15)  
**RANGE, xxvi**  
 reflection in a line, 131  
 scatter plot, 210, 689 (Exs. 9–12), 705  
 solving equations, xxx–xxxi  
 square screen, xxvii  
 statistical graphs, xxxii  
 systems of equations, 3, 5 (Cl. Ex. 10), 40, 41 (Exs. 21–24), 245 (Exs. 9–22)  
 tangent lines, 41 (Exs. 25, 26), 42 (Ex. 41a)  
**TRACE, xxix**  
 translations, 145 (Ex. 15)  
 trigonometric equations and inequalities, 280 (Exs. 21–23), 303, 306 (Exs. 19–26), 326–327 (Exs. 17–34), 340–341, 343–344 (Exs. 23, 24, 32–36), 386–388, 389–391 (Exs. 11–32, 36–39)  
 trigonometric functions, 285 (Ex. 10), 310–311, 314–315 (Exs. 17c, 19b, 21c), 384 (Exs. 27–30)  
 trigonometric identities, 319, 321 (Exs. 27, 28), 374 (Exs. 31, 32), 378 (Exs. 15, 16)  
**ZOOM, xxviii**

**Graphs**  
 absolute value, 96  
 asymptotes, 232  
 circle, 220  
 cosine, 278  
 cubic functions, 62, 69–70  
 ellipse, 227  
 of an equation, 1  
 exponential functions, 193  
 functions, 120–121  
 histogram, 640  
 horizontal and vertical lines, 2  
 hyperbola, 233–234  
 inequalities in two variables, 104–106

intersecting lines, 2–3  
 inverse trigonometric functions, 287–288  
 linear equations, 1–2  
 linear inequalities in one variable, 95, 100–102  
 logarithmic functions, 193  
 normal curve, 662  
 parabola, 238  
 periodic functions, 138–142  
 of plane, 453  
 polar, 396  
 polynomial functions, 62–65, 764–767  
 quadratic functions, 37–40, 68–69  
 quartic functions, 63–65  
 quintic functions, 64–65  
 rational functions, 726–727  
 reflecting, 131–133  
 secant, 284  
 sign, 100  
 sine, 278  
 standard normal curve, 663  
 stretching or shrinking, 139–141, 142  
 summary of methods, 142  
 tangent, 284  
 tests for symmetry, 134  
 translating, 141–142  
 trigonometric functions, 278, 283–284  
 visual glossary of functions, 858–859

**Greatest integer function,**  
 124

## H

**Half-angle formulas,** 382, 383  
**Half-life,** 182  
**Harmonic series,** 735  
**Histogram,** xxxii, 640  
**Historical notes,** 88, 187, 237, 246, 251, 266 (Ex. 16), 323, 375, 401 (Ex. 36), 411, 414 (Ex. 16b), 484 (Ex. 32), 499, 590, 660, 702 (Ex. 16), 703, 736, 751, 754  
**Horizontal-line test,** 148  
**Hyperbolas,** 231–234  
 asymptotes, 232  
 axes, 231  
 equation, 231

foci, 231  
 geometric definition, 231  
 graphs, 233–234  
 vertices, 233  
**Hyperboloid,** 236

## I

*i*, 26  
**Identity matrix,** 530  
**Imaginary unit *i*,** 26  
**Inclination of line,** 296  
**Inclusion-exclusion principle,**  
 566  
**Increasing function,** 138 (Exs. 37–39), 150 (Ex. 28), 182 (Ex. 3), 765  
**Index in sigma notation,** 506  
**Inequalities,** 95–117  
 absolute value in, 96  
 combined, 96  
 exponential, 207 (Exs. 31, 32, 42)  
 graphs, 95, 100–102, 104–106  
 horizontal-line test, 148  
 linear, 95–96  
 linear programming, 108–111  
 logarithmic, 202 (Exs. 47–50), 207 (Exs. 41, 42)  
 in one variable, 95, 100–102  
 polynomial, 100–106  
 solving, 95–96  
 triangle, 99 (Ex. 36)  
 in two variables, 104–106  
**Infinite sequences,** 493–496  
**Infinite series,** 500–507  
**Infinity,** 495–496  
**Interpolation,** 800  
**Interquartile range,** 649  
**Intersection**  
 of circle and line, 221  
 of lines, 2–3  
 of sets, 565  
**Inverse functions,** 146–148  
 trigonometric, 286  
**Investigations,** 57, 99, 186, 229, 236, 241, 249, 273, 327, 378, 402, 423, 424, 526, 537, 559, 582, 593, 618, 643, 669, 743, 747, 748  
**Iterated functions,** 737–746  
 applications, 744–748  
 finance, 747  
 personal finance, 746

## J

**JULIA, GASTON,** 751  
**Julia set,** 751

## K

**KEMPF, MARTINE,** xxxvi  
**KEPLER, JOHANNES,** 702 (Ex. 16)  
**KHAYYAM, OMAR,** 88  
**KITAGAWA DE LEON, MIDORI,**  
 xxxv

## L

**Law of cosines,** 350  
**Law of illumination,** 131  
**Law of sines,** 345  
**Least-squares line,** 684  
**Lemniscate,** 401 (Exs. 33, 36)  
**Light year,** 267 (Ex. 22)  
**Limaçon,** 401 (Exs. 19, 20, 35, 36)  
**Limit(s),** 181, 186, 187  
 formal definitions, 722–723  
 of functions, 717–722  
 infinite, 495  
 of infinite sequence, 493  
 left-hand, 719  
 of quotient, 720  
 right-hand, 719  
 of summation, 506  
**Line(s)**  
 finding equations, 14–15  
 intersection, 2–3  
 least-squares, 684  
 median-median, 830–834  
 normal, 844–845  
 parallel, 9  
 parametric equation, 433  
 perpendicular, 10  
 slope, 7–10, 296  
 tangent, 761, 842–845  
 vector equation, 432, 448  
**Line segment**  
 equation of perpendicular bisector, 15  
 length, 4  
 midpoint, 4  
**Linear equations,** 1  
 general form, 1, 14  
 graphs, 1–2  
 intercept form, 14  
 point-slope form, 14  
 slope-intercept form, 9, 14  
 solving simultaneously, 2–3



system solved by  
determinants, 461–462

**Linear functions**, 1–25

**Linear inequalities**  
combined, 95  
graphing, 95  
involving absolute value, 96  
properties used in solving, 95

**Linear interpolation**, 78 (Ex. 17)

**Linear polynomial**, 53

**Linear programming**, 108–111

**Linear transformation**, 551

**Lissajous curve**, 435

**Lituus**, 401 (Ex. 32)

**Local maximum and minimum**,  
69, 765, 769

**Location principle**, 76

**Logarithm(s)**, 191–209  
applications  
compound interest, 204,  
205  
conservation, 201  
decibel level, 192, 195  
magnitude of stars, 196  
pH, 196  
population growth, 204,  
205  
Richter scale, 202, 203  
base, 193  
changing bases, 205  
common, 191  
definition, 193  
laws of, 197  
natural, 193

**Logarithmic function(s)**, 191–  
193  
domain and range, 193  
graph, 193  
inverse, 193

**Logarithmic scale**, 192

## M

**MANDELBROT, BENOIT**, 736,  
754

**Mandelbrot set**, 754–755

**Markov chain**, 545

**Mathematical connections**. *See*  
Connections.

**Matrix (matrices)**, xxxiii, 517–  
561  
addition and subtraction, 518  
additive inverse, 518, 531  
applications  
communications, 537–  
542

transformation, 551–  
559  
transition, 543–550  
biology, 549, 550  
business, 546,  
547, 548  
geography, 546  
psychology, 547  
statistics, 547,  
548  
transportation,  
546

coefficient, 533  
communication, 537–539  
determinant, 531  
dimensions, 517  
dominance, 542 (Ex. 12)  
element, 517  
entry, 517  
equality of, 518  
equation, 533  
identity, 530  
multiplication, 523–525  
multiplicative inverse, 531  
product, 524  
properties, 530–532  
scalar multiplication, 518  
square, 530  
transformation, 551–555  
transition, 543–545  
transpose, 517

**Maximum, minimum**, 68–70,  
157, 765, 769  
in linear programming, 110  
quadratic function, 37, 68–69

**Mean**, 478, 641  
arithmetic, 478  
geometric, 478

**Median**, 642

**Median-median line**, 830–834

**Midpoint of line segment**, 4  
with vectors, 447

**Minute**, 257

**Mode**, 642

**Model(s)**  
choosing the best, 705  
and curve fitting, 683–710  
exponential, 691–692  
mathematical, 20–21  
of periodic behavior, 308–310  
quadratic, 43–45  
trigonometric, 310–312

**Monte Carlo simulation**, 617  
(Exs. 13, 14), 618

**Multiplication principle**, 571

**Mutually exclusive actions**, 573

## N

**Natural exponential function**,  
187

**Newton (unit of force)**, 420

**Newton's law of cooling**,  
210

**NOETHER, EMMY**, 323

**Normal distribution**, 661, 662  
standard, 663

**Normal line**, 844–845

**Numbers**

complex, 26–27  
counting, 25  
imaginary, 26  
irrational, 25  
pure imaginary, 27  
rational, 25  
real, 788  
zero, 26

## O

**Octant**, 446

**Odds**, 601

**Orbit**, 737

periodic, 743 (Ex. 26)

**Ordered pair**, 1

**Ordered triple**, 446

**Origin**, 1

**Orthocenter**, 17 (Exs. 23, 24)

**Outlier**, 649

## P

**Parabola(s)**, 37, 238–239

axis, 37  
definition, 238  
directrix, 238  
equation, 37  
focus, 238  
graphs, 37–40, 238  
line of symmetry, 238  
vertex, 37, 238  
x-intercepts, 38

**Parameter**, xxxi, 433, 842

**Parametric equations**, xxxi–  
xxxii, 397, 433–435  
finding Cartesian equations,  
434, 435

**Partial fractions**, 838–841

**Partial sums**, 500

**PASCAL, BLAISE**, 590

**Pascal's triangle**, 590

alternate form, 593 (Exs. 25,  
26)



- Percentiles**, 665
- Period of function**, 138
- Periodic functions**, 138–139, 271
- Permutations**, 578–579  
circular, 584–585  
with repetition, 583–584
- Perpendicular bisector equation**, 15
- Plane(s)**, 1  
Cartesian equation of, 453  
distance from point to, 457  
graph, 453  
parallel, 456  
perpendicular, 456  
transformation of, 551  
vector equation of, 453
- Point(s)**  
coordinates of, 1  
local maximum, 765  
local minimum, 765  
in the plane, 1  
in space, 446
- Polar axis**, 395
- Polar coordinates**, 395  
converting to rectangular, 396
- Polar equation**, 396
- Polar graph**, 396
- Pole**, 395
- Polynomial(s)**, 53–93  
coefficients, 53  
constant, 53  
degree, 53  
leading coefficient, 53  
leading term, 53  
terms, 53  
zero, 53
- Polynomial equations**, 64–65, 75–91  
double root, 64  
gaining a root, 33  
losing a root, 32  
root, 53  
approximating, 75–77  
solving by factoring, 80–83  
triple root, 65
- Polynomial functions**, 53  
derivative, 758  
graphing, 62–65, 764–767  
maximums and minimums, 68–70
- Polynomial inequalities**, 100–117  
graphing, 100–102, 104–106  
in one variable, 100–102  
linear, 95–96  
in two variables, 104–106
- Population**, 669
- Power series**, 733–735  
interval of convergence, 733
- Probability**, 597–637  
applications  
business, 635  
consumer economics, 635  
expected value, 630–635  
farming, 634  
insurance, 627, 634  
involving conditional probability, 624–629  
manufacturing, 623, 627  
medicine, 628  
meteorology, 602  
occupational safety, 602  
sports, 612, 613, 617  
test taking, 634  
conditional, 607  
empirical, 597  
of an event, 598  
theoretical, 598
- Problem solving**, xviii–xix  
*See also* Activities, Applications, Investigations, and Projects.
- Product, roots of negative numbers**, 26
- Programmable calculator**, 70, 72–74 (Exs. 1–9), 76, 480–481, 483–484 (Exs. 29–31), 485 (Exs. 33–35), 618 (Exs. 19, 20), 729–733, 745–748, 749 (Ex. 3a), 750–755, 783 (Exs. 3, 7)
- Projects**  
Chaos in the Complex Plane, 750  
Newton's Law of Cooling, 210  
The People of Mathematics, 366  
Preference Testing, 714  
Uniform Circular Motion, 470
- Proofs**  
by coordinate geometry, 214–217  
by mathematical induction, 510–512
- p-series**, 735
- Ptolemy's theorem**, 357 (Ex. 33b), 374 (Ex. 44)
- Pythagorean relationships**, 317
- Q**
- Quadrant**, 1
- Quadratic equations**, 30–33  
discriminant, 31  
extraneous root, 33  
finding from roots, 87  
losing or gaining a root, 32–33  
methods of solving, 30–33  
root, 30  
standard form, 30
- Quadratic functions**, 37–51  
maximums and minimums, 37, 68–69
- Quadratic polynomial**, 53
- Quartic polynomial**, 53
- Quartiles**, 649
- Quintic polynomial**, 53
- Quotient theorem for limits**, 720
- R**
- Radian**, 258
- Range**  
of data, 649  
of a function, 20, 119
- Rational functions**, 726  
graphing, 726–728
- Rational root theorem**, 82
- Ray**  
initial, 257  
terminal, 257
- $r \text{ cis } \theta$** , 403
- Reading**, 246, 623  
exercises focused on, 71, 91, 188, 262, 286, 333, 346, 361, 401, 405, 423, 568, 574, 585, 604, 609, 611, 625, 632, 658, 678, 706, 723, 735, 741
- Real number system**, 26  
properties of real numbers, 788
- Reasoning, mathematical**  
constructing proofs, *see for example*, 12 (Ex. 29), 137 (Exs. 35, 36), 218–219 (Exs. 1–18), 354 (Exs. 23, 24), 451 (Exs. 29, 30), 515 (Ex. 14)  
constructing valid arguments, *see for example*, 29 (Exs. 42–46), 93 (Ex. 14), 149 (Cl. Exs. 10, 11), 279 (Exs. 16, 18), 322 (Ex. 39), 535 (Ex. 19)

- following a logical argument,  
*see for example*, 90 (Ex. 29), 267 (Ex. 25), 385 (Ex. 46), 446 (Ex. 30), 571 (Ex. 27), 660 (Ex. 18)
- formulating counterexamples,  
*see for example*, 290 (Exs. 19–21), 526 (Ex. 11), 537 (Ex. 28), 767 (Ex. 3)
- judging the validity of an argument, *see for example*, 34 (Exs. 16, 19), 199 (Ex. 21), 504 (Ex. 40), 603 (Exs. 7, 8), 710 (Ex. 2)
- making and testing conjectures,  
*see for example*, 99 (Ex. 36), 201 (Exs. 35, 36), 458 (Ex. 39), 513 (Exs. 19–21), 701 (Exs. 9–15)
- Reciprocal, negative**, 10, 15 (Ex. 4)
- Reciprocal relationships**, 282
- Recursion equation (formula)**, 479
- Reference angle**, 275
- Refresher Exercises**  
 for use before: Ch. 1, 854; Ch. 2, 854; Ch. 3, 855; Ch. 4, 855; Ch. 5, 856; Ch. 6, 856; Ch. 7, 857; Ch. 8, 857; Ch. 9, 858; Ch. 10, 858; Ch. 11, 859; Ch. 12, 859; Ch. 13, 860; Ch. 14, 860; Ch. 15, 861; Ch. 16, 861; Ch. 17, 862; Ch. 18, 862; Ch. 19, 863; Ch. 20, 863
- Relation**, 121
- Relationships among trigonometric functions**  
 cofunction, 282, 318  
 negatives, 317  
 Pythagorean, 317  
 reciprocal, 317
- Remainder theorem**, 59
- Repeating decimals**, 501–502
- Research**, 124, 156, 157, 179, 262, 307, 316, 343, 364, 401, 522, 543, 577, 646, 653, 674, 695, 704, 707, 708, 709
- Resultant force**, 420
- Reviews**  
 cumulative, 116, 254–255, 416–417, 562–563, 711–712, 783
- of geometry, 784–787  
*See also* Chapter Summaries, Chapter Tests, and Refresher Exercises.
- Revolution**, 257
- Richter scale**, 202–203
- Root(s)**, 53  
 bounds for real, 835–837  
 of complex numbers, 412–413  
 double, 64  
 finding by factoring, 80–83  
 gaining a, 33  
 losing a, 32  
 nature of, in polynomial equations, 85–86  
 of quadratic equation, 30  
 sum and product of, 86  
 triple, 65  
 using technology to approximate, 75–77
- Rule of 72**, 182
- S**
- Sample**, 669
- Sample points**, 600
- Sample space**, 598
- Sampling**, 669–671  
 convenience, 670  
 judgment, 670  
 nonprobability, 670  
 probability, 670  
 by questionnaire, 670  
 random, 670  
 stratified random, 671
- Sampling theory**, 669
- SANDOVAL VALLARTA, MANUEL**, 499
- Scalar**, 422, 518
- Scalar multiplication**  
 of a matrix, 518  
 of a vector, 422
- Scatter plot**, 653 (Ex. 8), 683
- Scientific calculator**, 13, 29, 74, 150, 187, 498, 506, 618
- Secant**, 282  
 graph, 284
- Second**, 257
- Sector of circle**, 263  
 arc length, 263  
 area, 263
- Seed**, 737
- Sequences**, 181, 473–481, 493–496  
 applications  
 annuity, 483 (Ex. 29)  
 area under a curve, 498 (Exs. 35, 36)
- compound interest, 478 (Ex. 52)  
 half-life, 478 (Ex. 51)  
 height of dropped ball, 504 (Exs. 33, 34)  
 retirement fund, 483 (Ex. 30)
- arithmetic, 473  
 geometric, 473  
 infinite, 493–496  
 limit of, 181, 493  
 of partial sums, 500  
 recursive definition, 479
- Series**, 486–488, 500–507  
 divergent, 501  
 finite, sum of, 486–488  
 harmonic, 735  
 infinite, sum (or limit) of, 500  
 interval of convergence, 501  
 $p$ -, 735  
 power, 733  
 repeating decimal as, 501–502
- Set(s)**  
 complement of, 566  
 intersection, 565  
 Julia, 751  
 Mandelbrot, 754–755  
 union, 565
- Sigma notation**, 506
- Sign graph**, 100
- Signs of trigonometric functions by quadrant**, 282
- Sine(s)**, 268  
 applications, 308–312  
 domain and range, 270  
 evaluating, 275–276  
 fundamental period, 271  
 graph, 278  
 law of, 345  
 special angles, 276–277
- Sine waves**, 301–304, 308–312  
 applications  
 daylight curves, 314  
 electricity, 303–304, 306, 307  
 height of rotating point, 316  
 motion of pendulum, 308  
 music, 308  
 periodic temperature change, 315, 316  
 spring displacement, 307  
 tides, 314  
 time of sunset, 315
- axis of, 310  
 equations, 309

**Slope of a curve**, 757

**Slope of a line**, 7–10

- determined from equation, 8
- horizontal, 8
- negative, 8
- nonvertical, 8
- perpendicular, 10
- positive, 8
- vertical (no slope), 8

**Solution(s)**

- common, 3
- exponential equation, 203–204
- higher-degree equation, 75–77
- linear equation, 1
- linear inequality, 95–98
- logarithmic equation, 193
- polynomial inequality, 100–106
- simultaneous equations, 3, 221
- systems of second-degree equations, 242–244
- triangle, 331–333
- trigonometric equation, 295, 303, 323–326, 386–388

**Sphere**, 448

- Dandelin, 251 (Ex. 21b)

**Spiral**

- of Archimedes, 401 (Ex. 29)
- equiangular, 409, 411 (Exs. 17, 18)
- hyperbolic, 401 (Ex. 30)
- logarithmic, 401 (Ex. 31), 411 (Ex. 18)

**Square matrix**, 530

**Square root of a negative number**, 26

**Standard deviation**, 654

**Standard value**, 656

**Statistic**, 653

**Statistics**, 639–681

- applications
  - consumer economics, 646, 659
  - ecology, 673, 674
  - economics, 648
  - geography, 647
  - horticulture, 667
  - manufacturing, 667, 668, 673, 680
  - meteorology, 644
  - nutrition, 647, 652
  - operations research, 667
  - physiology, 645, 668
  - polling, 678–680
  - sports, 644, 668
  - transportation, 667

descriptive, 639–669

inferential, 639, 669–681

**Stem-and-leaf plot**, 639

**Stratum (strata)**, 671

**STROUD-LEE, AGNES**, 713

**Summand**, 506

**Symbols, list of**, 856–857

**Symmetry**

- axis of, 133
- point of, 133
- tests for, 134

**Synthetic division**, 58–60

**Synthetic substitution**, 54, 59

**Systems of second-degree equations**, 242–244

## T

**Tables**

- common logarithms, 803–804
- $e^x$  and  $e^{-x}$ , 821
- frequency, 640
- natural logarithms, 802
- squares/square roots, 801
- standard normal, 664
- trigonometric
  - in decimal degrees, 810–817
  - in degrees and minutes, 805–809
  - in radians, 818–821
- using, 276, 800

**Tangent**, 282

- graph, 284

**Tangent line**, 41 (Exs. 25, 26),

42 (Ex. 41), 761, 842–845

**TAUSSKY-TODD, OLGA**, 537

**Technology**, xxiv–xxxiii

- See also Computer exercises and uses, Graphing calculator, Programmable calculator, and Scientific calculator.

**Theorems**

- binomial, 591
- binomial probability, 614
- complex conjugates, 85
- composition of functions and multiplication of matrices, 555
- convergence and divergence of the  $p$ -series, 735
- De Moivre's, 408, 593 (Exs. 29, 30)
- factor, 60
- for finding derivatives, 759–760

fundamental, of algebra, 85

limits, 494

linear transformations and determinants, 553

product of two complex numbers in polar form, 404

Ptolemy's, 357 (Ex. 33b), 374 (Ex. 44)

quotient, for limits, 720

rational root, 82

remainder, 59

roots of polynomial equations, 85–86

slope and inclination, 296

slope of a line, 8

slopes of parallel lines, 9

slopes of perpendicular lines, 10

sum of finite series, 487, 488

sum of infinite series, 500

transformation matrices, 554

**Three dimensions**, vectors in, 446–449

**TING, SAMUEL CHAO** ㄊㄩㄥ, xxxvii

**Transformation of the plane**, 551

**Transformation matrices**, 555

**Transition matrices**, 543–545

**Translating graphs**, 141

**Transpose of a matrix**, 517

**Tree diagram**, 572

**Triangle(s)**

- altitude, 17 (Ex. 23)
- area formulas, 6 (Ex. 32), 339
- centroid, 17 (Exs. 21, 24)
- circumcenter, 17 (Exs. 22, 24)
- median, 17 (Ex. 21)
- orthocenter, 17 (Exs. 23, 24)
- situations for using law of sines, cosines, 352
- solving, 331–344
- solving right, 331–333
- solving using law of cosines, 350–352
- solving using law of sines, 345–347

**Triangle trigonometry**, 331–365

right, 331–333

**Triangular numbers**, 593 (Ex. 25)

**Trigonometric equations**, 295–298, 323–326

**Trigonometric formulas**, 369–393, 789

**Trigonometric functions**, 256–293  
 cofunction, 282, 318  
 definitions, 268, 282  
 identities, 317–326  
 inverse, 286–288  
 negatives, 317  
 period and amplitude, 302  
 Pythagorean relationships, 317  
 reciprocal, 317  
 signs in quadrants, 282  
**Trigonometric identities**, 318–320  
**Trigonometric inverses**, 286–288  
 applications  
   aviation, 338  
   optics, 300  
   space science, 338  
 domain and range, 287, 288  
 graphs, 287, 288  
**Trigonometry applications to navigation and surveying**, 359–361

## U

**Union of sets**, 565

## V

**Variability**, 653–657  
**Variables, functions of two**, 151–153  
**Variance**, 654  
**Vector sum**, 419  
**Vectors**, 419–469  
   absolute value, 419  
   addition, 419–421  
   algebraic representation, 426–428  
   angle between nonzero, 442  
   applications  
     aviation, 424, 426  
     navigation, 424, 426

  physics, 424, 425, 429, 438, 439, 445  
   sports, 424  
   component form, 426  
   cross product, 465  
   direction, 432  
   dot product, 441, 442  
   magnitude, 419  
   multiples, 421–422  
   negative of, 421  
   operations, 427  
   as ordered pairs, 426  
   orthogonal, 441  
   parallel, 441  
   perpendicular, 441  
   properties, 448, 465  
   scalar product, 441  
   subtraction, 421  
   sum, 419  
   in three dimensions, 446–449, 465–466  
   zero, 421

### Velocity

  average, 775  
   instantaneous, 775

### Venn diagram

### Vertex (vertices)

  ellipse, 227  
   hyperbola, 233  
   parabola, 37, 238

### Vertical-line test

### Visual glossary of functions

  858–859

### Visual thinking

  exercises focused on, 45, 89, 128, 129, 135–136, 138, 153, 186, 244, 245, 261, 279, 281, 304, 322, 337, 343, 354, 361, 373, 378, 385, 390, 406, 407, 414, 430, 476, 483, 592, 669, 688, 697, 724, 725, 731, 732, 735, 741, 749, 761, 762, 767, 768, 778, 779–780, 783

## W

### Web diagram

**WHITMAN, ROBERT K.**, xxxvi

### Wrapping function

  274 (Ex. 43)

### Writing, exercises focused on

  22, 24, 50, 78, 93, 99, 112, 115, 124, 167, 209, 223, 240, 253, 267, 274, 285, 290, 293, 306, 316, 322, 329, 365, 393, 407, 415, 430, 431, 436, 469, 477, 483, 503, 504, 515, 521, 522, 535, 543, 548, 558, 559, 560, 577, 595, 636, 645, 647, 652, 681, 690, 701, 703, 706, 710, 729, 747, 748, 749, 782, 783

## X

**x-axis**, 1

**x-coordinate**, 1

**x-intercept**, 1

**xy-coordinate plane**, 446

**xz-coordinate plane**, 446

## Y

**YAMASAKI, MINORU**, 117

**y-axis**, 1

**y-coordinate**, 1

**y-intercept**, 1

**yz-coordinate plane**, 446

## Z

### Zero

  of a function, 19

  as polynomial, 53

**Zero matrix**, 530

**ZUNIGA, MARTHA C.**, xxxiv



## Answers to Selected Exercises

**Try This, pages xxvi–xxxiii** 2.b. A sideways “S.”

3.a.  $-2.3$  b.  $(-1.28, -2.57)$  4.a.  $-0.9$

b.  $-1.8, 2.8$  5. 2 6.  $1.25$  s

**Class Exercises, pages 4–5** 1. 10;  $(4, 3)$  2.  $2\sqrt{5}$ ;

$(5, 4)$  3.  $6\sqrt{2}$ ;  $(0, 1)$  4. 8;  $(7, -5)$  5.  $\overline{AM}$ ;

$(3, 4)$ ;  $\overline{MB}$ :  $(5, 6)$  6. a, b, d 7.a. For example,

$(0, 4)$ ,  $(1, 4)$ ,  $(2, 4)$  b.  $y = 4$  8.a. For example,

$(8, -2)$ ,  $(8, 0)$ ,  $(8, 8)$  b.  $x = 8$  9.  $(0, -6)$ ,

$(4.5, 0)$  10.a.  $(2, 3)$  b.  $(2, 3)$  c. They are =.

**Written Exercises, pages 5–7** 1. 10;  $(4, 4)$  3. 17;

$(-\frac{1}{2}, 1)$  5.  $\frac{13}{2}$ ;  $(-\frac{3}{4}, \frac{3}{2})$  7. 5;  $(4.8, -0.3)$

9. a, c 11. Area of  $\triangle OPQ = 3$  13.  $(5, 3)$ ;

$y = 3$ ,  $x = 5$  15.  $(3, 0)$  17.  $(-\frac{5}{4}, -\frac{7}{4})$

19. Parallelogram 21. Parallelogram

25.a.  $PA = PB = 5$  b.  $-2$  27.  $(9, 0)$ ,  $(-15, 0)$

29.  $(10, 3)$  31.a.  $\angle C$  is a rt.  $\angle$ . b.  $(0, 5)$

33.  $A(2, -1)$ ,  $B(12, 7)$ ,  $C(8, 11)$

**Class Exercises, page 11** 1.  $\frac{7}{5}$  2. 6 3.  $-1$

4.  $\frac{1}{2}$  5. 0 6. No slope 7.  $m = 3$ ; y-int. 4

8.  $m = \frac{3}{5}$ ; y-int.  $-3$  9.  $m = -\frac{4}{3}$ ; y-int. 3

10.  $m = 0$ ; y-int.  $-2$  11. The change in  $x$ -values is zero, so def. of slope would involve division by zero.

12.  $\frac{1}{2}$  13.a.  $-\frac{3}{2}$  b.  $\frac{2}{3}$  14.a. Slope of  $\overline{AB}$

and  $\overline{CD} = \frac{4}{3}$ ; slope of  $\overline{BC}$  and  $\overline{DA} = -\frac{3}{4}$

b.  $(\frac{4}{3})(-\frac{3}{4}) = -1$

**Written Exercises, pages 11–13** 1.  $\frac{3}{5}$  3.  $-\frac{2}{3}$  5. 0

7.  $-4$  9.  $-1$  11.  $m = 3$ ; y-int. 5 13.  $m = 2$ ;

y-int.  $-4$  15.  $m = \frac{11}{3}$ ; y-int. 0 17.  $a \parallel b$ ,  $a \perp c$ ,  $b \perp c$

19. Slope of line 1 = 1, slope of line 2 =  $-1$ .

21.a. 0 b.  $\frac{17}{2}$  23.a. Slope of  $\overline{AB}$  and  $\overline{CD} = \frac{5}{3}$ ;

slope of  $\overline{BC}$  and  $\overline{AD} = \frac{1}{3}$  b. Both have midpt.

$(2, 0)$ . 25. Label the four points  $A$ ,  $B$ ,  $C$ , and  $D$ .

Slope of  $\overline{BC}$  and  $\overline{AD} = \frac{2}{3}$ ; slope of  $\overline{AB} \neq$  slope of

$\overline{CD}$ .  $AB = CD = \sqrt{26}$ . 29.a. Slope of  $l_1 = \frac{BC}{AC}$ ;

slope of  $l_2 = \frac{EF}{DF}$  31.a.  $OA = \sqrt{m_1^2 + 1}$ ,

$OB = \sqrt{m_2^2 + 1}$ ,  $AB = |m_2 - m_1|$  33.  $R(-3, 6)$ ,  
 $S(-1, 0)$  or  $R(-9, 4)$ ,  $S(-7, -2)$

**Calculator Exercises, page 13**

1. Area  $\triangle ABC = 6.37$  2. Midpt.  $M$  of  $\overline{BC}$  is

$(1.0, 1.95)$ .  $AM = BM = CM = \sqrt{7.9625}$

**Class Exercises, page 16** 1.  $y = \frac{5}{3}x - 2$

2.  $\frac{y-6}{x+4} = 3$  3.  $\frac{y-2}{x-1} = -\frac{1}{4}$  4.  $-x + \frac{y}{6} = 1$

5.  $\frac{y+1}{x-2} = \frac{2}{3}$  6.a.  $y = mx$  b.  $\frac{y}{x} = m$  c. No sol.

**Written Exercises, pages 16–18 Exs. 1–16:**

Ans. are given in general form. 1.  $2x + y = 8$

3.  $2x - y = -4$  5.  $2x - 3y = -14$  7.  $y = -7$

9.  $x = 2$  11.  $0.25x - y = -1.8$  13.  $x - 2y = 12$

15.  $2x - y = -8$  19.a.  $3x - 4y = -9$

b.  $3x - 4y = -9$  c. Yes; yes 21.a.  $y = 3$ ,

$y = x + 1$ ,  $y = -2x + 7$  b.  $G(2, 3)$

23.a.  $x = 6$ ,  $x - y = 0$ ,  $x + 2y = 18$  b.  $O(6, 6)$

25. The first  $\triangle$  27.a.  $3\sqrt{10}$ ;  $x + 3y = 3$

b.  $3x - y = 19$  c.  $(6, -1)$  d.  $2\sqrt{10}$  e. 30

29.a.  $bx - ay = bx_1 - ay_1$  d. 5

31.b.  $x + y = -2$ ,  $7x - 7y = -10$

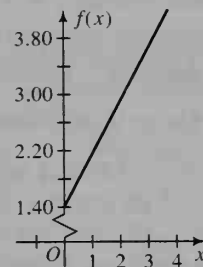
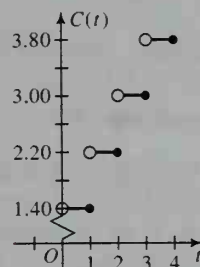
**Class Exercises, pages 21–22** 1.a.  $f(3) = 5$ ;

$f(-3) = -25$  b. 2 c.  $(0, -10)$ ,  $(2, 0)$

d. At the intersection with the horiz. axis 2. a, b

3.  $t(s) = -\frac{s}{2} + 4$

4.a, b.



5.  $f(a) - f(b) = m(a - b)$ ;  $f(12.8) - f(12.3) = 25$

**Written Exercises, pages 22–25** 1.a.  $f(2) = 1$ ;

$f(-2) = -2$  b.  $\frac{2}{3}$  3. No 5.a. 0

b.  $g(x) = 0x + 2$ ,  $h(x) = 0x - 1$  7.a. 1.5 b.  $\frac{4}{3}$

c. x-int.:  $\frac{4}{3}$ , y-int.:  $-2$  9.  $f(n) = -2n + 12$



11. b.  $f(x) = 2x + 3$  13. a.  $D(t) = -0.16t + 4$   
 b.  $\approx 25$  min 15. a.  $C(m) = 0.15m + 280$  b. 0.15  
 17. a. 51.8 mi/h;  $d(t) = 185 + 51.8t$   
 b.  $\{t | 0 \leq t \leq 4.25\}$  c. Slope is 51.8. d. They are =.  
 21. b. Answers may vary:  $C(m) = 1.6m + 1.2$   
 23.  $f(x) = \frac{25}{31}x + \frac{840}{31}$

**Class Exercises, page 28** 1.  $7 + 2i$  2.  $-1 - 5i$   
 3. 10 4.  $28 - 6i$  5. 17 6. 34 7. 3  
 8.  $a^2 + b^2$  9.  $x = 2, y = -5$  10. If  $n$  is a pos.  
 multiple of 4,  $i^n = 1$ .  
 11. a.  $\sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6 = \sqrt{36} = \sqrt{4 \cdot 9}$   
 b.  $\sqrt{-4} \cdot \sqrt{9} = 2i \cdot 3 = 6i = \sqrt{-36} = \sqrt{-4 \cdot 9}$   
 c.  $-6 \neq 6$  d. No

**Written Exercises, pages 28–29** 1.  $7i$  3.  $-3$   
 5. 2 7.  $-2 + 5i$  9.  $8 + 32i$  11. 37 13. 30  
 15.  $31 - 34i$  17.  $-9 - 40i$  19.  $\frac{2}{29} - \frac{5}{29}i$   
 21.  $\frac{12}{13} + \frac{5}{13}i$  23.  $\frac{19}{51} + \frac{10\sqrt{2}}{51}i$  25.  $-5i$  27.  $i$   
 29.  $i$  31.  $i$  33.  $x = 2, y = -3$   
 35.  $(a + bi) + (a - bi) = (a + a) + (b - b)i = 2a$   
 37. a. Calculate  $79^2$  and compare with  $6241$ .  
 b. Calculate  $(3 - i)^2$  and compare with  $8 - 6i$ .  
 39.  $\left[\frac{\sqrt{2}}{2}(1 + i)\right]^2 = \frac{1}{2}(2i) = i$  41. Real no.

**Calculator Exercise, page 29**  
 42,660,913 + 30,082,416i

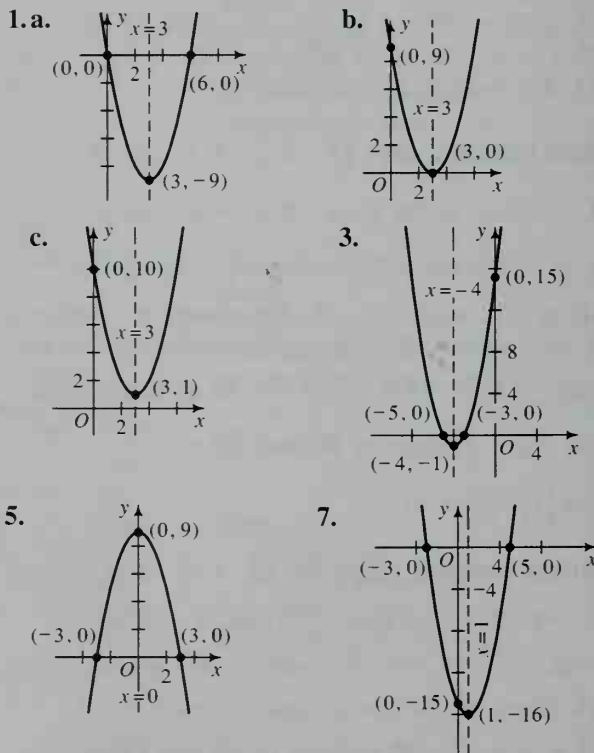
**Class Exercises, page 34** 1. 16 2. 25 3.  $\frac{49}{4}$   
 4.  $a^2$  5. Complete square 6. Factor 7. Factor  
 8. Quad. formula 9. Quad. formula  
 10. Complete square 11.  $\sqrt{b^2 - 4ac}$  is imag. if  
 $b^2 - 4ac < 0$ . 12.  $\sqrt{b^2 - 4ac} = 0$  if  
 $b^2 - 4ac = 0$ , so (double) root is  $-\frac{b}{2a}$ .  
 13.  $\sqrt{b^2 - 4ac}$  is a whole no. if  $b^2 - 4ac$  is the  
 square of an integer. 14. No;  $b = -\sqrt{5}$  isn't an  
 integer. 15. Yes; the derivation of the quad.  
 formula puts no restrictions (other than  $a \neq 0$ ) on  
 the coefficients. 16. Dividing by  $x + 4$  causes the  
 root  $-4$  to be lost. 17.  $\pm 6$  18.  $2, \frac{1}{4}$   
 19. Squaring introduces the extraneous root  $-2$ ,  
 which does not satisfy the original equation.

**Written Exercises, pages 35–36** 1.  $-1, \frac{7}{3}$   
 3.  $2, -\frac{9}{2}$  5. 45,  $-35$  7. 34,  $-26$  9.  $-3 \pm i$

11.  $\frac{-1 \pm \sqrt{6}}{5}$  13.  $2 \pm i$  15. 2, 8 17.  $\frac{1}{2}, -\frac{7}{4}$   
 19.  $-3, \frac{13}{3}$  21.  $1, -\frac{5}{4}$  23.  $0, \frac{1}{4}$  25. 0 27. 3  
 29. 16 31. 6 33.  $\frac{-7 + \sqrt{1169}}{2}$  35. a.  $64 - 16k$   
 b.  $k = 4$  c.  $k < 4$  d.  $k > 4$  e. For example,  $k = 0$ ,  
 $k = -12, k = -5$  37.  $2\sqrt{2}, \frac{\sqrt{2}}{2}$  39.  $3i$  41.  $\frac{2}{9}$   
 43. (41) 0.22 (42) No sol.; No 45. a. 1, 2, 5, 7,  
 12 b. For example, 155, 222

**Class Exercises, page 40** 1. a. Upward  
 b. None 2. a. Downward b. One  
 3. a. Downward b. Two 4. (5, 4) 5.  $(-2, -3)$   
 6. (0,  $-10$ )

**Written Exercises, pages 41–43**



9.  $V(1, -8)$ ; axis of sym.  $x = 1$ ; x-ints.  $1 \pm 2\sqrt{2}$ ;  
 y-int.  $-7$  11.  $V(1, -2)$ ; axis of sym.  $x = 1$ ;  
 x-ints.  $\frac{2 \pm \sqrt{2}}{2}$ ; y-int. 2 13.  $V(-4, 0)$ ; axis of  
 sym.  $x = -4$ ; x-int.  $-4$ ; y-int. 8 15.  $V(2, -1)$ ;  
 axis of sym.  $x = 2$ ; x-ints.  $\frac{3}{2}, \frac{5}{2}$ ; y-int. 15

17.  $V(-1, 16)$ ; axis of sym.  $x = -1$ ; no  $x$ -ints.;  $y$ -int. 17  
 19.  $V(2, -9)$ ; axis of sym.  $x = 2$ ;  $x$ -ints.  $\frac{1}{2}, \frac{7}{2}$ ;  $y$ -int. 7  
 21.  $(2, 0)$  23.  $(-1, -5), (-6, 0)$   
 25.  $y = 2x - 5$  27.  $f(x) = -3x^2 + 3x + 6$   
 29.  $f(x) = -\frac{1}{2}x^2 + 4x$  31.  $f(x) = \frac{7}{4}(x - 3)^2 - 5$   
 33.  $f(x) = \frac{3}{8}(x - 4)^2$

35.a.  $y = -\frac{29}{3600}(x - 60)^2 + 30$  b. Yes

37.a.  $f(1 + k) = f(1 - k) = 2k^2 + 5$  41.a. 2

b.  $y = 2x - 1$

**Class Exercises, page 45** 1. Quad. 2. Linear  
 3. Quad. 4. Other

### Written Exercises, pages 45–48

1.  $f(x) = 2x^2 + 3x + 5$  3.  $f(x) = \frac{1}{2}x^2 + 5x + 6$

5.b. Yes; \$8.75 c. No

7.a.  $D(s) = -0.02s^2 + 1.6s + 2$  b. 21.5 mi

c. 55 mi/h 9.a. 8 in. b.  $f(x) = \frac{1}{2}x^2 - 5x + 20$

11.a.  $h(t) = -4.9t^2 + 14t + 30$  b.  $\frac{10}{7}$  s later

c.  $\frac{30}{7}$  s later 13.  $\frac{10}{7}$  s

15.a.  $R(x) = -100x^2 + 1400x + 24,000$  b. \$17

17.a.  $f(t) = \pi L\left(\frac{D}{2} - t\right)^2$

b.  $f\left(\frac{1}{4}D\right) = \frac{1}{4} \cdot f(0) = \frac{1}{16}\pi LD^2$

**Chapter Test, pages 50–51** 1.a.  $2\sqrt{17}$

b.  $(-3, -2)$  2.  $-3$  3.  $\left(-\frac{1}{2}, 1\right)$

4.  $m = 2$ ,  $y$ -int.:  $-\frac{7}{2}$  5.  $b \parallel c$ ,  $a \perp b$ ,  $a \perp c$

6.  $x + 3y = 0$  7.  $4x + 3y = 35$

8.  $5x + 3y = -15$  9.  $x = 4$  10.a. Find the slope  $m$  and the midpt.  $M$  of the seg. Write an eq. of the line that contains  $M$  and has slope  $-\frac{1}{m}$ .

b.  $3x - 4y = -4$  11.a.  $f(t) = \frac{1}{9}t$

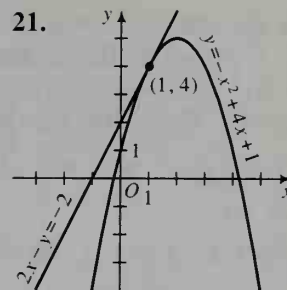
b. 207 min = 3 h 27 min 12.  $3i\sqrt{2}$

13.  $-5 + 12i$  14.  $\frac{2}{13} - \frac{3}{13}i$  15.  $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

16.  $7 + 13i$  17.  $i$  18.a.  $1, -\frac{2}{7}$  b.  $2 \pm \sqrt{13}$

c.  $-1 \pm i$  19. disc. =  $-20$ ;  $x = \frac{1 \pm i\sqrt{5}}{3}$

- 20.a.  $V(1, 8)$ ;  
 axis of sym.  $x = 1$ ;  
 $x$ -ints.  $-1, 3$ ;  
 $y$ -int. 6  
 b.  $V(3, -4)$ ;  
 axis of sym.  $x = 3$ ;  
 $x$ -ints.  $1, 5$ ;  
 $y$ -int. 5



22.a.  $f(x) = 3x^2 - 11x + 35$  b. \$1.05

**Class Exercises, page 55** 1. Quad.;  $5x^2$ ; 5; 2

2. Quartic;  $5x^4$ ; 5; 4 3. Cubic;  $-4x^3$ ;  $-4$ ; 3

4. Quintic;  $-2x^5$ ;  $-2$ ; 5 5. 23 6.  $-13$

7.  $18n^2 + 5$  8.  $2n^2 + 12n + 23$  9.  $-5, 3$

10.  $\pm 4, \pm 5i$  11. 3 12.  $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

13. 0,  $-108$

**Written Exercises, pages 56–57** 1. Yes;  $\frac{17}{3}$

3. Yes; none 5. No;  $-1, 4$  7. Yes;  $-3, 0, 3$

9. Yes;  $\frac{-q \pm \sqrt{q^2 - 4pr}}{2p}$  11. Yes;  $-1, 0$

13.c. The first is undef. for  $x = -2$ ; the second is never undef.

15.a. 13 b.  $-2 - 10i$  c.  $1 - i$  d.  $18a^2 - 15a + 6$

17.a.  $\frac{79\sqrt{2}}{27}$  b.  $-12i\sqrt{3}$  c.  $\frac{x^3}{27} - 3x$

d.  $x^3 - 9x^2 + 18x$  19.a. 17 b. 75 c.  $-75$

d.  $-183$  21.a. 3 b.  $-\frac{7}{3}$  23.  $-47.5$  25.  $\pm\sqrt{3}$

27.a. 1, 4 b.  $f(x) = a(x - 2)(x - 3)$  for any real  $a \neq 0$  c.  $g(x) = ax(x - 2)(x - 3)$  for any real  $a \neq 0$  29.a. 7 b. 7

**Computer Exercises, page 58** 1. Times may vary. The second program is faster. 2. Faster than the first, but slower than the second

**Class Exercises, page 60** 1.a. 6 b. 4

2.a.  $x^4 - 8x^2 + 5x - 1$ ;  $x + 3$

b.  $x^3 - 3x^2 + x + 2$ ;  $-7$  c.  $-7$

3.a.  $2x^3 - 5x^2 + 8x - 4$ ;  $x - \frac{1}{2}$  b.  $2x^2 - 4x + 6$ ;

$-1$  c.  $-1$  4. Yes; use the quotient of  $P(x) \div (x + 1)$  5. If  $P(a) = 0$ , then by the remainder theorem, the remainder of  $P(x) \div (x - a)$  is 0. Thus  $(x - a)$  is a factor of  $P(x)$ . The converse is also true.

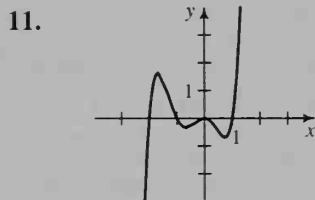
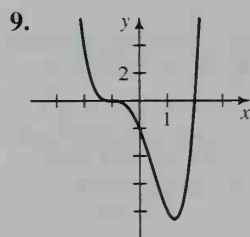
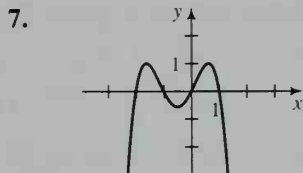
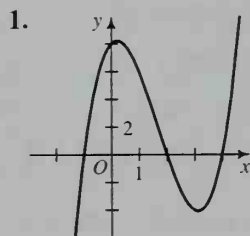
**Written Exercises, page 61** 1.a.  $-4$  b.  $-2$  c. 16

d.  $-16$  3.  $x^2 - x + 4$ ; 5

5.  $x^3 - 3x^2 + 3x + 2$ ; 0  
 7.  $x^4 + 3x^3 + 10x^2 + 30x + 91$ ; 273  
 9.  $3x^2 - 8x + 21$ ;  $-41x + 1$  11.  $x - 1$  is;  $x + 1$  is not. 13. c 15.  $P(a) = a^n - a^n = 0$   
 17.  $2x^3 - x^2 + 7x + 7$  19.  $-1, \frac{1}{2}$  21.  $\frac{1}{2}, 4$   
 23.  $\pm i$

**Class Exercises, page 65** 3.(1) 2; double root  $x = 3$  (2) 3; double root  $x = 3$  (3) 3; double roots  $x = 0, x = 3$  (4) 1; triple root  $x = 2$  (5) 1; triple root  $x = 2$  (6) 2; triple root  $x = 1$   
 5.a.  $y = (x - 1)(x - 4)^2$  b.  $y = -x(x + 1)(x - 3)$   
 c.  $y = -(x - 1)^2$  d.  $y = x(x - 2)^3$

### Written Exercises, pages 66-68



13.  $x(x + 2)(x - 2)$  15.  $x^2(x + 1)(x - 1)$   
 17.  $(x + 1)(x - 1)^3$  19.b.  $(-1, 1), (0, 0), (1, 1)$   
 21.  $y = -(x + 3)(x + 1)(x - 1)$   
 23.  $y = (x + 3)^2(x + 1)(x - 1)$  25.b.  $P(2) = 24$ ,  $P(2.1) = 24.339$ ; highest point at  $x \approx 2.3$   
 27. Intersect at  $(-1, 3), (0, 0)$ , and  $(1, -3)$   
 29.  $y = -\frac{3}{8}(x + 4)(x - 2)$  31.  $y = -\frac{1}{3}x(x + 2)^2$   
 33.d. Impossible 35. 0, 1, 2, 3, 4, 5, 6  
 37.  $y = -2x^2(x - 3)$  39.  $-(x + 3)(x + 1)(x - 2)$

**Class Exercises, pages 70-71** 1.a. Min. b. 4

- 2.a. Max. b. 2 3.a. Min. b.  $\frac{3}{2}$  4.a. Max. b.  $-\frac{2}{3}$   
 5. No; for example,  $f(x) = x^3$ .

7. Change line 20 to: 20 FOR X = 3 TO 5 STEP 0.01

Change line 40 to: 40 IF V >= M THEN 70  
 Change line 80 to: 80 PRINT "MINIMUM VALUE IS APPROXIMATELY "; M; "AT X = "; X1

### Written Exercises, Quadratic Functions, pages 71-72

1.a.  $A(x) = x(120 - 2x)$ ; Dom.:  $\{x | 0 < x < 60\}$

b. 30 3. 433.5 m<sup>2</sup> 5.  $\frac{x_1 + x_2}{2}$

9.a. 6 s b. Dom.:  $\{t | 0 \leq t \leq 6\}$  c. 45 m

11. \$2.40 13. Width = 9, height = 6

### Written Exercises, Cubic Functions, pages 72-74

1.b. Dom.:  $\{x | 0 < x < 4\}$  c.  $x \approx 1.64$  cm; max. vol.  $\approx 82.98$  cm<sup>3</sup> 3.b. Dom.:  $\{x | 0 < x < 2.5\}$

c.  $x \approx 1.67$  ft; max. vol.  $\approx 9.26$  ft<sup>3</sup>

5.a.  $V(x) = \frac{x^2(25 - x)}{4\pi}$ ; Dom.:  $\{x | 0 < x < 25\}$

b.  $x \approx 16.67$  cm; max. vol.  $\approx 184.21$  cm<sup>3</sup>

7a.  $V(x) = 2\pi x(25 - x^2)$ ; Dom.:  $\{x | 0 < x < 5\}$

b.  $x \approx 2.89$ ; max. vol.  $\approx 302.30$  9.  $r \approx 3.33$ ;  $h \approx 3.33$ ; max. vol.  $\approx 116.36$

### Calculator Exercises, page 74

1. 17.6 cm for circle, 22.4 cm for square 2. Use all of the wire for the circle.

### Class Exercises, page 77

1. Between 1 and 2  
 2. Between 1 and 2 3. Between  $-1$  and  $0$ , or  $2$  and  $3$   
 4. Between  $-4$  and  $-5$ , or  $-1$  and  $0$ , or  $0$  and  $1$   
 5. Ans. may vary.  $x \approx -2.2, 0.4, 2.8$   
 6. There is no change in sign at a double root.

### Written Exercises, pages 78-79

1. 1.5  
 3.  $-1, -0.7, 2.7$  5.  $\pm 2.2$  7. 5.7 9. 0, 1.6  
 11. Quot.:  $x^2 - 3x + 1$ ;  $x = \frac{3 \pm \sqrt{5}}{2} \approx 0.38, 2.62$   
 15. 0.46

### Class Exercises, page 83

1.  $-4, \pm 3$  2.a. Let  $y = x^2$ . Then the eq. becomes  $y^2 - 5y + 4 = 0$ .  
 b. Solve for  $y$ ; substitute  $x^2$  for  $y$  and solve for  $x$ .  
 3. a. b 4.a. Factor out  $x$ ; factor (or use quad. formula) on the remaining quad. b. Cubic eqs. with no constant term. 5.a.  $\pm 4, \pm 2, \pm 1, \pm \frac{1}{2}$   
 b.  $\pm 5, \pm 1, \pm \frac{5}{2}, \pm \frac{1}{2}, \pm \frac{5}{3}, \pm \frac{1}{3}, \pm \frac{5}{6}, \pm \frac{1}{6}$   
 6. No. There must be a real root, but it may be irrational. 7.a.  $P(-2) = -1 < 0$ ;  $P(-1) = 1 > 0$   
 b. Irrational c.  $P(x) > 1$  for all  $x > 0$

**Written Exercises, pages 83–85** 1. b;  $\pm\sqrt{6}$ ,  $\pm i\sqrt{2}$  3. a;  $\pm 2$ ,  $\frac{16}{3}$  5. b;  $\pm\frac{\sqrt{6}}{2}$ ,  $\pm i\sqrt{5}$  7. a;  $-2$ ,  $\pm\sqrt{6}$  9. a;  $\frac{3}{5}$ ,  $\pm\frac{\sqrt{2}}{2}i$  11.  $-1$ ,  $2$ ,  $-1 \pm i\sqrt{3}$ ,  $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$  13.  $\pm 1$  15.  $\pm 1$ ,  $\pm 3$

17.  $-1$ ,  $\frac{1}{3}$ ,  $2$  19.  $-1$ ,  $\pm\sqrt{3}$  21.  $\frac{2 \pm \sqrt{2}}{2}$ ,  $-2$   
 23.  $f(x) = (x+3)(x+2)(x-3)$   
 25.  $h(x) = (x+2)(x-2)(x+\sqrt{7})(x-\sqrt{7})$   
 27.  $m(x) = (x-2)(2x+1)(2x+3)$  31. Intersect at  $(2, 2)$ ; tang. at  $(-1, 2)$  33. Tang. at  $(\sqrt{2}, 4)$ ,  $(-\sqrt{2}, 4)$  35. Intersect at  $(2, 24)$ ; tang. at  $(-3, 9)$   
 39. b.  $2$ ,  $-1 + \sqrt{6}$  41.  $7 \text{ cm} \times 10 \text{ cm} \times 16 \text{ cm}$

**Class Exercises, page 89** 1.  $\sqrt{3} - 2i$   
 2.  $-3 - \sqrt{2}$ ,  $2i$  3. a. sum =  $-\frac{5}{2}$ ; prod. =  $-\frac{3}{2}$   
 b. sum =  $\frac{1}{2}$ ; prod. =  $\frac{3}{2}$  c. sum =  $-2$ ; prod. =  $0$   
 d. sum =  $0$ ; prod. =  $-32$  4. The graph must cross the  $x$ -axis at least once. 5. The graph may lie entirely above or below the  $x$ -axis.

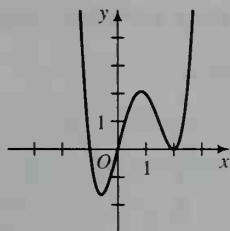
**Written Exercises, pages 89–91** 1. F 3. F 5. F  
 7. T 9. sum =  $\frac{3}{4}$ ; prod. =  $\frac{3}{2}$  11. sum =  $-\frac{5}{3}$ ; prod. =  $\frac{2}{3}$  Exs. 13, 15: Ans. may vary.

13.  $x^2 - 2x + 2 = 0$  15.  $x^2 - 6x + 7 = 0$   
 17.  $-4$ ,  $2 - i\sqrt{5}$  Exs. 19–23: Ans. may vary.  
 19.  $x^3 - 10x^2 + 33x - 34 = 0$   
 21.  $4x^3 - 12x^2 + 3x + 19 = 0$   
 23.  $x^4 - 10x^3 + 29x^2 - 10x + 28 = 0$   
 25.  $c = -6$ ,  $d = -4$  27.  $-27$  39.  $-3 - i$

**Chapter Test, page 93** 1. a. 20 b. 0 c.  $-2 - 9i$

2.  $-2$   
 3.  $x^3 + 2x^2 + 2x + 17$   
 4.  $-2$  (double)  
 5. See graph at right.

6.  $y = \frac{1}{2}(x+2)(x-1)(x-3)$   
 7.  $800 \text{ m}^2$



8. a. Use the location principle:  $P(1) = -2 < 0$  and  $P(2) = 7 > 0$ . b.  $-0.6$ ,  $1.6$  9.  $\frac{3}{2}$ ,  $\pm\sqrt{3}$   
 10.  $-\frac{2}{3}$ ,  $1$ ,  $\pm i\sqrt{2}$  12. sum =  $\frac{5}{2}$ ; prod. =  $-4$   
 13. Ans. may vary.  $x^3 + 5x^2 + 9x + 45 = 0$   
 14. If an imag. number is a double root, then so is

the number's conjugate. This gives the cubic eq. at least four roots, when it can only have three.

**Class Exercises, page 98** 1.  $x > -4$  2.  $x < 4$   
 3.  $x > -5$  4.  $-2 < x < 2$  5.  $3 < x < 7$   
 6.  $x < -4$  or  $x > -2$  7. f 8. a 9. e 10. c  
 11. b 12. d

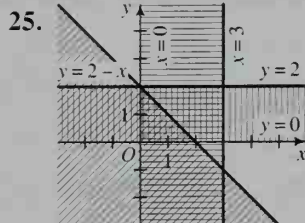
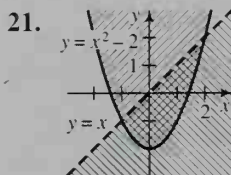
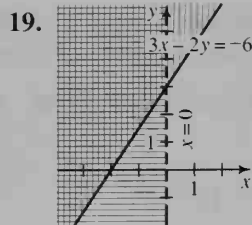
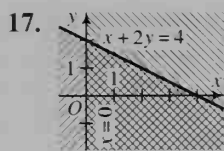
**Written Exercises, pages 98–99** 1.  $x < 3$  3.  $x < 0$   
 5.  $x \leq 11$  7.  $x > \frac{1}{5}$  9.  $x < 4$  11.  $-3 < x < 3$   
 13. No sol. 15.  $1 < x < 7$  17.  $x \leq -10$  or  $x \geq -4$  19.  $x = 4$  or  $x = 12$  21.  $-\frac{1}{2} \leq x \leq \frac{9}{2}$   
 23.  $-\frac{17}{4} \leq x \leq \frac{1}{4}$  25.  $15\frac{5^\circ}{9} \leq C \leq 18\frac{1^\circ}{3}$   
 27.  $1 \leq x \leq 3$  or  $5 \leq x \leq 7$  29.  $5 < x < 9$   
 31.  $x < 0$  or  $x > \frac{1}{2}$  33.  $0 \leq x \leq 2$

**Class Exercises, page 103** 1.  $-2 < x < 2$  or  $x > 4$   
 2.  $x \leq 1$  3.  $1 < x < 2$  or  $x > 4$  4.  $x < 2$  or  $4 < x < 7$  5.  $x \leq 5$  6. No sol.

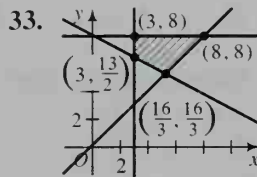
**Written Exercises, pages 103–104** 1.  $x < -4$  or  $x > 3$  3.  $1 < x < 2$  or  $x > 4$  5.  $x < 4$ ,  $x \neq 2$ ,  $x \neq 3$  7.  $-3 < x < 5$  9.  $x \leq -1$  or  $x \geq 1.5$   
 11.  $-3.5 \leq x \leq 1$  13.  $x < -\sqrt{5}$  or  $x > \sqrt{5}$   
 15.  $a > 2$  17.  $n < -3$  or  $1 < n < 2$   
 19.  $x < -2$  or  $\frac{1}{2} < x < 1$  21.  $-2 < y < 2$   
 23.  $x < -1$  or  $x > 1$  25.  $x < 3$  or  $4 < x < 5$   
 27.  $-4 < x \leq \frac{5}{2}$  or  $x > 7$  29. All real nos.  
 31.  $-2 < x < -1.73$  or  $x > 1.73$   
 33.  $-0.41 \leq x \leq 0.50$  or  $x \geq 2.41$  35.  $x < -4$  or  $x > 1$  37. All real nos. 39.  $x \geq 0.70$   
 41.  $-4.12 \leq x \leq -3$  or  $3 \leq x \leq 4.12$  43. No sol.

**Class Exercises, page 106** 1. Above 2. Below  
 3. On 4. On 5. Above 6. Below  
 7. a.  $x > 0$ ,  $y > 0$  b. II:  $x < 0$ ,  $y > 0$ ; III:  $x < 0$ ,  $y < 0$ ; IV:  $x > 0$ ,  $y < 0$  8.  $x \geq 0$ ,  $y \geq 0$ ,  $y \leq -x + 2$  9.  $0 \leq y \leq 3$ ,  $x \geq 0$ ,  $y \leq 4 - 2x$   
 10.  $y \geq x^2$ ,  $y < x + 2$

**Written Exercises, pages 106–108** 1. Pts. on or above the line  $y = x$  3. Pts. below the line  $3x + 4y = 12$  5. Pts. on or below the parabola  $y = x^2$  7. Pts. below the parabola  $y = 2x^2 - 4x + 1$  9. Pts. on or between the lines  $x = 0$  and  $x = 2$  11. Pts. below the line  $y = -1$  or above the line  $y = 1$  13. Pts. between the lines  $x = 1$  and  $x = 5$  15. Pts. above the curve  $y = x^3 - 9x$

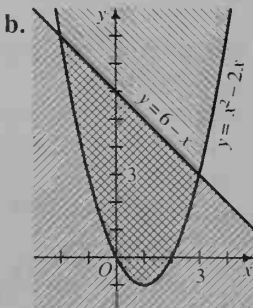
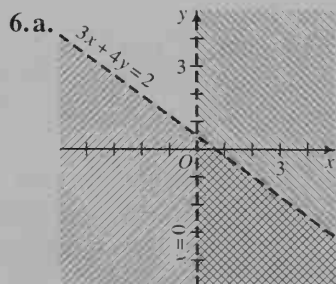


31.  $1.5 \leq x \leq 5.5$   
 $10x + 30 \leq y \leq 10x + 50$



**Written Exercises, pages 112–114** 1.a. 5 consoles, 4 portables 3. 5 consoles, 3 portables 5. (0, 2)  
 7.a. (7, 3) b. (2, 8), (7, 3) 9. 300 tons alfalfa, 200 tons corn 11.a. 12 racing skis, 30 free-style skis b. A max. profit of \$960 occurs for 12 racing skis and 30 free-style skis, or for 20 racing skis and 18 free-style skis. 13.a. 13 b. 32 c. 18

**Chapter Test, page 115** 1.a.  $x > 3$  b.  $x \leq 6$   
 2.a.  $-2 < x < 8$  b.  $x < -10$  or  $x > -2$   
 c.  $x = 0$  or  $x = -\frac{8}{3}$  d.  $1 \leq x \leq \frac{9}{5}$  3.a.  $x \leq -4$  or  $x \geq \frac{1}{2}$  b.  $-2 < x < \frac{3}{4}$  c.  $-\frac{1}{2} < x < 0$  or  $x > 1$   
 d.  $-1 \leq x < 3$  or  $3 < x \leq 6$  4. The pts. below the line  $y = 2x - 5$  5.a. Pts. on or below the line  $2 - 3y = 3x$  b. Pts. between the lines  $x = -3$  and  $x = 3$



7.  $1 \leq x \leq 3$ ,  $y \geq 2$ ,  $y \leq -\frac{1}{2}x + \frac{9}{2}$  8. 20 trucks, 10 automobiles

**Cumulative Review, page 116** 1.  $AB = \sqrt{58}$ ;

midpt. =  $(-\frac{11}{2}, \frac{3}{2})$  2.a.  $x - y = 6$

b.  $2x - 3y = -15$  3.a.  $P(d) = 12.5d - 1027$

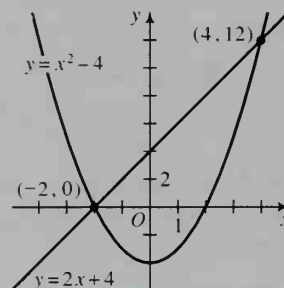
b. 123 4.a.  $-2i$  b.  $-12\sqrt{3}$  c.  $\frac{2}{5} - \frac{3}{10}i$

5.a.  $3 \pm 3i$  b.  $\frac{3 \pm \sqrt{2}}{2}$  c. 4, 12

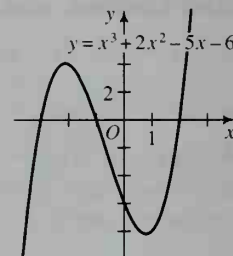
6. See graph below. 7.  $f(x) = -2x^2 + 20x - 42$

8.  $f(x) = \frac{1}{4}x^2 + x - 6$  9.a. 15 b. 0 c. -6

d.  $4 - 2i$  10.a. 2, -3 b. See graph below.



Ex. 6



Ex. 10.b.

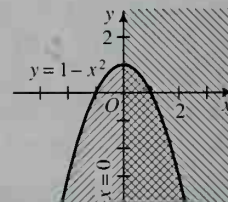
11.  $y = \frac{1}{6}(x + 2)^2(x - 3)$  12.  $625 \text{ cm}^2$

13. -1.5, -0.3, 1.9 14.  $\pm 2i$ , 3

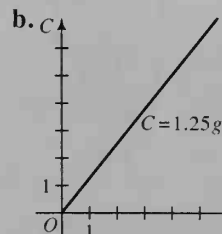
15.  $3x^3 - 11x^2 + 20x + 8 = 0$ , for example

16.a.  $x < -7$  or  $x > 17$  17.

b.  $x \leq -3$  or  $x \geq 1$



**Class Exercises, page 122** 1.a.  $g \geq 0$ ;  $C \geq 0$



2.a.  $x \neq 2$  b.  $t \geq 0$  c.  $s \geq 4$

3.a.  $f(2) = 3 \cdot 2^2 - 12 = 0$

b. -2 c.  $f(x) \geq -12$

4. Dom.: all real nos.;

Range:  $\{y | y \geq -2\}$ ;

zeros: -3, 0, 4

5.  $x^2y = 4x - y$  passes;  $y^2 = 4x^2$  fails.

6.  $y^2 - x = 1$  7. Yes 8. Yes 9. No 10. No

11. No; fails vert. line test 12. Yes; passes vert.

line test 13. No; fails vert. line test



**Written Exercises, pages 122–124** 1. Yes; Dom.:  $\{x|-2 \leq x \leq 5\}$ ; Range:  $\{y|-1 \leq y \leq 2\}$  3. Yes; Dom.:  $\{0, 2, 4, 6\}$ ; Range:  $\{0, 1, 2, 3\}$  5. No

7.  $y = \pm\sqrt{1-x^2}$ , so there are two y-values for each x-value. 9.a.  $x \neq 0$  b.  $x \neq 9$  c.  $x \neq \pm 2$

11.a. Dom.: all real nos.; Range:  $\{y|y \geq 0\}$ ; zero: 0

b. Dom.: all real nos.; Range:  $\{y|y \geq 0\}$ ; zero: 2

c. Dom.: all real nos.; Range:  $\{y|y \geq -2\}$ ;

zeros:  $\pm 2$  13. Range:  $\{y|y \geq -1\}$ ; zeros: 2, 4

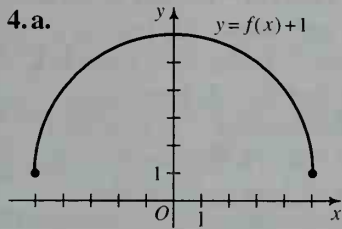
15. Range: all real nos.; zero: 2

17. Range:  $\{h(u)|-2 < h(u) \leq 4\}$ ; zeros: 0, 2

19.a.  $36\pi$  b. Dom.: all solids; Range: nonnegative real nos. 21.b. Dom.: all real nos.; Range: all integers 25. d

**Class Exercises, pages 127–128** 1.a. 5 b. 8

2.a.  $\pm 3$  b.  $-3 < x < 3$



5.  $x^2 + 2x + 1$

6.  $x^2 - 1$

7.  $x^3 + 2x^2 + x$

8.  $x$ , for  $x \neq -1$

9.a. 12

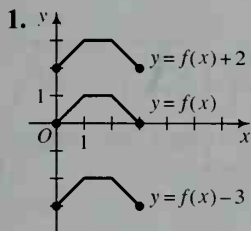
b.  $x^2 + 3x + 2$

10.a. 7

b.  $x^2 + x + 1$

12.  $F(M(X)); M(F(X))$  13.  $B \circ F$

**Written Exercises, pages 128–131**



3.a. 1 b. Pos.:  $0 < x < 4$ ;

neg.:  $-1 \leq x < 0$  or

$4 < x \leq 6$ ; zero:  $x = 0, 4$

c. 2 5.  $x^3 + x - 2$

7.  $x^4 - x^3 - x + 1$  9.a. 0

b.  $x^3 - 3x^2 + 3x - 2$

15. 0.64

17.a.  $f(g(x)) = g(f(x)) = x$

b.  $f(h(x)) = 6x + 1$ ;  $h(f(x)) = 6x - 7$  19.a. 3

b.  $\sqrt{2x-3}$  21.a. 0 b.  $\frac{\sqrt{6x-3}}{3}$  23.  $j(f(h(x)))$

25.  $f(j(h(x)))$  27.a. 1.5 m/s; 9 L/min b. 30 s

29.a.  $r = \frac{C}{2\pi}$  b.  $A = \frac{C^2}{4\pi}$

31.  $s(F) = 331 + \frac{1}{3}(F - 32)$

33.  $(f \circ g)(x) = 2\sqrt{16 - x^2}$ ;

Dom.:  $\{x|-4 \leq x \leq 4\}$ ;

$(g \circ f)(x) = 2\sqrt{4 - x^2}$ , Dom.:  $\{x|-2 \leq x \leq 2\}$

35.  $(f \circ g)(x) = 1 - x$ , Dom.:  $\{x|x \leq 1\}$ ;

$(g \circ f)(x) = \sqrt{1 - x^2}$ , Dom.:  $\{x|-1 \leq x \leq 1\}$

37. 2.5, 2.75, 2.875 39.a.  $E = \frac{130}{(t+1)^2}$  b. 9 s

**Class Exercises, pages 135–136** 2.a.  $x = 3$

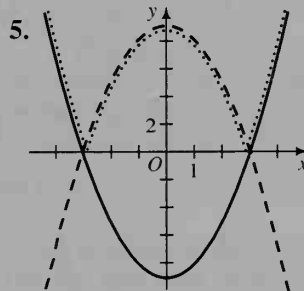
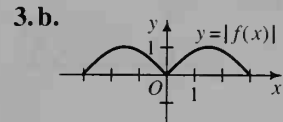
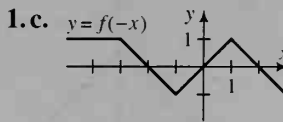
b.  $y = \pm x$ ; (0, 0) c.  $x = 0$ ,  $y = 0$ ; (0, 0)

3.a. i, ii, iii, iv b. iv c. iv 4. In the origin

5. No 7.a.  $x = 4$  b.  $x = 1$  c.  $x = 0$  8.a. (2, 1)

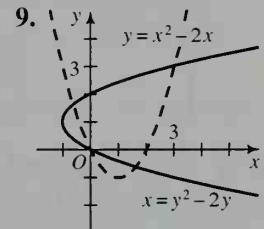
b. (-1, -5) c. (0, 7)

**Written Exercises, pages 136–138**

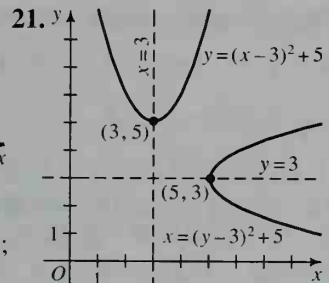
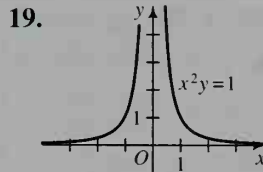


Key: —  $y = x^2 - 9$   
--  $y = 9 - x^2$   
....  $y = |9 - x^2|$

7.  $y = \frac{x+4}{3}$



11.  $y = \sqrt[3]{x}$  13.  $x = |y| + 2$  or  $y = \pm x \mp 2$  for  $x \geq 2$  15.a. iv b. i, ii, iii, iv c. iv



23.a.  $V(4, 3)$ ; axis of sym.  $x = 4$  b.  $V(3, 4)$ ; axis of sym.  $y = 4$

25.  $V(-1, -3)$ ; axis of sym.  $y = -3$  27. (-5, 11)

29.a. (0, 0) b. Pt. of sym. (1, 2); local max. (2, 4)

31.a. Even b. Odd c. Neither d. Even e. Neither

f. Odd 33.a. In the y-axis b. In the origin

37.a.  $m > 0$ ;  $m < 0$ ;  $m = 0$  39.a. dec. b. dec.

c. inc.

**Class Exercises, page 142** 1.a. 4 b.  $\frac{1}{2}$

c.  $f(25) = 1$ ;  $f(-25) = 0$  2.  $p = 4$ ,  $A = 1$ ;  $p = 4$ ,

$A = \frac{1}{4}$  3.  $p = 2$ ,  $A = \frac{1}{2}$ ;  $p = 8$ ,  $A = \frac{1}{2}$

4.  $p = 4, A = \frac{1}{2}; p = 4, A = \frac{1}{2}$

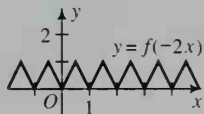
5. If  $(a, 0)$  is on  $y = f(x)$ , then  $(a, 0)$  is on  $y = c \cdot f(x)$ , since  $c \cdot f(a) = c \cdot 0 = 0$ .

6. If  $(0, a)$  is on  $y = f(x)$ , then  $(0, a)$  is on  $y = f(cx)$ , since  $f(c \cdot 0) = f(0) = a$ .

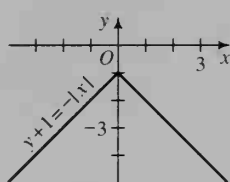
**Written Exercises, pages 143–146** 1. Yes;  $p = 6, A = 1; f(1000) = -1, f(-1000) = 1$  3. Yes;

$p = 3, A = \frac{1}{2}; f(1000) = 2, f(-1000) = 3$

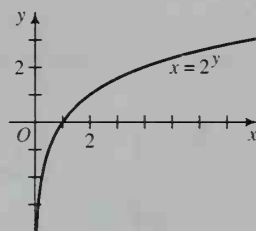
5. c.



7. e.



9. d.



11.  $(x - 8)^2 + (y - 4)^2 = 9$  13. a. x-ints. 0, 3; local max. (2, 32) b. x-ints. 0, 6; local max. (4, 64)  
c. x-ints. 2, 8; local max. (6, 32) d. x-ints. -2, 4; local max. (2, 32) e.  $f(x) = -x^2(x - 6)$   
15. a.  $y = |x - 4| + 2$  b.  $y = -|x| + 4$   
c.  $y = \frac{1}{2}|x| - 2$  d.  $y = -2|x|$  e.  $y = 2|x + 4|$

**Class Exercises, pages 148–149** 1. No; no 2. a. 2

b. 3 c. 2 3. a.  $g^{-1}(x) = \frac{x}{4}$  b.  $g^{-1}(x) = \frac{x - 2}{3}$

c.  $g^{-1}(x) = \frac{x + 1}{2}$  d.  $g^{-1}(x) = \frac{4 - x}{5}$  4. a. (8, 2),

(1, 1), (-1, -1), (-8, -2) b.  $f^{-1}(x) = \sqrt[3]{x}$

5. a. Yes; it passes the horiz. line test. b. No;  $g(-1) = g(1) = 0$ . 6. Solve for the desired variable. 7. g; g 8. b; d 9. No;

$T(A) = T(B) = 2$ , so  $T$  is not one-to-one.

**Written Exercises, pages 149–150** 1. a. 2 b. 3 c. 7

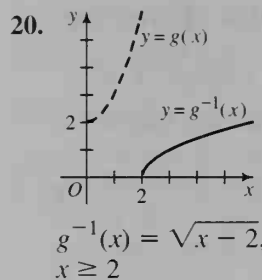
3. g is not one-to-one. 5. b.  $h^{-1}(x) = \frac{x + 3}{4}$

7. Yes 9. No 11. Yes;  $f^{-1}(x) = \frac{x + 5}{3}$ ;

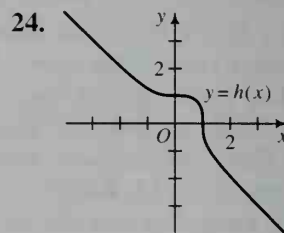
$f(f^{-1}(x)) = 3\left(\frac{x + 5}{3}\right) - 5 = x; f^{-1}(f(x)) =$

$\frac{(3x - 5) + 5}{3} = x$  13. Yes;  $f^{-1}(x) = x^4, x \geq 0$

15. No 17. No 19. Yes;  $f^{-1}(x) = \sqrt[3]{x^3 - 1}$



22.  $g^{-1}(x) = 1 - \sqrt{x - 1}, x \geq 1$



29. ii 31.  $P^{-1}(x) = \frac{-b - \sqrt{b^2 - 4a(c - x)}}{2a}, b^2 - 4a(c - x) \geq 0$

**Calculator Exercises, page 150** 1.  $f$  and  $g$  are inverse functions. 2. It is. 3.  $x > 0$

**Class Exercises, pages 153–154** 1. Area of a rect. is a function of length and width. 2. Dist. is a function of rate and time. 3. Vol. of a cone is a function of base radius and ht. 4. Force is a function of mass and acceleration. 5. Density is a function of mass and vol. 6. Surface area of a cylinder is a function of base radius and ht. 7. First-quadrant rays with vertices at the origin. 8. Ans. may vary. a. Tampa, San Francisco b. Chihuahua, Ciudad Juarez, Phoenix c. Quebec, Montreal, Boston

10. a. 5; 5; 5

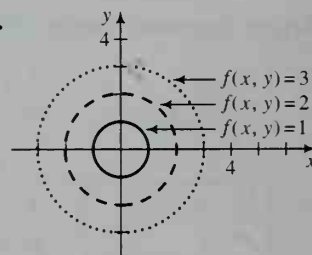
b.

11. For example:

$(-1, -1),$

$(0, 0),$

$(2, 2)$



**Written Exercises, pages 154–157** 1. a.  $r, t$

b.  $A = 100(1.12)^t, A = 100(1.16)^t$  c. 9 yr; 6 yr; 5 yr 3. a. Wind-chill temp. is a function of

recorded temp. and wind speed. b. 16 mi/h c. The latter feels warmer. d. Between 5 mi/h and 10 mi/h

5. a.  $A(8, 3) = 12; A(16, 6) = 48$  b. Area of a  $\Delta$  is a function of base and ht. c. For example:

(2, 5), (1, 10), (20, 0.5) 7. b. If the base and ht.

of a  $\Delta$  are each tripled, then the area is multiplied by 9. 9. a.  $d(r, t) = rt$  13.  $\{(x, y) | x \neq \pm y\}$

17. a.  $A(4, 3, 5) = 94; A(6, 4, 7) = 188$  b. Surface area of a rect. prism is a function of length, width, and height. 21. No;  $R(37, 10, 8.5) \approx 5.85 > 5.5$

**Class Exercises, pages 160–161** 1.a.  $d(s) = s\sqrt{2}$

b.  $s(d) = \frac{d\sqrt{2}}{2}$  c.  $A(d) = \frac{d^2}{2}$  2.a.  $V(e) = e^3$

b.  $V(d) = \frac{d^3\sqrt{3}}{9}$

3.a.  $AP + PB = \sqrt{16 + x^2} + 8 - x$

b.  $\{x|0 \leq x \leq 8\}$  4.a.  $d(x, y) = \sqrt{(x-2)^2 + y^2}$

b.  $d(x) = \sqrt{(x-2)^2 + x^4}$  c. With a computer or graphing calculator, find the y-value of the lowest point on the graph of  $y = d(x)$ . 5.  $d = 10t$

**Written Exercises, pages 161–165** 1.  $A(h) = \frac{h^2\sqrt{3}}{8}$

3.  $t(n) = \frac{11n}{36}$  5.  $P(n) = 6n$  7.  $A(h) = \frac{5}{8}\pi h^2$

9.  $s(d) = 1.5d$  11.  $V(w) = \frac{3w - 2w^3}{4}$

13.a.  $C(t) = \pi t$  b.  $A(t) = \frac{\pi}{4}t^2$

15.a.  $C(w) = \frac{8w^3 + 192}{w}$  b.  $\approx \$126$

17.b. 2:06 P.M. c. 4 km 19.a.  $V(h) = \frac{\pi}{48}h^3$

b.  $h(t) = \sqrt[3]{\frac{240t}{\pi}}$

21.a.  $d(x) = \sqrt{5x^2 - 40x + 100}$  b. Dom.: all real numbers; Range:  $\{d(x)|d(x) \geq 2\sqrt{5}\}$

23.a.  $P(x) = -2x^2 + 4x + 18$  b.  $\{x|0 < x < 3\}$

c. 1 25.a.  $t(x) = \sqrt{2500 + x^2} + \frac{1}{3}(100 - x)$

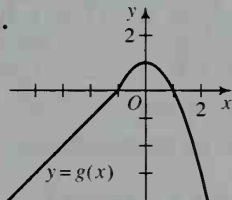
b.  $\approx 80.5$  s 27.a.  $C(x) = \frac{12x^2 + 480}{x}$  b.  $\approx \$152$

29.a.  $V(r) = \frac{2\pi r^4}{3(r^2 - 1)}$  b.  $r \approx 1.41, h \approx 4.02$

**Chapter Test, pages 166–167** 1. Dom.:

$\{x| -3 \leq x \leq 3\}$ ; Range:  $\{f(x)|0 \leq f(x) \leq 3\}$ ; zeros:  $\pm 3$

2.a.



b. Range:  $\{g(x)|g(x) \leq 1\}$ ; zeros:  $\pm 1$

3.a.  $x^2 + 3x + 2$

b.  $x^2 + x - 2$

c.  $x^3 + 4x^2 + 4x$

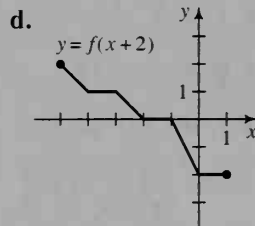
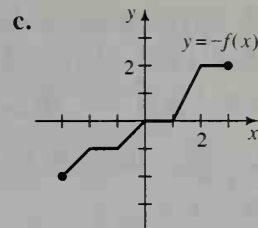
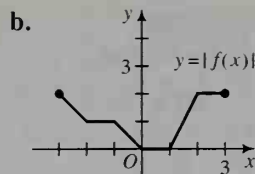
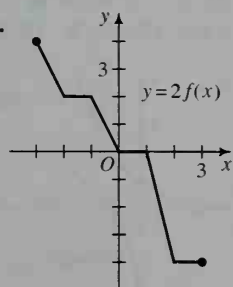
d.  $x$ , for  $x \neq -2$

4.a.  $x^2 + 6x + 8$

b.  $x^2 + 2x + 2$

5. iv

6.a.



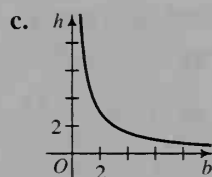
7.a. 4

b. max. 4; min. 1

c.  $\frac{3}{2}$

8. The translated graph coincides with the original graph. Since for all  $x$ ,  $f(x+p) = f(x)$ , the graphs of  $y = f(x)$  and its horiz. translation  $y = f(x+p)$  are identical. 9.a.  $g$ ;  $g^{-1}(x) = x - 3$  b.  $f$  does not have an inverse because  $f$  is not one-to-one; for example,  $f(-1) = f(1) = 2$ .

10.a.  $A(b, h) = \frac{1}{2}bh$  b.  $A(3, 4) = 6$ ;  $A(6, 5) = 15$



11.  $h(t) = \frac{5t}{2\pi}$

12.a.  $A(x) = x(9 - x^2)$

b.  $\{x|0 < x < 3\}$

c.  $\approx 10.4$

**Class Exercises, page 172** 1.a. 10.5% b. 1.03;

1.15; 1.046; 2.2 2.a. 32% b. 0.88; 0.925; 0.20; 0

3.a.  $200(1.05)^t$  b.  $20(1.08)^t$  c. 6%

4.a.  $9800(0.8)^t$  b.  $2200(0.85)^t$  c. 25% 5.a.  $\frac{1}{8}$

b.  $\frac{1}{64}$  6.a.  $\frac{3}{2}$  b.  $\frac{9}{4}$  7.a.  $\frac{4}{9}$  b.  $\frac{1}{144}$  8.a. 8 b.  $\frac{4}{3}$

9. 8 10.  $6^n$  11. -2 12.  $x^2 + 1$  13.  $18a^{10}$

14.  $3a^4 - 6a^7$  15.  $5b^5 + 10b^8$  16.  $50a^3b^8$

**Written Exercises, pages 173–175** 1.a.  $\frac{1}{16}$  b.  $-\frac{1}{16}$

3.a.  $\frac{5}{8}$  b.  $\frac{1}{1000}$  5.a.  $x^2$  b. 1 7.a.  $\frac{a}{3}$  b.  $\frac{a}{3a+1}$

9.a.  $\frac{8}{3}$  b. 32 11.a.  $\frac{ab}{b-a}$  b.  $ab$  13. \$1214;

\$4910 15. \$1.97; \$3.87 17. \$27,422; \$11,569

19. \$0.83; \$0.69 21.  $\frac{81}{a}$  23.  $3^6n^{12}$  25.  $\frac{16}{a^4}$

27.  $\frac{b^{10}}{a^5}$  29.  $2x^2 - 4$  31.  $2 + 3a^4$

33. \$8000; \$6400; \$5100; \$4100; \$3300

35. \$112.68 37.a. 3 b. 5 c.  $\sqrt{2^{5n+7}}$  39.a.  $b^n$

b.  $b^{-2}$  c.  $b^{1-2n}$  41.a.  $\frac{4}{x}$  b.  $\frac{1}{x}$  43.a.  $\frac{y+x}{y}$

b.  $\frac{1}{y+1}$  45.  $\approx \$55,000,000,000$  47. 1, -1

49. -1, 1

**Class Exercises, page 177** 1.a. 2 b.  $\frac{1}{2}$  2.a. 8

b.  $\frac{1}{8}$  3.a. -3 b.  $-\frac{1}{3}$  4.a. 15 b.  $8 + 2\sqrt{15}$

5.  $\frac{5}{7}$  6.  $\frac{8}{27}$  7. 2 8. 64 9.  $\frac{8}{x}$  10.  $\frac{5}{x^2}$  11.  $\frac{x}{2}$

12.  $8x$  13.a. \$62 b. 2.5 c. -0.75 14. 6 15.  $\frac{5}{2}$

16. 27 17.  $\frac{1}{16}$

**Written Exercises, pages 178-180** 1.a.  $\sqrt[3]{x^2}$

b.  $\sqrt{x^3}$  c.  $\sqrt{\frac{5}{x}}$  d.  $\sqrt[3]{6x^2}$  3.a.  $x^{5/2}$  b.  $y^{2/3}$

c.  $(2a)^{5/6}$  d.  $x$  5.a.  $\frac{3}{5}$  b.  $\frac{5}{3}$  c.  $\frac{243}{3125}$  d.  $\frac{125}{27}$  7.  $\frac{1}{8}$

9. 36 11.  $\frac{a^4}{4}$  13.  $2x^2$  15.a. \$182 b. \$108

17.  $a^2 - 2a$  19.  $x^2 - 2x$  21.  $x - 2$  23.  $n - 2$

25.  $3n - 4n^2$  27.  $2\sqrt{a}$  29. 2 31. 2 33.  $\frac{25}{3}$

35.a.  $\frac{1}{32}$  b.  $\frac{1}{2}$  c.  $-\frac{31}{4}$  37. 12% 39. 4.3%

41.  $\approx 7\%$ ; \$24,400 43.a.  $\frac{81}{4}$  b. 3 45. -2, 3

47. -2 49.a.  $a^{1/2}b^{1/2}(a-b)$  b.  $a^{1/2}b^{-1/2}(1-ab)$

51.a.  $(x^2 + 1)^{1/2}$  b.  $2(x^2 + 2)^{-1/2}$

53. D  $220 \cdot 2^{5/12}$ ; D#  $220 \cdot 2^{6/12}$ ; E  $220 \cdot 2^{7/12}$ ;

F  $220 \cdot 2^{8/12}$ ; F#  $220 \cdot 2^{9/12}$ ; G  $220 \cdot 2^{10/12}$ ;

G#  $220 \cdot 2^{11/12}$ ; A 440 55. 27, 64 57.  $\frac{1}{2}$

**Class Exercises, pages 182-183** 1. 7 2.  $a = 5$ ;

$b = 3$  3.  $b > 1$ ;  $0 < b < 1$  4. Dom.: all real

numbers; Range:  $\{y | y > 0\}$  5. 125;

$5\pi \approx 156.9925$ ; diff.  $\approx 32$  6. 8 yr; 1 yr

7.a. \$1000 b. 10 yr 8.a. 90 b. 8 hr 9.a. 1000

b. 500 c. 4 days 10. Yes

**Written Exercises, pages 183-186** 1.  $6\pi \approx 278.4$ ;

$\pi^6 \approx 961.4$  3.  $f(x) = 3 \cdot 5^x$  5.  $f(x) = 64\left(\frac{1}{4}\right)^x$

7.a. 1.6 kg b. 0.8 kg c. 0.1 kg d.  $3.2\left(\frac{1}{2}\right)^{t/4}$  kg

9.a. 4 days b. 20 g c.  $A(t) = 320\left(\frac{1}{2}\right)^{t/4}$

11.a. 15% b. 4 yr 13.a. \$6727 b. 7 yr

15. 12 yr 17.  $A_0\left(\frac{1}{2}\right)^{0.617}$

19.b. The graphs are reflections of each other in the y-axis. 21.b. The graph of  $y = 2^{x-1}$  is the graph of  $y = 2^x$  shifted one unit to the right. 23.a. Ans. may vary.  $a = 18.924$ ;  $b = 1.014$  b. In 2000, 305.2 million; in 2050, 611.6 million. 29. For example:  $f(x) = 2^x$

**Class Exercises, pages 188-189** 1. 7.3891

2. 24.5325 3. 0.0183 4. 4.1133 5. 2.7183

6.  $e^\pi$  7.a.  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$  b. 2.718

8.a.  $(1.02)^4$  b.  $(1.0067)^{12}$  c.  $e^{0.08}$

9. 5% compounded daily yields same interest as 5.13% annually.

**Written Exercises, pages 189-190** 1. 2.71801;

2.718282 b.  $\approx$  equal 3.  $e^{\sqrt{2}}$  5.a. \$1.0614

b. \$1.0617 c. \$1.0618 7. 7.5% 9. Plan A

11. \$1419.07 13.b.  $f$  is one-to-one. 19. 2.7083

23.  $e^2$

**Class Exercises, page 194** 1.a. 1.4771 b. 2.7686

c. 3.8325 d. -0.5229 e. -3.5229 2. Error

message; the domain of the log function is pos.

reals. 3.a. 1.9031 b. 2.9031 c. 4.9031

d. -0.0969 e. -1.0969 4.a.  $2^4$  b.  $10^{1.49}$

c.  $10^{1.79}$  5.a. 2 b. 4 c. -3 d. -1 e.  $\frac{1}{2}$

6.a. 0.6931 b. 1.0986 c. 0.9933 d. 1.0296 e. 1

7.a. 1.70 b. 3.91 8.a. 25 b. 36 c. 100 d.  $e^2$

9.a. 11 b. 4 c. 2 d. 6

**Written Exercises, pages 194-197** 1.  $4^2 = 16$

3.  $6^{-2} = \frac{1}{36}$  5.  $10^3 = 1000$  7.  $e^{2.1} \approx 8$

9.a.  $10^x = N$  b. 0.85; -0.25 c.  $e^x = N$  d. 2.48;

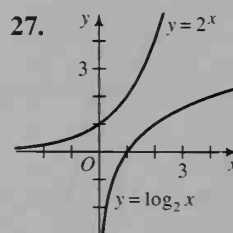
-2.81 11.a. 2 b. 4 c. -2 d. -4 13.a. 2 b. 3

c. 5 d. 8 15.a. 3 b. -3 c.  $\frac{1}{4}$  d. 0 17.a. 1 b. 2

c. -1 d.  $\frac{1}{2}$  19.a. 2.6201 b. -0.3799 c. -1.3799

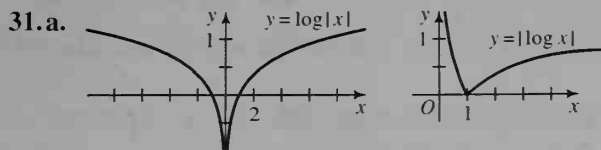
21.a. -2.3026 b. -4.6052 c. 4.6052

23.a. 68 dB b. 15 dB 25.a. 65 dB b. 67 dB



Domain  $f = \text{Range } f^{-1} =$   
all real numbers  
Range  $f = \text{Domain } f^{-1} =$   
pos. real numbers

29.a.  $(f \circ g)(x) = e^{\ln x} = x, x > 0$ ;  
 $(g \circ f)(x) = \ln(e^x) = x, x \text{ any real no.}$



b. For  $y = \log|x|$ , dom. is  $\{x|x \neq 0\}$ ; range is all real nos. For  $y = |\log|x||$ , dom. is  $\{x|x \neq 0\}$ ; range is  $\{y|y \geq 0\}$ . 33. For  $y = \log x + 3$ , dom. is  $\{x|x > 0\}$ ; the range is all real nos.; zero is 0.001. For  $y = \log(x + 3)$ , dom. is  $\{x|x > -3\}$ ; the range is all real nos.; zero is  $-2$ .

35.a. 1000 b.  $\pm 1000$  c. 1001,  $-999$  37.a. 8  
b.  $e^{3/2}$  c. 1 39.a. 3 b.  $10^{10}$  c.  $e^e$  41.a. 5.01  
b. 5011.87 c. 0.50 43.a. 66.69 b. 0.22 c. 1.61

45. gastric juices: 2, acidic; acid rain: 4.52, acidic;  
pure water: 7, neutral; soil: 6.30, acidic; sea

water: 8, alkaline 47.a.  $\log_4 16 = 2$ ;  $\log_{16} 4 = \frac{1}{2}$

b.  $\log_9 27 = \frac{3}{2}$ ;  $\log_{27} 9 = \frac{2}{3}$  c.  $\log_a b = \frac{1}{\log_b a}$

49.b.  $y \approx 10(3.16)^x$  51.a. (1) 145; (2) 72,382  
b. 0.145; 0.072 c. Approaches zero

**Class Exercises, page 199** 1.  $2 \log M + \log N$

2.  $2 \log M - \log N$  3.  $\frac{1}{2}(\log M - \log N)$

4.  $\frac{1}{3}(\log M + \log N)$  5.  $\log M + \frac{1}{2} \log N$

6.  $2 \log M - 3 \log N$  7.  $\log_5 6$  8.  $\log_3 20$

9.  $\log 4$  10.  $\log 9$  11.  $\ln 36$  12.  $\ln \frac{5}{2}$

13.  $\log MN^2$  14.  $\log \left(\frac{P^2}{Q}\right)$  15.  $\log_b MNP$

16.  $\log_b \frac{MN}{P^3}$  17.  $\ln \left(\frac{a}{b}\right)^{1/2}$  18.  $\ln cd^{1/3}$  19. For

example,  $\log 10 + \log 1 = 1 + 0 = 1 \neq \log(11)$ .

20. For example,  $\frac{\log 1}{\log 10} = \frac{0}{1} = 0 \neq \log \frac{1}{10}$ .

21.  $\log \frac{1}{2} < 0$ ; dividing by neg. reverses the order of the inequality.

**Written Exercises, pages 200–203**

1.  $2(\log M + \log N)$  3.  $\frac{1}{3}(\log M - \log N)$

5.  $2 \log M + \frac{1}{2} \log N$  7.  $\log 24$  9.  $\log_6 15$

11.  $\ln 12$  13.  $\log \left(\frac{M}{N^3}\right)$  15.  $\log \frac{AB^2}{C^3}$

17.  $\log_b \sqrt[3]{\frac{M^2}{NP}}$  19.  $\log \pi r^2$  21.  $\ln 4$  23.a. 2

b. 3 c.  $-\frac{1}{2}$  d.  $\frac{1}{2}$  25.a.  $x$  b.  $x$  c.  $x^2$  d.  $\frac{1}{x}$

27.a. 6 b. 36 c. 4000 d.  $4e^3$  29.a.  $y = x^2$

b.  $y = 5x^3$  31.a.  $y = \frac{1}{x}$  b.  $y = 2x^2$

33.a.  $y = 0.1(10^{1.2})^x$  b.  $y = \frac{1}{e}(e^{1.2})^x$

35.a. Reflections in the  $x$ -axis, since

$\log_b \frac{1}{x} = -\log_b x$ . 37.b. Shifting it up one unit

39.  $y = 10^b(10^a)^x$ ;  $\log y$  in terms of  $x$  is linear;  $y$  in terms of  $x$  is exponential. 41.a.  $r + 2s$

b.  $2r + 2s$  c.  $r - 2s$  d.  $r - 2$  43.a.  $\frac{6}{5}$  45.a. 5

47.a.  $0 < M < 1$  b.  $1 < x < 3$  c.  $M > 9$

d.  $x < -2$  or  $x > 2$  e.  $0 < x < 5$  49.a. No sol.

51.a.  $\approx 16$  b. (1)  $\approx 25$ ; (2)  $\approx 50$  53.  $\approx 1230$  km

**Class Exercises, page 205** 1.  $\sqrt[3]{81}$  2. 4 3. 3.17  
4.  $\pm \sqrt[4]{8}$  5. 0.48 6. 0.91 7. 2.41 8. 1.24

**Written Exercises, pages 205–207** 1. 2.26 3. 18.85

5. 2.89 7. 7.82 9.a.  $\frac{9}{4}$  b. 2.16 11.a. 0 b. 0.22

13. 2007 15.a.  $A(t) = 10,000(1.08)^t$

b. 14.3 yr 19.a. 177 g b. 95.7 h 23.b. none

c. 2; 2 d. 0 or 2 25.  $\log_2 3$  27.  $\ln 2, \ln 3$

29.  $\log_3 2, -1$  31.a.  $m > n$  b.  $m < n$

33.a.  $N(t) = N_0 \left(\frac{1}{2}\right)^{t/5700}$  b. 18,900 yr 37. 3

39. 1 41.  $x > 3$  43. 49.5 cm; 279.4 cm

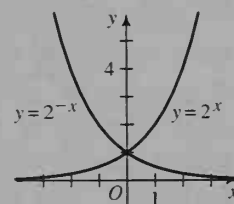
**Chapter Test, page 209** 1.  $v(t) = 150,000(1.09)^t$

2.a. 8 b.  $\frac{25}{26}$  c. 256 d.  $\frac{4}{3}$

3.a.  $\frac{2}{3}$  b.  $\frac{5}{8}$  4. 5%

5. See graph at right.

The graphs are reflections in the  $y$ -axis.



6. Ans. will vary. An initial amount  $P_0$ , growing at a rate of  $r\%$ , yields  $P(t) = P_0(1 + r)^t$  at time  $t$ ; decaying at a rate of  $r\%$ ,  $P(t) = P_0(1 - r)^t$ . For example, \$1000 invested at 5% annual interest yields  $1000(1 + 0.05)^2$  dollars after two yr. A \$10,000 car depreciating at 7% per yr will be worth  $10,000(1 - 0.07)^2$  dollars after two yr.

7.a. \$1701.70 b. \$1703.98 8.a. 0.3979 b. 3.3979



- c.  $-1.3979$  9.a. 3 b.  $\frac{1}{3}$  c. 8 10.a.  $y = 4x^2$   
 b.  $y = 4x$  11.a.  $\frac{2}{3} \log_b M - \frac{1}{3} \log_b N$   
 b.  $2 \log_b M + 3 \log_b N$  12. 5 13.a. 1 and 2  
 b.  $\frac{\log 21}{\log 5} \approx 1.892$  14. 1.29

**Class Exercises, pages 217–218** 1. a, c 2. a, b, c

**Class Exercises, page 222** 1.  $C(0, 0); r = 4$

2.  $C(2, 7); r = 6$  3.  $C(4, -7); r = \sqrt{7}$   
 4.  $C(0, -6); r = 6$  5.  $C(1, 3); r = \sqrt{19}$   
 6.  $C(0, 0); r = 3$  7.  $(x-7)^2 + (y-3)^2 = 36$   
 8.  $(x+5)^2 + (y-4)^2 = 2$  9.  $x^2 + y^2 = 169$   
 10.a. All pts. less than 5 units from  $(0, 0)$ . b. All  
 pts. more than 5 units from  $(0, 0)$ . 12. The line  
 and the circle do not intersect.

**Written Exercises, pages 222–225**

1.  $(x-4)^2 + (y-3)^2 = 4$   
 3.  $(x+4)^2 + (y+9)^2 = 9$  5.  $(x-6)^2 + y^2 = 15$   
 7.  $(x-2)^2 + (y-3)^2 = 18$   
 9.  $(x-5)^2 + (y+4)^2 = 16$   
 11.  $(x-4)^2 + (y+5)^2 = 25$   
 13.  $(x-1)^2 + (y-4)^2 = 1; C(1, 4), r = 1$   
 15.  $x^2 + (y-6)^2 = 11; C(0, 6), r = \sqrt{11}$   
 17.  $(x-\frac{5}{2})^2 + (y-\frac{9}{2})^2 = 27; C(\frac{5}{2}, \frac{9}{2}), r = 3\sqrt{3}$   
 27.  $(8, 15), (15, 8)$  29. Fail to intersect  
 31.  $(-5, 12), (\frac{135}{29}, -\frac{352}{29})$  33.  $(0, 0), (2\sqrt{3}, 6)$   
 39.a.  $x^2 + y^2 = 13$  b.  $2x + 3y = 13$   
 41.a.  $AC = BC = 13$  b.  $M = (\frac{21}{2}, \frac{25}{2});$  slope of  
 $\overline{CM} = 1$ , slope of  $\overline{AB} = -1$  43.a. The circle is  
 $x^2 + y^2 = 169; (-5)^2 + 12^2 = 169$  b. slope of  
 $\overline{PA} = -\frac{2}{3}$ ; slope of  $\overline{PB} = \frac{3}{2}$   
 45.  $(x-6)^2 + y^2 = 36; C(6, 0), r = 6$   
 47.a.  $y = \sqrt{16-x^2}$  b.  $A(x) = 4\sqrt{16-x^2}$   
 c.  $\{x | -4 \leq x \leq 4\}$  e. 0 f. 16  
 49.a.  $A(x) = (x+4)\sqrt{16-x^2}$  b. 2  
 51.  $(x-3)^2 + (y-4)^2 = 25$   
 53.  $(x-2)^2 + y^2 = 50$  55. No pts. 57. The  
 $x$ - and  $y$ -axes, and the circle centered at  $(0, 0)$  with  
 $r = 1$  59.b.  $\frac{8\pi}{3} - 2\sqrt{3}$

**Class Exercises, page 228** 1.a. Horiz.

- b.  $V(\pm 10, 0); F(\pm 6, 0)$  c. 20 d.  $\frac{x^2}{100} + \frac{y^2}{64} = 1$   
 2.a. Vert. b.  $V(0, \pm 13); F(0, \pm 12)$  3. The region

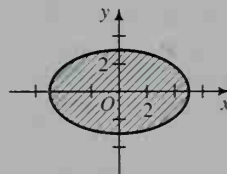
inside the ellipse  $\frac{x^2}{9} + y^2 = 1$ ; the region outside  
 the ellipse  $\frac{x^2}{9} + y^2 = 1$ . 4.a. The surface of an  
 elliptical solid b. The interior of an elliptical solid

**Written Exercises, pages 228–230** 1.  $V(\pm 6, 0);$

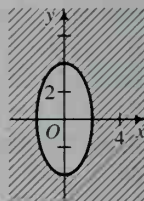
$F(\pm 2\sqrt{5}, 0)$  3.  $V(0, \pm 5); F(0, \pm 3)$

5.  $V(\pm 5, 0); F(\pm 4, 0)$

7.a.



b.



9. Dom.:  $\{x | -6 \leq x \leq 6\}$ ; Range:  
 $\{y | -2 \leq y \leq 0\}$  11.b. Circle  $= \pi$ ; ellipse  $\approx 2\pi$   
 13.  $\frac{x^2}{9} + \frac{y^2}{81} = 1$  15.  $\frac{x^2}{289} + \frac{y^2}{225} = 1$   
 19.  $(12, -8), (16, -6)$  21.  $(2, \sqrt{3}), (-2, \sqrt{3}),$   
 $(-2, -\sqrt{3}), (2, -\sqrt{3})$  23.a. The graph is shifted  
 down 6 units. 25.  $C(-5, 4); V_1(-10, 4), V_2(0, 4)$   
 27.  $C(-2, 5); V_1(-2, 7), V_2(-2, 3)$  29.b.  $\frac{4}{3}\pi a^2 b$   
 31.  $V_1(-6, 0), V_2(-6, 8); F_1(-6, 2), F_2(-6, 6)$   
 33.  $V_1(-4, -3), V_2(2, -3); F(-1 \pm \frac{3\sqrt{3}}{2}, -3)$   
 35.  $V_1(-1, 6), V_2(-1, 0); F(-1, 3 \pm 2\sqrt{2})$   
 37.a.  $y = \pm \frac{2}{3}\sqrt{36-x^2}$  b. Yes  
 39.  $\frac{4(x-4)^2}{39} + \frac{(y+1)^2}{16} = 1$   
 41.  $\frac{(x-5)^2}{25} + \frac{(y-6)^2}{36} = 1$

**Class Exercises, page 234** 1.a.  $V(\pm 2, 0);$

$F(\pm \sqrt{13}, 0)$  b. Horiz. c.  $\frac{x^2}{4} - \frac{y^2}{9} = 1; y = \pm \frac{3}{2}x$

2.a.  $y = -\frac{2}{x}$  b.  $y = \frac{2}{x}$  3.a. Vert. b.  $V(0, \pm 5);$

$F(0, \pm \sqrt{26})$  c.  $y = \pm 5x$  d.  $\frac{(y+5)^2}{25} - \frac{(x-6)^2}{1} = 1$

**Written Exercises, pages 235–237** 1.  $V(\pm 3, 0);$

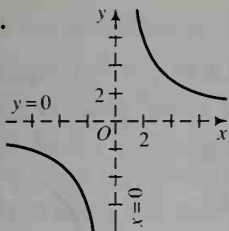
asymptotes:  $y = \pm \frac{2}{3}x$  3.  $V(\pm 1, 0);$  asymptotes:

$y = \pm x$  5.  $V(0, \pm 1);$  asymptotes:  $y = \pm \frac{1}{2}x$

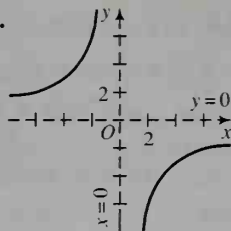
7.  $V_1(2, 2), V_2(-2, -2);$  asymptotes:  $x = 0, y = 0$

9. Dom.:  $\{x | |x| \geq 2\}$ ; Range:  $\{y | y \geq 0\}$

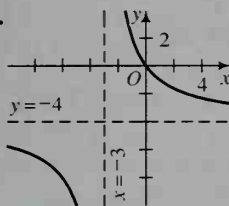
11.a.



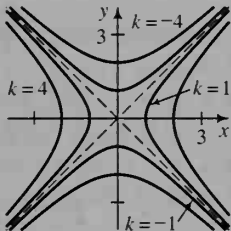
b.



c.



13.



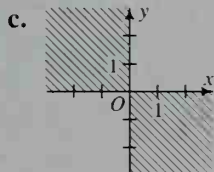
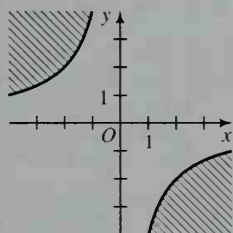
15. (4.1, 0.9) 17. (-4, 5), (10, -2)

19.  $\frac{x^2}{36} - \frac{y^2}{64} = 1$  21.  $\frac{y^2}{4} - \frac{x^2}{4} = 1$  23.b. 0.209,

0.101, 0.010 27.c. Equal

29.a.  $\sqrt{x^2 + (y - 10)^2} - \sqrt{x^2 + (y + 10)^2} = \pm 12$ b.  $\frac{y^2}{36} - \frac{x^2}{64} = 1$  33.  $\frac{y^2}{36} - \frac{x^2}{9} = 1$ ;  $y = \pm 2x$ 

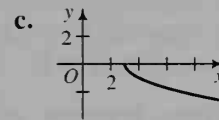
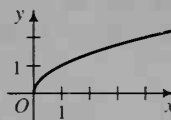
35.a.

39.  $V_1(12, 8)$ ,  $V_2(0, 8)$ ;  $F_1(16, 8)$ ,  $F_2(-4, 8)$ ;asymptotes:  $-4x + 3y = 0$ ,  $4x + 3y = 48$ 41.  $V_1(2, 2)$ ,  $V_2(2, 0)$ ;  $F(2, 1 \pm \sqrt{2})$ ; asymptotes: $y = x - 1$ ,  $y = -x + 3$  43.  $\frac{(x - 5)^2}{16} - \frac{y^2}{9} = 1$ 45.  $a = b = \sqrt{2}$ 

Class Exercises, pages 239–240 1.a. 1; 1 b. 2; 2

c. 5; 5 d.  $\sqrt{x^2 + y^2}$ ,  $y + 2$ 2.a. (1)  $\sqrt{(x + 4)^2 + y^2}$ ; (2)  $|x|$  b. Set (1) and (2) from Ex. 2a equal and simplify. 4.  $F(0, 1)$ ;dir.  $y = -1$ ; up 5.  $F(0, -1)$ ; dir.  $y = 1$ ; down6.  $F(\frac{1}{4}, 0)$ ; dir.  $x = -\frac{1}{4}$ ; right 7.  $F(-\frac{1}{4}, 0)$ ;dir.  $x = \frac{1}{4}$ ; leftWritten Exercises, pages 240–241 1.a.  $V(0, 0)$ ; $F(0, 2)$ ; dir.  $y \neq -2$  b.  $V(0, 0)$ ;  $F(2, 0)$ ;dir.  $x = -2$  3.a.  $V(0, 0)$ ;  $F(0, -\frac{1}{8})$ ; dir.  $y = \frac{1}{8}$ b.  $V(0, 0)$ ;  $F(-\frac{1}{8}, 0)$ ; dir.  $x = \frac{1}{8}$  5.  $V(2, 1)$ ; $F(2, 2)$ ; dir.  $y = 0$  7.  $V(4, 7)$ ;  $F(4\frac{1}{4}, 7)$ ;dir.  $y = 3\frac{3}{4}$  9.  $x = \frac{1}{16}y^2$  11.  $x = -\frac{1}{4}y^2$ 13.  $y = -x^2$  15.  $x = -\frac{1}{16}y^2$  17.  $y = \frac{1}{4}x^2 + 1$ 

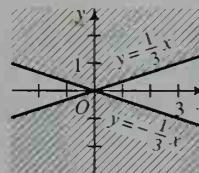
19.a.

21.a.  $\sqrt{x^2 + (y - 6)^2} = y$  b.  $y - 3 = \frac{1}{12}(x - 0)^2$ 23.  $x = \frac{1}{4}y^2 + \frac{3}{2}y + \frac{5}{4}$  25.  $y = \frac{1}{4}x^2 + \frac{7}{2}x + \frac{25}{4}$ 27.  $V(-1, 8)$ ;  $F(-1, \frac{95}{12})$ ; dir.  $y = \frac{97}{12}$ 29.  $V(4, -1)$ ;  $F(4, 0)$ ; dir.  $y = -2$  31.  $V(-1, 3)$ ; $F(-5, 3)$ ; dir.  $x = 3$ 

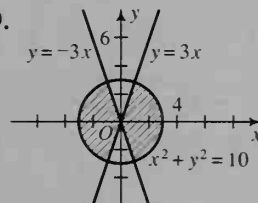
Class Exercises, page 244 1. Ans. may vary.

No sol.:  $(x + 4)^2 + y^2 = 9$ ;  $y^2 = x$ .1 sol.:  $(x + 3)^2 + y^2 = 9$ ;  $y^2 = x$ .2 sols.:  $(x + 2)^2 + y^2 = 9$ ;  $y^2 = x$ .3 sols.:  $(x - 3)^2 + y^2 = 9$ ;  $y^2 = x$ .4 sols.:  $(x - 4)^2 + y^2 = 9$ ;  $y^2 = x$ .2. The graph of  $x^2 + y^2 = 9$  is a circle. The graph of  $(x + y)^2 = 9$  is two  $\parallel$  lines. 3. The graph of  $x^2 - y^2 = 4$  is a hyperbola. The graph of  $(x - y)^2 = 4$  is two  $\parallel$  lines.Written Exercises, pages 244–246 1.  $(2, \pm 2\sqrt{3})$ 3.  $(4, 2)$ ,  $(0.8, -4.4)$  5. Tangent at  $(2, -1)$ 7.a. 4 b.  $9x^2 + 9y^2 = 162$ ; circle c. An intersection pt. must satisfy both equations, and  $\therefore$  satisfies the eq. of their sum. 9.  $(0, \pm 2)$ 11.  $(0, -2)$ ,  $(\pm \frac{2\sqrt{17}}{3}, \frac{16}{9})$  13.  $(4, -3)$ ,  $(-4, 3)$ , $(3, -4)$ ,  $(-3, 4)$  15. No solution 17.  $(-1, -4)$ 19. No solution 21.  $(-2, \pm \sqrt{5})$  23.a.  $(4, 2)$ ; $y = -2x + 10$  b.  $y = -2x + 10$ 

25.



29.

27.  $(5, -3)$ 31.  $\approx 10.6$  cm

**Class Exercises, page 249** 1.a. A: 4; B: 6; C: 8;

P:  $|x - 3|$  b. A: 8; B: 10; C: 5; P:  $|y + 6|$

2.a.  $\sqrt{x^2 + y^2} = 2\sqrt{(x - 7)^2 + (y - 2)^2}$

b.  $\sqrt{x^2 + y^2} = \frac{1}{2}\sqrt{(x + 3)^2 + (y - 4)^2}$

c.  $\sqrt{(x + 5)^2 + (y + 1)^2} = |y + 6|$

d.  $\sqrt{(x + 3)^2 + (y - 4)^2} = 2|y + 6|$

e.  $\sqrt{(x - 7)^2 + (y - 2)^2} = \frac{1}{2}|y + 6|$

3. Hyperbola 4.a. Hyperbola b. Ellipse

c. Parabola d. Circle 5. The pt. (0, 0) 6. Two

lines,  $x \pm 3y = 0$  7. No graph

**Written Exercises, pages 250–251**

1.a.  $\frac{x^2}{4} + \frac{y^2}{3} = 1$  b. (1, 0)

c.  $\sqrt{(x - 1)^2 + y^2} = 0.5|x - 4|$  simplifies to

$\frac{x^2}{4} + \frac{y^2}{3} = 1$  3.  $\sqrt{(x + 6)^2 + y^2} = 2|x + \frac{3}{2}|$

simplifies to  $\frac{x^2}{9} - \frac{y^2}{27} = 1$  5. Ellipse

7. Hyperbola 9. Ellipse 11. No graph

13. 0.016 15.  $(\pm 3, 0)$ ,  $(0, \pm 3)$  17.  $(2\sqrt{5}, 2\sqrt{5})$ ,

$(-2\sqrt{5}, -2\sqrt{5})$ ,  $(6, -2)$ ,  $(-6, 2)$  19.  $(\pm\sqrt{3}, 0)$ ,

$(2, -1)$ ,  $(-2, 1)$

**Chapter Test, page 253** 1. Let the vertices of the

square be  $A(a, 0)$ ,  $B(a, a)$ ,  $C(0, a)$  and  $O(0, 0)$ .

Slope of  $\overline{AC} = -1$ ; slope of  $\overline{OB} = 1$ ;  $\therefore \overline{AC} \perp \overline{OB}$ .

$AC = a\sqrt{2}$ ;  $OB = a\sqrt{2}$ ;  $\therefore AC = OB$ . 2. (0, -2),

(2, 0) 3.  $x^2 + (y - 2)^2 = 13$  4.  $V(0, \pm 3)$ ;

$F(0, \pm 2)$  5. (3, 0), (0, 2) 6.  $\frac{x^2}{9} + \frac{y^2}{16} = 1$

7.a.  $\frac{y^2}{9} - \frac{x^2}{4} = 1$

b.  $\frac{x^2}{4} - \frac{y^2}{16} = 1$

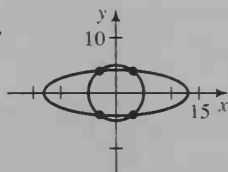
8. See graph at right.

9.  $V(0, 0)$ ;  $F(0, \frac{3}{2})$ ;

$y = -\frac{3}{2}$

10.  $x = \frac{1}{12}y^2$

11.



Intersect at  $(3, \pm 4)$ ,  
and  $(-3, \pm 4)$

12.a. Circle

b. Ellipse

c. Parabola

13.  $\sqrt{(x - 1)^2 + (y - 3)^2} = |y + 4|$ ;

$y + \frac{1}{2} = \frac{1}{14}(x - 1)^2$  14. If  $B = C = 0$ , the graph

is the y-axis, and if  $A = B = 0$ , the graph is the

x-axis. If  $A = C = 0$ , the graph is the x- and

y-axes. If only  $A = 0$ , the graph is two lines,  $y = 0$

and  $Bx + Cy = 0$ ; if only  $C = 0$ , the graph is two

lines,  $x = 0$  and  $Ax + By = 0$ .

**Cumulative Review, pages 254–255**

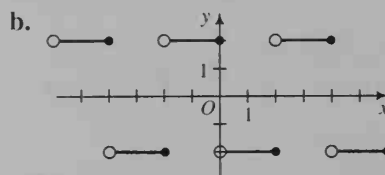
1. Range:  $\{y | y \geq -1\}$ ; zeros: 0, 2

2.a. 0 b.  $2x - 2$  c.  $-x^2 + 2x - 1$  d.  $-1, x \neq 1$

e.  $-x$  f.  $2 - x$  3.a. iii b. ii c. i, ii, iv

4.a.  $p = 6$ ;  $A = 2$  b.  $p = 4$ ;  $A = 2$

5.a. Shift the graph of  $y = f(x)$  left one unit.



6.a.  $f^{-1}(x) = \frac{8 - x}{2}$ ;

$f(f^{-1}(x)) = 8 - 2\left(\frac{8 - x}{2}\right) = 8 - 8 + x = x$ ;

$f^{-1}(f(x)) = \frac{8 - (8 - 2x)}{2} = \frac{2x}{2} = x$  b. No inverse

c.  $f^{-1}(x) = \frac{1 + 2x}{x}$ ;

$f(f^{-1}(x)) = \frac{1}{\frac{1 + 2x}{x} - 2} = \frac{x}{1 + 2x - 2x} = x$ ;

$f^{-1}(f(x)) = \frac{1 + 2\left(\frac{1}{x - 2}\right)}{\frac{1}{x - 2}} = \frac{x - 2 + 2}{1} = x$

7.a.  $S(x, y) = 6x + 4y$

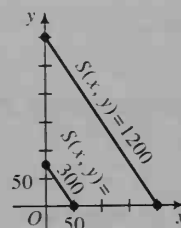
b.  $S(40, 25) = 340$ ;

$S(32, 48) = 384$

c. See graph at right.

8.a.  $\frac{9}{16}$  b.  $2y^{10}$  c.  $\frac{x}{18}$

d.  $\frac{8}{9}$  e. 9 f.  $1 - a$



9.a.  $\frac{3}{4}$  b. 1000 c.  $\frac{6}{5}, \frac{4}{5}$  10.  $f(x) = \frac{1}{2}(\sqrt{3})^x$

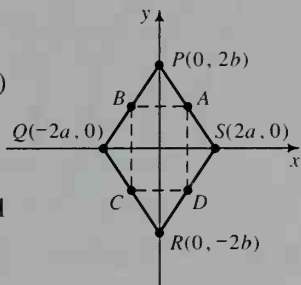
11.a. 3 h b.  $N(t) = 150 \cdot 2^{t/3}$  c. 38,400

12.a. \$107.19 b. \$107.23 c. \$107.25 13. Plan A

14.a. -2 b. 4 c.  $\frac{1}{3}$  d. -5 e. 9 f. 50

15.a.  $2 \log M + 3 \log N$  b.  $\frac{3}{2} \log M - \frac{1}{2} \log N$

- c.  $2 + \log M + \frac{1}{2} \log N$  16.a.  $y = 100x$  b.  $y = \frac{2}{x^3}$   
 c.  $y = 10 \cdot 10^{2.5x}$  17.a. 2.08 b. 4.64 c. 17.67  
 18. See diagram below.  $PQRS$  is a rhombus.  
 $A = (a, b)$ ,  $B = (-a, b)$ ,  $C = (-a, -b)$ ,  
 $D = (a, -b)$ . Slope of  $\overline{AB} = 0$ , slope of  $\overline{BC}$  is  
 undef.;  $\therefore \overline{AB} \perp \overline{BC}$ .  $AB = DC = 2a$ ,  
 $BC = AD = 2b$ ;  $\therefore ABCD$  is a parallelogram. A parallelogram  
 with 1 rt.  $\angle$  is a rect.;  
 $\therefore ABCD$  is a rect.



Ex. 18

**Class Exercises, page 260** 1.a.  $\pi$  b.  $\frac{\pi}{2}$  c.  $\frac{7\pi}{4}$

- d.  $\frac{\pi}{3}$  e.  $\frac{2\pi}{3}$  f.  $\frac{4\pi}{3}$  g.  $\frac{\pi}{6}$  h.  $\frac{\pi}{180}$  2.a.  $360^\circ$   
 b.  $180^\circ$  c.  $90^\circ$  d.  $45^\circ$  e.  $135^\circ$  f.  $300^\circ$  g.  $330^\circ$   
 h.  $150^\circ$  3. Ans. may vary. a.  $370^\circ$ ,  $-350^\circ$   
 b.  $460^\circ$ ,  $-260^\circ$  c.  $355^\circ$ ,  $-365^\circ$  d.  $40^\circ$ ,  $-320^\circ$   
 e.  $3\pi$ ,  $-\pi$  f.  $\frac{5\pi}{2}$ ,  $-\frac{3\pi}{2}$  g.  $\frac{5\pi}{3}$ ,  $-\frac{7\pi}{3}$  h.  $\pm 2\pi$   
 4.a.  $60^\circ$  b.  $\theta = \frac{\pi}{3} + 2n\pi$  5.a. 1 b. 2 c. 0.75  
 6.a.  $600^\circ$  b.  $-990^\circ$

**Written Exercises, pages 261–262** 1.a.  $\frac{7\pi}{4}$  b.  $\frac{5\pi}{4}$

- c.  $\frac{\pi}{12}$  d.  $-\frac{\pi}{4}$  3.a.  $-\frac{2\pi}{3}$  b.  $-\frac{4\pi}{3}$  c.  $\frac{5\pi}{3}$  d.  $2\pi$   
 5.a.  $-90^\circ$  b.  $240^\circ$  c.  $-135^\circ$  d.  $-30^\circ$  7.a.  $180^\circ$   
 b.  $-270^\circ$  c.  $120^\circ$  d.  $210^\circ$  9.a. 1.2 b. 0.75  
 11.a. 1.66 b. 1.92 c. 1.66 d. 2.08 13.a.  $91^\circ 40'$   
 or  $91.7^\circ$  b.  $97^\circ 20'$  or  $97.4^\circ$  c.  $69^\circ 20'$  or  $69.3^\circ$   
 d.  $75^\circ 40'$  or  $75.6^\circ$  15. 0.5 Exs. 17–21: Ans. may  
 vary. 17. a.  $140^\circ$ ,  $-220^\circ$  b.  $300^\circ$ ,  $-420^\circ$  c.  $\frac{9\pi}{4}$ ,  
 $-\frac{7\pi}{4}$  d.  $\frac{4\pi}{3}$ ,  $-\frac{8\pi}{3}$  19.a.  $388.5^\circ$ ,  $-331.5^\circ$   
 b.  $476.3^\circ$ ,  $-243.7^\circ$  c.  $299.6^\circ$ ,  $-420.4^\circ$  d.  $44.7^\circ$ ,  
 $-675.3^\circ$  21.a.  $0^\circ 30'$ ,  $-359^\circ 30'$  b.  $269^\circ 20'$ ,  
 $-450^\circ 40'$  c.  $363^\circ 21'$ ,  $-356^\circ 39'$  d.  $475^\circ 15'$ ,  
 $-244^\circ 45'$  23.  $29.7^\circ + n \cdot 360^\circ$  25.a.  $12,600^\circ$   
 b. 219.91 27.a.  $900^\circ$  b. 15.71 29.a.  $5256^\circ$   
 b. 91.73 33.a. 69 mi

**Class Exercises, page 264** 1.  $s = \pi$ ; area =  $2\pi$   
 2.  $s = 4\pi$ ; area =  $12\pi$  3.  $s = 8$ ; area = 8  
 4.a. 100 m b. In a large circle, a small arc is very  
 nearly a straight line.

**Written Exercises, pages 264–267** 1.  $s = 3$  cm;  
 area =  $9 \text{ cm}^2$  3.  $r = 5$  cm; area =  $27.5 \text{ cm}^2$   
 5.  $r = 10$  cm;  $s = 5$  cm 7.  $12 \text{ cm}^2$  9. 1.5 cm,  
 2 cm 11. 402,000 km 13.a.  $72,000^\circ$ ; 1257  
 b. 7477 cm c.  $\approx 125 \text{ cm/s}$  15.  $1.3 \times 10^6 \text{ km}$   
 17.  $1885 \text{ m}^2$  19. 350 m 21.  $4.1 \times 10^{13} \text{ km}$   
 25.b.  $r$ ;  $s$

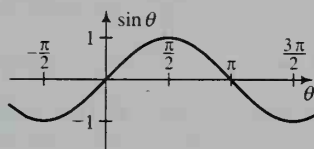
**Class Exercises, page 271** 1.  $\sin \theta = \frac{\sqrt{10}}{10}$ ;  
 $\cos \theta = -\frac{3\sqrt{10}}{10}$  2.  $\sin \theta = -\frac{2\sqrt{5}}{5}$ ;  $\cos \theta = \frac{\sqrt{5}}{5}$   
 3.  $\sin \theta = -\frac{4\sqrt{41}}{41}$ ;  $\cos \theta = -\frac{5\sqrt{41}}{41}$  4.a. Pos.  
 b. Neg. c. Neg. d. Pos. e. Pos. f. Neg. g. Neg.  
 h. Pos. i. Pos. j. Neg. k. Neg. l. Neg. 5.a. Dec.  
 b. Dec. c. Inc. d. Inc. 6.a. Inc. b. Dec. c. Dec.  
 d. Inc. 7. The terminal ray of  $\theta$  intersects the  
 unit circle,  $x^2 + y^2 = 1$ , at  $(\cos \theta, \sin \theta)$ .  
 8.  $\pm 90^\circ$ ,  $\pm 270^\circ$  9.a.  $\theta$  is coterminal with an  
 angle of  $45^\circ$ . b.  $\theta = \frac{\pi}{4} + 2n\pi$

**Written Exercises, pages 272–274** 1.a. 0 b. -1  
 c. -1 d. 0 3.a. 0 b. -1 c. -1 d. 0 5.a. II  
 b. III 7.a.  $\frac{\pi}{2} + 2n\pi$  b.  $\pi + 2n\pi$  c.  $n\pi$   
 d. No sol. 9.a. 0 b. Neg. c. Neg. d. Neg.  
 11.a. Pos. b. Neg. c. Pos. d. Pos. 13.a. Neg.  
 b. Neg. c. 0 d. Pos. 15.a. Pos. b. Neg. c. 0  
 d. Pos. 17.  $\sin \theta = \frac{4}{5}$ ;  $\cos \theta = \frac{3}{5}$  19.  $\sin \theta = \frac{12}{13}$ ;  
 $\cos \theta = -\frac{5}{13}$  21.  $\frac{4}{5}$  23.  $-\frac{7}{25}$  25.  $-\frac{2\sqrt{6}}{5}$   
 27.  $-\frac{\sqrt{7}}{4}$  29.a.  $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ ;  $Q\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$   
 31.c.  $\sin z \approx z$  33.  $>$  35.  $=$  37.  $=$  39.  $>$   
 41.  $\sin 4$ ,  $\sin 3$ ,  $\sin 1$ ,  $\sin 2$  43.a.  $W(2) = (0, 1)$ ;  
 $W(3) = (0, 0)$ ;  $W(4) = (1, 0)$ ;  $W(5) = (1, 1)$   
 b.  $p = 4$

**Computer Exercises, page 274** 1. For  $x = 1$ ,  
 .84147; for  $x = 2$ , .90935; for  $x = 1.5708$ ,  
 1.00000.  $\text{SIN}(1) = .84147$ ;  $\text{SIN}(2) = .90930$ ;  
 $\text{SIN}(1.5708) = 1.00000$ . 2. For  $x = 1$ , .54030; for  
 $x = 2$ , -.41587; for  $x = 3.1416$ , -.97602  
 $\text{COS}(1) = .54030$ ;  $\text{COS}(2) = -.41615$ ;  
 $\text{COS}(3.1416) = -1$

- Class Exercises, pages 278–279** 1.a.  $10^\circ$  b.  $50^\circ$   
 c.  $25.1^\circ$  d.  $\pi - 3 \approx 0.14$  2.  $110^\circ$  3.  $320^\circ$   
 4.a.  $\sin 10^\circ$  b.  $-\sin 30^\circ$  c.  $-\sin 15^\circ$  d.  $\sin 40^\circ$   
 5.a.  $-\cos 20^\circ$  b.  $-\cos 2^\circ$  c.  $-\cos 80^\circ$  d.  $\cos 5^\circ$   
 6.a.  $-0.1392$  b.  $-0.9397$  c.  $0.9848$  d.  $-0.6428$   
 7.a.  $0.9842$  b.  $0.4787$  c.  $-0.8161$  d.  $0.8471$   
 8.a.  $0.1411$  b.  $-0.6536$  c.  $-0.8415$  d.  $-0.4161$   
 9.a.  $-0.8011$  b.  $0.7452$  c.  $0.2555$  d.  $0.3802$   
 10.a.  $\frac{\sqrt{2}}{2}$  b.  $\frac{\sqrt{2}}{2}$  c.  $-\frac{\sqrt{2}}{2}$  d.  $-\frac{\sqrt{2}}{2}$  11.a.  $\frac{1}{2}$   
 b.  $-\frac{1}{2}$  c.  $-\frac{1}{2}$  d.  $\frac{1}{2}$  12.a.  $\frac{1}{2}$  b.  $-\frac{1}{2}$  c.  $\frac{\sqrt{3}}{2}$  d.  $\frac{\sqrt{3}}{2}$   
 13.a.  $-\frac{1}{2}$  b.  $\frac{\sqrt{3}}{2}$  c.  $-\frac{1}{2}$  d.  $-\frac{\sqrt{3}}{2}$  14.a.  $\frac{\sqrt{2}}{2}$   
 b.  $-\frac{\sqrt{3}}{2}$  c.  $-\frac{\sqrt{3}}{2}$  d.  $\frac{\sqrt{3}}{2}$  15. See graph below.

16. The graph of  $y = \cos \theta$  translated  $90^\circ$  to the right coincides with the graph of  $y = \sin \theta$ .



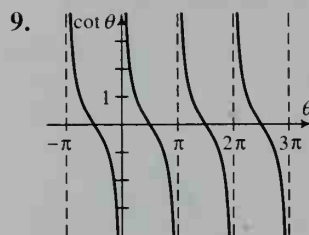
Ex. 15

- 17.a. Symmetry in the origin b. Symmetry in the y-axis 18.  $\cos(90^\circ - \theta) = \cos(-(\theta - 90^\circ)) = \cos(\theta - 90^\circ) = \sin \theta$

- Written Exercises, pages 279–282** 1.a.  $\sin 52^\circ$   
 b.  $-\cos 52^\circ$  c.  $-\sin 37^\circ$  d.  $-\cos 40^\circ$   
 3.a.  $-\cos 44.5^\circ$  b.  $\cos 62^\circ$  c.  $\sin 34.3^\circ$  d.  $\sin 21^\circ$   
 5.a.  $0.4695$  b.  $0.7193$  c.  $0.4179$  d.  $0.4706$   
 7.a.  $0.3420$  b.  $-0.9041$  c.  $0.5314$  d.  $-0.4586$   
 9.a.  $0.9320$  b.  $0.8085$  c.  $-0.9365$  d.  $0.2837$   
 11.a.  $\frac{\sqrt{2}}{2}$  b.  $-\frac{\sqrt{2}}{2}$  c.  $-\frac{1}{2}$  d.  $-\frac{\sqrt{3}}{2}$  13.a.  $\frac{1}{2}$   
 b.  $-\frac{1}{2}$  c.  $-\frac{\sqrt{2}}{2}$  d.  $\frac{\sqrt{3}}{2}$  15.a.  $\frac{\sqrt{3}}{2}$  b.  $\frac{\sqrt{3}}{2}$  c.  $-\frac{1}{2}$   
 d.  $\frac{\sqrt{2}}{2}$  17.a. 1 b.  $-\frac{1}{2}$  c.  $-\frac{\sqrt{3}}{2}$  d.  $-\frac{\sqrt{2}}{2}$   
 21.a. 3 b.  $-1.90, 0, 1.90$  23.a. 2 b.  $-0.82, 0.82$   
 25.a. 2 b.  $45^\circ, 225^\circ$  27. 5395 mi 29. 353 mi  
 31.a. 503 mi/h b. 955 mi/h 33. 771 mi/h

- Class Exercises, page 285** 1.a.  $90^\circ + n \cdot 180^\circ$   
 b.  $90^\circ + n \cdot 360^\circ$  c.  $135^\circ + n \cdot 180^\circ$  d.  $n \cdot 180^\circ$   
 2.a. 1.035 b.  $-1.035$  c.  $-1.035$  d. 1.035  
 3.a.  $-2.185$  b. 11.43 c. 7.086 d. 1.079 4. III  
 5.a.  $-\frac{3}{5}$  b.  $-\frac{4}{3}$  c.  $-\frac{3}{4}$  d.  $-\frac{5}{3}$  e.  $\frac{5}{4}$

- Written Exercises, pages 285–286** 1.a.  $-5.671$   
 b.  $-0.1051$  c.  $-1.043$  d.  $-1.855$  3.a.  $-\sin 15^\circ$   
 b.  $\sec 80^\circ$  c.  $\tan 40^\circ$  d.  $-\sec(\pi - 2) \approx -\sec 1.14$   
 5.a.  $-\tan 80^\circ$  b.  $\sec 70^\circ$  c.  $\cot 5^\circ$   
 d.  $-\csc(4 - \pi) \approx -\csc 0.86$  7.a.  $n\pi$  b. None  
 c.  $\frac{\pi}{2} + 2n\pi$  d.  $\frac{3\pi}{2} + 2n\pi$



9. 11. Infinitely many  
 13.  $\cos x = -\frac{12}{13}$ ;  
 $\tan x = -\frac{5}{12}$ ;  
 $\csc x = \frac{13}{5}$ ;  
 $\sec x = -\frac{13}{12}$ ;  
 $\cot x = -\frac{12}{5}$   
 15.  $\sin x = -\frac{3}{5}$ ;  $\cos x = -\frac{4}{5}$ ;  $\csc x = -\frac{5}{3}$ ;  
 $\sec x = -\frac{5}{4}$ ;  $\cot x = \frac{4}{3}$  17.  $\sin x = \frac{2\sqrt{2}}{3}$ ;  
 $\cos x = -\frac{1}{3}$ ;  $\tan x = -2\sqrt{2}$ ;  $\csc x = \frac{3\sqrt{2}}{4}$ ;  
 $\cot x = -\frac{\sqrt{2}}{4}$  19.  $\cot x$  is undef. for  $x = n\pi$ , and  
 as  $x$  gets close to  $n\pi$ ,  $|\cot x|$  increases. 21.b. All  
 real nos. except for odd multiples of  $\frac{\pi}{2}$  23.a.  $-\frac{1}{2}$   
 b.  $-2$  c.  $\frac{\sqrt{3}}{2}$  d.  $-\sqrt{3}$  25.a. 1 b.  $-1$  c.  $\sqrt{3}$   
 d. Undef. 27.a. Undef. b.  $-\sqrt{3}$  c. 0 d.  $-\frac{2\sqrt{3}}{3}$

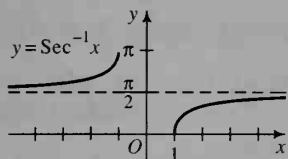
- Class Exercises, page 289** 1.  $50.2^\circ$  2.  $-17.5^\circ$   
 3.  $115.2^\circ$  4.  $-1.24$  5. 0.85 6. 1.51 7. An  
 error message; 1.7 is not in the domain of  
 $y = \sin^{-1} x$ . 8.a.  $\frac{3}{5}$  b.  $\frac{4}{3}$  c.  $\frac{3}{4}$

- Written Exercises, pages 289–291** 1.a.  $64.2^\circ$   
 b.  $-64.2^\circ$  c.  $41.4^\circ$  d.  $138.6^\circ$  3.a. 0.23 b.  $-0.23$   
 c. 1.22 d. 1.92 5.a. 0 b.  $\frac{\pi}{2}$  c.  $\frac{\pi}{4}$  d.  $-\frac{\pi}{4}$   
 7.a.  $\frac{\pi}{6}$  b.  $-\frac{\pi}{6}$  c.  $\frac{\pi}{3}$  d.  $\frac{2\pi}{3}$  9. 3 is not in the  
 domain of  $y = \cos^{-1} x$ . Error message appears.  
 11.a.  $0.42$ ;  $\frac{5}{12}$  b.  $-0.42$ ;  $-\frac{5}{12}$  13.a. 1.38;  $\frac{29}{21}$   
 b. 1.15;  $\frac{2\sqrt{3}}{3}$  15.a.  $-1 \leq x \leq 1$ ;  $-1 \leq y \leq 1$ ;  
 $y = x$ , for  $-1 \leq x \leq 1$  c. No 19.a. True b. Not  
 true 21.a. True b. Not true



23. Dom.:  $\{x|x \leq -1 \text{ or } x \geq 1\}$ ;

Range:  $\{y|0 \leq y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y \leq \pi\}$



25.a.  $\text{Sec}^{-1} 2 = \text{Cos}^{-1} \frac{1}{2} = \frac{\pi}{3}$

b.  $\text{Sec}^{-1} x = \text{Cos}^{-1} \frac{1}{x}$  27.  $\pi$

Chapter Test, page 293 1.a.  $\frac{3\pi}{2}$  b. 3.35

2.a.  $300^\circ$  b.  $143^\circ 10'$  or  $143.2^\circ$  3. Ans. may vary.

a.  $160^\circ$ ,  $-560^\circ$  b.  $673.2^\circ$ ,  $-46.8^\circ$  c.  $\frac{17\pi}{6}$ ,  $-\frac{7\pi}{6}$

d.  $502^\circ 10'$ ,  $-217^\circ 50'$  4.  $s = 4.2$  cm;

area =  $10.5 \text{ cm}^2$  5. 3480 km 6.a.  $\sin \theta = \frac{\sqrt{2}}{2}$ ;

$\cos \theta = -\frac{\sqrt{2}}{2}$  b.  $\sin \theta = -\frac{\sqrt{3}}{2}$ ;  $\cos \theta = -\frac{1}{2}$

c.  $\sin \theta = 1$ ;  $\cos \theta = 0$  7.a.  $<$  b.  $>$  c.  $=$  d.  $<$

8.a.  $-\frac{\sqrt{2}}{2}$  b. 0 c.  $\frac{1}{2}$  d.  $\frac{\sqrt{3}}{2}$  9.a.  $-1$  b.  $-\frac{\sqrt{3}}{3}$

c. 2 d. Undef. 10.  $\sin x = \frac{\sqrt{10}}{10}$ ;  $\cos x = -\frac{3\sqrt{10}}{10}$ ;

$\csc x = \sqrt{10}$ ;  $\sec x = -\frac{\sqrt{10}}{3}$ ;  $\cot x = -3$

11.a.  $\frac{\pi}{3}$  b.  $\sqrt{3}$  c.  $-\frac{5}{3}$  12. The restriction is

necessary to make  $f(x) = \sin x$  one-to-one.

Class Exercises, page 299 1.  $60^\circ$ ,  $300^\circ$

2. No sol. 3.  $30^\circ$ ,  $150^\circ$  4.  $135^\circ$ ,  $315^\circ$

5.  $\frac{5\pi}{6}$ ,  $\frac{7\pi}{6}$  6.  $\frac{\pi}{4}$ ,  $\frac{5\pi}{4}$  7.  $\frac{2\pi}{3}$ ,  $\frac{5\pi}{3}$  8. No sol.

9.  $(180 + 360n)^\circ$  10.  $(225 + 360n)^\circ$ ,  $(315 + 360n)^\circ$  11.  $(45 + 180n)^\circ$  12.  $-0.8391$

13.  $31^\circ$  14.  $9^\circ$

Written Exercises, pages 299–301 1.  $224.4^\circ$ ,

$315.6^\circ$  3.  $50.2^\circ$ ,  $230.2^\circ$  5.  $101.5^\circ$ ,  $258.5^\circ$

7.  $70.5^\circ$ ,  $289.5^\circ$  9.  $146.4^\circ$ ,  $213.6^\circ$  11.  $41.8^\circ$ ,  $138.2^\circ$  13. 2.16, 5.30 15. 3.94, 5.49

17. 0.17, 3.31 19. 0.84, 5.44 21. 2.50, 5.64

23.  $m = -\sqrt{3}$ ;  $y = -\sqrt{3}x + 3 + 2\sqrt{3}$  25.  $149^\circ$

27.  $169^\circ$  29.  $37^\circ$  31.  $\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$  33.  $\frac{\pi}{2}$  35.  $\frac{\pi}{3}$ ,  $\frac{4\pi}{3}$

37. Ellipse;  $\frac{\pi}{4}$  39. Ellipse;  $\frac{\pi}{4}$

41. Hyperbola;  $\frac{\pi}{4}$  43. Parabola;  $\frac{\pi}{4}$

45.a.  $0^\circ \leq \alpha \leq 90^\circ$ ,  $0^\circ \leq \beta \leq 48.6^\circ$  b. No

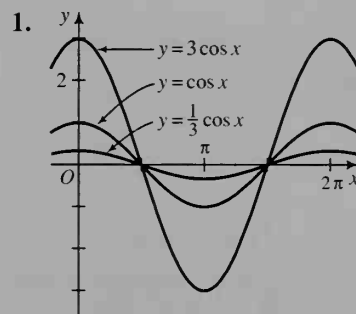
Class Exercises, page 304 1.  $p = \pi$ ;  $A = 4$

2.  $p = 4\pi$ ;  $A = 3$  3.  $p = 7$ ;  $A = 5$  4.  $p = 3$ ;

$A = 6$  5.  $p = 4$ ;  $A = 3$ ;  $y = 3 \sin \frac{\pi}{2}x$  6.  $p = 4\pi$ ;

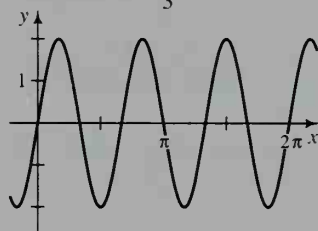
$A = 2$ ;  $y = -2 \cos \frac{1}{2}x$  7.a. 1 b. 4 c. 3

### Written Exercises, pages 305–308



5.  $A = 2$ ;  $p = \frac{2\pi}{3}$

7.  $A = 2$ ;  $p = \pi$



9.  $A = \frac{1}{2}$ ;  $p = 1$

11.  $A = 4$ ;  $p = \pi$ ;  
 $y = 4 \sin 2x$

13.  $A = 3$ ;  $p = 4\pi$ ;  
 $y = 3 \cos \frac{1}{2}x$

15.  $A = 3$ ;  $p = 4$ ;  $y = 3 \sin \frac{\pi}{2}x$

17.  $y = \pm 4 \sin \frac{\pi}{6}x$  19.a.  $\pi$  or 3.14 b.  $\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$  or

1.57, 4.71 c.  $\frac{\pi}{3}$ ,  $\pi$ ,  $\frac{5\pi}{3}$  or 1.05, 3.14, 5.24

21. 0.72, 2.42, 3.86, 5.56 23. No sol. 25. 0.16, 1.84, 4.16, 5.84 31.a. 200 MHz

b. Max. 200.01 MHz; min. 199.99 MHz

33.b.  $\approx 3263$  c. 162 g

Class Exercises, page 312 1. 2 2. 4 3.  $y = 5$

4.a. Horiz. 0; vert. 5 b.  $y = 5 + 2 \sin \frac{\pi}{2}x$

5.a. Horiz. 1; vert. 5 b.  $y = 5 + 2 \cos \frac{\pi}{2}(x - 1)$

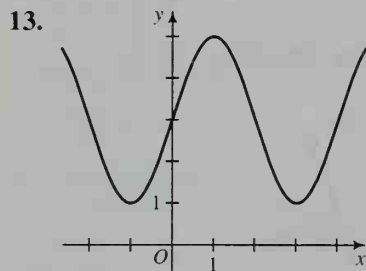
6. 3:  $p = \frac{2\pi}{3}$ ; 4:  $A = 4$ ; 5: horiz. shift is left 5;

6: vert. shift is up 6.

# Written Exercises, pages 313–316

1.  $y = 2 + 2 \cos \frac{\pi}{2}x$  or  $y = 2 + 2 \sin \frac{\pi}{2}(x - 3)$
3.  $y = 2 + \cos \frac{\pi}{3}(x - 1)$  or  $y = 2 + \sin \frac{\pi}{3}(x + \frac{1}{2})$
5.  $y = 3 \cos 2(x - \frac{5\pi}{12})$  or  $y = 3 \sin 2(x - \frac{\pi}{6})$
7.  $y = 3 + 2 \cos 8(x - \frac{\pi}{8})$  or

$$y = 3 + 2 \sin 8(x - \frac{\pi}{16})$$



15.  $y = 30 + 28 \cos(\frac{5\pi}{31}t)$  17.a.  $p = 365$ ,  $A = 3$ ,

$y = 12 + 3 \sin \frac{2\pi}{365}(x - 80)$  b. 9.1 h; 14.9 h

c. 123rd day (May 3) through 220th day (Aug. 8)

d. Greater 19.a.  $T = 20 + 8 \cos \frac{2\pi}{365}(x - 197)$

b. 1st day (Jan. 1) through 66th day (Mar. 7),  
328th day (Nov. 24) through 365th day (Dec. 31)

21.a.  $y = -0.03x + 18.38$

b.  $y = 17.83 + 1.65 \cos \frac{2\pi}{365}(x - 172)$

25.  $h = 7 + 6 \cos \frac{2\pi}{15}(t - 7.5)$

**Class Exercises, page 320** 1.a.  $\cos^2 \theta$  b.  $\sin^2 \theta$

2.a.  $\tan^2 \theta$  b. 1 3.a.  $\cot^2 \theta$  b. 1 4.a.  $\sin \theta$

b.  $\cot A$  c.  $\sin x$  5.a.  $\cos^2 x$  b.  $-\cos^2 x$  c.  $\tan^2 x$

6.a. 1 b.  $\cos y$  c.  $-1$  7.a. 1 b. 1 c. 1 8. Mult.

num. and den. by: a.  $t$ ;  $\tan A$  c.  $xy$ ;  $\cos \theta \sin \theta$

d.  $x$ ;  $\cos \theta$

**Written Exercises, pages 321–322** 1.a. 1 b.  $\sin^2 \theta$

c.  $-\cos^2 \theta$  3.a.  $\csc^2 A$  b.  $\cot^2 A$  c. 1 5.a.  $\sin \theta$

b. 1 c.  $\sin^2 \theta$  7.  $\sec A$  9. 1 11.  $\tan x$

13.  $\cot x$  15. 2 17.  $\csc \theta$  19.  $\cos y$

21.  $2 \sec \theta$  23. 1 27. Dom.: all real numbers;

$y = 1$  41.  $\pm \frac{1}{\sqrt{1 - \sin^2 \theta}}$

**Written Exercises, pages 326–327** 1.  $70.5^\circ$ ,  $109.5^\circ$ ,

$250.5^\circ$ ,  $289.5^\circ$  3.  $30^\circ$ ,  $150^\circ$ ,  $210^\circ$ ,  $330^\circ$  5.  $30^\circ$ ,

$150^\circ$ ,  $41.8^\circ$ ,  $138.2^\circ$  7.  $60^\circ$ ,  $70.5^\circ$ ,  $289.5^\circ$ ,  $300^\circ$

9.  $\frac{\pi}{4}$ ,  $\frac{5\pi}{4}$  11. 0,  $\frac{\pi}{2}$ ,  $\pi$  13.  $\frac{\pi}{3}$ ,  $\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$ ,  $\frac{5\pi}{3}$

15.  $\frac{3\pi}{4}$ ,  $\frac{7\pi}{4}$  17. 0,  $\frac{\pi}{3}$ ,  $\pi$ ,  $\frac{5\pi}{3}$  19.  $\frac{\pi}{6}$ ,  $\frac{5\pi}{6}$ ,  $\frac{7\pi}{6}$ ,  $\frac{11\pi}{6}$

21. 0.67, 2.48 23. 0.34, 2.80 25. 0,  $\frac{\pi}{3}$ ,  $\pi$ ,  $\frac{5\pi}{3}$

27. 0, 5.36 29. 0.74 31. 0, 1.90 33. 1.52

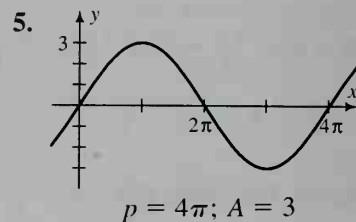
## Chapter Test, pages 328–329

1.  $18^\circ$

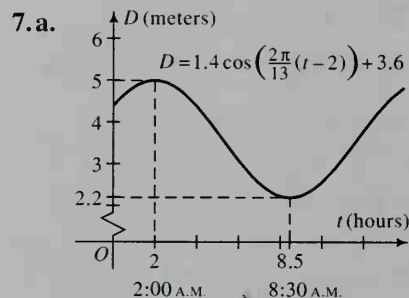
2.  $101.5^\circ$ ,  $258.5^\circ$

3. 3.39, 6.03

4.  $\frac{\pi}{12}$ ,  $\frac{\pi}{4}$ ,  $\frac{3\pi}{4}$ ,  
 $\frac{11\pi}{12}$ ,  $\frac{17\pi}{12}$ ,  $\frac{19\pi}{12}$



6.  $y = 1 + 2 \sin \frac{\pi}{2}x$



b. Ans. may vary. For example: The difference in time between high and low tide is one-half the period. 8.a.  $\sin A$  b.  $\csc \theta$  c.  $\cos x$  d.  $\tan \alpha$

9.a. 1.11, 4.25 b.  $\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$  10.a.  $30^\circ$ ,  $90^\circ$ ,  $150^\circ$

b.  $26.6^\circ$ ,  $90^\circ$ ,  $206.6^\circ$ ,  $270^\circ$

**Class Exercises, page 333** 1.  $\sin = \frac{\text{opp.}}{\text{hyp.}}$ ;

$\cos = \frac{\text{adj.}}{\text{hyp.}}$ ;  $\tan = \frac{\text{opp.}}{\text{adj.}}$ ;  $\cot = \frac{\text{adj.}}{\text{opp.}}$ ;  $\sec = \frac{\text{hyp.}}{\text{adj.}}$ ;

$\csc = \frac{\text{hyp.}}{\text{opp.}}$  2.a.  $\sin \alpha = \frac{x}{z}$ ;  $\cos \alpha = \frac{y}{z}$ ;  $\tan \alpha = \frac{x}{y}$

b.  $\sin \beta = \frac{y}{z}$ ;  $\cos \beta = \frac{x}{z}$ ;  $\tan \beta = \frac{y}{x}$  c. (1) T;

(2) T; (3) T 3.  $\cot 28^\circ$  4.a. 85.2 b. Use the

Pythagorean theorem 5.a.  $\sin 50^\circ = \frac{x}{10}$ ;

$\csc 50^\circ = \frac{10}{x}$  b.  $\cos 32^\circ = \frac{5}{x}$ ;  $\sec 32^\circ = \frac{x}{5}$

c.  $\tan 55^\circ = \frac{x}{8}$ ;  $\cot 55^\circ = \frac{8}{x}$

d.  $\cos 70^\circ = \frac{3}{x}$ ;  $\sec 70^\circ = \frac{x}{3}$  6.a.  $\tan \theta = \frac{3}{2}$ ;  
 $\cot \theta = \frac{2}{3}$  b.  $\cos \theta = \frac{5}{8}$ ;  $\sec \theta = \frac{8}{5}$  c.  $\sin \theta = \frac{7}{12}$ ;  
 $\csc \theta = \frac{12}{7}$  d.  $\cos \theta = \frac{7}{20}$ ;  $\sec \theta = \frac{20}{7}$

**Written Exercises, pages 334–338** 1.  $b = 7.61$ ;  
 $c = 16.3$  3.  $d = 43.3$ ;  $f = 42.3$  5.a.  $\frac{5}{13}$  b.  $\frac{5}{13}$   
c.  $\frac{5}{12}$  d.  $\frac{5}{12}$  e.  $\frac{13}{12}$  f.  $\frac{13}{12}$  7.  $36.9^\circ$ ,  $53.1^\circ$  9.a.  $\sqrt{2}$   
b. (1) 1; (2)  $\frac{\sqrt{2}}{2}$ ; (3)  $\frac{\sqrt{2}}{2}$  c. (1) 1; (2) 0.707;  
(3) 0.707 11. Each length is the tan of the  
corresponding  $\angle$ . 13.a. 333,000 ft b. 335,000 ft  
15. 38.2 ft 17.  $48.2^\circ$ ,  $48.2^\circ$ ,  $83.6^\circ$ ; area = 17.9  
19.  $PA = PB = 15.6$  21.  $126.9^\circ$  23. 21.5 m  
25.a. 76.4 ft b.  $AH = 196$  ft;  $HC = 97.1$  ft  
c. 8.3 ft/s d. 34.7 ft/s; too small

27.a.  $\sec A = \frac{13}{12}$ ;  $\csc A = \frac{13}{5}$  b.  $\frac{1}{\sqrt{1-x^2}}$   
31.  $x = a \sin \beta \cot \alpha$ ;  $y = a \sin \beta \csc \alpha$  33.  $3 \text{ cm}^2$   
37.a. For example,  $\theta = \tan^{-1} \left( \frac{30000}{d} \right)$  b.  $5.4^\circ$   
39.  $35.3^\circ$  41.  $A = nr^2 \sin \theta \cos \theta$

**Class Exercises, page 341** 1.a.  $10 \text{ cm}^2$  b.  $10 \text{ cm}^2$   
2. 10 3. 10 4.  $\frac{5\sqrt{2}}{2}$  5.a. 0.5 b.  $30^\circ$ ,  $150^\circ$   
6.  $\frac{\pi}{3} - 1 \approx 0.047$

**Written Exercises, pages 342–344** 1.a. 5 b. 5  
3.a.  $3\sqrt{2}$  b.  $3\sqrt{2}$  5.  $K = \frac{1}{2}ab$  7. 158 9.  $30^\circ$ ,  
 $150^\circ$  11.  $3200\sqrt{2} \text{ cm}^2$  13.  $21 \text{ cm}^2$   
15.  $K = 6 \sin \theta$ ; Dom.:  $\{\theta | 0 < \theta < \pi\}$ ; Range:  
 $\{K | 0 < K \leq 6\}$  17.b.  $2\frac{2}{3}$  19. 73 21. 13.6  
23. 2.5 radians 27. 22.1 29. 6.64 31.a. area  
 $\triangle BCD = \frac{\sqrt{3}}{4}ax$ ; area  $\triangle ACD = \frac{\sqrt{3}}{4}bx$  b.  $\frac{\sqrt{3}}{4}ab$   
d. Yes 33. 0.56 m from bottom 35. 1.17

**Class Exercises, pages 346–347** 1. the ratio of the  
sine of any  $\angle$  to the length of the opp. side  
2. They all =  $\frac{1}{2}$  3.  $\angle B < 90^\circ$  4.  $b > c$   
5.  $\angle A > \angle B$  6.  $\frac{\sin 60^\circ}{8} = \frac{\sin 25^\circ}{x}$   
7.  $\frac{\sin 48^\circ}{9} = \frac{\sin x^\circ}{7}$  8.  $\frac{\sin 112^\circ}{7} = \frac{\sin 40^\circ}{x}$

**Written Exercises, pages 347–349** 1.a. 0 b. 1 c. 2

d. 1 3.  $\angle C = 75.0^\circ$ ;  $b = 7\sqrt{6}$ ;  $c = 19.1$   
5.  $\angle C = 15.0^\circ$ ;  $a = 4\sqrt{2}$ ;  $c = 2.07$   
7. Either  $\angle A = 115.7^\circ$ ;  $\angle B = 39.3^\circ$ ;  $a = 4.26$ , or  
 $\angle A = 14.3^\circ$ ;  $\angle B = 140.7^\circ$ ;  $a = 1.17$   
9.  $\angle B = 18.9^\circ$ ;  $\angle C = 85.1^\circ$ ;  $c = 12.3$   
11.  $\angle A = 4.0^\circ$ ;  $\angle B = 88.0^\circ$ ;  $a = 0.489$   
13. No sol. 15.  $\angle S = 28.8^\circ$ ;  $\angle T = 11.2^\circ$   
17. 15.5 km 19.  $\frac{25\sqrt{2}}{3}$  21.a. 1580 yd b. 1410 yd  
c. 193 yd/min d. 5.71 knots  
25.b.  $K = \frac{1}{2} \left( \frac{\sin A \sin C}{\sin B} \right) b^2$ ;  $K = \frac{1}{2} \left( \frac{\sin A \sin B}{\sin C} \right) c^2$

**Class Exercises, page 352**

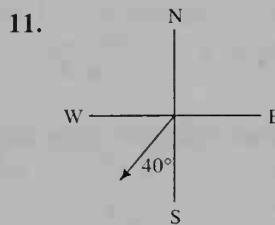
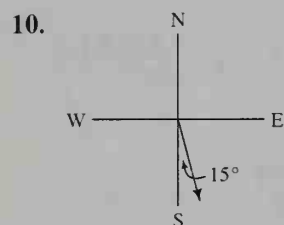
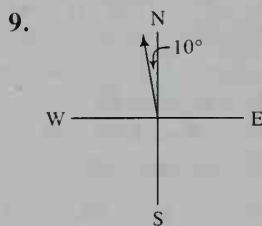
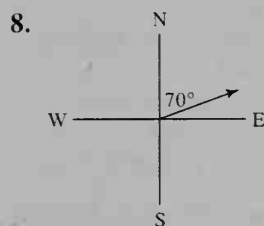
1.  $x^2 = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cos 35^\circ$   
2.  $x^2 = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos 115^\circ$   
3.  $\cos x^\circ = \frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6}$  4. 6.23 5.  $\angle Y = 100.4^\circ$ ;  
 $\angle X = 29.6^\circ$  6.  $\angle X < \angle Z < \angle Y$

**Written Exercises, pages 352–354** 1.  $\angle A = 81.8^\circ$ ;  
 $\angle B = 38.2^\circ$ ;  $c = 7$  3.  $\angle X = 12.7^\circ$ ;  $\angle Y = 77.3^\circ$ ;  
 $\angle Z = 90^\circ$  5.  $\angle P = 20.7^\circ$ ;  $\angle Q = 109.3^\circ$ ;  $r = 6.49$   
7.  $\angle A = 32.2^\circ$ ;  $\angle B = 27.8^\circ$ ;  $\angle C = 120.0^\circ$   
9. 7.07 11.  $\sqrt{10}$  13. 9.74 cm, 13.3 cm 15. 146  
17. 15.8 mi 21.a. 8; 12 b.  $3\sqrt{6}$  23.b.  $x = \frac{c}{2}$

**Mixed Trigonometry Exercises, pages 355–358**

1. Area = 16.1;  $p = 5.57$  3.  $90^\circ$  5.  $r \approx 3.35$ ;  
 $s = \sqrt{6} \approx 2.45$  9.  $\sin A = 0.8$ ;  $\tan A = -\frac{4}{3}$   
11. 17.8 ft 13.  $32.7^\circ$ ,  $147.3^\circ$  15.  $19.5^\circ$ ,  $160.5^\circ$   
17.a.  $a = 14.0$ ;  $e = 8.57$ ;  $f = 14.3$  b. 49:100  
19.  $6\sqrt{5} \approx 13.4$ ,  $2\sqrt{13} \approx 7.21$  21. \$478.77  
23. 1.15 nautical mi  $\approx 1.32$  mi 25.  $\sin A = \frac{\sqrt{2}}{2}$ ;  
 $\sin B = \frac{3}{5}$ ;  $a = 15\sqrt{2}$  27.  $\sin x = -\frac{\sqrt{26}}{26}$ ;  
 $\cos x = \frac{5\sqrt{26}}{26}$  29.a.  $\angle A = 38.2^\circ$ ;  $\angle B = 81.8^\circ$ ;  
 $\angle C = 60^\circ$  b.  $10\sqrt{3} \approx 17.3$  c.  $\frac{5\sqrt{3}}{2} \approx 4.33$   
d.  $\sqrt{21} \approx 4.58$  e.  $\frac{5\sqrt{7}}{3} \approx 4.41$  f.  $\frac{7\sqrt{3}}{3} \approx 4.04$   
g.  $\sqrt{3} \approx 1.73$  31.a.  $\angle A = 25.0^\circ$ ;  $\angle B = 28.1^\circ$ ;  
 $\angle C = 126.9^\circ$ ;  $c = 17$  b. 36 c. 7.2  
d.  $4\sqrt{10} \approx 12.6$  e.  $\frac{36\sqrt{17}}{13} \approx 11.4$  f.  $\frac{85}{8} \approx 10.6$   
g. 2 33.a.  $\approx 2.80$ ;  $\approx 3.92$  43.  $\frac{6}{7}$

**Class Exercises, pages 361–362** 2. \$146 3. 040°  
4. 115° 5. 200° 6. 300° 7.a. 090° b. 270°



12. NE = 045°; SE = 135°; NW = 315°;  
SW = 225° 13. 337.5° 14. 72°

**Written Exercises, pages 362–364** 5. 260°  
7. 2255 km 9. 030°; 17.3 km 11. 18.6 nautical  
mi 13. 26 nautical mi; 358° 15. 125,000 m<sup>2</sup>  
17. 29,600 m<sup>2</sup>

**Chapter Test, page 365** 1. 75.5°; 75.5°; 29.0°  
2. 53.2 m 3. 38.0 in.<sup>2</sup> 4. 448 5.a. 0 b. 1  
6. ≈32.0 km from A; ≈21.3 km from B 7. 97.9°  
8. 3.86 mi 9. Refer to the chart at the top of  
page 352. 10. 332 km

**Class Exercises, page 372** Exs. 1–4: Ans. may  
vary. 1. Let  $x = 60^\circ$ ,  $y = 30^\circ$ . 2. Let  $a = 60^\circ$ ,  
 $b = 0^\circ$ . 3. Yes; let  $b = 0^\circ$ . 4. Yes; let  $b = 0^\circ$ .

5.  $\sin 3^\circ$  6.  $\sin 5^\circ$  7.  $\cos \frac{\pi}{2}$  8.  $\cos 50^\circ$

9.  $\sin(-\alpha) = \sin(0 - \alpha)$

**Written Exercises, pages 373–374** 1. 1 3. 0

5.  $\sin x$  11.  $y = \sin x$  13.  $\frac{\sqrt{6} - \sqrt{2}}{4}$

15.  $\frac{\sqrt{2} - \sqrt{6}}{4}$  17.  $\frac{\sqrt{2} - \sqrt{6}}{4}$  19.  $\frac{\sqrt{6} + \sqrt{2}}{4}$

21.  $\cos \theta$  23.  $\cos x$  25. 0 27.  $\frac{3}{5}$  29.  $\frac{63}{65}$

31. Same graph as  $y = \sin(x + 1)$  33.  $2 \tan \alpha$

35.  $\sin(x + y)$  37.  $\tan x + \tan y$  41.b.  $-\frac{\sqrt{2}}{2}$

43. 0

**Class Exercises, page 377** 1.a.  $-\frac{1}{7}$  b.  $-\frac{1}{7}$

2.a. 1 b.  $\sqrt{3}$  3. The lines are  $\perp$ .

**Written Exercises, pages 377–379** 1.a.  $\frac{7}{4}$  b.  $\frac{1}{8}$

3. 1 5.  $-1$  7. 3 9. Use  $\tan(-\alpha) = \tan(0 - \alpha)$

11.  $2 + \sqrt{3}$ ,  $-2 + \sqrt{3}$  13.  $26.6^\circ$ ,  $153.4^\circ$

15. Graph of  $y = \tan x$  shifted one unit left

17.  $\tan(\alpha + \beta) = 1$  21. Both are true for  $\alpha$  or

$\beta = n\pi$ ,  $n$  an integer. 23.  $\frac{1}{8}$  25.a.  $\frac{63}{65}$  b.  $-\frac{16}{65}$

c.  $-\frac{63}{16}$  27.b.  $y = \frac{1 + \sqrt{10}}{3}x$

**Class Exercises, page 383** 1.  $\sin 20^\circ$  2.  $\frac{\sqrt{3}}{2}$

3.  $\cos 70^\circ$  4.  $\cos 50^\circ$  5.  $\tan 100^\circ$  6.  $\tan 80^\circ$

7.  $\cos^2 x$  8.  $\cos 2x$  9.  $\sin 6\alpha$  10.  $\cos 10\theta$

11. Use the formula for  $\cos \frac{\alpha}{2}$  with  $\alpha = 70^\circ$ .

**Written Exercises, pages 383–385** 1.  $\cos 20^\circ$

3.  $2 \tan 2\beta$  5.  $\sin 70^\circ$  7.  $\tan 50^\circ$  9.  $\cos x$

11.  $\sin 40^\circ$  13.  $\frac{\sqrt{2}}{2}$  15.  $\frac{\sqrt{3}}{2}$  17.  $\frac{1}{4}$

19.  $\sin 2A = \frac{120}{169}$ ;  $\cos 2A = \frac{119}{169}$  21.  $\sin 2A = \frac{24}{25}$ ;

$\sin 4A = \frac{336}{625}$  23.  $\cos 2A = -\frac{23}{25}$ ;  $\cos \frac{A}{2} = \frac{\sqrt{15}}{5}$

25.a.  $\frac{\sqrt{2} - \sqrt{6}}{4}$  b.  $-\frac{\sqrt{2} - \sqrt{3}}{2}$  27. Range:

$\{y | -2 \leq y \leq 1.13\}$ ;  $p = 2\pi$  29. Range:

$\{y | -6 \leq y \leq 4.13\}$ ;  $p = \pi$  39.  $\sin 2x$  41.  $\cos 2x$

43. 2 47.  $-1$  53.c. 4

**Class Exercises, page 389** 1. Substitute

$2 \cos^2 x - 1$  for  $\cos 2x$  and solve algebraically for  $x$ . 2. Solve algebraically for  $x$ .

3. Rewrite as  $\tan x = 1$ ,  $\cos x \neq 0$ , and solve for  $x$ .

4. Rewrite as  $\tan 2x = 1$ ,  $\cos 2x \neq 0$ , and solve for  $2x$ .

5. Rewrite as  $\tan 3x = 1$ ,  $\cos 3x \neq 0$ , and solve for  $3x$ . 6. Solve for  $x - 10^\circ$ . 7.  $2x$  8.  $2x$

9.b.  $\frac{\pi}{4}$  c.  $\frac{\pi}{4} < x \leq \frac{\pi}{2}$  d.  $0 \leq x < \frac{\pi}{4}$

**Written Exercises, pages 389–391** 1.  $0^\circ$ ,  $120^\circ$ ,  $240^\circ$

3.  $45^\circ$ ,  $225^\circ$  5.  $15^\circ$ ,  $75^\circ$ ,  $135^\circ$ ,  $195^\circ$ ,  $255^\circ$ ,  $315^\circ$

7.  $0^\circ$ ,  $30^\circ$ ,  $90^\circ$ ,  $150^\circ$ ,  $180^\circ$ ,  $210^\circ$ ,  $270^\circ$ ,  $330^\circ$  9.b. 0,

$\pi$ ,  $\frac{\pi}{4}$ ,  $\frac{3\pi}{4}$ ,  $\frac{5\pi}{4}$ ,  $\frac{7\pi}{4}$  c.  $0 < x < \frac{\pi}{4}$ ,  $\frac{\pi}{2} < x < \frac{3\pi}{4}$ ,

$\pi < x < \frac{5\pi}{4}$ ,  $\frac{3\pi}{2} < x < \frac{7\pi}{4}$  11.  $0 \leq x < 1.11$ ,

$4.25 < x \leq 6.28$  13.  $0.52 < x < 1.57$ ,

$2.62 < x < 3.67$ ,  $4.71 < x < 5.76$

15.  $0.52 \leq x \leq 1.57$ ,  $2.62 \leq x \leq 4.71$  17.  $15^\circ$ ,  $255^\circ$  19.  $19.1^\circ$ ,  $199.1^\circ$  21.  $0^\circ$ ,  $60^\circ$ ,  $180^\circ$ ,  $300^\circ$   
 23.  $\frac{\pi}{4}$ ,  $\frac{5\pi}{4}$  25.  $0$ ,  $\pi$ ,  $\frac{\pi}{6}$ ,  $\frac{5\pi}{6}$ ,  $\frac{7\pi}{6}$ ,  $\frac{11\pi}{6}$  27.  $\frac{\pi}{6}$ ,  $\frac{5\pi}{6}$ ,  $\frac{7\pi}{6}$ ,  $\frac{11\pi}{6}$  29.  $\frac{\pi}{6}$ ,  $\frac{5\pi}{6}$  31.  $0$  33. a. No c. 3  
 d.  $0.9$  35.  $5$  37.  $\pm 5.41$ ,  $\pm 3.80$ ,  $\pm 2.14$ ,  $0$   
 39.  $-0.42$ ,  $0.62$ ,  $2.40$ ,  $3.91$ ,  $5.50$  41.  $\frac{\pi}{4}$ ,  $\frac{3\pi}{4}$ ,  $\frac{5\pi}{4}$ ,  $\frac{7\pi}{4}$   
 43.  $\frac{\sqrt{5}}{5}$

### Chapter Test, page 393

1. a.  $\frac{1}{2}$  b.  $1$  c.  $\sqrt{3} \cos x$   
 d.  $-\sqrt{2} \sin x$  2.  $\frac{\sqrt{6} + \sqrt{2}}{4}$  3.  $2$  4. Use sum

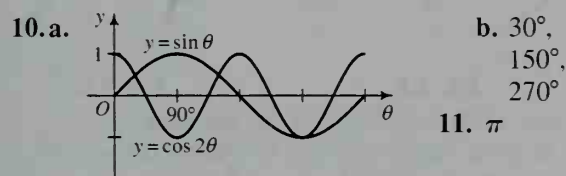
and diff. formulas to show they both  $= \frac{1}{2}$ .

5. Ans. may vary. Use the slopes of the lines and the formula  $\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$  to find one  $\angle$ . Find the second  $\angle$  using the fact that the  $\angle$ s are supplementary. 6. a.  $\frac{3}{5}$  b.  $\frac{7}{25}$  c.  $\frac{24}{25}$  d.  $\frac{336}{625}$

7. a.  $\cot x$  b.  $-2 \tan^2 y$  c.  $\tan \frac{t}{2}$  d.  $\cos x$

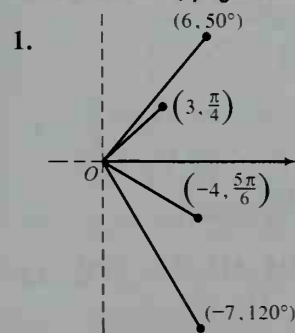
8. a.  $\frac{\sqrt{3}}{2}$  b.  $\sqrt{3}$  9. a.  $(1 + \cot^2 x)(1 - \cos 2x) = \csc^2 x(1 - (1 - 2 \sin^2 x)) = \csc^2 x(2 \sin^2 x) = 2$

$$\begin{aligned} \text{b. } \frac{\sin \theta \sec \theta}{\tan \theta + \cot \theta} &= \frac{\sin \theta}{\cos \theta} \div \left( \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) = \\ &= \frac{\sin \theta}{\cos \theta} \div \left( \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \right) = \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta \sin \theta}{1} = \\ &= \sin^2 \theta = 1 - \cos^2 \theta = \cos^2 \theta + \sin^2 \theta - \cos^2 \theta = \\ &= \cos^2 \theta - \cos 2\theta \end{aligned}$$



11.  $\pi$

### Class Exercises, page 399



Exs. 1–3.: Ans. may vary.

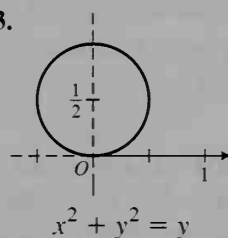
- a.  $(6, 410^\circ)$ ,  $(-6, -130^\circ)$   
 b.  $(7, 300^\circ)$ ,  $(7, -60^\circ)$   
 c.  $(3, \frac{9\pi}{4})$ ,  $(-3, \frac{5\pi}{4})$   
 d.  $(4, \frac{11\pi}{6})$ ,  $(4, -\frac{\pi}{6})$

2. a.  $(5, 315^\circ)$ ,  $(-5, 135^\circ)$  b.  $(2, 120^\circ)$ ,  $(-2, 300^\circ)$   
 c.  $(1, \frac{3\pi}{2})$ ,  $(-1, \frac{\pi}{2})$  d.  $(\frac{3}{2}, 0)$ ,  $(-\frac{3}{2}, \pi)$   
 3.  $(\sqrt{2}, 315^\circ)$  4.  $(-4, -4\sqrt{3})$   
 5.  $(x-1)^2 + y^2 = 1$ ; a circle with  $r = 1$ ,  $C(1, 0)$ ; both graphs are the same.

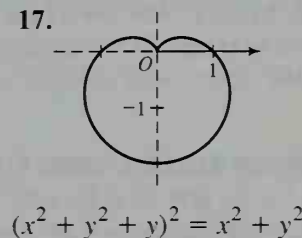
### Written Exercises, pages 400–402

1. a.  $(2, 400^\circ)$ ,  $(-2, 220^\circ)$  b.  $(3, 260^\circ)$ ,  $(-3, 80^\circ)$  c.  $(5, -\frac{\pi}{2})$ ,  $(-5, \frac{\pi}{2})$  d.  $(4, 190^\circ)$ ,  $(-4, 370^\circ)$   
 3. a.  $(2\sqrt{2}, 135^\circ)$  b.  $(5, 0^\circ)$  c.  $(2, -45^\circ)$   
 d.  $(2, 150^\circ)$  5. a.  $(\sqrt{2}, \frac{5\pi}{4})$  b.  $(12, \frac{\pi}{2})$  c.  $(1, \frac{\pi}{3})$   
 d.  $(2, \pi)$  7. a.  $(-2, 2\sqrt{3})$  b.  $(0, -3)$   
 c.  $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$  d.  $(-\sqrt{2}, \sqrt{2})$  9. a.  $(0.940, 0.342)$   
 b.  $(1.879, 0.684)$  c.  $(-0.416, 0.909)$   
 d.  $(-0.416, -0.909)$  11. a.  $(5, 53.1^\circ)$   
 b.  $(2.2, 63.4^\circ)$  c.  $(3.6, 123.7^\circ)$  d.  $(7.2, -56.3^\circ)$

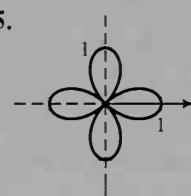
13.



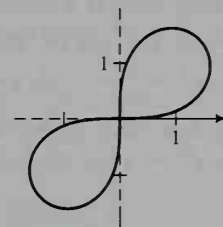
17.



25.



33.



### Computer Exercises, page 402

1. b. Roses with  $k$  leaves d. Roses with  $2k$  leaves e. Roses with  $n$  leaves if  $n$  is odd and  $2n$  leaves if  $n$  is even  
 3. The graph will have  $a$  leaves and will cross the horiz. axis,  $t = n\pi$ ,  $b$  times.

### Class Exercises, page 405

1.  $\sqrt{2} \text{ cis } 45^\circ$   
 2.  $\text{cis } 90^\circ$  3.  $3 \text{ cis } 180^\circ$  4.  $2 \text{ cis } 330^\circ$   
 5.  $2(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$  6.  $5i$  7.  $-3$   
 8.  $2\sqrt{2} + 2i\sqrt{2}$  9.  $3\sqrt{3} + 3i$  10.  $24 \text{ cis } 60^\circ$   
 11.  $10 \text{ cis } \pi$



**Written Exercises, pages 406–407** 1.  $\sqrt{2} \text{ cis } 135^\circ$

3.  $2 \text{ cis } 60^\circ$  5.  $7 \text{ cis } 180^\circ$  7.  $5 \text{ cis } 307^\circ$

9.  $-1.04 + 5.91i$  11.  $-\frac{9}{2} - \frac{9\sqrt{3}}{2}i$

13.  $10 \text{ cis } 90^\circ$ ,  $10i$  15.  $4 \text{ cis } \frac{5\pi}{3}$ ,  $2 - 2i\sqrt{3}$

17.a.  $4\sqrt{3} + 4i$  b.  $z_1 = 4 \text{ cis } 60^\circ$ ,  $z_2 = 2 \text{ cis } 330^\circ$ ,  
 $z_1 z_2 = 8 \text{ cis } 30^\circ = 4\sqrt{3} + 4i$  19.a. 8

b.  $z_1 = 2\sqrt{2} \text{ cis } 45^\circ$ ,  $z_2 = 2\sqrt{2} \text{ cis } 315^\circ$ ,

$z_1 z_2 = 8 \text{ cis } 0^\circ = 8$  21.a.  $-22 + 32i$

b.  $z_1 = 7.211 \text{ cis } 326.31^\circ$ ,  $z_2 = 5.385 \text{ cis } 158.199^\circ$ ,  
 $z_1 z_2 = 38.831 \text{ cis } 484.509^\circ \approx -22 + 32i$  23.  $45^\circ$

25. If  $z = r \text{ cis } \alpha$ ,  $z^2 = r^2 \text{ cis } 2\alpha$ .

27.  $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$

**Class Exercises, page 409** 1.  $4 \text{ cis } 90^\circ$

2.  $8 \text{ cis } 135^\circ$  3.  $4 \text{ cis } (-72^\circ)$  4.  $1 \text{ cis } 360^\circ$

5.  $64 \text{ cis } \frac{\pi}{2}$  6.  $27 \text{ cis } 5\pi$  7.a.  $z = \sqrt{2} \text{ cis } 135^\circ$ ,

$z^6 = 8 \text{ cis } 90^\circ$  b.  $8i$

8.a.  $32 \text{ cis } 3150^\circ = 32 \text{ cis } 270^\circ = -32i$

b.  $32 \text{ cis } (-450^\circ) = 32 \text{ cis } 270^\circ = -32i$  c. Part  
 (b) is easier;  $3150^\circ$  involves a larger multiple of  
 $360^\circ$  than  $-450^\circ$  does.

**Written Exercises, pages 410–411** 1.  $z = \text{cis } 30^\circ$ ,

$z^2 = \text{cis } 60^\circ$ ,  $z^3 = \text{cis } 90^\circ$  3.a.  $z = \sqrt{2} \text{ cis } 315^\circ$

b.  $(\sqrt{2})^{12} \text{ cis } [12(-45^\circ)] = 64 \text{ cis } (-540^\circ) = -64$

5.a.b.  $-8$  7. Yes;  $z^2$ ,  $z^4$ ,  $z^5$ ,  $z$

9.a.  $z^0 = r^0 \text{ cis } (0 \cdot \alpha) = 1 \text{ cis } 0^\circ = 1$

b.  $z^{-1} = \frac{1}{z} = \frac{1}{r} \text{ cis } (-\theta) = r^{-1} \text{ cis } (-1 \cdot \theta)$

11. Ans. may vary.  $z = 1 \text{ cis } 90^\circ$  or  $z = -i$

13.  $e^{i\pi} = 1 \text{ cis } \pi = \cos \pi + i \sin \pi = -1$

**Written Exercises, pages 413–414** 1.  $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ ,

$-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ ,  $-i$  3. 2,  $-1 + i\sqrt{3}$ ,  $-1 - i\sqrt{3}$

5. 2,  $-2$ ,  $2i$ ,  $-2i$  7.b. sum = 0, prod. =  $8i$

9. 2,  $-1 + i\sqrt{3}$ ,  $-1 - i\sqrt{3}$

11.a.  $(z^2 + 2z + 2)(z^2 - 2z + 2)$  b.  $1 + i$ ,  $1 - i$ ,  
 $-1 + i$ ,  $-1 - i$  13.a. Regular pentagon inscribed

in a circle with  $C(0, 0)$  and  $r = 10$  b. Regular  
 nonagon (9-gon) inscribed in a circle with  $C(0, 0)$   
 and  $r = 1$  15.  $1.7895 + 0.3155i$ ,

$-1.1680 + 1.3920i$ ,  $-0.6215 - 1.7075i$

17.  $P(n) = 2n \sin \left( \frac{180^\circ}{n} \right)$

**Chapter Test, page 415**

1.a.  $(3\sqrt{2}, 45^\circ)$  2.a.

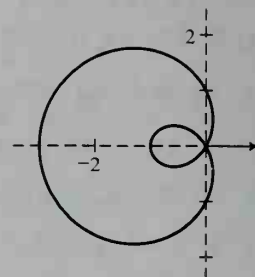
b.  $(0, 6)$

c.  $(2, 270^\circ)$

d.  $(-8, 0)$

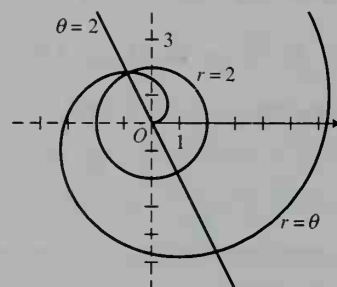
e.  $(-1, -\sqrt{3})$

f.  $(2, 120^\circ)$



b.  $(x^2 + y^2 + 2x)^2 = x^2 + y^2$

3.a,b,c.

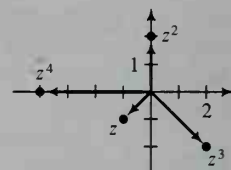
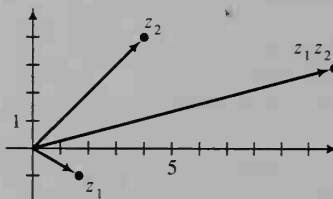


4.a.  $z_1 = 2 \text{ cis } 330^\circ$ ;  $z_2 = 4\sqrt{2} \text{ cis } 45^\circ$ ;  
 $z_1 z_2 = 8\sqrt{2} \text{ cis } 15^\circ$  b. See graph below.

5.a.  $4 \text{ cis } 240^\circ$ ;  $-2 - 2i\sqrt{3}$

b.  $4 \text{ cis } 240^\circ = 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -2 - 2i\sqrt{3}$

6.a.  $\sqrt{2} \text{ cis } 225^\circ$  b. See graph below. c.  $32i$



**Ex. 4.b.**

7. Ans. may vary. Express the complex  
 no. in polar form,  $z = r \text{ cis } \theta$ . The  $n$ th roots are  
 $\sqrt[n]{r} \text{ cis } \left( \frac{\theta}{n} + k \cdot \frac{360^\circ}{n} \right)$  for  $k = 0, 1, 2, \dots, n-1$ .

Since  $\sqrt[n]{z} = z^{1/n}$ , by DeMoivre's Theorem,

$\sqrt[n]{z} = z^{1/n} = r^{1/n} \text{ cis } \left( \frac{1}{n} \cdot \theta \right) = \sqrt[n]{r} \text{ cis } \frac{\theta}{n}$ .

8.  $(\sqrt[6]{2} \text{ cis } 135^\circ)^3 = \sqrt{2} \text{ cis } 405^\circ = \sqrt{2} \text{ cis } 45^\circ =$   
 $\sqrt{2} \left( \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) = 1 + i$ ; the other two cube  
 roots are  $\sqrt[6]{2} \text{ cis } 15^\circ$  and  $\sqrt[6]{2} \text{ cis } 255^\circ$ .

**Ex. 6.b.**

**Cumulative Review, pages 416–417** 1.a.  $325^\circ$ ,

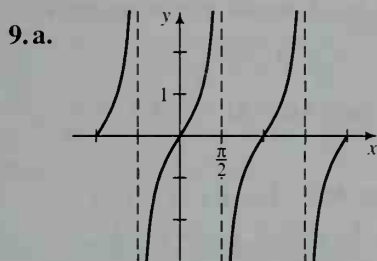
$-395^\circ$  b.  $\frac{17\pi}{6}$ ,  $-\frac{7\pi}{6}$  2.a.  $\frac{11\pi}{12}$  b. 3.63

3. a.  $-210^\circ$  b.  $103^\circ 10'$  or  $103.1^\circ$  4.  $r = 9$  cm;  
 $s = 1.8\pi$  cm 5.  $-\frac{\sqrt{5}}{3}$  6. a.  $-\cos 56^\circ$  b.  $\sin 55^\circ$   
 c.  $-\sin 62^\circ$

7.		sin	cos	tan	csc	sec	cot
a.	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
b.	$225^\circ$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
c.	$-\frac{\pi}{2}$	-1	0	undef.	-1	undef.	0

8.  $\sin x = -\frac{12}{13}$ ;  $\cos x = -\frac{5}{13}$ ;  $\tan x = \frac{12}{5}$ ;

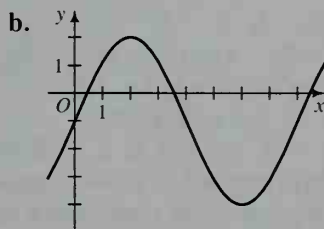
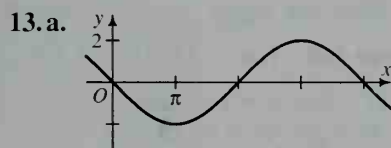
$\csc x = -\frac{13}{12}$ ;  $\sec x = -\frac{13}{5}$



10. a.  $\frac{\pi}{4}$  b.  $\frac{\pi}{3}$   
 c.  $\frac{5\pi}{6}$  d.  $\frac{\pi}{3}$  e. 0  
 f.  $\frac{4}{5}$  g.  $\frac{13}{12}$  h. 0  
 11. 1.16, 5.12  
 12.  $37^\circ$

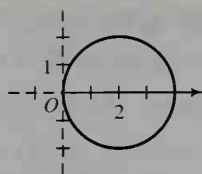
b. Dom.:  $\{x | x \neq \frac{\pi}{2} + n\pi\}$ ;

Range: all real nos.;  $p = \pi$



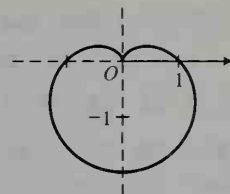
14.  $-\frac{\cos \theta}{1 + \cos \theta}$  15. 0,  $\frac{\pi}{4}$ ,  $\pi$ ,  $\frac{5\pi}{4}$  16.  $a \approx 7.42$ ;  
 $b \approx 11.9$  17.  $23.1 \text{ cm}^2$  18. 6.0 mi 19. 10.5  
 20. 136 m 21.  $\frac{56}{65}$  22.  $\frac{\sqrt{2} + \sqrt{6}}{4}$  23. -3  
 24.  $\sin 2\alpha = \frac{4\sqrt{2}}{9}$ ;  $\cos \frac{1}{2}\alpha = \frac{\sqrt{6}}{3}$  25. (2,  $120^\circ$ );  
 (2,  $-240^\circ$ ), (2,  $480^\circ$ ) 26.  $(-3\sqrt{3}, -3)$

27. a.



b.  $(x - 2)^2 + y^2 = 4$

28.



cardioid

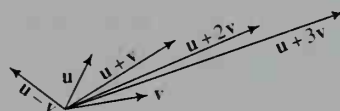
29. a.  $2 + 2i\sqrt{3}$  b.  $z_1 = 2 \text{ cis } 300^\circ$ ;  
 $z_2 = 2 \text{ cis } 120^\circ$ ;  $z_1 z_2 = 4 \text{ cis } 420^\circ$   
 c.  $4 \text{ cis } 420^\circ = 4 \text{ cis } 60^\circ = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$   
 $2 + 2i\sqrt{3}$  30.  $-512i$  31.  $\sqrt[6]{2} \text{ cis } 105^\circ$ ;  
 $\sqrt[6]{2} \text{ cis } 225^\circ$ ,  $\sqrt[6]{2} \text{ cis } 345^\circ$

**Class Exercises, pages 422–423** 1. T 2. F 3. F

4. T 5. T 6. T 7. T 8. T 9. Yes; no  
 10.  $\vec{BC}$  11. 6 12.  $\vec{BO}$  or  $\vec{OD}$  13.  $\vec{AC}$  14.  $\vec{AC}$   
 15.  $\vec{AB}$  16.  $\vec{OD}$ ;  $\vec{AD}$  17.  $\vec{DB}$ ;  $\vec{DC}$

**Written Exercises, pages 423–426** 1. b. 200 mi/h to the NW c. 100 mi/h to the SE 3. a.  $\vec{PR}$  b. 5 c. 0

5.

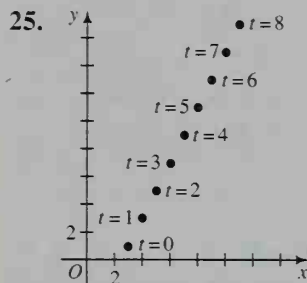


7.  $2(\mathbf{u} + \mathbf{v}) = 2\mathbf{u} + 2\mathbf{v}$  9.  $219.8^\circ$  11. 436.8  
 knots;  $85.4^\circ$  15. c.  $\sqrt{97}$  mi 17. a.  $\mathbf{a} + \mathbf{b}$   
 b.  $2(\mathbf{a} + \mathbf{b}) = 2\mathbf{a} + 2\mathbf{b}$  c. The seg. joining the  
 midpts. of 2 sides of a  $\triangle$  is  $\parallel$  to the third side and  
 is half as long as the third side. 19. a.  $\frac{2}{3}\mathbf{v}$  b.  $\frac{1}{3}\mathbf{v}$   
 c.  $-\frac{1}{3}\mathbf{v}$  21. a.  $\mathbf{u} + \mathbf{v}$  b.  $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$  c.  $\frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$   
 d.  $\frac{1}{2}\mathbf{u} + \frac{7}{2}\mathbf{v}$  23. a.  $\mathbf{w} - \mathbf{v}$  b.  $\frac{1}{3}(\mathbf{w} - \mathbf{v})$   
 c.  $\frac{2}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$  d.  $\frac{4}{9}\mathbf{v} + \frac{2}{9}\mathbf{w}$  e.  $\frac{2}{9}\mathbf{w} - \frac{5}{9}\mathbf{v}$   
 25.  $158.9^\circ$

- Class Exercises, pages 428–429** 1. a. (1, 4);  
 (4, -4) b.  $|\vec{AB}| = \sqrt{17}$ ;  $|\vec{CD}| = 4\sqrt{2}$  c.  $(\frac{17}{4}, 1)$   
 d. (0, -2) 2. a. (3, 6) b. (4, 2) c. (-2, 2)  
 d. (11, 4) 3. a.  $(-\sqrt{3}, 1)$  b. 2  
 4.  $2\mathbf{v} = (6 \cos 40^\circ, 6 \sin 40^\circ)$ ;  
 $-\mathbf{v} = (-3 \cos 40^\circ, -3 \sin 40^\circ)$  5. a. (7, 2)  
 b. (4, 9) 6. Horiz.;  $40\sqrt{3} \text{ lb} \approx 69.3 \text{ lb}$ ; yes

**Written Exercises, pages 429–432** 1. (2, 0); 2  
 3. (-2, 6);  $2\sqrt{10}$  5. (1.85, 5.71)

7.  $(-1, -1.73)$  9. a.  $(-5, 5)$  b.  $(11, -3)$   
 c.  $(3, 1)$  d.  $\sqrt{10}$  11. a.  $0 = (0, 0)$  b.  $|0| = 0$   
 c.  $2\sqrt{10}$  13.  $(3, \frac{3}{2})$  15.  $(3, 6)$  17.  $(-\frac{17}{5}, -\frac{11}{5})$   
 21. a.  $(20, 20\sqrt{3})$  b. Horiz. c. More easily; the  
 horiz. component is greater with a  $50^\circ$   $\angle$  than with  
 a  $60^\circ$   $\angle$ . 23. 4.6 knots S; 14.3 knots E

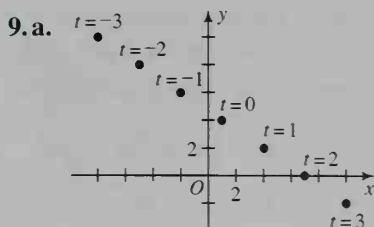


29.  $(7, 4)$   
 31. a.  $r = 3, s = 2$   
 33.  $\pm \frac{1}{5}$   
 35. Circle:  $r = 2$ ;  
 center at A  
 39. b. 14

- Class Exercises, page 435** 1. a.  $(2, -5), (3, -2),$   
 $(4, 1)$  b.  $(1, 3)$  c.  $x = 2 + t, y = -5 + 3t$   
 2.  $(x, y) = (-3, 1) + t(4, 3)$

**Written Exercises, pages 435–440** Exs. 1–8: Ans.  
 may vary.

1.  $(x, y) = (1, 5) + t(2, -1); x = 1 + 2t, y = 5 - t$   
 3.  $(x, y) = (1, 0) + t(2, -4); x = 1 + 2t,$   
 $y = -4t$  5.  $(x, y) = (-2, 3) + t(7, -2);$   
 $x = -2 + 7t, y = 3 - 2t$   
 7.  $(x, y) = (\pi, e) + t(1, 0); x = \pi + t, y = e$



- b.  $\mathbf{v} = (3, -2);$   
 $|\mathbf{v}| = \sqrt{13}$   
 c.  $x = 1 + 3t,$   
 $y = 4 - 2t$

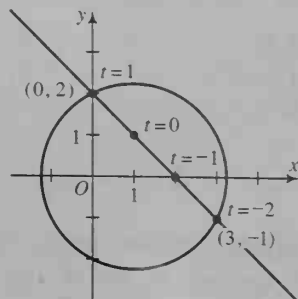
11.  $(x, y) = (2, 3) + t(3, -1); x = 2 + 3t,$   
 $y = 3 - t$  13.  $x = 3 + 2t, y = 2 + 4t; 2x - y = 4$   
 15. a. Vert. line through  $(2, 0)$  b.  $(0, 1)$  c. Slope is  
 undef. 17. a. 1.5 b. 1.5 c. Slopes are =.  
 d.  $(x, y) = (7, 9) + t(2, 3)$

19. a.  $\mathbf{v} = (-3, 2);$   
 $|\mathbf{v}| = \sqrt{13}$

- b.  $t = -1;$   
 $(5, -3)$

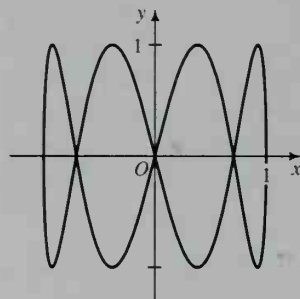
21.  $t = 1, (0, 2);$   
 $t = -2, (3, -1)$   
 See graph at right.

23. a. Yes, at  $(-1, 3)$   
 b. No c. No



25. Circle: radius  $r$ ; center  $(0, 0)$   
 27. a. Ellipse:  $V(0, \pm 5)$   
 b.  $25x^2 + 4y^2 = 100$  c.  $x = 3 \cos t, y = 6 \sin t$   
 29. a.  $(x - 3)^2 + (y - 5)^2 = 36$  b. Before:  
 $x = 6 \cos t, y = 6 \sin t$ ; after:  $x = 3 + 6 \cos t,$   
 $y = 5 + 6 \sin t$  31. b. (1) I (2) III (3) II (4) IV

35.



39. a.  $y = \frac{\sqrt{3}}{3}x - \frac{4}{675}x^2$  b. No c. For example,  
 $x = 90, y = 5t, 0 \leq t \leq 2$  d.  $36^\circ$

**Computer Exercise, page 440** 1. 7.999672  
 2. 9.42271

- Class Exercises, page 443** 1.  $\mathbf{u} \parallel \mathbf{v}; \mathbf{v} \perp \mathbf{w}; \mathbf{u} \perp \mathbf{w}$   
 2. a. 2 b. 4 c. 4 d. 4 3. a. 5 b. 25 c. 25  
 4. a.  $5\sqrt{2}$  b.  $5\sqrt{2}$  c. 40 d.  $37^\circ$  5. 0.8

- Written Exercises, pages 444–446** 1. a.  $-7$  b. 1  
 3. a. 2 b.  $-4.5$  5. a. 13 b. 13 13.  $106.3^\circ$   
 15.  $53.1^\circ$  19. a.  $\cos C = 0.6, \sin C = 0.8$  b. 20  
 21.  $73.7^\circ$  25. c. 350

**Class Exercises, page 449** 1. a.  $\sqrt{14}; (\frac{1}{2}, -1, \frac{3}{2})$

- b.  $2\sqrt{14}; (4, 2, -1)$  2. a.  $A = (4, 0, 0),$   
 $B = (4, 5, 0), C = (0, 5, 0), D = (4, 0, 3),$   
 $E = (0, 0, 3), F = (0, 5, 3)$  b.  $5\sqrt{2}$   
 3. a.  $x^2 + y^2 + z^2 = 25$  b.  $(x - 1)^2 +$   
 $(y - 2)^2 + (z - 3)^2 = 25$  4. a.  $(5, 9, 4)$   
 b. 8 5. If  $\mathbf{u} \cdot \mathbf{v} = 0, \mathbf{u} \perp \mathbf{v}$ . If  $\mathbf{u} = k\mathbf{v}, \mathbf{u} \parallel \mathbf{v}$ .  
 6. Ans. may vary. a.  $(2, 5, 1), (8, 12, 9)$   
 b.  $x = 2 + 6t, y = 5 + 7t, z = 1 + 8t$  c.  $(6, 7, 8)$   
 d. Dir. vectors are =. e. Dot prod. of dir. vectors = 0.  
 7. Subst. parametric eqs. obtained from vector eq.  
 of line into eq. of sphere. 8. The vectors are  $\parallel$ .

- Written Exercises, pages 450–452** 1.  $2\sqrt{6};$   
 $(1, 4, -1)$  3.  $2\sqrt{14}; (1, -2, 1)$  5.  $A = (5, 0, 0),$   
 $B = (5, 6, 0), C = (0, 6, 0), D = (5, 0, 4),$   
 $E = (0, 0, 4), F = (0, 6, 4); |\overrightarrow{OG}| = \sqrt{77}$   
 7. a.  $(11, 6, 2)$  b.  $-5$  c.  $\sqrt{35}$  9. Yes  
 11. a.  $x^2 + y^2 + z^2 = 4$   
 b.  $(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 4$

13.  $(-1, 2, 3)$ ; 5    15.  $70.5^\circ$     17.b. 9

19.a.  $\cos A = \frac{\sqrt{3}}{2}$ ,  $\sin A = \frac{1}{2}$     b.  $\frac{\sqrt{3}}{2}$

21.a.  $x = -2 + 4t$ ,  $y = -t$ ,  $z = 1 + t$

b.  $(-2, 0, 1)$ ,  $(2, -1, 2)$     c.  $(-10, 2, -1)$

d.  $(x, y, z) = (1, 2, 3) + t(4, -1, 1)$

23.  $(x, y, z) = (4, 2, -1) + t(2, 1, 3)$ ;  $x = 4 + 2t$ ,  
 $y = 2 + t$ ,  $z = -1 + 3t$

27.a.  $(x, y, z) = (1, 2, 3) + t(1, 4, -6)$ ;  $x = 1 + t$ ,  
 $y = 2 + 4t$ ,  $z = 3 - 6t$

b.  $(x, y, z) = (5, 3, 0) + t(6, -3, -1)$ ;  $x = 5 + 6t$ ,  
 $y = 3 - 3t$ ,  $z = -t$     31.a. The  $\perp$  bis. of  $\overline{AB}$

b. The plane  $\perp$  to  $\overline{AB}$  and passing through its  
midpt.    33.a. The circle with diameter  $\overline{AB}$     b. The

sphere with diameter  $\overline{AB}$     35.a. The ellipse with  
foci at  $A$  and  $B$  and major axis 10    b. The ellipsoid  
with foci at  $A$  and  $B$  and major axis 10

37.a. Ans. may vary.  $4x + 6y + 12z = 49$

39.a.  $x^2 + y^2 + z^2 = 9$     b. Both are spheres.

41. Circle:  $C(1, 2, 0)$ ,  $r = 4$ ;

$(x - 1)^2 + (y - 2)^2 = 16$     43.b. No    45.b. Ans.  
may vary.  $(x, y, z) = (3, -1, 5) + r(3, -10, 6)$

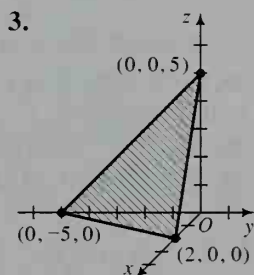
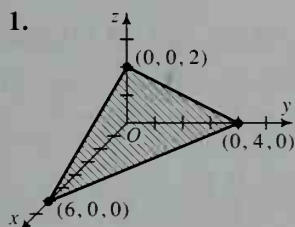
**Class Exercises, pages 454–455**    1.  $x$ -int. 2;

$y$ -int. 6;  $z$ -int. 3    2.a.  $z = 4$     b.  $z = 0$     c.  $x = 3$

d.  $y = 6$     3.a.  $(2, 3, 4)$     b.  $(3, 0, -4)$     c.  $(0, 0, 1)$

4.  $x + y + z = 12$     5.  $6x + 7y + 8z = 0$

**Written Exercises, pages 455–458**



7.a.  $z = 1$     b.  $x = 4$     c.  $y = 6$     d.  $y = 0$

Exs. 9–13: Ans. may vary.    9.  $(3, 4, 6)$

11.  $(1, 1, 0)$     13.  $(0, 0, 1)$     15.  $2x + 3y + 5z = 44$

17.  $z = 5$     19.a.  $x + 2y + 3z = 26$

21.b.  $2x + y + 2z = 9$     23.  $16\pi$     27.  $90^\circ$

29.a.  $(2, 3, -1)$     b. Yes    c. Yes    d. Yes

31.a.  $4x + y - 3z = 0$     b.  $3x + 4z = 0$

33.a.  $(4, -5, 2)$     35.b.  $(2, 1, 0)$

37.a.  $(x, y, z) = (3, 1, 5) + t(2, 2, 1)$

b.  $(1, -1, 4)$     c. 3    39.  $\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$

**Class Exercises, page 460**    1. -2    2. 12    3. 65

4. 31    5. 3; 2; 4

6.  $4 \begin{vmatrix} 5 & 7 \\ 8 & 6 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 8 & 6 \end{vmatrix} + 9 \begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$     7. -3

**Written Exercises, pages 460–461**    1. -48

3. -3125    5. 435    7. 90    9. -30    11. No

13. 78

**Class Exercises, page 463**

1.  $x = \frac{\begin{vmatrix} 7 & 4 \\ 8 & 6 \\ 3 & 4 \\ 5 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}}$ ;  $y = \frac{\begin{vmatrix} 3 & 7 \\ 5 & 8 \\ 3 & 4 \\ 5 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}}$

2.  $x = \frac{\begin{vmatrix} 4 & 1 & -2 \\ 1 & -1 & 4 \\ 3 & 2 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix}}$ ;  $y = \frac{\begin{vmatrix} 3 & 4 & -2 \\ 2 & 1 & 4 \\ 1 & 3 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix}}$

$z = \frac{\begin{vmatrix} 3 & 1 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -2 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{vmatrix}}$

3. The den. = 0. There  
is no sol. because  
the lines are  $\parallel$ .

4. 2

**Written Exercises, pages 463–464**    1.  $(1, 1)$

3.  $(1, -2)$     5.  $(\frac{1}{a+b}, \frac{1}{a+b})$     7.a. No sol.

b.  $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$     9. 4    11. 8    13. 6    15. They are  
collinear.    17.  $(3, 2, 1)$     19.  $(2, 0, 1)$     21. 5

**Class Exercises, page 466**

1.a.  $\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$     b.  $\begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

2.a. Vector    b. Scalar    c. Vector    d. Scalar  
e. Vector    f. Scalar

**Written Exercises, page 467** 1.  $(-1, -5, 4)$ ;  $(1, 5, -4)$ ; yes 5.  $\sqrt{42}$  9.a.  $(-1, 4, 1)$   
b.  $-x + 4y + z = 3$  c.  $\frac{3\sqrt{2}}{2}$  11.a.  $(4, -4, -2)$   
b.  $2x - 2y - z = 2$  c. 3 13.a.  $\frac{\sqrt{5}}{3}$  b.  $\frac{2}{3}$  15. 73

**Chapter Test, page 469** 1.  $112.6^\circ$ ; 13 N  
2.a.  $-u$  b.  $v - u$  c.  $\frac{1}{3}v - \frac{1}{3}u$  d.  $\frac{2}{3}u + \frac{1}{3}v$   
3. 367.7 mi/h N; 308.5 mi/h W 4.  $(6, 6)$   
5.a.  $v = (1, 4)$ ;  $|v| = \sqrt{17}$  b.  $x = -2 + t$ ,  
 $y = -2 + 4t$  c.  $t = \frac{3}{2}$ ,  $(-\frac{1}{2}, 4)$ ;  $t = 5$ ,  $(3, 18)$   
6. Since  $\cos \theta = \frac{u \cdot v}{|u||v|} = -1$ ,  $\theta = 180^\circ$  and  $\therefore u$   
and  $v$  are opp. in direction. 7.  $136.8^\circ$   
8.a.  $2\sqrt{62}$  b.  $(-1, -1, 1)$   
9.a.  $(x, y, z) = (2, 1, 7) + t(4, -1, -3)$   
b.  $(-4, 0, 9)$  10.  $2x - 2y - 5z = 20$  11.a. 22  
b. 80 12.  $(3, -\frac{1}{2})$  13.a. Ans. may vary.  
 $(-16, 46, -24)$  b.  $\frac{\sqrt{2948}}{2} \approx 27.1$

**Class Exercises, page 476** 1. Arith.; 5 2. Geom.; 2  
3. Neither 4. Arith.;  $-6$  5. Geom.;  $-\frac{2}{3}$   
6. Neither 7. 7, 12, 17, 22; arith. 8.  $\frac{2}{3}$ ,  $\frac{3}{4}$ ,  $\frac{4}{5}$ ,  $\frac{5}{6}$ ;  
neither 9. 3, 9, 27, 81; geom. 10. 1, 8, 27, 64;  
neither 11.a. Yes;  $d = 0$  b. Yes;  $r = 1$  12. The  
graph of the seq. is a set of discrete pts.; the graph  
of the line is continuous.

**Written Exercises, pages 476-479** 1. 5, 7, 9, 11;  
arith. 3. 6, 12, 24, 48; geom. 5. 2,  $\frac{5}{2}$ ,  $\frac{10}{3}$ ,  $\frac{17}{4}$ ;  
neither 7.  $-1, 2, -3, 4$ ; neither 9. 1, 0,  $-1, 0$ ;  
neither 11.  $t_n = -2 + 3n$  13.  $t_n = 16(0.5)^n$   
15.  $t_n = 34 - 4n$  17. Arith.;  $t_n = 13 + 4n$   
19. Geom.;  $t_n = \frac{16}{3}(\frac{3}{2})^n$  21. Neither;  $t_n = n^2$   
23. Neither;  $t_n = 10^n + 1$   
25. Arith.;  $t_n = (a - 3b) + n(a + b)$   
27. Geom.;  $t_n = 2^{-1/3}(2^n)$  29. 129 31. 149  
33. 128 35.  $-\frac{1}{2}$  37. 54 39. 38 41. 75 43. F  
45.  $t_n = \log a + (n - 1) \log r$  47.a. 3 b. 9  
49.  $x = 3$ ,  $y = 2$  51.  $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{1}{32}$ ,  $\frac{1}{2^{d/2}}$   
53. Ans. may vary. 1, 43, 85, 127, 169, ...  
55. 58, 135, 212, 289, 366 57.  $\triangle$  with sides  $3n$ ,  
 $4n$ ,  $5n$  59.a. Arith.: 6.5; geom.: 6 b. Arith.: 7.5;  
geom.:  $5\sqrt{2}$  61.b.  $y$ ;  $c$

**Class Exercises, page 481** 1. 5, 8, 11, 14 2. 10,  
12, 15, 19 3. 3, 6, 12, 24 4. 4, 7, 13, 25 5. 5,  
11, 24, 51 6. 4, 16, 20, 36

**Written Exercises, pages 481-485** 1. 14, 18, 22  
3. 5, 14, 41 5. 9, 16, 25 7. 6, 10, 16 9. 9, 1,  
64 11.  $t_n = 2 + 4n$  13.  $t_1 = 9$ ;  $t_n = t_{n-1} + 4$   
15.  $t_1 = 1$ ;  $t_n = t_{n-1} + 2^{n-1}$  17.  $t_1 = 1$ ;  
 $t_n = t_{n-1} + n$  19.a. 3, 5, 2,  $-3, -5, -2, 3, 5$   
b.  $-3$  21.a.  $P_0 = 5,000,000$  and  
 $P_n = (1.02)P_{n-1} + 50,000$  b. 9,644,889  
23.  $S_{n+1} = S_n + 2n + 1$  25.a. 5  
b.  $n - 2$  additional diagonals;  $d_4 = 2$  and  
 $d_n = d_{n-1} + n - 2$   
27.a. 6, 10 b.  $p_1 = 0$  and  
 $p_n = p_{n-1} + n - 1$  29.a.  $A_0 = 5000$  and  
 $A_n = 1.1A_{n-1} + 2000$  b. \$118,997.93  
31.a.  $M_2 = 3$ ;  $M_3 = 7$  c.  $M_4 = 15$ ;  $M_5 = 31$ ;  
 $M_6 = 63$ ;  $M_7 = 127$ ;  $M_8 = 255$  d.  $M_n = 2^n - 1$   
e.  $5.8 \times 10^{11}$  yr 33.a.  $-0.05$ ; neg. sign indicates  
that the coffee is cooling. c. After  $\approx 7$  min

**Class Exercises, page 488** 1. Ans. may vary. 1, 3,  
5, ...;  $1 + 3 + 5 + \dots$  2. Sum of Finite Geom.  
Series; 2047 3.a. Neither b.  $S_0 = 0$ ,  
 $S_n = S_{n-1} + t_n$ , where  $t_n = 2^n + 1$  ( $n \geq 1$ )  
c. 20 FOR N = 1 TO 15; 30 LET T =  $2^N + 1$

**Written Exercises, pages 489-492** 1. 210 3. 6375  
5. 294 7. 500,500 9.a.  $\frac{255}{16}$  b.  $\frac{85}{16}$  17.  $t_n = 2n$ ;  
 $S_0 = 0$ ,  $S_n = S_{n-1} + 2n$  19.  $t_n = 8(\frac{3}{2})^{n-1}$ ;  $S_0 = 0$ ,  
 $S_n = S_{n-1} + 8(\frac{3}{2})^{n-1}$  21. 25,333 23. 33,835,000  
25.b. 20 FOR N = 1 TO 25; 30 LET T =  $2^{-(N)}$   
c. 0.99999997 d. 1 27.  $t_1 = 5$ ;  $t_2 = 7$ ;  $t_3 = 9$   
29. 166,833 31. 49,320 33. 501  
35. 305,175,780 37.a. 2046 b. 19  
39.a.  $T_1 = 1$ ;  $T_2 = 3$ ;  $T_3 = 6$ ;  $T_4 = 10$ ;  $T_5 = 15$   
b.  $T_n + T_{n+1} = (n + 1)^2$   
41.  $M(n) = \frac{n(n^2 + 1)}{2}$   
45.b. 9753.90; 97.54; 124.90  
c.  $A_n = 1.01A_{n-1} - 222.44$

**Class Exercises, page 496** 1. 1 2. 1 3.  $\frac{2}{3}$  4.  $\frac{8}{5}$   
5. 1 6. 0 7. 0 8.  $\infty$  9.  $\infty$  10. 0 11. No; the  
terms do not "home in" on a single value.

**Written Exercises, pages 496-498** 1. 1 3. 1  
5.  $\frac{3}{8}$  7. 0 9. 0 11. 0 13. 0  
15. Does not exist 17. Does not exist 19.  $\infty$



21.  $-\infty$  23. 0 25. 1 27. 0 29. 1 31. b.  $\frac{3}{2}$

33.  $e^2$  35. b.  $\frac{1}{4}$  c.  $\frac{1}{4}$

**Calculator Exercises, pages 498–499** 1. a. 0

b.  $\frac{1}{\sqrt{n+1} + \sqrt{n}}$  c. The value of the expression approaches zero. 2. a. 1, 1.5, 1.4166667, 1.4142157, 1.4142136 b.  $\sqrt{2}$  3. a. 1.618034  
b. Same

**Class Exercises, page 502** 1. 1,  $\frac{4}{3}$ ,  $\frac{13}{9}$ ,  $\frac{40}{27}$ ;

sum =  $\frac{3}{2}$  2.  $\frac{1}{2}$ ,  $\frac{1}{4}$ ,  $\frac{3}{8}$ ,  $\frac{5}{16}$ ; sum =  $\frac{1}{3}$  3. 1, 4, 13,

40; diverges 4. 1, 1.1, 1.11, 1.111; sum =  $\frac{10}{9}$

5. a.  $-1 < x < 1$  b.  $-\frac{1}{2} < x < \frac{1}{2}$  6. a. 0.3 b. 0.1

c.  $\frac{1}{3}$  8. Diverges

**Written Exercises, pages 502–505** 1. 2 3. 16

5.  $\frac{125}{24}$  7. 2 9.  $\frac{1}{2}$  11.  $-\frac{1}{3}$  13. a.  $-1 < x < 1$

b.  $\frac{1}{1-x^2}$  15. a.  $2 < x < 4$  b.  $\frac{1}{4-x}$

17. a.  $x < -2$  or  $x > 2$  b.  $\frac{x}{x+2}$

21.  $|r| = 1.5 > 1$ ; thus, series would diverge.

23.  $\frac{7}{9}$  25.  $\frac{400}{9}$  27.  $\frac{1}{7}$  29.  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{3}{4}$ ,  $\frac{4}{5}$ ;

$S_n = \frac{n}{n+1}$ ;  $\lim_{n \rightarrow \infty} S_n = 1$  31.  $\frac{1}{4}$ ,  $\frac{2}{7}$ ,  $\frac{3}{10}$ ,  $\frac{4}{13}$ ;

$S_n = \frac{n}{3n+1}$ ;  $\lim_{n \rightarrow \infty} S_n = \frac{1}{3}$  33. 56 m 35. a. 288

b.  $96 + 48\sqrt{2}$  37. 15 39. 10, 1, 0.1;

sum =  $\frac{100}{9}$  s 41.  $5 + 5i$ ,  $7 + 6i$ ,  $9 + 7i$ ;

$t_{25} = 49 + 27i$ ;  $S_{25} = 625 + 375i$  43. a.  $\frac{4}{5} + \frac{2}{5}i$

b.  $\frac{243}{10} - \frac{81}{10}i$  45. Odd:  $\frac{1}{1+r}$ ;

even:  $\frac{r}{1+r}$  47.  $\frac{2\sqrt{3}}{5}$

**Calculator Exercise, page 506** a. 6 b. 8 c. 11

**Class Exercises, page 508**

1.  $5 + 10 + 15 + 20$  2.  $9 + 16 + 25 + 36$

3.  $1 - 1 + 1 - 1 + 1 - 1 + 1$

4.  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

5.  $\sum_{k=2}^6 k^2$  6.  $\sum_{k=1}^4 \frac{k}{k+1}$  7.  $\sum_{k=1}^{100} 3k$  8.  $\sum_{k=1}^{\infty} 3^{-k}$

**Written Exercises, pages 508–510**

1.  $2 + 3 + 4 + 5 + 6$  3.  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

5.  $3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots$  7.  $\frac{1}{16} + \frac{1}{4} + 1 + 4 + 16$

9.  $\sum_{k=1}^5 4k$  11.  $\sum_{k=1}^{25} 4k + 1$  13.  $\sum_{k=1}^{\infty} \frac{1}{k^2}$

15.  $\sum_{k=1}^{\infty} \sin kx$  19. 0 21. a. 0 b. 8

25. a.  $\sum_{k=0}^5 \left(-\frac{1}{2}\right)^k$  b.  $\sum_{k=0}^5 -1 \left(-\frac{1}{2}\right)^k$

27.  $\sum_{k=0}^{49} (-1)^k (2k+1)$  33. 343,400 35. 53,130

37. a. 204 b.  $\frac{n(n+1)(2n+1)}{6}$

39.  $\frac{1}{6}(n(n+1)(n+2))$

**Computer Exercise, page 510** a. 11 b. 31

c. 12,367

**Written Exercises, pages 513–514** 1. For  $n = 1$ ,

$\frac{1(1+1)}{2} = \frac{2}{2} = 1$ ; if  $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ ,

then  $(1 + 2 + \cdots + k) + (k+1) =$

$\frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2} =$

$\frac{(k+1)[(k+1)+1]}{2}$ . 11. For  $n = 1$ ,  $11^1 - 4^1 = 7$ ,

and 7 is a multiple of 7; if  $11^k - 4^k$  is a multiple of 7, then  $11^{k+1} - 4^{k+1} = 11 \cdot 11^k - 4 \cdot 4^k = (7+4)11^k - 4 \cdot 4^k = 7 \cdot 11^k + 4(11^k - 4^k) =$  the sum of 2 multiples of 7 = a multiple of 7.

13.  $n \geq 3$  19.  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\dots$ ;  $a_n = \frac{1}{n+1}$

21.  $\frac{1}{4}$ ,  $\frac{2}{7}$ ,  $\frac{3}{10}$ ,  $\frac{4}{13}$ ,  $\dots$ ;  $a_n = \frac{n}{3n+1}$

23. 1, 5, 23, 119,  $\dots$ ;  $S_n = (n+1)! - 1$

**Chapter Test, page 515** 1. a. Arith.;  $t_n = 14 - 4n$

b. Neither;  $t_n = n^2 + 2$  c. Geom.;  $t_n = \frac{2}{3}(-3)^n$

2. 53.5 3.  $\frac{9}{512}$  4. 128 5.  $t_1 = 1$ ,  $t_n = 4t_{n-1} + 1$

6. a.  $t_n = 94 + 10n$  b.  $S_n = 99n + 5n^2$  7.  $\frac{182}{9}$

8. Ans. will vary. An infinite seq. has a limit  $L$  if the terms get arbitrarily close to  $L$  as  $n$  becomes larger and larger.

9. a.  $\frac{3}{4}$  b. Does not exist c. 0 10.  $\sum_{n=1}^{\infty} 13(10)^{-2n}$

$\frac{13}{99}$  11. a.  $-\frac{2}{3} < x < \frac{2}{3}$  b.  $\frac{2}{2-3x}$

12. a.  $\sum_{n=0}^4 \frac{(-1)^n}{(2n+1)^2}$  b.  $\sum_{n=1}^7 (11-4n)$  13. 9870

14. For  $n = 1$ ,  $\frac{1(5 \cdot 1 - 1)}{2} = 2 = 5(1) - 3$ ; if

$2 + 7 + 12 + \cdots + (5k - 3) = \frac{k(5k - 1)}{2}$ , then

$2 + 7 + 12 + \cdots + (5k - 3) + [5(k + 1) - 3] =$   
 $\frac{k(5k - 1)}{2} + \frac{2(5k + 2)}{2} = \frac{5k^2 + 9k + 4}{2} =$

$\frac{(k + 1)(5k + 4)}{2} = \frac{(k + 1)[5(k + 1) - 1]}{2}$

**Class Exercises, page 520** 1.  $A_{2 \times 1}$ ;  $B_{1 \times 2}$ ;  $C_{1 \times 2}$ ;  $D_{1 \times 2}$  2.  $A$  and  $B$ ,  $A$  and  $D$  3.  $B$  and  $D$

4.  $A_{3 \times 2}$ ;  $B_{3 \times 2}$ ;  $C_{3 \times 3}$ ;  $D_{2 \times 3}$

5.  $A$  and  $B$ ;  $\begin{bmatrix} 8 & 6 \\ 5 & 5 \\ -3 & 14 \end{bmatrix}$

6. a.  $2 \times 3$  b.  $\begin{bmatrix} 5 & 3 & 0 \\ -1 & -4 & 6 \end{bmatrix}$

7. a. 10.4 g of carbohydrate in a serving of tomato juice b. 0.002 g of sodium in a serving of orange juice c. Nutritional information for 2 servings d.  $N^t$

### Written Exercises, pages 520–522

1.  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$  3.  $\begin{bmatrix} 5 & 3 \\ -5 & 6 \end{bmatrix}$  5.  $\begin{bmatrix} 20 & 5 & 8 \\ -3 & 1 & 8 \end{bmatrix}$

7.  $\begin{bmatrix} 40 & -16 \\ 32 & 0 \end{bmatrix}$  9.  $\begin{bmatrix} 4 & -2 \\ -5 & 2 \\ 6 & 9 \end{bmatrix}$

11. a.  $A: 3 \times 4$ ;  $B: 4 \times 3$ ;  $C: 4 \times 3$

b.  $B + C$ ;  $\begin{bmatrix} 17 & 1 & 9 \\ 7 & 3 & 8 \\ 5 & 17 & 13 \\ 12 & 4 & 0 \end{bmatrix}$

13.  $\begin{bmatrix} 16 & 2 & 8 \\ 6 & 0 & 14 \\ -4 & 10 & 22 \\ 12 & 8 & -6 \end{bmatrix}; \begin{bmatrix} 25 & 2 & 13 \\ 10 & 3 & 15 \\ 3 & 22 & 24 \\ 18 & 8 & -3 \end{bmatrix}$

15. Ans. may vary. a.  $\begin{matrix} l & m & h \\ D & \begin{bmatrix} 31 & 42 & 18 \end{bmatrix} \\ M & \begin{bmatrix} 22 & 25 & 11 \end{bmatrix} \end{matrix}$

b. Location by mower-type. c.  $\begin{matrix} l & m & h \\ D & \begin{bmatrix} 28 & 29 & 20 \end{bmatrix} \\ M & \begin{bmatrix} 20 & 18 & 9 \end{bmatrix} \end{matrix}$

d.  $\begin{matrix} l & m & h \\ D & \begin{bmatrix} 59 & 71 & 38 \end{bmatrix} \\ M & \begin{bmatrix} 42 & 43 & 20 \end{bmatrix} \end{matrix}$ ; combined sales for April and May

e. 20; up; 1.08 f. 9%; 15% 17.  $x = 2$ ,  $y = 3$ ,  $z = -1$  19.  $A$  21. Yes

**Class Exercises, page 525** 1. Yes; no;  $AB_{3 \times 2}$

2. No; yes;  $BA_{2 \times 6}$  3. No; yes;  $BA_{3 \times 4}$  4. Yes; no;  $AB_{5 \times 3}$  5. No; no 6. Yes; yes;  $AB_{3 \times 3}$

$BA_{4 \times 4}$  7.  $\begin{bmatrix} 9 & 8 \\ 23 & 22 \end{bmatrix}$  8.  $\begin{bmatrix} 4 & 10 \\ 0 & 2 \end{bmatrix}$  9.  $CM$

10. No; yes

### Written Exercises, pages 526–529

1.  $\begin{bmatrix} 23 \\ -7 \end{bmatrix}$  3.  $\begin{bmatrix} 4 & -9 \\ 8 & -5 \end{bmatrix}$  5. Not def.

7.  $\begin{bmatrix} 30 & 7 & 20 \\ -14 & -33 & -9 \end{bmatrix}$  9.  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

11. b. Yes 13. b. Yes c. Yes 15.  $PS$ ; [37,200 35,050]; profit for each dealer in March 17. a. No. of cols. of  $P$  = no. of rows of  $M$ , but cols. of  $P$  and rows of  $M$  do not represent the same things. b. 2400 c. There are 3 cols. in  $M$  and 2 rows in  $R$ .

$\begin{matrix} s & b & g & \text{Jan.} & \text{Feb.} & \text{Mar.} \\ d. & \begin{matrix} x & \begin{bmatrix} 17 & 21 & 37 \\ 22 & 28 & 50 \end{bmatrix} \\ y & \end{matrix} & e. & \begin{matrix} x & \begin{bmatrix} 16,400 & 25,500 & 29,200 \\ 21,600 & 33,800 & 38,800 \end{bmatrix} \\ y & \end{matrix} \end{matrix}$

19. a. 8.3 b. 1st: Joyce; 2nd: Eduardo; 3rd: Stephanie 21. c. \$10,072 d.  $A$

### Class Exercises, page 534

1.  $-B = \begin{bmatrix} -7 & 1 \\ -6 & -2 \end{bmatrix}$ ;  $-C = \begin{bmatrix} -3 & -2 & -5 \\ -4 & -3 & 0 \end{bmatrix}$

2.  $A^{-1} = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ ;  $B^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{1}{20} \\ -\frac{3}{10} & \frac{7}{20} \end{bmatrix}$

3.  $C$  is not a square matrix, and  $|D| = 0$ .

4.  $\begin{bmatrix} 79 & -30 \\ -50 & 19 \end{bmatrix}$ ; both are correct because mult. of matrices is associative.
5. Mult. of matrices is not commutative.
6.  $3X = \begin{bmatrix} -9 & -6 \\ -3 & -15 \end{bmatrix}$  (Perform the subtraction.)
- $X = \begin{bmatrix} -3 & -2 \\ -1 & -5 \end{bmatrix}$  (Simplify using scalar div.)

### Written Exercises, pages 534–537

- 1.a.  $\begin{bmatrix} -\frac{7}{3} & \frac{4}{3} \\ 2 & -1 \end{bmatrix}$  b.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  3.a.  $\begin{bmatrix} -9 & -3 \\ -12 & -4 \end{bmatrix}$
- b. Not def. 5.a.  $\begin{bmatrix} -3 & -1 \\ -4 & -\frac{4}{3} \end{bmatrix}$  b. Not def.

7.a.  $X = D^{-1}E$  b.  $X = ED^{-1}$  9.  $\begin{bmatrix} 6 & -8 \\ -3 & -20 \end{bmatrix}$

11.  $\begin{bmatrix} 27 & 13 \\ -36 & -18 \end{bmatrix}$  13.  $\begin{bmatrix} 6 & 0 \\ -12 & -2 \end{bmatrix}$

15.a.  $x = -\frac{1}{2}$ ,  $y = 5$  b.  $x = 4$ ,  $y = -3$

17.a. Cannot be done; coef. matrix has no inverse.

b. There is no sol. because the lines are  $\parallel$ .

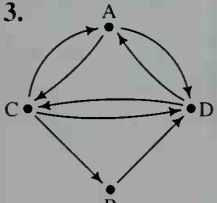
21.  $IX$ ;  $X = (I - A)^{-1}D$  23.  $x = -2$ ,  $y = 1$ ,

$z = -3$  25. I: 4 oz; II: 6 oz; III: 8 oz; IV: 6 oz

27.  $y = \frac{3}{4}x^2 + \frac{7}{2}x - 15$  29.  $w = \frac{d}{ad - bc}$ ,

$x = -\frac{b}{ad - bc}$ ,  $y = -\frac{c}{ad - bc}$ ,  $z = \frac{a}{ad - bc}$

### Class Exercises, page 540

1. C and D 2. C
3. 
4. No. of ways a message can be sent using one relay
5. No; a station can send a message to itself using another station as a relay.

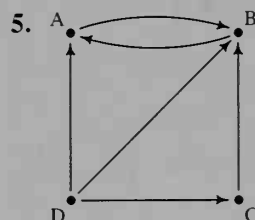
### Written Exercises, pages 540–543

1. from B to A B C

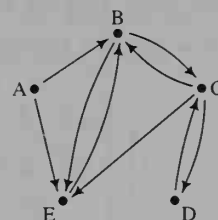
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

3. from to A B C D E

$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$



7.a.



7.c. There is no route from E to B using one relay. e. 1 9.a. Transmitters: Kim, Bob;

receivers: Ron, Sue; relays: Dot, Ted, Les, Neal

11. Zeros:  $n$ ; ones:  $n^2 - n$  13.b. Wolves feed on 2 animals that feed on grass. c. Put a "1" in the grass row, salmon col.

### Class Exercises, pages 545–546

2. Associative

3.a.  $\begin{bmatrix} 0.9 & 0.1 \\ 0.03 & 0.97 \end{bmatrix}$  b.  $\begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$  c.  $\begin{bmatrix} 0.552 & 0.448 \end{bmatrix}$ ;

population distribution in one year d. Population distribution in 2 years e. Calculate  $M_n$  for large values of  $n$  to see if its values stabilize.

### Written Exercises, pages 546–550

1.a.  $\begin{bmatrix} 0.82 & 0.18 \\ 0.05 & 0.95 \end{bmatrix}$  b.  $\begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$

c.  $\begin{bmatrix} 0.358 & 0.642 \end{bmatrix}$ ; next year, 36% will commute by car and 64% by public transportation.

d.  $\begin{bmatrix} 0.326 & 0.674 \end{bmatrix}$ ; in 2 years, 33% will commute by car and 67% by public transportation.

3.a. 62%; 63% b.  $\frac{2}{3}$  at Dixie's,  $\frac{1}{3}$  at Sargent's

c.  $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

5.a.  $\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.4 & 0.6 \\ 1 & 0 & 0 \end{bmatrix}$  b. 14% c. 12.2%

d. 73%; 73%; 73% e. 73%; 73%; 73% f. 73%; 12%; 15%

$$\begin{matrix} & w & f & i \\ \text{7.a. } & \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} & \text{b. } 19.5\%; 12.5\% \end{matrix}$$

c. 10 days: 54% well, 23% flu, 22% immune;  
30 days: 22% well, 14% flu, 64% immune; 50 days:  
9% well, 6% flu, 85% immune 9.  $x = 0.125$ ;

$$S = [0.125 \quad 0.875] \quad 11. \begin{bmatrix} 5 & 3 & 5 \\ 12 & 8 & 24 \end{bmatrix}$$

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ 13.a. & \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} & \text{b. } 21.4\%; 21.4\%; \\ & & & & & 14.3\%; 28.6\%; 14.3\% \\ & & & & & \text{c. } 21.4\%; 21.4\%; \\ & & & & & 14.3\%; 28.6\%; 14.3\% \\ & & & & & \text{d. No} \end{matrix}$$

15. In the first col. of row 3 change 0 to 20.

b.  $\begin{bmatrix} 6000 & 320 & 150 & 60 \end{bmatrix}$

c.  $\begin{bmatrix} 4200 & 2400 & 160 & 45 \end{bmatrix}$

d.  $\begin{bmatrix} 444,768 & 45,824 & 25,190 & 6286 \end{bmatrix}$

$$\begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 17.a. & \begin{bmatrix} 329 & 128 & 70 & 48 & 13 \end{bmatrix} & \text{b. In 38 yr} \end{matrix}$$

**Class Exercises, page 556** 1. 3 2.  $-2x - 2$

$$3. \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad 4. T = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix};$$

$$|T| = 11 \quad 5. \frac{\text{area } \triangle A'B'C'}{\text{area } \triangle ABC} = 11 \quad 6. \text{ Same}$$

7. No; no; no 8. No; no; no 9. No; yes; no

**Written Exercises, pages 557-559**

$$1.b. T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; |T| = 6 \quad c. 6 \quad d. \text{ Same}$$

$$3.c. T = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}; |T| = -1; \text{ the areas are } = \text{ and}$$

the  $\triangle$  have opp. orientations.

$$5.a. (36, 4) \quad b. \begin{bmatrix} 36 \\ 4 \end{bmatrix} \quad 7. y = \frac{1}{2}x; T = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix},$$

$$|T| = 0 \quad 9. \text{ Right 5, down 1} \quad 11. \text{ Down 2}$$

$$13. \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad 15. \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$17. \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad 19. \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad 21. \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$23. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; I_{2 \times 2} \quad 25. \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; R_{180}$$

$$27. X^2 = L^2 = M^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$29.a. \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \quad 31.a. \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

**Chapter Test, pages 560-561**

$$\begin{matrix} & s & m & l \\ 1.a. & \begin{bmatrix} M & 7 & 14 & 10 \\ P & 8 & 20 & 9 \\ B & 15 & 20 & 12 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} & s & m & l \\ & \begin{bmatrix} M & 10.5 & 21 & 15 \\ & 12 & 30 & 13.5 \\ & B & 22.5 & 30 & 18 \end{bmatrix} & 2. \begin{bmatrix} 5 & 0 & -6 \\ -6 & 3 & 8 \end{bmatrix} \end{matrix}$$

$$3. \text{ Ans. may vary. } A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix};$$

the 1st row, 1st col. entry of a product matrix  $XY$  comes from the product of the 1st row of  $X$  and the 1st col. of  $Y$ . These products are different for  $AB$  and  $BA$ . 4.a. NP;  $NP = [5.8 \quad 4 \quad 2.8 \quad 3.4]$

$$b. 2.8 \text{ lb} \quad 5. x = 5, y = -5$$

$$\begin{matrix} & \text{to } B & L & P & S & W \\ & \begin{bmatrix} B & 0 & 1 & 0 & 0 & 0 \\ L & 1 & 0 & 1 & 0 & 0 \\ & P & 0 & 0 & 0 & 1 & 1 \\ S & 0 & 1 & 0 & 0 & 0 \\ W & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} & \text{to } B & L & P & S & W \\ & \begin{bmatrix} B & 1 & 0 & 1 & 0 & 0 \\ L & 0 & 1 & 0 & 1 & 1 \\ & P & 0 & 1 & 1 & 1 & 0 \\ S & 1 & 0 & 1 & 0 & 0 \\ W & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{matrix};$$

Lookout Peak, Park Entrance, Scenic Overlook, and Waterfall have two 2-step paths to other sites.

7. a.  $c \begin{bmatrix} 0.92 & 0.08 \\ 0.03 & 0.97 \end{bmatrix}$

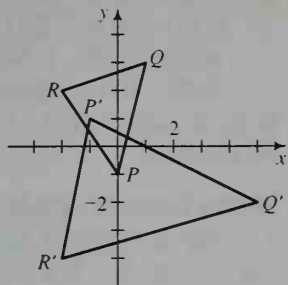
b.  $[0.6 \quad 0.4]$

c. City: 56.4%;  
suburbs: 43.6%

8. a, b. See graph at right.

c.  $T = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$|T| = -3$ ;  $\frac{\text{area } \triangle P'Q'R'}{\text{area } \triangle PQR} = 3$  and the  $\triangle$  have opp. orientations.



21. a.  $\begin{bmatrix} -4 & 7 \\ 15 & 8 \\ -9 & -4 \end{bmatrix}$  b.  $AB = \begin{bmatrix} -30 & -7 \\ -6 & -3 \end{bmatrix}$

$BA = \begin{bmatrix} -5 & 8 & -4 \\ 3 & -18 & 6 \\ 7 & 46 & -10 \end{bmatrix}$ ; no 22.  $\begin{bmatrix} \frac{2}{3} & -\frac{5}{3} \\ 0 & 1 \end{bmatrix}$

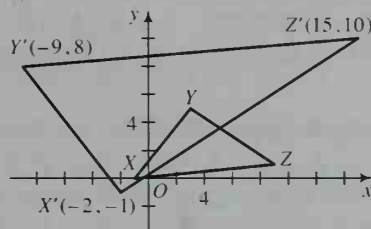
23.  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$  24. a.  $\begin{matrix} & A & B & C & D & E \\ A & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ B & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\ C & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix} \\ D & \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\ E & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$

$\begin{matrix} & A & B & C & D & E \\ A & \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \end{bmatrix} \\ B & \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \end{bmatrix} \\ C & \begin{bmatrix} 0 & 1 & 2 & 2 & 2 \end{bmatrix} \\ D & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ E & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$

25. a.  $\begin{matrix} & A & B & C \\ A & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ B & \begin{bmatrix} 0.35 & 0.65 & 0 \end{bmatrix} \\ C & \begin{bmatrix} 0.45 & 0 & 0.55 \end{bmatrix} \end{matrix}$

b. A: 51%; B: 32.5%; C: 16.5% c. A: 69.8%; B: 21.1%; C: 9.1%

26. a, b.



c.  $T = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$ ,  $|T| = 5$ ;  $\frac{\text{area } \triangle X'Y'Z'}{\text{area } \triangle XYZ} = 5$  and the  $\triangle$  have the same orientation.

### Cumulative Review, pages 562–563

1. a.  $P$  300 km 200° 120 km b. 414.8 km; 185.7°  
2. a.  $(-6, 8)$  b.  $(3, 1)$  c.  $5\sqrt{2}$   
3. a.  $(x, y) = (-2, -4) + t(-2, 5)$   
b.  $x = -2 - 2t$ ,  $y = -4 + 5t$   
4. a. Vel. =  $(2, -4)$ , speed =  $2\sqrt{5}$   
b.  $t = 1$ ;  $(3, -5)$

5.  $62.3^\circ$  6.  $(x, y, z) = (2, 0, 5) + t(-5, -1, -1)$ ;  
 $x = 2 - 5t$ ,  $y = -t$ ,  $z = 5 - t$  7.  $2x + y - z = 0$   
8. 336 9.  $(3, 2)$  10.  $(6, 9, -1)$ ;

$6x + 9y - z = 1$  11. a. Geom.;  $t_n = -\frac{4}{9}\left(-\frac{3}{2}\right)^n$

b. Neither;  $t_n = \frac{2n}{2n+1}$  c. Arith.;  $t_n = n - \frac{1}{3}$

12. a.  $-2, 0, 4, 12, 28$ ; b.  $t_n = 2^n - 4$  c. 1020

13.  $-180$  14.  $\frac{364}{3}$  15. a. 1 b. Does not exist.

16. a. 3 b.  $\infty$  c. Does not exist. 17.  $-4 < x < 4$ ;

$\frac{8}{x+4}$  18.  $\frac{115}{333}$  19.  $\sum_{n=1}^{20} (5n+2)$ ; 1090

20. For  $n = 1$ ,  $\frac{1}{2(1)+1} = \frac{1}{3} = \frac{1}{1 \cdot 3}$ ; if  
 $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ , then

$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} +$

$\frac{1}{(2(k+1)-1)(2(k+1)+1)} =$

$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} =$

$\frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)} =$

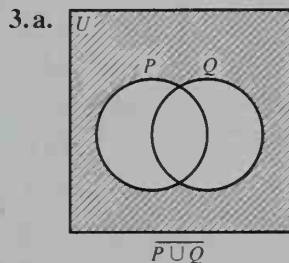
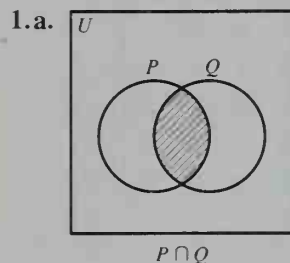
$\frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2(k+1)+1}$

**Class Exercises, page 568** 1. The set of freshmen music majors; 15 2. The set of students who are either freshmen or music majors; 350 3. The set of students who are not freshmen; 700 4. The set of students who are not music majors; 935 5. The set of music majors who are not freshmen; 50



6. The set of freshmen who are not music majors; 285 7. The set of students who are neither freshmen nor music majors; 650 8. The set of students who are either freshmen or are not music majors; 950 9.a.  $B \cap \bar{A}$  b.  $A \cap B$  c.  $\overline{A \cup B}$  or  $\bar{A} \cap \bar{B}$  10.a.  $A$  b.  $\emptyset$  c.  $U$  d.  $\emptyset$  e.  $U$  f.  $A$

### Written Exercises, pages 568–571



- 5.a. Teachers of either math or physics b. Teachers of math but not of physics 7.a. Teachers of biology or of both physics and chemistry b. Teachers of both biology and chemistry 9.a. Teachers of neither math nor physics nor chemistry b. Same as part a. 11. 28 13. 12 19.a. 11 b. 12 21.  $\emptyset$  25.b.  $\overline{A \cap B} = \bar{A} \cup \bar{B}$  c.  $\bar{A} \cup (\bar{B} \cap \bar{C})$  27.  $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

- Class Exercises, page 574** 1.a. 2 b. 6 c. 24 2. 3,628,800 3. 608,400 4.a. 60 b. 180 5.a. 8 b. 6 6. 720 7.a. 362,879 b. 1,133,836,704,000

- Written Exercises, pages 575–577** 1.a. 120 b. 720 c. 5040 d. 1 3. 120 5. 1024 7.a. 17,576 b. 13,800 9. 11,880 11.b. 14 branches, 8 leaves c. 2046 branches, 1024 leaves 13.a. 362,880 b. 40,320 c. 5040 15. 156 17. 2 digits: 1,757,600; 3 digits: 17,576,000; total: 19,333,600 19.a. 648 b. 252 21. 542 23.a. 160 b. 640 c. 9999 d. 6,399,360 e. 1,023,897,600 25.a. 362,880 b. 30,240 27.  $26^2 = 676 < 677$  31.  $\log_{10} 9! \approx 5.56$ ;  $\log_{10} 10! \approx 6.56$  33.a.  $\frac{10!}{6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$

- Class Exercises, page 580** 1.a. 20 b. 10 2.a. 120 b. 20 3.a. 720 b. 120 4.a. 24 b. 1 5. 720 6. 120 7. The order of the three nos. is important.

- 8.a. ABC ACB BAC BCA CAB CBA  
ABD ADB BAD BDA DAB DBA  
ACD ADC CAD CDA DAC DCA  
BCD BDC CBD CDB DBC DCB  
b. A, B, and C: 6; A, C, and D: 6 c. ABC, ABD, ACD, BCD

- Written Exercises, pages 580–582** 1.a. 380 b. 190 3.a. 210 b. 5040 5.a. 1,313,400 b. 7,880,400 7.a. 336 b. 56 9.a. 665,280 b. 924 11.a. 35 b. 35 13.a.  ${}_{100}C_2 = {}_{100}C_{98} = \frac{100!}{2!98!}$  15.a. 1 b. 1 17. 22,764 19. 48 21.a. 792 b. 658,008 c. 1,940,952 23.a. 15 b. There will be fewer line segs. if three or more pts. are collinear. 25. 10 29.  $5.36 \times 10^{28}$

- Class Exercises, page 585** 1.a. 24 b. 12 2.a. 120 b. 60 3.a. 720 b. 180 4.a. 39,916,800 b. 1,663,200 7. 84 8. BCDAE

- Written Exercises, pages 585–587** 1. 10,080 3. 34,650 5. 840 7.  $9.71 \times 10^{14}$  years 9.a. 3003 b. 21 c. 35 d. 735 11. 24 13.  $1.37 \times 10^{11}$  15.b.  $8.25 \times 10^{12}$  17. 6188

### Mixed Combinatorics Exercises, pages 587–589

1. 161,700 3.a. 6561 b. 5832 c. 2000 5.a. 870 b. 435 7. 60 ways from A to D; 3600 round trips 9. 83 11.a. 1 min b. 13 h 13.a. 70 b. 35 c. 35 15. 15 17. 64 19.a. 86,400 b. 28,800 21. 37,512

### Class Exercises, page 592

2. 1 7 21 35 35 21 7 1 3.a. 1 8 28 56 b.  $x^8, 8x^7y, 28x^6y^2, 56x^5y^3$  c.  $x^8, -8x^7y, 28x^6y^2, -56x^5y^3$  4.  $a^3 - 3a^2b + 3ab^2 - b^3$  5.  $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$  6.  $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$  7.a.  $15x^4y^2$  b.  $84x^6y^3$

### Written Exercises, pages 592–594

- 1.a.  $a^3 + 3a^2b + 3ab^2 + b^3$  b.  $8000 + 1200 + 60 + 1 = 9261$  c.  $8000 - 1200 + 60 - 1 = 6859$  3.a.  $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$  b.  $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$  c.  $32a^5 + 80a^4 + 80a^3 + 40a^2 + 10a + 1$  5.a.  $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$  b.  $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

- c.  $x^{14} - 14x^{12}y + 84x^{10}y^2 - 280x^8y^3 + 560x^6y^4 - 672x^4y^5 + 448x^2y^6 - 128y^7$   
 7.  $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$   
 9.  $x^{10} + 5x^8 + 10x^6 + 10x^4 + 5x^2 + 1$   
 11.  $x^6 + 6x^4 + 15x^2 + 20 + 15x^{-2} + 6x^{-4} + x^{-6}$   
 13.  $(a^2)^{100} + 100(a^2)^{99}(-b)^1 + \frac{100 \cdot 99}{2}(a^2)^{98}(-b)^2 + \frac{100 \cdot 99 \cdot 98}{3 \cdot 2}(a^2)^{97}(-b)^3$   
 15.  $(\sin x)^{10} + 10(\sin x)^9(\sin y) + \frac{10 \cdot 9}{2}(\sin x)^8(\sin y)^2 + \frac{10 \cdot 9 \cdot 8}{3 \cdot 2}(\sin x)^7(\sin y)^3$   
 17. 1.05 19.a. 495 b. 495 21. 126,720  
 23.a.  ${}_{n+1}C_k$  b.  ${}_nC_{k-1}$  c.  ${}_nC_k$   
 d.  ${}_{n+1}C_k = {}_nC_{k-1} + {}_nC_k$  25. Third  
 29.b.  $\cos 3\theta + i \sin 3\theta$

**Chapter Test, page 595** 1. 36 2. 720 3. 363,840  
 4. 2520 5. 15 6. 560 7. Ans. may vary. A linear permutation is like a football team of  $n$  players seated on a bench. There is a first player, then a second, etc., and the  $n$  positions can be filled in  $n!$  ways. A circular permutation is like this same team in a huddle. There is no first player, second player, etc. For each circular arrangement there are  $n$  possible linear ones. Thus, there are  $\frac{n!}{n} = (n-1)!$  circular permutations.  
 8.  $x^{13}$ : 420;  $x^{12}$ : -3640

**Class Exercises, page 602** 1.a.  $\frac{1}{3}$  b. 1 c. 0

- 2.a.  $\frac{1}{52}$  b.  $\frac{1}{4}$  c.  $\frac{1}{13}$  d.  $\frac{1}{2}$  e.  $\frac{3}{13}$  f.  $\frac{3}{26}$  3.  $\frac{1}{6}$   
 4.a.  $\frac{1}{18}$  b.  $\frac{1}{12}$  c.  $\frac{5}{36}$  5. 60% 6. 0.18 8.a. Yes  
 b. The sample pts. are not equally likely.  
 9. Some outcomes, such as the queen of hearts, correspond to more than one element of  $S$ .  
 10. This is incorrect because population is not evenly distributed among the states.

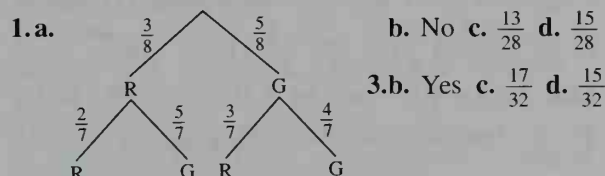
**Written Exercises, pages 603–605** 1.a.  $\frac{1}{2}$  b.  $\frac{1}{4}$

- c.  $\frac{3}{4}$  3.a.  $\frac{1}{4}$  b. 0 c. 1 5.a.  $\frac{1}{214}$  b.  $\frac{4}{535}$  c.  $\frac{531}{535}$   
 7.a. There is not an equal number of phone nos. in each borough. b. The no. of phone nos. in Manhattan and the no. of phone nos. in all of New York City 9.a.  $\frac{5}{36}$  b.  $\frac{1}{6}$  c.  $\frac{5}{36}$  11.  $\frac{5}{6}$  13.b.  $\frac{1}{4}$

- 15.a.  $\frac{4}{7}$  b.  $\frac{3}{7}$  17.a.  $\frac{1}{64}$  b.  $\frac{1}{32}$  c. 9 19.b.  $\frac{1}{6}$  c.  $\frac{1}{2}$   
 23.  $\frac{1}{120}$  25.  $\frac{3}{8}$  27.  $\frac{1}{10}$  29.a.  $\frac{81}{100}$  b.  $\frac{19}{100}$

**Class Exercises, page 609** 2. No 3.a.  $\frac{8}{15}$  b.  $\frac{9}{16}$   
 c. No

**Written Exercises, pages 609–613**



- 5.b.  $P(HH) = \frac{1}{17}$ ;  $P(HN) = \frac{13}{68}$ ;  $P(NH) = \frac{13}{68}$ ;  
 $P(NN) = \frac{19}{34}$  7.a.  $P(F) = P(F|D) = \frac{3}{13}$   
 b.  $P(R) = \frac{1}{2}$ ,  $P(R|D) = 1$ ;  $P(R) \neq P(R|D)$   
 9.b.  $P(3) = \frac{1}{8}$ ;  $P(2) = \frac{3}{8}$ ;  $P(1) = \frac{3}{8}$ ;  $P(0) = \frac{1}{8}$   
 11.a.  $\frac{1}{56}$  b.  $\frac{5}{28}$  13.a.  $\frac{147}{400}$  b.  $\frac{1}{20}$  15.a.  $\frac{29}{90}$  b.  $\frac{29}{147}$   
 17. Yes 21.a.  $\frac{11}{50}$ ,  $\frac{11}{850}$  b.  $\frac{2}{17}$  c.  $\frac{1}{5525}$  d.  $\frac{4324}{5525}$   
 e.  $\frac{1201}{5525}$  f.  $\frac{48}{221}$  25.a.  $\frac{30}{49}$  b.  $\frac{19}{49}$  27.  $\frac{3}{5}$   
 29.a. 0.064 b. 0.784 c. 0.8208

**Class Exercises, page 615** 1.  $4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) =$

- $P(3 \text{ H and } 1 \text{ T}); 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = P(2 \text{ H and } 2 \text{ T});$   
 $4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 = P(1 \text{ H and } 3 \text{ T}); \left(\frac{1}{2}\right)^4 = P(4 \text{ T})$  3.  $\frac{1}{2}$   
 4.a.  $\frac{49}{198}$  b. The 1st and 2nd drawings are not independent trials. c.  $\frac{1}{4}$

**Written Exercises, pages 616–618** 1.a.  $\frac{1}{8}$  b.  $\frac{3}{8}$  c.  $\frac{3}{8}$

- d.  $\frac{1}{8}$  3.a.  $\frac{1}{16}$  b.  $\frac{1}{4}$  c.  $\frac{3}{8}$  d.  $\frac{1}{4}$  e.  $\frac{1}{16}$  5.  $\frac{25}{216}$   
 7.a.  $\frac{1}{4}$  b.  $P(0) = \frac{9}{16}$ ;  $P(1) = \frac{3}{8}$ ;  $P(2) = \frac{1}{16}$   
 9.a.  $\frac{1}{4096}$  b.  $\frac{9}{2048}$  11.a.  $\frac{64}{125}$  b.  $\frac{1}{125}$  c.  $\frac{124}{125}$   
 13.a. Emp. prob.  $(0.3125) >$  theor. prob.  $(0.125)$   
 b. 0.16 c. 83;  $\approx 0.13253$  15.a. 0.396 17. 0.387  
 19.a. 366 b. 0.117 c. 23

**Class Exercises, pages 620–621** Exs. 1 and 2:  
Probabilities in column match with events: 1. c,  
b, a, d 2. b, a, d, c 3. a. No aces b. 0.3412

**Written Exercises, pages 621–623** 1. a.  $\frac{13! 47!}{8! 52!}$

b.  $\frac{39! 47!}{34! 52!}$  c.  $1 - \frac{39! 47!}{34! 52!}$  3.  $P(0 R) = \frac{3}{28}$ ;

$P(1 R) = \frac{15}{28}$ ;  $P(2 R) = \frac{5}{14}$  5. a.  $\frac{33}{323} \approx 0.102$

b.  $\frac{352}{969} \approx 0.363$  c.  $\frac{616}{1615} \approx 0.381$  d.  $\frac{224}{1615} \approx 0.139$

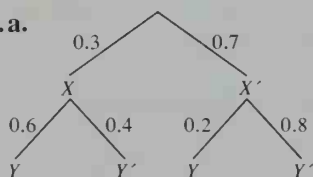
e.  $\frac{14}{969} \approx 0.014$  7.  $\frac{1}{2}$  9.  $\frac{4}{5}$  11.  $\frac{3}{10}$  13.  $\frac{2}{5}$

15. a.  $1.64 \times 10^{-5}$  b.  $4.64 \times 10^{-6}$  c. 0.981 d. 0

17.  $\frac{8}{21}$  19.  $\frac{8279}{23,765} \approx 0.348$  21. a. 0.0681

**Class Exercises, page 625**

1. a.



b. Lower:  $P(Y|X')$ ;  
upper:  $P(X')$

c. 0.18

d. 0.14

e. 0.32

f. 0.5625

2. a. 0.2 b. 0 c. 0.05 d. 0.36 e. 0.41 f. 0.122

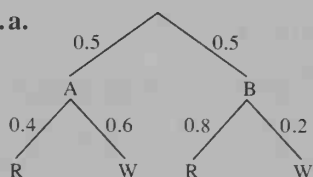
3.  $P(F'|T) = 1 - P(F|T)$  or  $\frac{P(F' \text{ and } T)}{P(T)}$

**Written Exercises, pages 626–629** 1. a. 0.6 b. 0.7

c. 0.5 d. 0.12 e. 0.3 f. 0.42 g. 0.286 h. 0.714

3. a. 0.5 b. 0.4 c. 0.1

5. a.



b. 0.6

c. 0.333

7. b. 3.8%

c. 0.789

9. a. 4.41%

b. 0.549

11. c. 0.853 d. 0.938 13.  $\frac{1}{7} \approx 0.143$

15.  $\frac{9}{13} \approx 0.692$  17. b. 0.74 d. 0.75 e. 2 f. 0.7

**Class Exercises, pages 632–633** 1. 7 2. 1.2

3. Yes 4. Ans. may vary. 5.  $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ ;  
payoff = \$1

**Written Exercises, pages 633–635** 1. 0 3. 55

5. Not fair; B 7. Fair 9. 11¢ loss 11. 0

13. 50¢ gain 15. Expected value =  $\sum_{i=1}^n x_i P(x_i)$

17. \$20,850 21. 15¢ loss 23. \$75

**Chapter Test, pages 636–637** 1. Empirical  
probability is determined by observing what  
previously happened. Theoretical probability is  
determined by reasoning about the events.

2. a.  $\frac{1}{24}$  b.  $\frac{1}{8}$  c.  $\frac{5}{8}$  3. a.  $\frac{5}{18}$  b.  $\frac{1}{9}$  c.  $\frac{1}{8}$

4. a. 0.59 b. 0.991

5. a. 0.034 b. 0.081

6. a. See diagram

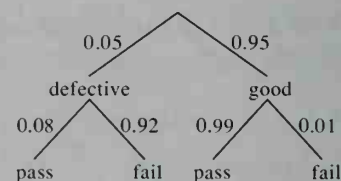
at right.

b. 94.5%

c. 0.004

7. a. Payoffs: \$1, \$2, \$3, \$4, \$-20;

probabilities:  $\frac{5}{32}, \frac{10}{32}, \frac{10}{32}, \frac{5}{32}, \frac{1}{32}$  b. \$1.09



**Class Exercises, page 643** 1. a. Mean = 3.2;

median = 3; mode = 5 b. Mean = 6.875;

median = 7.5; modes = 4 and 8 c. Mean = 2;

median = 2; no mode d. Mean = 2.2;

median = 2.5; mode = 3 3. a. Yes b. Yes c. No

4. Mean: decreases; median and mode: no change

**Written Exercises, pages 643–648**

1. Mean = 5.4; median = 6; mode = 6

3. Mean = 14.1; median = 6; mode = 5

5. a. Median = 13.5; modes = 9, 13, and 16

7. a. 24, 25, 26, 27 b. 24 | 25

c. .2605

25 | 2 6 7 9

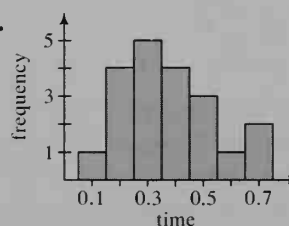
26 | 0 1 1 3 9

27 | 1 6 7

24 | 2 represents .242

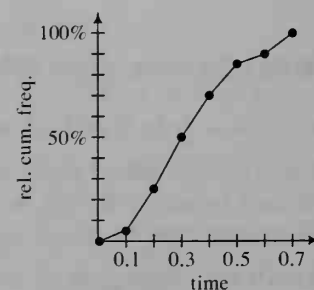
9. a. 60% of scores were  $\leq 7$ ; 20% were  $> 8$ . b. 6

11. a.



b. Mean = 0.375;  
median = 0.35;  
mode = 0.3

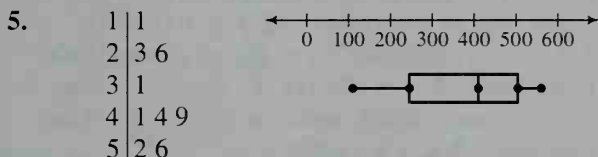
c. See diagram at  
right. Note that  
frequencies can be  
expressed as  
integers, instead of  
as percents.



13.a. Mean = \$9.40; median = \$8.50; mode = \$8.00 b. The mean rises to \$11.00; the median and mode do not change.  
 17.b. Mean = \$4.98; median = \$5.00; mode = \$4.50 25.a. Mean = \$17.950; median = \$14,500 b. Mean = \$18,600; median = \$15,500

**Class Exercises, page 651** 1. Extremes = 137, 225; range = 88 2. Lower = 170; upper = 190; iq range = 20 3. 178 4.  $225 > 1.5(20) + 190$ ; other outliers are 137 and 222.

**Written Exercises, pages 651–653** 1.b. Vermont's no. is not; California's no. is.



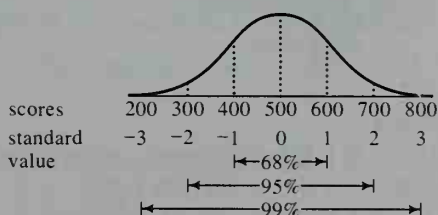
5 | 2 = 520 calories

**Class Exercises, pages 657–658** 1.a. Team A's hts. are more closely clustered around 192 cm than Team B's. b. B 2.  $x_i$  sum = 32;  $x_i^2$  sum = 356; mean = 8; variance = 25;  $s = 5$  3. The mean would dec. by 1; the standard deviation would not change. 4.a. 1 b. 2 c. 2.5 d. -0.5 e. 0 5. 1a 6. Yes, if all the data are equal.

**Written Exercises, pages 658–660** 1. Mean = 7;  $s = 1.90$  3.a. Row A:  $\bar{x} = 3$ ,  $s = 1.41$ ; Row B:  $\bar{x} = 13$ ,  $s = 1.41$  b.  $\bar{x} = 23$ ,  $s = 1.41$  5.  $\bar{x} = 78$ ,  $s = 14$  7.a. 2 b. -0.5 c. 0 d. -1.3 9. Math: 1.1; verbal: 0.8 13. Smallest = 0; largest = 4 15.  $\bar{x} = 10$ ,  $s = 1.45$

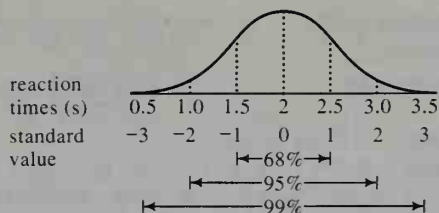
**Class Exercises, page 666** 2. 88.49% 3. 3.59% 4. 27.43% 5. 90.32% 6. 34.13% 7. 68.26% 8.  $P(2) - P(-2) = 0.9772 - 0.0228 = 0.9544 \approx 95\%$

9.a.



b. (1) 2.28%; (2) 2.28%; (3) 68.26% c. 630 d. 430

10.a.



b. (1) 0.13%; (2) 2.28%; (3) 68.26% c. 1.6 s d. 2.4 s

**Written Exercises, pages 667–669** 1.a. 30.85% b. 6.68% c. 68.26% d. 30.23% 3.a. 9.68% b. 30.85% c. 43.32% d. 1.66% 5.a. 2.28% b. 1.31% c. 3.59% 7. 11.96% 9.a. 580 b. 430 11. A: 84; B: 75 13.a. 100% b. 2.0074 L c. 2.0084 L 15. 11.51% 17.b. 0.1473 c. 0.6473 d. Too small;  $P(0.4) = 0.6554$

**Class Exercises, page 672** 1. c 2. b 3. a 4. d 5. c 6. c 7. 44%

**Written Exercises, pages 673–674** 5. 44% 7. 2400 yr 9. 1040 11. 47%

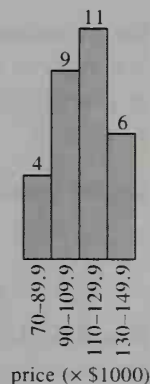
**Class Exercises, pages 677–678** 1.a. 0.1 b. 95%:  $0.2 < p < 0.6$ ; 99%:  $0.1 < p < 0.7$  2.a. 0.1 b. 95%:  $0.3 < p < 0.7$ ; 99%:  $0.2 < p < 0.8$  3.a.  $\bar{p} = 0.5$ ,  $s = 0.05$  b.  $0.4 < p < 0.6$  4. Sample size 5. No

**Written Exercises, pages 678–680** 1.  $\bar{p} = 0.7$ ,  $s = 0.145$ ;  $0.41 < p < 0.99$  3.  $\bar{p} = 0.36$ ,  $s = 0.048$ ;  $0.264 < p < 0.456$  5.  $\bar{p} = 0.0625$ ,  $s = 0.012$ ;  $0.0385 < p < 0.0865$  7.  $0.015 < p < 0.045$  9.  $0.004 < p < 0.176$  11.  $0.724 < p < 0.876$  13. 4 15. 95% 17. 95%:  $0.537 < p < 0.663$ ; 99%:  $0.505 < p < 0.695$  19.b.  $\frac{1}{4}$  21. 10,000 23. 400

### Chapter Test, page 681

1. Descriptive statistics involves collecting, organizing, and summarizing data. Inferential statistics involves drawing conclusions about a population based on a sample.

2.a. See diagram at right.  
 b. Mean = \$112,667;  
 median = \$120,000;  
 mode = \$120,000

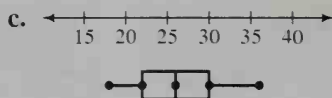




3.a. Median = 26;

first = 22;

third = 30



b. range = 18; iq range = 8

4. English 5. 7.83 6.a. 15.87% b. 2.28%

c. 1330 d. 1270 7. 44% 8. 95%:

$0.778 < p < 0.842$ ; 99%:  $0.762 < p < 0.858$

### Class Exercises, pages 687-688

1. Ans. may vary. a.  $\frac{1}{2}$  b. 2 c.  $y = \frac{1}{2}x + 2$

2.a.  $\bar{x} = 11$ ;  $\bar{y} = 7$ ;  $\bar{xy} = 83$ ;  $s_x = 3.571$ ;  $s_y = 2$

b. (11, 7) c. 0.471 d.  $y = 0.471x + 1.82$  e. 0.84

4.a. 0.6 b. 0.8 c. -0.6 d. -0.8 e. 0 f. 0.2

5.a. Pos. b. Pos. c. Neg. d. Neg. e. 0

6.a.  $\approx 1.75$  b. Thousands of dollars per yr

c.  $\approx \$1750$  7. No; the nos. of firefighters and students are both related to the population of Chesterfield.

### Written Exercises, pages 688-691

1. Pos. 3. Neg. 5. Pos. 7. Neg. 9.  $r = 0.89$ ;  $y = x + 0.5$

11.  $r = -0.93$ ;  $y = -1.5x + 8.5$  13.a. B b. C

c. F 15.  $y = 0.018x + 0.007$ ; 388.5

19.  $y = 0.17x - 1.075$ ;  $r = 0.8246$

### Class Exercises, page 694

1.a.  $\log y = (\log 3)x + \log 2$  b.  $\log 3$  c. Vert. int. is  $\log 2$  2.  $y \approx (6.31)19.95^x$  3.  $y \approx (100)3.162^x$

4.  $y \approx (316.2)0.0398^x$  5.  $y \approx (54.6)4.482^x$

### Written Exercises, pages 694-697

1.  $y \approx (63.1)39.81^x$  3.  $y \approx (199.5)0.3981^x$

5.  $y \approx (24.53)1.649^x$  7.  $y = (2)1.5^x$

9.a.  $y = (15,070)0.7989^x$  b. \$1596 c.  $\approx \$15,000$

13.a.  $\log V = -0.1766t + 0.1811$

b.  $V = (1.517)0.6659^t$  15.a.  $74^\circ\text{F}$  b.  $T = 74$

c.  $T = (103.3)0.9627^t + 74$  17.b. Ans. may vary.

$\log(y - 100) = -0.0075x + 1.225$ ;

$y = (16.79)0.9829^x + 100$

### Class Exercises, page 701

1.a.  $\log y = \log 4 + 3 \log x$  b. 3 2.  $y = 100x^5$

3.  $y = 3162x^{2.5}$  4.  $y = ex^{-3}$  5.  $y = 1.649x^2$

6. 2 7. 3

### Written Exercises, pages 701-704

1.  $y = 10x^4$  3.  $y = \frac{5.012}{x}$  5.  $y = 24.53x^{1.6}$  7. Ans. may vary.

$y = \frac{1}{2}x^3$  9.a. 2 b. 3 11. 2 13.  $\frac{1}{2}$  15.  $\frac{2}{3}$

Exs. 17-21: Ans. may vary.

17.  $T = (1.553 \times 10^{-7})a^{3/2}$  21.  $y = (3.065)x^{0.6141}$

### Mixed Exercises, pages 705-709 Exs. 1-13: Ans.

may vary. 1.a.  $y = (8.385 \times 10^{-4})x^{2.69}$

b.  $\approx 0.7$  lb 3.a.  $T = (13.13)P^{0.3061}$  b.  $\approx 94^\circ\text{C}$

5.a.  $W = aC^3$  b.  $W = 0.394C^3$  c.  $\approx 41$  lb

7.b.  $G = (17.15)1.063^x$  c.  $\approx 7720$  9.a.  $\approx 19^\circ\text{C}$

b.  $T = 19$  c.  $\log(19 - T) = -0.0166t + 1.4037$

d.  $T = 19 - (25.33)0.9625^t$

11.  $y = (3.371)0.9995^x + 5$  13.a. 5; 8 = B

b.  $y = 0.9054x + 0.4932$ ;  $\approx 7.7$

### Chapter Test, page 710

1.a.  $\approx 0.71$  b.  $y = 0.82x - 4.9$  c. Since the corr. coeff. is only 0.71, the model probably has limited accuracy in predicting precipitation. 2. No; both scores are probably related to other factors, perhaps I.Q., rather than to each other. 3.  $y = 100(1000)^x$

4.a.  $y = (57.86)1.065^x$ ,  $x$  = no. of yr after 1900

b. \$22,940 5.  $y = (0.01)x^{1.7}$  6.  $m = 2$ ; cost

varies as area, which varies as the square of the diagonal. 7.a.  $y = (4.099 \times 10^{-10})d^{1.5}$

b.  $\approx 92,560,000$  mi

b.  $\approx 92,560,000$  mi

### Cumulative Review, pages 711-712

1. 27 2. 2100

3. 126 4. 504 5. 60

6.  $16x^4 - 16x^3 + 6x^2 - x + \frac{1}{16}$  7.a.  $\frac{1}{4}$  b.  $\frac{3}{4}$  c. 0

d. 1 e.  $\frac{1}{13}$  f.  $\frac{2}{13}$  8.a.  $\frac{11}{850}$  b. None 9.  $\frac{625}{3888}$

10. Diamond:  $\frac{4}{17}$ ; spade:  $\frac{13}{51}$  11.a. 5.2% b.  $\frac{6}{13}$

12. \$250

13.

mean  $\approx 6.63$

median = 7

mode = 10

13.

number of customers

day of month

14.a. 4 | 3 4

• 7 8 9

5 | 0 0 0 1 1 1 1 2 2 3 4

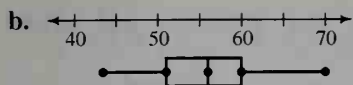
• 5 5 5 5 6 6 6 6 7 7 7 8 8 8 9

6 | 1 1 2 2 3 4

• 5 6 8

7 | 0





15. Mean = 38.5;  $s = 32.4$  16. a. 4% b. 90%  
 c. 48% 17. Ans. may vary. The people polled are not representative of the general population. The factors which keep them at home may influence their feelings about taxes. Opinions of those who work during business hours would not be included.  
 18.  $0.383 < p < 0.477$

19. a. b.  $y = -2.4x + 12$

20. a. Power; vol. varies as the square of the diameter. b. Linear; sales tax is a % of cost.  
 c. Asymptotic; the function's value does not go beyond the boiling pt. of water.  
 21.  $y = 1.97x^{3.016}$  22. a. Exponential; populations tend to double at regular intervals.  
 b.  $P = (1.156)1.786^t$  c.  $\approx 382$

### Class Exercises, page 723

1. Meanings are given:  
 a. For large values of  $x$ , the value of  $\frac{2x^2}{x^2 + 1}$  is close to 2. b. For neg. values of  $x$  where  $|x|$  is large, the value of  $\frac{3x + 1}{2x - 5}$  is close to  $\frac{3}{2}$ . c. For values of  $x$  near 2, but greater than 2, the value of  $f(x)$  is close to 3. d. For values of  $x$  near 2, but less than 2, the value of  $g(x)$  is arbitrarily large.  
 2. 0, -1, 4, 1 3.  $\lim_{x \rightarrow 2} f(x) \neq f(2)$   
 4. a.  $\lim_{x \rightarrow 0^+} f(x) = 0$ ;  $\lim_{x \rightarrow 0^-} f(x) = -3$   
 b.  $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$  5. The limit of the quotient of two functions is the quotient of the limits of the functions. 6. a.  $\frac{5}{7}$  b.  $\infty$  c.  $-\infty$  d. 4

### Written Exercises, pages 724–725

1.  $\frac{3}{4}$  3. 2  
 5.  $\frac{1}{2}$  7. 2 9.  $\frac{5}{2}$  11.  $\infty$  13.  $-\infty$  15. 0 17. 2  
 19. a. 1 b. -1 c. Does not exist 21. a. 0 b. Does not exist c. Does not exist 23. a. 1 b. 0 c.  $\infty$

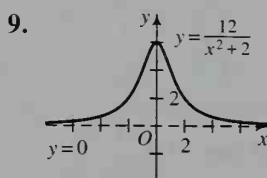
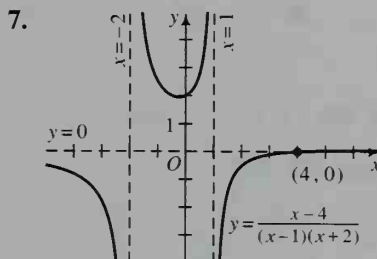
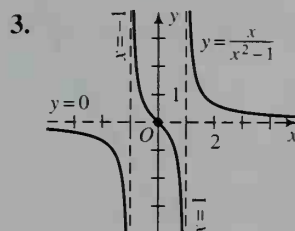
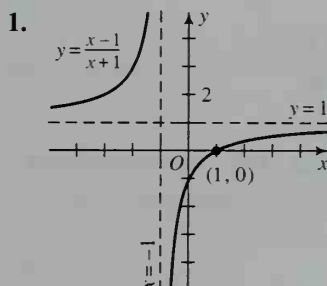
- d.  $-\infty$  25.  $-\frac{1}{2}$  27. 0 29. a. 2 b. 4 c.  $2x$

31. Cont. 33. Cont. 35.  $a = -1$ ,  $b = 0$  37. 1  
 39. 0 41. Does not exist 45. 2 47. 1

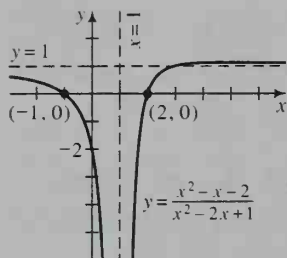
### Class Exercises, page 728

1. x-int. 2; vert. asymptotes:  $x = 1$ ,  $x = 3$ ; horiz. asymptote:  $y = 0$   
 2. x-int. 0; vert. asymptotes:  $x = \pm 2$ ; horiz. asymptote:  $y = 0$  3. x-ints.  $\pm 2$ ; vert. asymptote:  $x = 0$  4. x-ints.  $\pm 3$ ; horiz. asymptote:  $y = 1$  5. x-int. 0; vert. asymptotes:  $x = \pm \sqrt{6}$ ; horiz. asymptote:  $y = 2$   
 6. x-int. 0; vert. asymptotes:  $x = \pm 1$   
 7. If  $c$  is a zero of greater multiplicity for  $g$  than for  $f$ , there is a vert. asymptote at  $x = c$ ; otherwise, there is a hole in the graph at  $x = c$ .

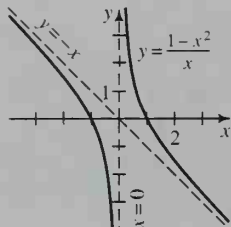
### Written Exercises, pages 728–729



15.



17.

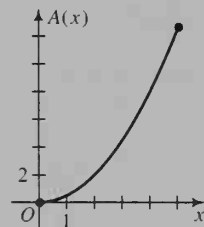
**Class Exercises, page 731**

1. 20 FOR X = .1 TO 3.1 STEP .1  
30 LET Y = SIN(X)  
2. 20 FOR X = .1 TO 3.1 STEP .1  
30 LET Y = 2 \* SIN(X)  
3.b. Greater

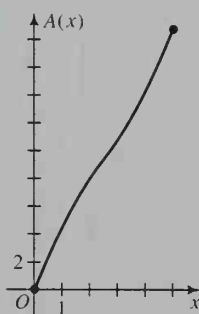
**Written Exercises, pages 731-733** 1. 0.3461589263. 0.33835001 5.a.  $\frac{1}{2}$  b.  $\frac{1}{5}$ ; 0.20503334

7. 2.72

9.



11.



13. 2 15. 2 17. 2

**Class Exercises, page 735** 1. 6 2.  $-\sin x$ **Written Exercises, pages 735-736**

1.  $\cos \frac{\pi}{4} = 1 - \frac{\pi^2}{4^2 \cdot 2!} + \frac{\pi^4}{4^4 \cdot 4!} - \frac{\pi^6}{4^6 \cdot 6!} + \cdots$   
 $\sin \left(-\frac{\pi}{4}\right) = -\frac{\pi}{4 \cdot 1!} + \frac{\pi^3}{4^3 \cdot 3!} - \frac{\pi^5}{4^5 \cdot 5!} + \frac{\pi^7}{4^7 \cdot 7!} - \cdots$  3.a.  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$   
 b. 2 is not in the interval of convergence.  
 5.  $1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$ ; all real  $x$

7.  $2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \cdots$ ;  $-\frac{1}{2} \leq x \leq \frac{1}{2}$

9.  $x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$ ; all real  $x$

13. 1 15.  $e^{i\pi} = \cos \pi + i \sin \pi = -1$ ;  
 $e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1$

**Class Exercises, page 741** 1. 5, 7, 9, 11 2. 0, 1,

0, 1 3. -1, -1, -1, -1 4. 1, 1.1, 1.11, 1.111

5. Ans. may vary. If  $f(x) = x$ , then  $f^2(x_0) = x_0$ ;but  $(f(x_0))^2 = x_0^2$ . 6.a. 0 b. Slowly 7. At thegraph's intersection with the graph of  $y = x$ 8.  $1 - \sqrt{3}$  is attracting;  $1 + \sqrt{3}$  is repelling.9.a. 0, 1 b. Yes; -1 c. If  $|x_0| > 1$ ,  
 $\lim_{n \rightarrow \infty} f^n(x_0) = \infty$ ; if  $|x_0| < 1$ ,  $\lim_{n \rightarrow \infty} f^n(x_0) = 0$ . d. 0

is attracting; 1 is repelling. 10.a. -1, 0, 1

b. No c. If  $x_0 > 1$ ,  $\lim_{n \rightarrow \infty} f^n(x_0) = \infty$ ; if  $x_0 < -1$ , $\lim_{n \rightarrow \infty} f^n(x_0) = -\infty$ ; if  $|x_0| < 1$ ,  $\lim_{n \rightarrow \infty} f^n(x_0) = 0$ .

d. 0 is attracting; -1 and 1 are repelling.

**Written Exercises, pages 741-743** 1. 5, 9, 17, 333. 2.5, 0.4, 2.5, 0.4 5. 4 7.  $\frac{1}{2}$  9. 4 11.  $x_4$  isnot def. because  $x_3 < 0$ . 13.a.  $\lim_{n \rightarrow \infty} f^n(x_0) = 2$ 

b. No change 15.a. 0.739 17.a. 0 b. Undef.

19. 0 and 1 are fixed pts.; 2 is an eventually fixed

pt.;  $\lim_{n \rightarrow \infty} f^n(x_0) = \begin{cases} 1 & \text{if } 0 < x_0 < 2 \\ -\infty & \text{if } x_0 < 0 \text{ or } x_0 > 2 \end{cases}$ 21.  $|m| < 1$ ;  $m = 1$  when  $k = 0$ . 23.a. 1.62b. No; 1.62 is an attracting pt. 25.a. 3.17, 3.16,  
3.16, 3.16**Class Exercises, page 746** 1.a.  $x_0 = 100$ , $x_n = 1.01(x_{n-1}) - 20.6$  b. The orbit of  $x_0 = 100$ for iterations of  $f(x) = 1.01x - 20.6$  2. 0.32,

0.435, 0.492, 0.500; 0.500 is equilibrium pt.

3. 0.08, 0.037, 0.018, 0.009; 0 is equilibrium pt.

**Written Exercises, pages 746-748** 1.a.  $x_0 = 100$ , $x_n = 1.06(x_{n-1})$  b. The orbit of  $x_0 = 100$  foriterations of  $f(x) = 1.06x$  3.a.  $x_0 = 400,000$ , $x_n = 1.065(x_{n-1}) - 30,000$  b. The orbit of $x_0 = 400,000$  for iterations of $f(x) = 1.065x - 30,000$  5. Equilibrium pt. at

0.333 7. Period-4 cycle: 0.501, 0.875, 0.383,

0.827 9. Period-3 cycle: 0.505, 0.957, 0.156

11.a. \$425,678.19 b. Because you would also

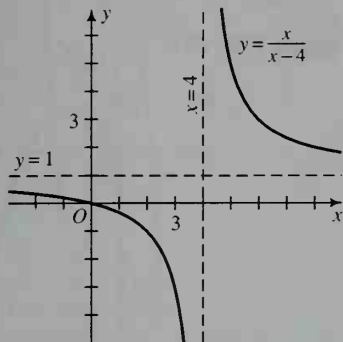
receive the interest, you would have an additional

\$3,863,749.98. 15.  $f(x) = (1 + c(1 - x))x + hx$

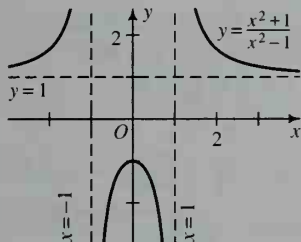
**Chapter Test, page 749** 1. a. 1 b.  $\frac{1}{2}$  c.  $\frac{1}{6}$  d. Does

not exist:  $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$ , but  $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$ .

2. a.



b.



3. a. 5.31330004  
b. 10.62660008

4. a.  $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \dots$

b.  $x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \frac{(x^2)^4}{4} + \dots; -1 \leq x \leq 1$

5. -1.841 6. 0 is a repelling fixed pt.;

$\lim_{n \rightarrow \infty} f^n(x_0) = \begin{cases} \infty & \text{if } x_0 > 0 \\ -\infty & \text{if } x_0 < 0 \end{cases}$

7.  $\lim_{n \rightarrow \infty} f^n(x_0) = 0$ . The population will not survive regardless of the initial population.

**Class Exercises, page 761** 1.  $8x^7$  2.  $12x^{11}$

3.  $-\frac{4}{x^5}$  4.  $21x^6$  5.  $\frac{3}{2}\sqrt{x}$  6.  $-\frac{6\sqrt{x}}{x^3}$  7.  $\frac{\sqrt[3]{x}}{3x}$

8.  $-\frac{\sqrt[3]{x^2}}{3x^2}$  9.  $24x^2 - 14x + 3$

10.  $-\frac{20}{x^6} + \frac{6}{x^4} - \frac{9}{x^2}$  11. a.  $f'(x) = 8x; f'(1) = -8$

b. -8 12. a.  $f'(1) = f'(2) = f'(3) = m$

**Written Exercises, pages 762-763** 1. a. 7 b. 4.75

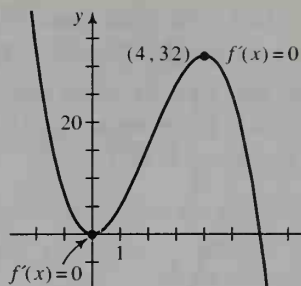
c. 3.31 d.  $f'(x) = 3x^2; f'(1) = 3$  3.  $10x^4$

5.  $-6x^2$  7.  $-\frac{6}{x^3}$  9.  $\frac{\sqrt{5x}}{2x}$  11.  $7x - 5 + \frac{1}{x^2}$

13. 3 15. -2 17. -13 19.  $\frac{1}{2}$  21. a.  $2ax + b$

b. 0 23.  $y = 12x + 16$  25.  $y = 2x + 3$

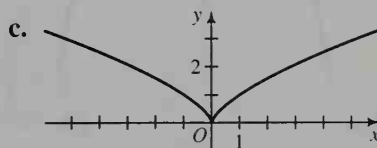
27. a, c.



b. 0 or 4

29. a.  $\frac{2}{3}x^{-1/3}$

b. 0



**Exs. 31-35:** Ans. may vary by a constant. 31.  $x^4$

33.  $\frac{1}{2}x^6$  35.  $2x^{3/2}$

37. a.  $x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + h^n$

**Class Exercises, page 767** 1. a. -5, -1, 2, 6

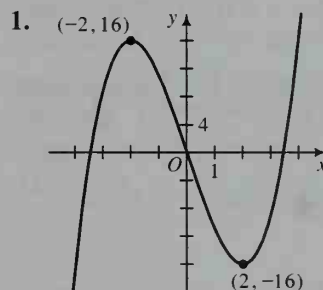
b.  $-5 < x < -1, 2 < x < 6$  c.  $x < -5,$

$-1 < x < 2, x > 6$  2. a. -4, 1, 4 b.  $x < -4,$

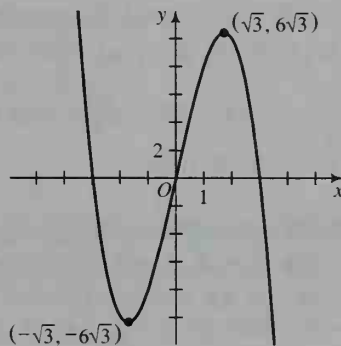
$-4 < x < 1, x > 4$  c.  $1 < x < 4$  3. Yes

4. Local max. 5. Local max. at  $x = -1$ ; local min. at  $x = 2$  6. Local max. at  $x = -2$ ; local min. at  $x = 2$

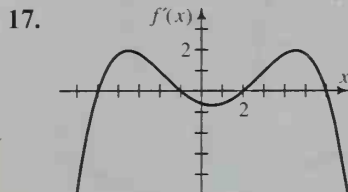
**Written Exercises, pages 768-769**



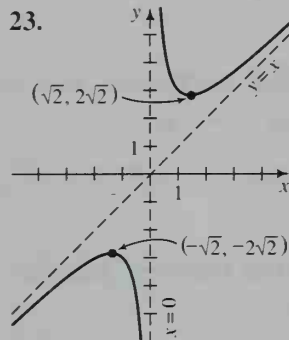
3.  $(\sqrt{3}, 6\sqrt{3})$



5. Local max.:  $(-1, 1)$ ,  $(1, 1)$ ; local min.:  $(0, 0)$   
 7. Local max.:  $(0, 8)$ ; local min.:  $(-\sqrt{3}, -1)$ ,  $(\sqrt{3}, -1)$  9. Local max.:  $(0, 0)$ ; local min.:  $(-1, -5)$ ,  $(2, -32)$  11. Local max.:  $(1, 1)$ ; local min.:  $(0, 0)$  13.  $(3, -5)$  15.  $(a, 0)$



21. a. 1  
 b. 1; 0; -1; 0; 1  
 c.  $\cos x$



25. Local min.:  $(\frac{1}{2}, 3\frac{3}{4})$ ; asymptotes:  $y = x^2 + 3x$ ;  
 $x = 0$

**Class Exercises, page 772** 1. Global max. 5 at  $x = 3$ , global min. -1 at  $x = 0$  2. Global max. 2 at  $x = 1$ , global min. -1 at  $x = 4$  3. Global max. 6 at  $x = 2$ ; global min. 2 at  $x = 0$  4. Global max. 2 at  $x = 0$ ; global min. -2 at  $x = -2$  5. Global max. 2 at  $x = 2$ ; global min. -2 at  $x = 1$  6. Global max. 14 at  $x = 2$ ; global min. -14 at  $x = -2$  7. Global max. 2 at  $x = \frac{\pi}{2}$ ; global min. 0 at  $x = 0$ ,  $x = \pi$  8. Global max. 1 at  $x = 0$ ; global min. 0 at  $x = \pi$

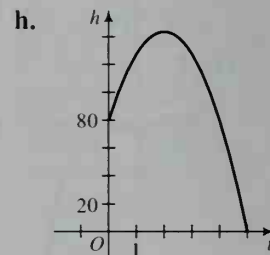
**Written Exercises, pages 772-774** 3. 8 sq. units  
 5. a.  $V(x) = x(8 - 2x)(15 - 2x)$ ,  $0 \leq x \leq 4$   
 b.  $\frac{5}{3}$  cm 7. a.  $(-1, -1)$ ,  $(0, 0)$ ,  $(1, 1)$  b.  $\frac{\sqrt{3}}{3}$   
 9. 400 11. 2 ft by 2 ft by 2 ft 13. b.  $\frac{32\pi}{3}$  cubic units 15. a. \$6.60; \$6.52 b. 60 mi/h

**Class Exercises, page 778** 1. a.  $(48 - 32t)$  ft/s; 16 ft/s b.  $-32$  ft/s<sup>2</sup> 2. a. 2 m b. 1 m/s c. Right:  $0 \leq t < 2$ ; left:  $t > 2$  d.  $t = 4$  s e. 1 m/s; 0 m/s; -1 m/s f. Right:  $t = 0$  s; left:  $t = 4$  s g. 2 m

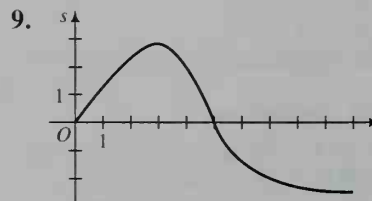
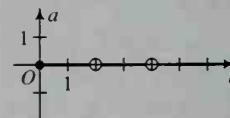
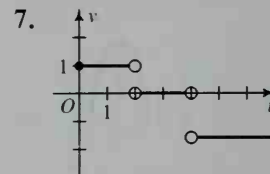
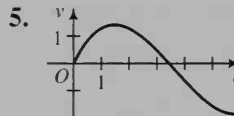
3. a. Left b. 4 m to the left of the origin c. 6 m  
 d. -3 m/s e.  $t = 5$  s f.  $2$  m/s<sup>2</sup>;  $2$  m/s<sup>2</sup> 4. a. to the left of the origin b. moving to the right  
 c. decelerating

### Written Exercises, pages 778-781

1. a. 32 ft/s  
 b.  $(64 - 32t)$  ft/s;  
 32 ft/s  
 c.  $t = 2$  s  
 d. 144 ft  
 e.  $t = 5$  s  
 f.  $2 < t < 5$   
 g.  $-32$  ft/s<sup>2</sup>



3. a. 2 m b. 1 m/s c. Right:  $0 \leq t < 3.5$ ; left:  $t > 3.5$  d.  $t = 6$  s e. 1 m/s; 1 m/s; 0.5 m/s  
 f. Right:  $t = 1.5$  s; left:  $t = 5$  s g. 3 m



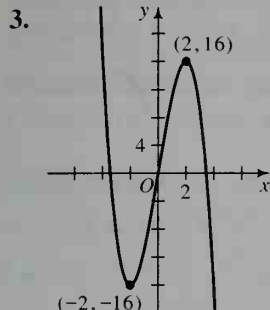
11. a. At the present time  $t$ ,  $P'(t) > 0$ . b. At the present time  $t$ ,  $P''(t) < 0$ . c. At some future time  $t$ ,  $P'(t) = 0$ . 13. a.  $t = 2$  s b.  $t = 0$  s,  $t = 2$  s  
 c.  $t = 1$  s 15. c.  $\approx 101.5$  ft/s d. At  $t = 2$  s, speed  $\approx 102.4$  ft/s; the ball hits the ground at  $t = 3$  s at a speed of  $\approx 125.0$  ft/s.

**Chapter Test, page 782** 1. a.  $15x^2 - 6x + 1$

b.  $-\frac{6}{x^3} + \frac{2}{x^2}$  c.  $\frac{2\sqrt{3x^2}}{3x} + \frac{\sqrt{x^2}}{3x^2}$  d.  $-\frac{12}{x^7} + 18x^5$

2. Ans. may vary. Solve  $f'(x) > 0$  and  $f'(x) < 0$  to find the intervals on which the graph of  $f(x)$  is rising and falling, respectively. At a pt. where

$f'(x) = 0$ , the graph has a local max. or min. (if  $f'(x)$  changes sign), or it flattens out (if  $f'(x)$  doesn't change sign).



4.  $f(2)$  is the max. value of the function.

5.a.  $A(x, h) = 2x^2 + 4xh$  b.  $h = \frac{1000}{x^2}$

c.  $A(x) = 2x^2 + \frac{4000}{x}$ ,  $x > 0$

d.  $A'(x) = 4x - \frac{4000}{x^2}$ ;  $4x - \frac{4000}{x^2} = 0$ ;  $x^3 = 1000$ ;

$x = 10$ ;  $h = \frac{1000}{10^2} = 10$ . 6. Position: 54 units to

the right of the origin; velocity: 25 units/s;

acceleration: 6 units/s<sup>2</sup> 7.a. 3 s b. 400 ft c. 7 s

d. -64 ft/s

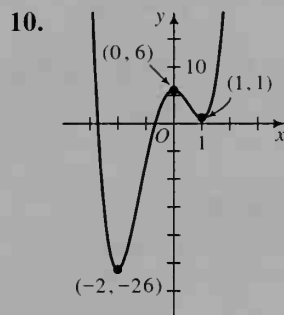
b.  $x \leq -1$  or  $x \geq 1$  5.a. 10, 1, -0.8, -1.16

b. -1.25 6.  $\frac{1}{2}$  is a fixed pt.;  $-\frac{1}{2}$  is an eventually

fixed pt.;  $\lim_{n \rightarrow \infty} f^n(x_0) = \begin{cases} \frac{1}{2} & \text{if } |x_0| < \frac{1}{2} \\ \infty & \text{if } |x_0| > \frac{1}{2} \end{cases}$

7. The population has an equilibrium pt. of 0.6.

8.a.  $6x - 1 - \frac{4}{x^2} + \frac{1}{x^3}$  b.  $\frac{\sqrt{2x}}{2x} + \frac{\sqrt{x}}{x^2}$  9. 4



11.  $6\frac{2}{3}$  in. by  $6\frac{2}{3}$  in. by  $1\frac{2}{3}$  in.

12.a.  $0 \leq t < 4$  b.  $t > 4$

SAT Sample Questions, pages 790–793 1. C 2. E

3. E 4. D 5. B 6. B 7. E 8. E 9. E

10. B 11. E 12. B 13. D 14. B 15. E

16. B 17. C 18. B 19. D 20. A 21. D

22. B 23. D

Mathematics Level I Test Sample Questions, pages

795–796 1. C 2. A 3. B 4. D 5. C 6. D

7. D 8. A 9. E 10. E 11. E 12. B 13. A

14. E 15. D 16. B 17. D 18. C 19. D

20. B

Mathematics Level II Test Sample Questions, pages

797–798 1. A 2. B 3. E 4. B 5. E 6. C

7. B 8. C 9. B 10. E 11. D 12. E 13. B

14. B 15. E 16. E

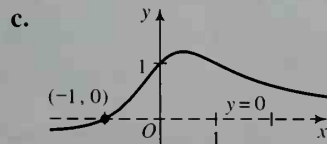
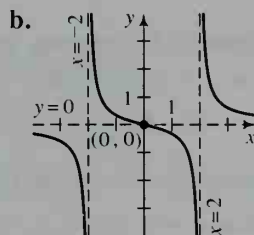
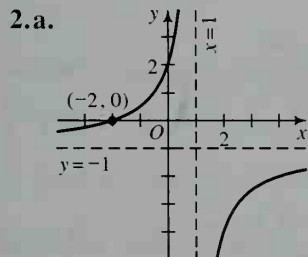
Mathematics Level IIC Test Sample Questions, pages

798–799 1. B 2. A 3. B 4. A 5. C 6. B

7. C 8. E 9. A 10. E 11. C 12. D 13. D

14. B 15. C 16. B

Cumulative Review, page 783 1.a. 3 b.  $\frac{3}{2}$  c.  $-\infty$



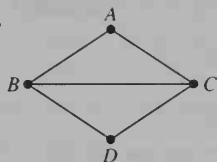
3.a.  $\approx 0.169$  b.  $\approx 1.169$ ; since the graph of  $y = 1 + \log x$  is the graph of  $y = \log x$  translated vertically 1 unit, the area between the two graphs from  $x = 1$  to  $x = 2$  is equivalent to the area of a rectangle 1 unit high and  $2 - 1 = 1$  unit wide.

4.a.  $\tan^{-1} \frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \dots$



# Appendix 1 Exercises, pages 823–824

1.



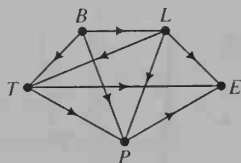
Valences: A: 2, B: 3  
C: 3, D: 2  
No Euler circuit

3. Ans. may vary. *BACBDC* 5.a. No b. Yes

7.a. Bears and Tigers

b.

c. First: Bears  
last: Eagles



## Appendix 1 Exercises, pages 826–827

1. No  
3.a. Yes b. No c. At least one of the values  $x$  or  $y$  must be odd. 5. *FGHEF*; yes

7.a. 20!

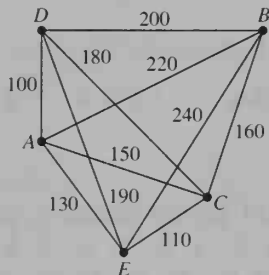
b. 20

c.  $20 \times 20!$

d.  $\approx 1542$  yr

9.a. See graph at right.

b. *ADCEBA*



## Appendix 1 Exercises, pages 828–829

1. *AH, HG, HF, FB, BC, CD, DE, EI* 3.  $n - 1$  5. b. *AB, BH, BI, IC, IF, FG, FE, ED* 7. \$9,900

## Appendix 2 Exercises, pages 832–834

1. (2.5, 3)  
3.a. 8; 8; 8 b. 6; 5; 6 c. 20; 21; 20

5.  $y = \frac{1}{2}x + \frac{4}{3}$  7.  $y = -\frac{1}{2}x + \frac{15}{4}$

9.a. No change b. No change c. Before replacement,  $(\bar{x}, \bar{y}) = (3.6, 2)$ ; after replacement,  $(\bar{x}, \bar{y}) = (3.6, 4)$ . 11.a. Boys:  $y = 1.75x + 29.75$ ; girls:  $y = 3.71x - 142.14$  b. Boy:  $\approx 166$  cm; girl:  $\approx 147$  cm 13.a.  $y = 0.4138x + 12.1471$  b. U.S.A. c. 49.0

## Appendix 3 Exercises, page 837

pos.	neg.	imag.
1	1	0

pos.	neg.	imag.
2	1	0
0	1	2

5.

pos.	neg.	imag.
2	2	0
2	0	2
0	2	2
0	0	4

7.

pos.	neg.	imag.
3	1	0
1	1	2

9.a.  $L = -1$ ;  $M = 2$  b.  $-1 < x < 2$

11.a.  $L = -2$ ;  $M = 3$  b.  $-2 < x < 3$

13.a.  $L = -1$ ;  $M = 0$  b.  $-1 < y < 0$

15.a.  $L = -2$ ;  $M = 3$  b.  $-2 < y < 3$

## Appendix 4 Exercises, pages 840–841

$$1. \frac{1}{x-1} + \frac{5}{x+5} \quad 3. \frac{3}{4x-3} + \frac{1}{x+3}$$

$$5. \frac{4}{x} - \frac{9}{x+1} + \frac{6}{x+2} \quad 7. \frac{2}{x-2} - \frac{1}{x+2}$$

$$9. \frac{3}{x+4} - \frac{13}{(x+4)^2}$$

$$11. \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{2}{(x+1)^3}$$

$$13. \frac{\frac{1}{2}}{x-1} + \frac{\frac{3}{2}}{x+1} - \frac{1}{(x+1)^2}$$

$$15. \frac{1}{x+3} - \frac{10}{(x+3)^2} + \frac{33}{(x+3)^3} - \frac{36}{(x+3)^4}$$

$$19. 4x - 7 + \frac{1}{x} + \frac{8}{x+1} \quad 21. \frac{3}{x-2} + \frac{2x-1}{x^2+x-1}$$

## Appendix 5 Exercises, page 845

1.a.  $y = mx + 5$  b.  $y = -2x + k$  3.  $y = x$ ,

$y = -x$  5.  $y = -x \pm 2$  7.  $y = \frac{2}{3}x - 1$ ,  $y = -1$

9.a.  $y = -2x - 9$  b.  $(-3, 3)$  c.  $y = \frac{1}{2}x - \frac{3}{2}$

11.a.  $y = x + 4$ ,  $y = -x + 4$  b.  $(-2, 2)$ ;  $(2, 2)$

c.  $y = -x$ ;  $y = x$

## Appendix 6 Exercises, page 847

3.a.  $Y_{2T} = 3 + 65T - 16T^2$  b. No.

5.  $v = 117$  ft/s,  $\theta = 31.0^\circ$

## Appendix 6 Exercises, pages 849–850

1. a. 10.35 ng/ml b. 0.0252 ng/ml 3. 77.5 ng/ml

5. The concentration increases, approaching a limiting value of 80 ng/ml. 7. a dose that increases the concentration by 200 ng/ml

**Appendix 6 Exercises, pages 851–852**

5. a. 0.53 ton b. 0.67 ton c. It increased from 0.49 ton to 0.79 ton. 7.  $L_5$  values (in lb/day): 1960, 2.68; 1965, 2.91; 1970, 3.28; 1975, 3.26; 1980, 3.64; 1985, 3.78; 1990, 4.31; 1995, 4.33

**Appendix 6 Exercises, page 853**

3. Tim: \$214,195; Tom: \$172,317 5. \$13,181

**Refresher Exercises, Chapter 1, page 854**

1. II 3.a.  $9\sqrt{7}$  b.  $2\sqrt{5} + 2\sqrt{10}$  5.  $-\frac{3}{4}$   
7.a.  $4x^2 - 20x + 25$  b.  $-1$  9.  $-3, 1$

**Refresher Exercises, Chapter 2, page 854**

1.a. Linear b. Quadratic c. Linear 3. Yes  
5.  $5x(x-2)(x+2)$  7.b. 0 9.  $V(r) = 2\pi r^3$

**Refresher Exercises, Chapter 3, page 855**

3.a. 5,  $-5$  b.  $-5 < x < 5$  c.  $-5 < x$  or  $x > 5$   
5.  $(0, 3)$ ,  $(\frac{9}{2}, \frac{21}{4})$  7.  $(a-2)(a+1)(a-1)$

**Refresher Exercises, Chapter 4, page 855**

1.  $-7$  3.  $3a+2$ ;  $6a+2$ ;  $3a+5$   
7.  $y = \sqrt[3]{x+4}$  9. Ans. may vary. For example,  $2\text{ cm} \times 3\text{ cm}$ ,  $6\text{ cm} \times 1\text{ cm}$ ,  $\sqrt{2}\text{ cm} \times 3\sqrt{2}\text{ cm}$

**Refresher Exercises, Chapter 5, page 856**

1.a.  $16x^{11}$  b.  $\frac{3x}{4}$  3. 3, 10, 4;  $\sqrt{x}$ ,  $x \geq 0$  5. 3  
7. 0.1

**Refresher Exercises, Chapter 6, page 856**

1.a.  $f(x) = 2(x+2)^2 - 3$  b.  $(-2, -3)$   
3.  $y = 2(x+4)^2 - 2$  5.  $4x + 3y = 20$   
7. Isosceles;  $AC = BC = 10$  9.  $(-3, -7)$ ,  $(\frac{2}{3}, \frac{1}{3})$

**Refresher Exercises, Chapter 7, page 857**

1.b.  $\approx 31.4$ ;  $\approx 78.5$  3.  $5\sqrt{3}$  5.a.  $x = -2$   
b.  $y = -3$  c.  $x = 0$ ,  $y = 0$  d.  $y = \pm \frac{1}{3}x$   
7.  $x \geq 0$ ;  $f^{-1}(x) = \sqrt{x-2}$

**Refresher Exercises, Chapter 8, page 857**

3.a.  $\frac{\pi}{2}$  b.  $-\frac{\pi}{6}$  c.  $\frac{\pi}{4}$  7.  $-1, \frac{1}{2}, 1$  9.  $\frac{a+b}{ab}$

**Refresher Exercises, Chapter 9, page 858**

1.  $6.43$  3. 20 5.  $26.4^\circ$  7.  $70^\circ$

**Refresher Exercises, Chapter 10, page 858**

1.  $\cos x = -\frac{8}{17}$ ,  $\tan x = \frac{15}{8}$ ,  $\sec x = -\frac{17}{8}$ ,

$$\csc x = -\frac{17}{15}, \cot x = \frac{8}{15} \quad 3. -1$$

$$5. \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

$$7. \csc \theta \quad 9. \text{Domain: } \{x | x \neq \frac{\pi}{2} + n\pi\},$$

Range: {all real numbers}

**Refresher Exercises, Chapter 11, page 859**

1.  $-\cos 36^\circ$  3.a. 8.1 b.  $60.3^\circ$  5.  $11 + 2i\sqrt{3}$   
7.  $-1 - 3i$  9.  $8 \cos^4 \theta - 8 \cos^2 \theta + 1$

**Refresher Exercises, Chapter 12, page 859**

1.a.  $AO = 5$ ,  $AB = 4$ ,  $OB = \sqrt{65}$  b.  $127^\circ$   
3.  $y = -3x - 2$  5.  $(x-5)^2 + (y+7)^2 = 25$   
7.  $(x-1)^2 + y^2 = (2 \cos^2 \theta - 1)^2 + (\sin 2\theta)^2 =$   
 $(\cos 2\theta)^2 + (\sin 2\theta)^2 = 1$  9.a.  $-\frac{5}{2}$   
b. Ans. may vary. For example,  $(5, 0, 0)$  and  $(0, 2, -1)$

**Refresher Exercises, Chapter 13, page 860**

3. 170 5. 10,  $-\frac{35}{3}$  7.a. 2.002 b. 0.002

**Refresher Exercises, Chapter 14, page 860**

1.a.  $(2, -3)$  b.  $(1, -1, 2)$  3.  $(5, -8, 6)$ ;  
 $(1, -13, -6)$ ;  $-13$  5.a.  $(1, 0) \rightarrow (1, 0)$ ;  
 $(0, 1) \rightarrow (0, -1)$  b.  $y = x - 1$   
7.a.  $(1, 0) \rightarrow (0, 1)$ ;  $(0, 1) \rightarrow (1, 0)$  b.  $y = -x + 1$

**Refresher Exercises, Chapter 15, page 861**

1. 6 3. 6 5.  $x^3 - 3x^2y + 3xy^2 - y^3$   
7.a.  $2\sqrt{2} \text{ cis } 45^\circ$  b.  $-64$

**Refresher Exercises, Chapter 16, page 861**

1.a.  $\frac{1}{4}$  b.  $\frac{3}{13}$  c.  $\frac{1}{13}$  5. 2,023,203 7.  $\frac{10}{21}$

**Refresher Exercises, Chapter 17, page 862**

1. 18 3.a. 9 b. 6.5  
5.  $f(g(x)) = x - 1$ ,  $x \geq -3$

**Refresher Exercises, Chapter 18, page 862**

1.  $y = 0.659x + 0.237$  5.a. 5.248 b. 0.792  
c. 2.974

**Refresher Exercises, Chapter 19, page 863**

1.a.  $+\infty$  b.  $\frac{2}{7}$  3.  $\sqrt{x-4}$  5.  $-3 < x < -1$  or  
 $1 < x < 5$  7.  $2.71\bar{6}$ ;  $e$

**Refresher Exercises, Chapter 20, page 863**

1.  $2x + 3y = -14$  3.  $4x + 2h$

# Credits

**Cover:** Concept by Ligature, Inc.; Photography by Ralph Brunke

**Book Design:** Morgan Cain & Associates

**Technical art:** Morgan Cain & Associates; **Illustrations:** Robert Grace: 237, 323, 375, 411, 499, 537, 660, 736

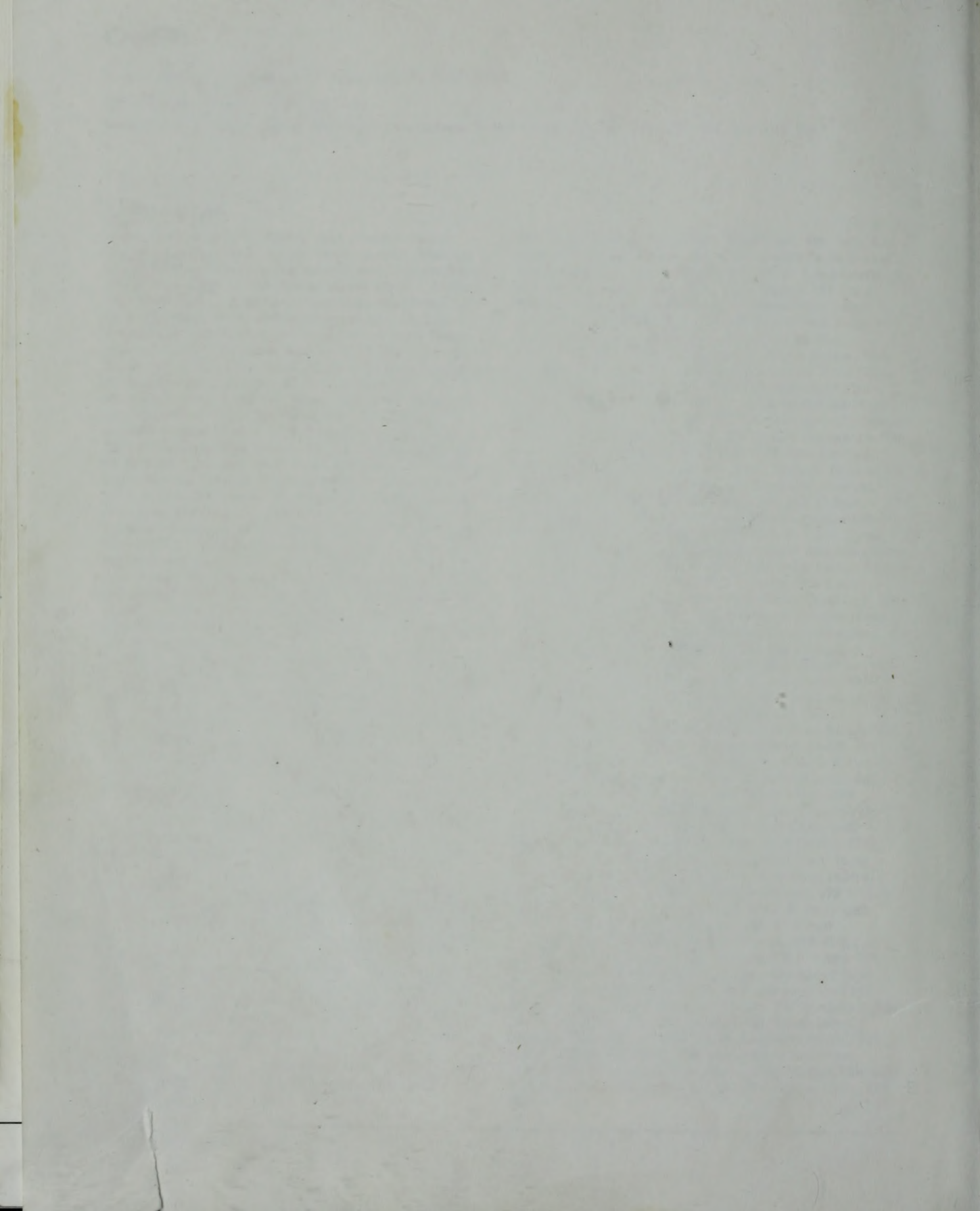
## PHOTOGRAPHS

xxiv courtesy, Martha C. Zuniga. xxxv courtesy, Midori Kitagawa De Leon. xxxv NASA. xxxvi courtesy, American Indian Science and Engineering Society. xxxvi Martine Kempf. xxxvii NASA. xxxviii Wide World. xxxviii David J. Maenza/The Image Bank. 6 John Lamb/Tony Stone Worldwide. 13 Susan Van Etten. 16 B. Gelburg/Sharpshooters. 19 Ken Straiton/The Stock Market. 20 Owen Franken/Stock Boston. 22 Malcolm Gilson/Tony Stone Worldwide. 23 Bill Gallery/Stock Boston. 29 © Susan Van Etten. 37 right, Ed Baverham Studio. 37 left, © Susan Van Etten. 42 © Bob Daemmrich. 45 Mauritius/Superstock. 46 Roy Morsch/The Stock Market. 48 Mark Segal/Tony Stone Worldwide. 52 Lew Merrim/Monkmeyer Press Photos. 71 Bob Daemmrich/Stock Boston. 72 Spencer Grant/Stock Boston. 88 The Granger Collection. 94 © 1980, David Em. 99 © Susan Van Etten. 107 Timothy Eagan/Woodfin Camp & Associates. 108 Chris Jones/The Stock Market. 109 Michael L. Abramson/Woodfin Camp & Associates. 114 Bachman/Instock. 117 courtesy, Minoru Yamasaki Associates. 117 © Susan Van Etten. 118 NOAA/NESOIS. 121 Cecile Brunswick/Peter Arnold, Inc. 126 Roger Dollarhide/Monkmeyer Press Photos. 138 C. Allan Morgan/Peter Arnold, Inc. 139 Mimi Forsyth/Monkmeyer Press Photos. 146 Frank Siteman/The Picture Cube. 156 Derek Van Etten, Jr. 157 © Susan Van Etten. 165 Bob Daemmrich/Stock Boston. 168 Werner Forman Archive. 172 © Bob Daemmrich. 174 © Susan Van Etten. 175 Steve Elmore/Tony Stone Worldwide. 177 Gabe Palmer/The Stock Market. 179 Kennedy/TexaStock. 183 Ted Horowitz/The Stock Market. 184 © Paul Conklin. 189 Jerry Howard/Stock Boston. 195 Al Satterwhite/The Image Bank. 196 Phil Degginger/National Optical Astronomy Observatories. 201 Ellis Herwig/Stock Boston. 203 Kevin Schafer/Peter Arnold, Inc. 204 © Paul Conklin. 207 Roger Dollarhide/Monkmeyer Press Photos. 212 Kitt Peak National Observatory. 223 Grant Heilman/Grant Heilman Photography. 225 Kindra Clineff. 240 Richard Surman/Tony Stone Worldwide. 250 Richard Pasley/Stock Boston. 256 Butch Powell/Stockphotos, Inc. 262 Villafuerte/TexaStock. 265 H/O Photographers, Inc. 266 Moore & Moore Publishing/Superstock. 271 Photri. 281 Scott Berner/The Picture Cube. 294 B. Christensen/Stock Boston. 300 George F. Riley/Stock Boston. 306 © Susan Van Etten. 307 Klaus Matwijow/Tony Stone Worldwide. 310 Leonard Harris/Stock Boston. 314 Steve Kaufman/Peter Arnold, Inc. 316 Peter Glass/Monkmeyer Press Photos. 327 Chris Lunski/Photo Researchers, Inc. 330 Joachim Messerschmidt/West Light. 332 Jim Anderson/Woodfin Camp & Associates. 336 NASA. 348 Michael D. Sullivan/TexaStock. 356 © Bob Daemmrich. 362 Tony Freeman/Photo Edit. 362 Richard Pasley/Stock Boston. 363 Grant Le Duc/Monkmeyer Press Photos. 367 Steve Elmore/Tony Stone Worldwide. 368 John Tesh/Instock. 369 David Joel/Tony Stone Worldwide. 380 Charles Gupton/Stock Boston. 386 Raymond G. Barnes/Tony Stone Worldwide. 390 C. Allan Morgan/Peter Arnold. 394 Kevin Vandivier/TexaStock. 397 Kindra Clineff. 401 Photri.

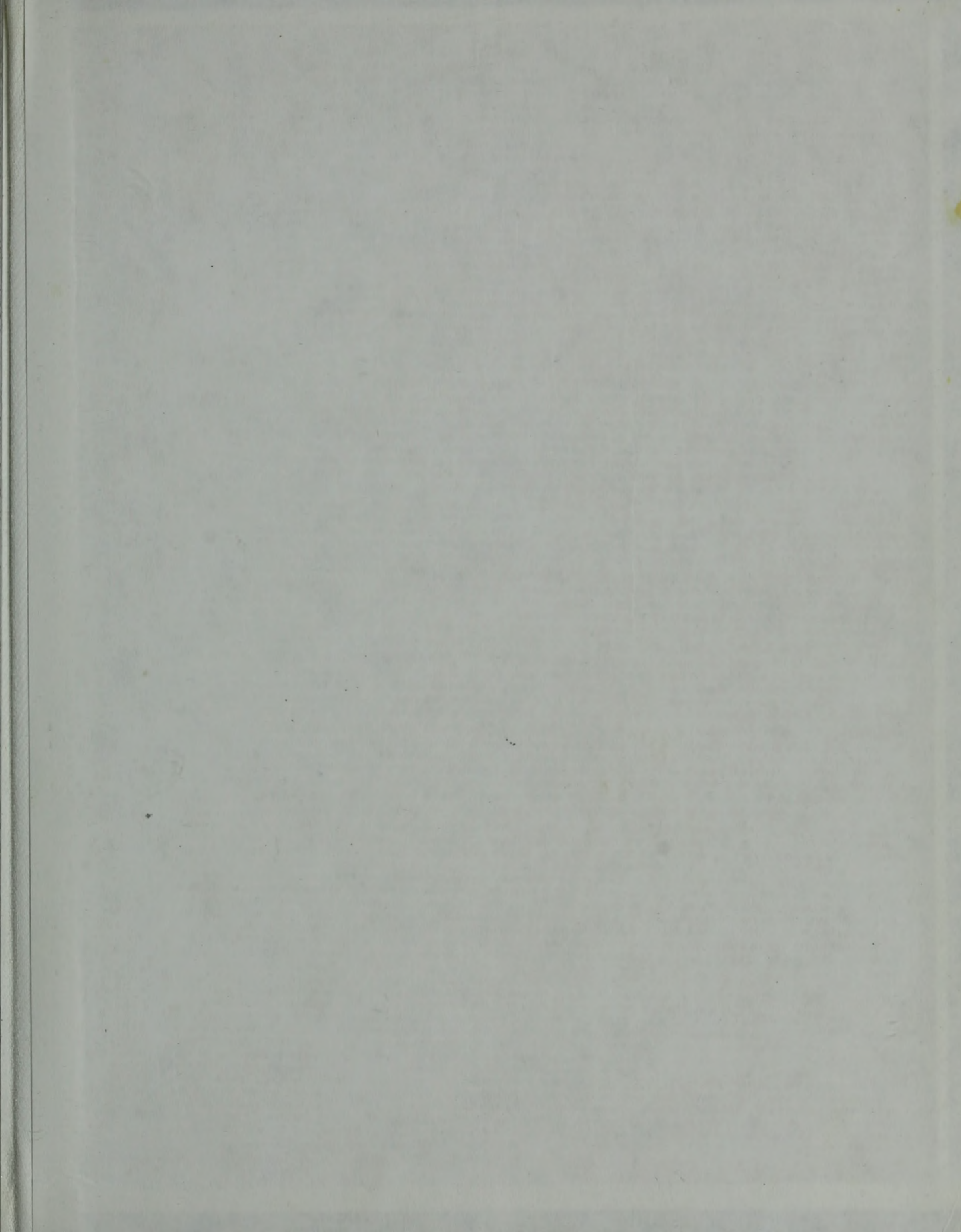
409 Right, Manfred Kage/Peter Arnold, Inc. 409 Left, Ray Pfortner/Peter Arnold, Inc. 418 Rick Smolan/Stock Boston. 424 Faerder/Tony Stone Worldwide. 430 © Bob Daemmrich. 437 Herbert Lanks/Monkmeyer Press Photos. 439 Greg Meadows/Stock Boston. 451 Masa Vemura/Tony Stone Worldwide. 457 Peter Menzel/Stock Boston. 464 Doug Armand/Tony Stone Worldwide. 472 Jim Chandler. 476 Kindra Clineff. 483 John Running/Stock Boston. 484 Kindra Clineff. 509 Bill Anderson/Monkmeyer Press Photo. 510 © Susan Van Etten. 511 Runk/Schoenberger/Grant Heilman Photography, Inc. 516 Evans & Sutherland/Photo Edit. 517 © Susan Van Etten. 522 Charles Gupton/Uniphoto. 528 Peter Menzel/Stock Boston. 529 © Bob Daemmrich. 536 Jim Cambon/Tony Stone Worldwide. 538 Alex Langley/DPL. 539 Hank Morgan/Rainbow. 541 Courtesy, Heirs of the Estate of Norman Rockwell, Norman Rockwell Museum, Sturbridge, MA. 542 Dr. E.R. Degginger. 543 © Bob Daemmrich. 549 Oxford Scientific Films/Animals, Animals. 550 Patty Murray/Animals, Animals. 551 © Orion Press/West Light. 564 George Hall/Woodfin Camp and Associates. 569 Bob Daemmrich/Stock Boston. 570 Brian Parker/Tom Stack and Associates. 575 © Paul Conklin. 576 Larry Lawfer/The Picture Cube. 579 A & R Photo/TexaStock. 581 Steve Hansen/Stock Boston. 586 © Ronald Sheridan/Tony Stone Worldwide. 587 Takeshi Takahara/Photo Researchers. 589 Bruce Curtis/Peter Arnold, Inc. 590 The Granger Collection. 596 Dr. Vic Bradbury/Photo Researchers, Inc. 601 Susan Van Etten. 602 Bob Daemmrich/Stock Boston. 603 Photri. 604 Peter Menzel/Stock Boston. 611 Mike Malyszko/Stock Boston. 615 Charles Feil/Uniphoto. 617 Bob Daemmrich/Stock Boston. 618 Ellis Herwig/Stock Boston. 621 © Bob Daemmrich. 623 P.A. Harrington/Peter Arnold, Inc. 627 Andrew Sacks/Tony Stone Worldwide. 634 © Robert Llewellyn. 638 Mitch Reardon/Tony Stone Worldwide. 647 Willie L. Hill/The Image Works. 648 Wide World. 659 Robert Percy/Animals, Animals. 661 Laura Dwight/Peter Arnold, Inc. 666 Larry Lawfer/The Picture Cube. 668 Martha Cooper/Peter Arnold, Inc. 669 Peter Gridley/FPG. 670 © Robert Llewellyn. 672 Barbara Filet/Tony Stone Worldwide. 673 Mimi Forsyth/Monkmeyer Press Photos. 675 © Bob Daemmrich. 677 Boroff/TexaStock. 679 © Bob Daemmrich. 682 Steve Elmore/Tom Stack & Associates. 689 Bob Daemmrich/Stock Boston. 690 © Robert Llewellyn. 692 Michael Stuckey/Comstock. 698 Bob Daemmrich/Stock Boston. 703 Camerique/E.P. Jones. 706 Eric Carle/Stock Boston. 708 Zao Sulle/The Image Bank. 713 courtesy, Agnes Stroud-Lee. 713 courtesy, Joe Fred Gonzalez. 716 Homer Smith/Art Matrix. 737 Comstock. 744 Leonard Lee Rue, III/Stock Boston. 748 Wouterloot-Gregoire/Valan Photos. 750 Homer Smith/Art Matrix. 753 Homer Smith/Art Matrix. 754 both, Homer Smith/Art Matrix. 755 Homer Smith/Art Matrix. 756 Morgan Howarth/Nawrocki Stock Photo. 774 J. Goell/The Picture Cube. 779 Photri.













2-07280